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Bistabilities in twisted nematics with a holding voltage

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Résumé. — On étudie la bistabilité entre un état en hélice à 180° et l'état planaire vertical topologiquement équivalent dans les nématiques purs soumis à une tension. On peut distinguer optiquement ces états bistables en mélangeant un colorant pléochroïque avec le cristal liquide. L'intervalle de tension ΔV pour lequel on observe la bistabilité s'accroît à mesure que décroissent le rapport k_1/k_3 et la valeur du paramètre d'anisotropie diélectrique. Avec des constantes élastiques plausibles et avec un paramètre d'anisotropie diélectrique $(\epsilon_{\parallel} - \epsilon_{\perp})/\epsilon_{\perp} = 0,05$, l'intervalle théorique de tension ΔV où la bistabilité existe est à peu près 7 % dans le cas idéal où l'alignement est à 180° et où le directeur est parallèle à la surface. Avec de telles conditions aux limites, la cellule serait sujette à des domaines en hélice, de sens inverse ainsi qu'à des domaines inclinés en sens inverse, dont l'élimination entraînerait une réduction importante de l'intervalle ΔV et une diminution de contraste. Bien que la bistabilité puisse être démontrée expérimentalement, il n'est pas probable qu'elle soit utilisée dans les affichages sans innovation majeure de matériau pour obtenir des valeurs de k_1/k_3 considérablement moins élevées.

Abstract. — We study the bistability between a 180° twisted state and its topologically equivalent planar vertical state in pure nematics in the presence of a holding voltage. These bistable states can be distinguished optically by incorporating a pleochroic dye guest into the liquid crystal host. The holding voltage range ΔV of the bistability is found to increase with decreasing k_1/k_3 and with decreasing dielectric anisotropy. With reasonable elastic constants and dielectric anisotropy parameter $(\epsilon_{\parallel} - \epsilon_{\perp})/\epsilon_{\perp} = 0.05$, the theoretical holding voltage range ΔV is about 7 % in the idealized case of perfect alignment and zero boundary tilt bias. With such boundary conditions, the cell would be subject to oppositely tilted and oppositely twisted domains, whose elimination would entail a drastic reduction of ΔV and some loss of contrast. While the bistability could probably be demonstrated experimentally, it is not likely to be practical in displays without major materials innovation to obtain significantly lower values of the elastic constant ratio k_1/k_3 .

1. **Introduction.** — A cholesteric twist cell that used bistabilities in the presence of a holding voltage was recently demonstrated by Berreman and Heffner [1]. Scheffer's investigation [2] of voltage-induced bistabilities in pure nematics concluded that the range of bistability is too narrow to be useful in 90° twist cells.

For 180° twist, there is a larger range of material parameters and voltage over which bistabilities exist in nematics. Such bistabilities are limiting cases of the cholesteric bistabilities studied by Berreman and Heffner when the natural chirality disappears. It seems important to investigate these bistabilities because a bistability between states of sufficient optical contrast could have application to displays. The purpose of this paper is to explore the range of material parameters and voltage over which these bistabilities exist, and to describe the bistable states.

Bistabilities that exist at zero voltage have been considered for displays previously. These include bistabilities between topologically equivalent states

in cholesterics used by Berreman and Heffner [3] and bistabilities between topologically distinct planar horizontal (H) and planar vertical (V) or 180° twisted (T) states in pure nematics, demonstrated by Boyd, Cheng, and Ngo [4, 5]. The V and T states are topologically equivalent to each other (but distinct from H). Ordinarily, only one of V or T is stable in the same cell.

The switching between topologically distinct states involves the motion of disclinations [6], and the energy barrier can be understood in terms of the energy required to detach disclinations [7, 8]. Because fast switching by disclination motion requires high fields, bistabilities between topologically equivalent states have a possible advantage.

Thurston showed that bistability exists at zero voltage between the topologically equivalent V and T states in pure nematics with appropriate elastic constants and boundary tilt bias [9]. However, with achievable elastic constants, the bistable range of boundary tilt bias angles is too narrow to be promising.

A bistability between a 180° twisted state (T) and its topologically equivalent untwisted planar « vertical » state (V) can be distinguished by incorporating a pleochroic dye guest into the liquid crystal host [10-11]. With parallel boundary conditions (zero boundary tilt bias), a sufficient condition for such a bistability is that the planar V state be stabilized by a voltage V below that of the Freedericksz transition for the 180° twisted T state. In principle, bistability between two different 180° twisted states can also be useful if in one of them the director is nearly vertical at the midplane and in the other is nearly horizontal.

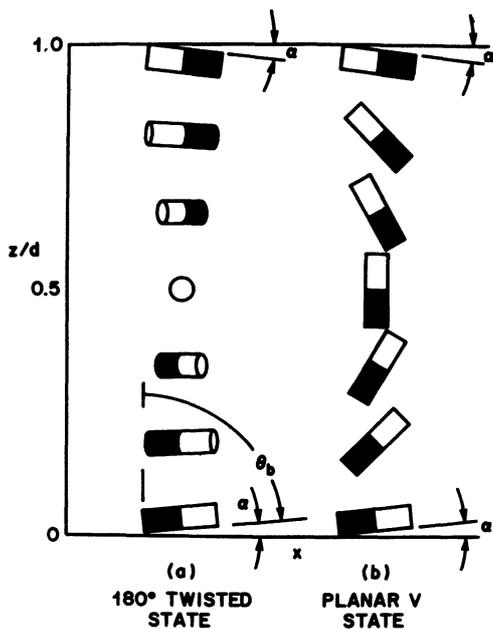


Fig. 1. — Pictorial representation of the cell boundary conditions and the 180° twisted and planar states. The director is represented by a right circular cylinder with one end shaded. The reader is looking along the y axis to see a projection of the cylinder in the (x, z) plane. At the upper and lower surfaces of the cell ($z/d = 0$ and $z/d = 1$), the director lies in the (x, z) plane with boundary tilt bias angle α , as shown. (a) Twisted state. To illustrate twist clearly, we show a special case in which the director is horizontal at the midplane, as would be observed only with $\alpha = 0$. (When $\alpha \neq 0$ the director tilt is typically greater at midplane than at the boundary, not smaller as pictured.) (b) Planar vertical (V) state. In the planar state, the director remains in the (x, z) plane, as illustrated by the fact that the cylinder is seen as an identical rectangle at each level of the cell.

Figure 1 illustrates the cell boundary conditions under consideration, and two states that are central to the present investigation : the planar vertical (V) state and the 180° twisted (T) state. The cell depth is d . The boundary tilt bias angle α is measured from the surface, and the opposite surfaces are assumed to be tilt biased in opposite senses, as indicated. With θ measured from the surface normal, $\alpha = 90^\circ - \theta_b$, where θ_b is the value of θ at the boundary.

Figure 2 indicates a coordinate system and angles that describe the director orientation at the midplane

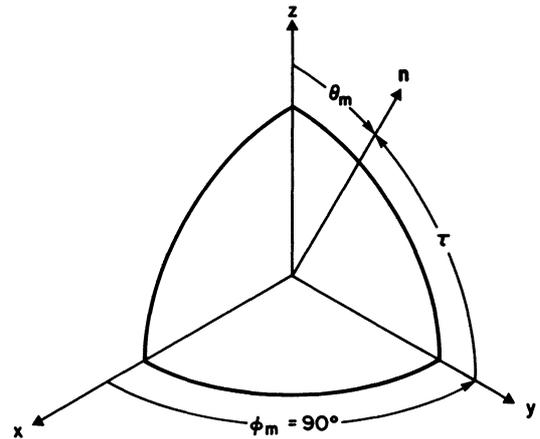


Fig. 2. — Coordinate system and angles to describe the director orientation at the midplane of the cell. For a 180° twisted state, $\phi_m = 90^\circ$. The z axis is normal to the major surfaces of the cell. At the surface, the director lies in the x, z plane. At the midplane, the angle of tilt, measured from the surface, is $\tau = 90^\circ - \theta_m$.

of the cell. Angles θ and ϕ are the conventional spherical polar coordinate angles of the director \mathbf{n} . The subscript m indicates their values at the midplane of the cell. The director tilt angle at the midplane, measured from the surface, is $\tau = 90^\circ - \theta_m$.

A 180° twisted equilibrium state may be characterized by the boundary tilt bias angle α and the angle of tilt τ at the midplane of the cell where the twist $\phi_m = 90^\circ$.

The relevant material parameters are the elastic constant ratios and the dielectric anisotropy at the frequency of the holding voltage. Following Scheffer [2], we use the dielectric anisotropy parameter

$$\gamma = (\varepsilon_{\parallel} - \varepsilon_{\perp})/\varepsilon_{\perp}$$

where ε_{\parallel} and ε_{\perp} are the principal dielectric permittivities parallel and perpendicular to the director. For any ratio of elastic constants and any boundary tilt bias angle α , there is a γ dependent holding voltage $V_p(\gamma)$, above which the planar vertical state is stable. Twist-vertical bistability occurs when a 180° twisted state is also stable in some voltage range above V_p .

Figure 3 illustrates the voltage *versus* τ curve for a nonbistable case, and figure 4 for a bistable case. The abscissa is V/V_F where V is the holding voltage and V_F is the Freedericksz voltage for the 180° twisted state with zero boundary tilt bias. The ordinate τ is the director tilt at the midplane of the cell. With zero tilt boundary conditions ($\alpha = 0$), the perfectly horizontal 180° twisted state with $\tau = 0$ ($\theta_m = 90^\circ$) is stable at zero voltage. As the voltage is increased, there is no electrical torque since the director is perpendicular to the field, and the configuration does not change until $V = V_F$. In the case of figure 3 (Merck E7, $\gamma = 2.78$) [12], there is only a single stable equilibrium state at each voltage, and no bistability. As the voltage is increased above V_F , τ increases rapidly with voltage until the planar vertical state ($\tau = 90^\circ$; $\theta_m = 0$) is reached at P_0 where $V = V_p$. For $V > V_p$, the planar vertical

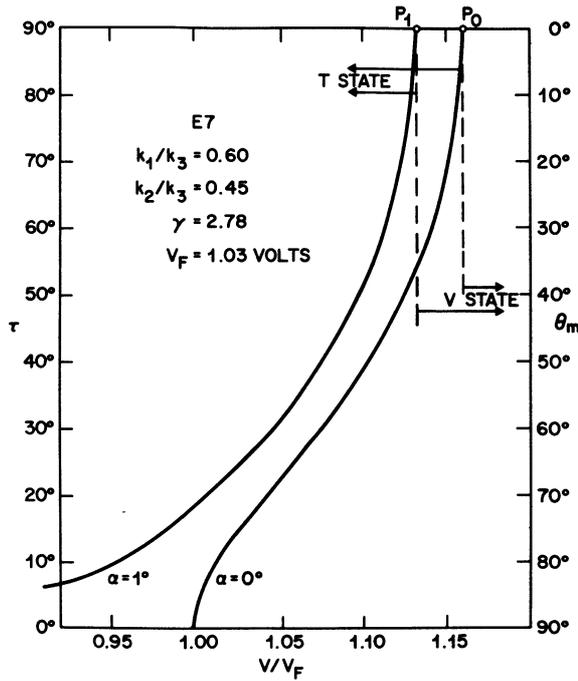


Fig. 3. — Reduced voltage *versus* director tilt at the midplane of the cell for 180° twisted (T) states, $\tau = 90^\circ - \theta_m$. $\alpha =$ boundary tilt bias angle measured from the surface. $\gamma = (\epsilon_{\parallel} - \epsilon_{\perp})/\epsilon_{\perp}$. The voltage at P_0 , and at P_1 is that required to stabilize the planar vertical state with $\alpha = 0$ and 1° , respectively. The curves show no bistability.

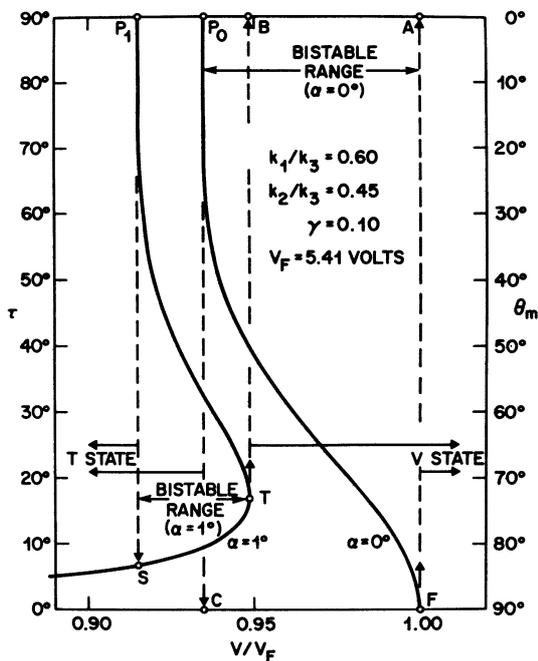


Fig. 4. — Same as figure 3 except $\gamma = 0.1$. Twist-vertical bistability exists for the holding voltage range over which the slope $d\tau/dV < 0$. At any voltage, a point of negative slope on the curve represents an unstable equilibrium 180° twisted barrier state between the V and T states that are stable at that voltage.

state is stable, and there is no stable twisted state. The same sequence of stable equilibrium states is traced in reverse when the voltage decreases.

Figure 3 also shows the curve for a boundary tilt bias angle α of 1° . Since the director is no longer perfectly horizontal, the Freedericksz threshold is no longer perfectly sharp : an electrical torque is exerted and the configuration begins to change as soon as any voltage is applied. However, the change is slight until a voltage comparable to V_F is reached, after which τ increases rapidly, reaching 90° (point P_1) at a voltage slightly less than that for the case of $\alpha = 0$.

Figure 4 applies to a liquid crystal material having the elastic constants of E7 but much lower dielectric anisotropy ($\gamma = 0.1$). Consider $\alpha = 0$ first. In this case, as the voltage is increased above V_F , the equilibrium state passes to the planar vertical state through a sequence of nonequilibrium twisted states, as suggested by the vertical dashed line FA at $V/V_F = 1$, since there is no stable equilibrium twisted state for $V > V_F$. As the voltage is decreased, the planar vertical state remains stable down to the point P_0 . As the voltage is further decreased, the configuration « falls » to the horizontally twisted state through a sequence of nonequilibrium states, as suggested by the dashed line $P_0 C$ at $V/V_F = 0.935$. The solid curve labelled $\alpha = 0^\circ$ connecting the twisted state at F to the planar vertical state at P_0 represents calculable but unobservable unstable equilibrium twisted states on the voltage range $V_P < V < V_F$. (The *stable* twisted state has $\tau = 0$.)

From the other curve in figure 4, it can be seen that in the case $\gamma = 0.1$, $\alpha = 1^\circ$, the twist-vertical bistability remains, but its holding voltage range is decreased from $V_F - V(P_0)$ to $V_T - V(P_1)$ where V_T is the maximum voltage that allows the 180° twisted state to exist. The portion of the solid curve above the point T represents calculable but unobservable unstable equilibrium states. The vertical dashed lines TB and $P_1 S$ suggest, respectively, a transition to the planar vertical state through a sequence of non-equilibrium states when the voltage exceeds V_T , and the inverse transition when the voltage is less than $V(P_1)$.

We can infer from figures 3 and 4 that bistability requires a relatively low dielectric anisotropy. This can be achieved either by operating the holding voltage at an appropriate frequency in a material having a frequency at which the anisotropy changes sign, or by mixing materials of positive and negative anisotropy [13]. Switching to the planar vertical state (V) is achieved by raising the voltage across a selected cell to a voltage above the bistable range. Switching to the T state takes place either if the holding voltage is temporarily removed or lowered to a voltage below the bistable range, or if a pulse is applied at a high frequency where the dielectric anisotropy is negative.

The electrically applied torque due to anisotropy is proportional to $V^2(\epsilon_{\parallel} - \epsilon_{\perp})$. Hence, it might be thought that a material of low dielectric anisotropy would switch slowly. This is true at a given voltage, but the permissible applied voltage scales with the

Freedericksz voltage V_F , which is proportional to $(\varepsilon_{\parallel} - \varepsilon_{\perp})^{-1/2}$. Hence, the switching speed at the operating voltages should be essentially independent of the dielectric anisotropy.

In order to observe the twist-vertical bistability under discussion, the planar horizontal state (H) [5] must be prevented from forming. With the low boundary tilt bias α needed to achieve the twist-vertical bistability, the nearly uniform H state is energetically favoured over the 180° twisted state (T) in a one-dimensional situation, and except at high voltages, it is also favoured over the V state. Since the H state is topologically distinct from T and V, the T or V state cannot be converted to the H state without the formation and movement of disclinations. We suggest preventing the spontaneous movement of disclinations and concomitant switching to the H state by surrounding each switchable region by an isolation region of perpendicular alignment ($\alpha = 90^\circ$).

Besides the possibility of H state formation, another potential disadvantage is the appearance of walls between left and right twisted regions. Since perfect alignment favours neither left nor right twist, both left and right twisted regions will form unless preventive measures are taken. With dyes to obtain contrast between T and V, the walls between left and right 180° twisted regions would appear as transmitting lines in the regions that are wanted to be opaque. Perhaps either a deviation from 180° or a small percentage of cholesteric additive can eliminate the oppositely twisted regions. The vertical state would then no longer be exactly planar, but would still have low absorption of light.

2. Equilibrium equations. — In this section we develop the equations needed for calculating the properties of twisted states in the presence of a holding voltage. The differential equations of equilibrium are the conditions for the free energy F to be stationary, subject to the constraint that the charge on the electrodes remains constant [14].

The free energy F of the system (whether in equilibrium or not) is the sum of an elastic part F_k and an electrical part F_e .

$$F = F_k + F_e \quad (2.1)$$

where F_k and F_e are the volume integrals of the respective densities f_k and f_e . F_k is the energy required to produce the configuration reversibly at constant temperature with zero charges and zero field. F_e , which equals $QV/2$ where Q is the charge and V the voltage, is the electrical energy input to produce the specified charges with the configuration fixed [14].

The elastic energy density of a nematic liquid crystal is [15]

$$f_k = \frac{1}{2} [k_1(\nabla \cdot \mathbf{n})^2 + k_2(\mathbf{n} \cdot \nabla \times \mathbf{n})^2 + k_3(\mathbf{n} \times \nabla \times \mathbf{n})^2]. \quad (2.2)$$

Under the assumption that the unit director field \mathbf{n}

depends only on the space coordinate z , and with the cartesian components of \mathbf{n} expressed in conventional spherical polar coordinates (θ, φ) as

$$n_x = \sin \theta \cos \varphi, \quad n_y = \sin \theta \sin \varphi, \quad n_z = \cos \theta, \quad (2.3)$$

we find

$$\begin{aligned} (\nabla \cdot \mathbf{n})^2 &= \theta_z^2 \sin^2 \theta, \quad (\mathbf{n} \cdot \nabla \times \mathbf{n})^2 = \varphi_z^2 \sin^4 \theta, \\ (\mathbf{n} \times \nabla \times \mathbf{n})^2 &= \cos^2 \theta (\theta_z^2 + \varphi_z^2 \sin^2 \theta) \end{aligned} \quad (2.4)$$

so that (2.2) reduces to

$$f_k = \frac{1}{2} k_3 [\theta_z^2 f(\theta) + \varphi_z^2 g(\theta)] \quad (2.5)$$

where

$$\begin{aligned} k_3 f(\theta) &= k_1 \sin^2 \theta + k_2 \cos^2 \theta, \\ k_3 g(\theta) &= \sin^2 \theta (k_2 \sin^2 \theta + k_3 \cos^2 \theta), \quad (2.6) \\ \theta_z &= d\theta/dz, \quad \varphi_z = d\varphi/dz. \end{aligned}$$

Here k_1 , k_2 , and k_3 are the elastic constants for splay, twist, and bend, respectively.

The electrostatic energy density is

$$f_e = \frac{1}{2} \mathbf{D} \cdot \mathbf{E} \quad (2.7)$$

where \mathbf{D} and \mathbf{E} are the usual field vectors of Maxwell's equations.

The relation between \mathbf{D} and \mathbf{E} may be taken as [16]

$$\mathbf{D} = \varepsilon_{\perp} \mathbf{E} + (\varepsilon_{\parallel} - \varepsilon_{\perp})(\mathbf{E} \cdot \mathbf{n}) \mathbf{n} \quad (2.8)$$

where ε_{\perp} and ε_{\parallel} are the principal dielectric permittivities perpendicular and parallel to the director \mathbf{n} .

Let us now consider an idealized situation in which the only nonvanishing component of the electric field is its z component, E_z . Then (2.8) and (2.3) yield

$$\begin{aligned} D_z &= \varepsilon_{\perp} E_z (1 + \gamma \cos^2 \theta), \\ \gamma &\equiv (\varepsilon_{\parallel} - \varepsilon_{\perp}) / \varepsilon_{\perp}. \end{aligned} \quad (2.9)$$

From the condition that $\nabla \cdot \mathbf{D} = 0$ in the absence of free charge, and the one-dimensional assumption that all quantities are independent of x and y , we conclude that D_z is a constant, independent of z . With the abbreviation

$$D = D_z, \quad (2.10)$$

(2.7) becomes

$$f_e = \frac{D^2}{2 \varepsilon_{\perp} (1 + \gamma \cos^2 \theta)}. \quad (2.11)$$

In the one-dimensional situation under consideration, constant charge means constant D . Hence, the equilibrium equations are the conditions for F to be stationary with D constant. With

$$q \equiv q[\theta_z(z), \varphi_z(z), \theta(z), \varphi(z)] \equiv f_k + f_e, \quad (2.12)$$

the Euler-Lagrange equations for this stationary value problem are [17]

$$\begin{aligned} -\frac{\partial q}{\partial \theta} + \frac{d}{dz} \left[\frac{\partial q}{\partial \theta_z} \right] &= 0, \\ -\frac{\partial q}{\partial \varphi} + \frac{d}{dz} \left[\frac{\partial q}{\partial \varphi_z} \right] &= 0. \end{aligned} \quad (2.13)$$

Since q does not depend on φ explicitly ($\partial q / \partial \varphi = 0$), the second equation of (2.13) can be integrated at once to yield

$$\varphi_z q(\theta) = m = \text{constant}. \quad (2.14)$$

The other equation gives

$$k_3 [\theta_z^2 f'(\theta) - \varphi_z^2 g'(\theta) + 2 \theta_{zz} f(\theta)] + 2 T_\varphi = 0, \quad (2.15)$$

where

$$T_\varphi = - [\partial f_\varepsilon / \partial \theta]_D. \quad (2.16)$$

The subscript D here indicates that D is constant in the differentiation of f_ε from (2.11) [18].

By virtue of (2.14) and (2.16), equation (2.15) may be expressed as

$$\frac{2}{\theta_z} \frac{d}{dz} [f_k - f_\varepsilon] = 0 \quad (2.17)$$

whence

$$f_k - f_\varepsilon = p_0 = \text{constant}. \quad (2.18)$$

The physical significance of (2.14) is that the elastic torque per unit area about the z axis, exerted across a plane perpendicular to that axis, is independent of the height z of the plane [19]. The physical significance of (2.18) is that the electric Gibbs function $G = F_k - F_\varepsilon$ has constant density, independent of z . Thus, an equilibrium configuration is characterized by constant torque (2.14) and constant density $f_k - f_\varepsilon$ of the electric Gibbs function (2.18).

From (2.5) and (2.18),

$$\theta_z^2 = \frac{1}{f(\theta)} \left[\frac{2}{k_3} (p_0 + f_\varepsilon) - \frac{m^2}{g(\theta)} \right]. \quad (2.19)$$

States that are truly twisted have $m \neq 0$. They can be characterized by a real value of θ , $\theta = \theta_m$, at which $\theta_z = 0$. In terms of θ_m ,

$$p_0 = \frac{k_3 m^2}{2 g(\theta_m)} - f_\varepsilon(\theta_m), \quad m \neq 0 \quad [20]. \quad (2.20)$$

It is expedient to define a function $h(\theta)$ by the relation

$$\frac{1}{h(\theta)} \equiv \frac{1}{g(\theta)} - \frac{2 f_\varepsilon(\theta)}{m^2 k_3} \quad (2.21)$$

so that

$$\theta_z^2 = \frac{m^2}{f(\theta)} \left[\frac{1}{h(\theta_m)} - \frac{1}{h(\theta)} \right]. \quad (2.22)$$

With $\theta = \theta_b$ at the upper and lower surfaces of the cell, the extremal value $\theta = \theta_m$ occurs at the midplane. The cell depth d , total twist Φ and voltage V must satisfy certain integrals over the cell thickness. When transformed to integrals over θ by the substitution $dz = d\theta/\theta_z$, these are

$$\frac{d}{2} = \int_{\theta_m}^{\theta_b} \frac{d\theta}{\theta_z}, \quad (2.23)$$

$$\frac{\Phi}{2} = \int_{\theta_m}^{\theta_b} \frac{\varphi_z d\theta}{\theta_z} = \int_{\theta_m}^{\theta_b} \frac{m d\theta}{\theta_z g(\theta)}, \quad (2.24)$$

$$\frac{V}{2} = \frac{2}{D} \int_{\theta_m}^{\theta_b} \frac{f_\varepsilon}{\theta_z} d\theta. \quad (2.25)$$

In these equations, θ_z is to be obtained from (2.22), and θ_b is the value of θ at the upper and lower surfaces of the cell.

Following a pattern set by Scheffer [2], we introduce a new variable of integration β , defined by

$$q \sin \beta = \cos \theta, \quad (2.26)$$

$$q \equiv \cos \theta_m. \quad (2.27)$$

We see that $\beta(\theta_m) = \pi/2$ and

$$\beta_b \equiv \beta(\theta_b) = \sin^{-1} (\cos \theta_b / q).$$

Following Scheffer [2], we denote elastic constant ratios by

$$\kappa = k_3/k_1, \quad \xi = k_3/k_2, \quad (2.28)$$

define a function of β with parameters q and y by

$$F(y; \beta) \equiv 1 + yq^2 \sin^2 \beta, \quad (2.29)$$

and define a parameter δ by

$$\delta^2 \equiv m^2 \varepsilon_1 k_3 \xi / D^2 \gamma. \quad (2.30)$$

Then the transformation of (2.23)-(2.25) to the variable of integration β yields

$$\frac{Dd}{(\varepsilon_1 k_3)^{1/2}} = 2 I_1 / (\kappa \gamma)^{1/2}, \quad (2.31)$$

$$\Phi = 2 \delta (\xi / \kappa)^{1/2} I_2, \quad (2.32)$$

$$V = 2(k_3 / \kappa \gamma \varepsilon_1)^{1/2} I_3, \quad (2.33)$$

where

$$I_n = \int_{\beta_b}^{\pi/2} i_n d\beta, \quad n = 1, 2, 3, \quad (2.34)$$

$$i_1 = [F(\kappa - 1; \beta) / S(\beta)]^{1/2}, \quad (2.35)$$

$$\begin{aligned} S(\beta) = & \frac{\delta^2 / (1 - q^2)}{F(\xi - 1; \pi/2)} - \frac{\delta^2 (\xi - 1)}{F(\xi - 1; \beta) F(\xi - 1; \pi/2)} + \\ & + \frac{F(-1; \beta)}{F(\gamma; \beta) F(\gamma; \pi/2)}, \end{aligned} \quad (2.36)$$

$$i_2 = \frac{i_1}{F(\xi - 1; \beta) F(-1; \beta)}, \quad (2.37)$$

$$i_3 = i_1/F(\gamma; \beta). \quad (2.38)$$

A convenient reference voltage is the Freedericksz transition voltage V_F for the Φ -twisted state with $\theta_b = 90^\circ$. V_F is given by [21]

$$V_F = \left[\frac{k_3}{\xi \gamma \varepsilon_1} (\pi^2 \xi / \kappa + \Phi^2(\xi - 2)) \right]^{1/2}. \quad (2.39)$$

From (2.33) and (2.39),

$$V/V_F = 2(\xi/\kappa)^{1/2} [\pi^2 \xi / \kappa + \Phi^2(\xi - 2)]^{-1/2} I_3. \quad (2.40)$$

Equations (2.32) and (2.40) are equivalent to equations given by Scheffer [2]. As explained by Scheffer, points on the equilibrium θ_m versus voltage curve can be calculated as follows: for fixed $\Phi \neq 0$ and $\theta_m \neq 0$, one first seeks the value of δ that satisfies (2.32), after which these values of θ_m and δ are substituted into (2.40) to obtain the reduced voltage V/V_F .

3. Voltage required to stabilize the planar vertical state. — In the absence of a holding voltage, the planar vertical state (V) is stable for $\alpha > \alpha_c$ and unstable for $\alpha < \alpha_c$, where α is the director tilt at the boundary and α_c is its critical value (measured from the surface) [9, 22-24]. In a material of positive dielectric anisotropy γ , the V state can be stabilized on $\alpha < \alpha_c$ by the application of a holding voltage to provide a vertical electric field. On $\alpha > \alpha_c$ in a material of negative γ , the application of a vertical field can destabilize the V state.

With $\alpha < \alpha_c$ and positive γ , how can we find the voltage $V_p(\gamma)$ required to stabilize the planar vertical state? We shall prove that V_p is just the voltage at $\tau = 90^\circ$ on the voltage versus τ curve, i.e., the equilibrium voltage in the limit as the 180° twisted state, whether stable or not, becomes planar. The proof is based on the idea that states in equilibrium at constant voltage have stationary values of the Gibbs function, and that the stationary value is a minimum for stable states [14]. Let $\frac{1}{2} G(\tau, V)$ denote the Gibbs function at voltage $V/2$ of the 90° twisted stable equilibrium state in a cell of thickness $d/2$ with boundary tilt bias angles α on one surface and τ on the other. Two such states

placed back-to-back with tilt τ in the middle form a 180° twisted state in a cell of thickness d at voltage V with boundary tilt bias α , but in which θ_z may jump at the midplane of the cell. (On the unit sphere, the path of the 180° twisted state formed in this way has a corner at its midpoint unless the voltage is adjusted to make $\theta_z = 0$ at the midplane of the cell where the tilt is τ .) Because the 90° twisted states in each half of the cell are stable equilibrium states for the prescribed (τ, V) , the « broken » 180° twisted state formed by joining them has the lowest possible Gibbs function at voltage V of any configurations that satisfy the same boundary conditions and have midplane tilt τ . Any variation of the configuration while preserving (τ, V) can only increase the Gibbs function. Thus, its definition makes $G(\tau, V)$ the lowest possible Gibbs function for the prescribed (τ, V) . Now points at which $\partial G(\tau, V)/\partial \tau = 0$ are equilibrium 180° twisted states because states in equilibrium at constant voltage have stationary values of the Gibbs function [14]. These are the points that lie on the equilibrium voltage versus τ curves such as are exhibited in figures 3 and 4, as well as points on the line $\tau = 90^\circ$ (planar vertical V state), and, if $\alpha = 0$, points on the line $\tau = 0$. These loci of zero $\partial G/\partial \tau$ partition the (τ, V) space into regions of positive or negative $\partial G/\partial \tau$. The sign of $\partial G/\partial \tau$ in any of these regions can be inferred as soon as the stability or instability is known for states on any part of its boundary. For example, for $\alpha < \alpha_c$, the stable state at low voltage is a low- τ , 180° twisted state. Hence, $\partial G/\partial \tau > 0$ to the left of the equilibrium curve, while $\partial G/\partial \tau < 0$ to the right of the curve. This establishes the stability limit of the V state as the intersection point of the curve with the line $\tau = 90^\circ$, as was to be proved, and shows also that points of positive or negative slope on the voltage versus τ curve represent stable or unstable states, respectively.

Let us now consider the calculation of the desired intersection point. One cannot use (2.32) and (2.40) directly to calculate V/V_F at $\tau = 90^\circ$ where $\theta_m = 0$ because both δ and $(1 - q^2)$ are zero, and $S(\beta)$ becomes an indeterminate form. It is found that if δ behaves like

$$\delta = (\sin \theta_m)/p \quad (3.1)$$

in the limit as $\theta_m \rightarrow 0$, then the limiting value of Φ is π for all p , and the condition $d\Phi/d\theta_m = 0$ can be used to determine the parameter p . We find

$$\lim_{\theta_m \rightarrow 0} \frac{d\Phi}{d\theta_m} = \frac{2\xi}{\kappa^{1/2}} \left[-\nu(\alpha) \tan \alpha - \int_{\alpha}^{\pi/2} B(\beta) \nu(\beta) \sin^2 \beta \, d\beta \right] \quad (3.2)$$

where

$$B(\beta) = \frac{\kappa - 1}{F(\kappa - 1; \beta)} - \frac{2(\xi - 1)}{F(\xi - 1; \beta)} + \frac{p}{\frac{[F(\gamma; \beta)]^2}{\xi} + \frac{pF(\gamma; \beta) \cos^2 \beta}{(1 + \gamma)}} \Bigg|_{q=1} \quad (3.3)$$

$$\nu(\beta) = \frac{R(\beta)}{1 + (\xi - 1) \sin^2 \beta}, \quad (3.4)$$

$$R(\beta) = \{ F(\kappa - 1; \beta) / [1 + (p\xi \cos^2 \beta) / (1 + \gamma) F(\gamma; \beta)] \}^{1/2} \Big|_{q=1}. \quad (3.5)$$

Once p is found as the value that makes (3.2) vanish, the limiting value of V/V_F can be found from

$$\lim_{\theta_m \rightarrow 0} \frac{V}{V_F} = \frac{2(\xi/\kappa)^{1/2}}{\pi(\xi/\kappa + \xi - 2)^{1/2}} \int_{\alpha}^{\pi/2} \frac{(p\xi)^{1/2} R(\beta)}{1 + \gamma \sin^2 \beta} d\beta, \quad (3.6)$$

which is the limit of (2.40) subject to (3.1).

Figure 5 shows the result of the calculation of V_P/V_F as a function of the dielectric anisotropy γ for boundary tilt $\alpha = 0$ and a material having the elastic constants of E7 [12]. We see that for these elastic constants, twist-vertical bistability ($V_P < V_F$) is achieved only for γ less than about 0.6. The voltage range of the bistability increases with decreasing γ , approaching about 8% in the limit as $\gamma \rightarrow 0$.

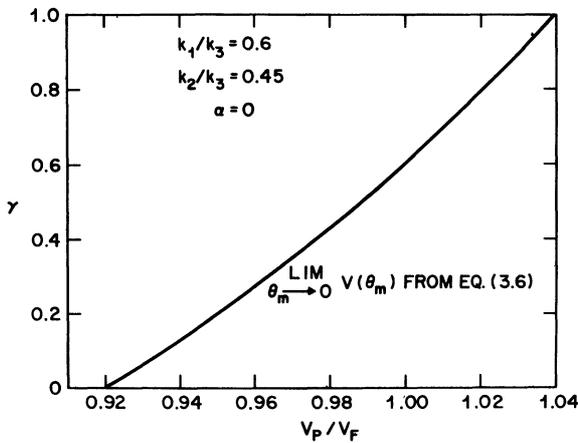


Fig. 5. — γ dependence of V_P/V_F with boundary tilt bias $\alpha = 0$, and elastic constants of E7. Twist-vertical bistability exists for $V_P/V_F < 1$.

In the case of 90° twist, Scheffer [2] found that a small tilt bias of about 0.2° destroyed the bistability completely. Figure 6 shows the effect of tilt bias in narrowing the voltage range of the bistability in the present case of 180° twist, with the elastic constants of E7 and $\gamma = 0.05$. In contrast to the 90° case, the 180° twist-vertical bistability persists out to large tilt bias angles α of more than 10°, although the voltage range is narrowed drastically as α departs from zero.

When $\alpha \neq 0$, the voltage range of the bistability no longer extends from V_P to V_F , but from V_P to $V_T < V_F$, as illustrated in figure 4, V_T being the maximum voltage that is attained on the voltage *versus* τ curve (point T in Fig. 4). The α dependence of V_T/V_F and V_P/V_F is illustrated in figure 7, along with the value of midband tilt τ of the point T.

Figure 8 shows the voltage *versus* τ curves for $\alpha = 0^\circ$ and $\alpha = 1^\circ$, for a material having the elastic constants of E7, but $\gamma = 0.1$, and $\gamma = 0.05$.

4. Range of material parameters for bistability. — To investigate bistability in pure nematics, the relevant

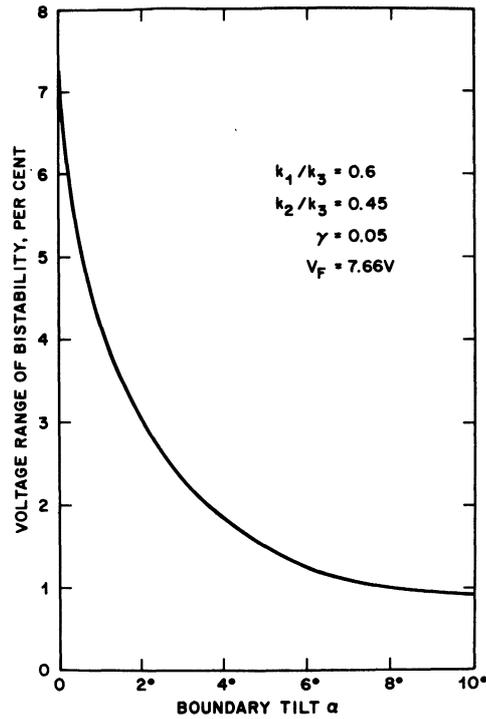


Fig. 6. — Effect of boundary tilt bias α on $(V_T - V_P)/V_T$, with $\gamma = 0.05$ and elastic constants of E7.

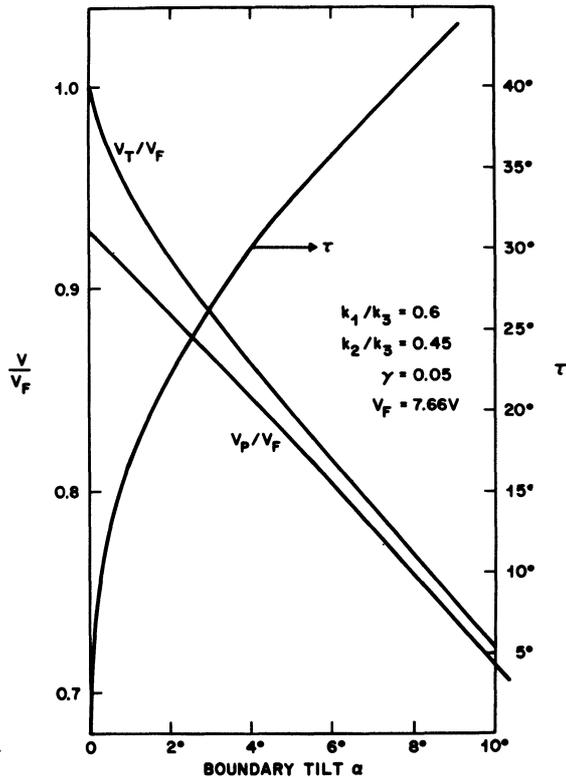


Fig. 7. — Effect of boundary tilt bias α on V_P/V_F , V_T/V_F , and τ at $V_T/V_F, \tau = 90^\circ - \theta_m$.

material parameters are the elastic constant ratios ($k_1/k_3, k_2/k_3$) and the dielectric anisotropy

$$\gamma = (\epsilon_{||} - \epsilon_{\perp})/\epsilon_{\perp}.$$

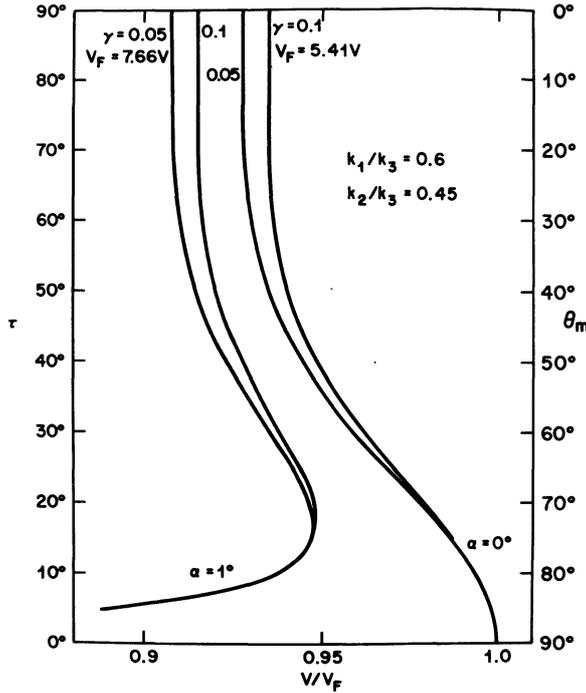


Fig. 8. — V/V_F versus τ for 180° twisted states with $\alpha = 0^\circ$ and $\alpha = 1^\circ$, for a material having the elastic constants of Merck E7, but $\gamma = 0.1$ and $\gamma = 0.05$. Midplane tilt $\tau = 90^\circ - \theta_m$.

In this section we find, with zero tilt boundary conditions, the range of these parameters for which bistability exists and for which the holding voltage range is a given percentage of V_F . Zero boundary tilt bias represents an idealized « best » case. The introduction of tilt bias (to avoid oppositely tilted domains) will narrow the range of bistability, as illustrated by figures 6 and 7.

By analysing (2.23)-(2.25) with θ limited to small deviations near $\pi/2$ and $\theta_b = \pi/2$ ($\alpha = 0$), one can derive an equation given by Scheffer [2],

$$V/V_F = 1 + \frac{1}{4} \tau^2 \left[\gamma + \frac{\pi^2 \xi + \Phi^2(\xi - 1 - \xi^2)}{\pi^2 \xi/\kappa + \Phi^2(\xi - 2)} \right]. \quad (4.1)$$

As before, V_F is the Freedericksz voltage of the Φ -twisted state, given by (2.39), τ is the director tilt at the midplane of the cell, measured from the surface, $\gamma = (\varepsilon_{\parallel} - \varepsilon_{\perp})/\varepsilon_{\perp}$, $\xi = k_3/k_2$, and $\kappa = k_3/k_1$. Since the voltage versus τ curve must extend to stable states with $V > V_F$, Scheffer pointed out that the existence of small angle solutions with $V < V_F$ is sufficient to indicate bistability. Hence, sufficient conditions for bistability with zero boundary tilt bias are (1) that V_F be real and positive, and (2) that the quantity in square brackets in (4.1) be negative.

For 180° twist, we set $\Phi = \pi$. From (2.39), the condition for positive Freedericksz threshold voltage V_F for the 180° twisted state then becomes (with $\varepsilon_{\parallel} > \varepsilon_{\perp}$),

$$k_2/k_3 < \frac{1}{2}(1 + k_1/k_3). \quad (4.2)$$

The condition for small-angle solutions with $V < V_F$ is, from (4.1),

$$\gamma(\xi/\kappa + \xi - 2) - (\xi - 1)^2 < 0. \quad (4.3)$$

If the elastic constants violate (4.2), the 180° twisted equilibrium state with $\theta \equiv \pi/2$ is unstable. If both (4.2) and (4.3) are satisfied, there is a voltage range below V_F for which there are bistable states, one of which is the 180° twisted state with $\theta \equiv \pi/2$.

In the $(k_1/k_3, k_2/k_3)$ -plane, the equality associated with (4.2) is the straight line of slope 1/2 and (k_2/k_3) -intercept $1/2$: $k_2/k_3 = (1 + k_1/k_3)/2$. The bistable region of elastic constant ratios is the region below this straight line and to the left of the equality associated with (4.3).

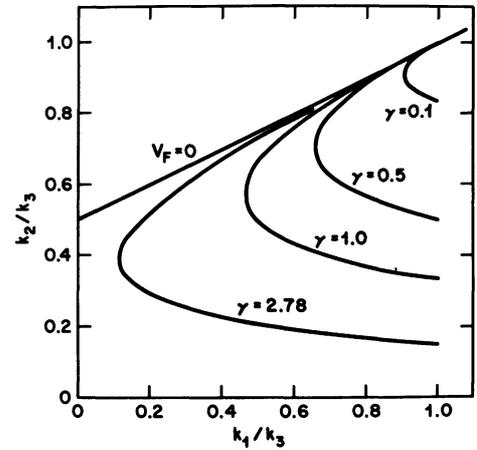


Fig. 9. — Region of elastic constants for nematic bistability with boundary tilt bias $\alpha = 0$ and various values of dielectric anisotropy $\gamma = (\varepsilon_{\parallel} - \varepsilon_{\perp})/\varepsilon_{\perp}$. Bistability of 180° twisted states (one of which is sometimes the untwisted planar « V » state) occurs under a bias voltage for elastic constants below the straight line and to the left of the γ dependent loop. This is Scheffer's sufficient condition (4.3).

Figure 9 shows the γ -independent straight line associated with (4.2) and the graph of (4.3) for several values of γ . For any γ , the curve makes a similar loop in the $(k_1/k_3, k_2/k_3)$ -plane, passing through (1,1) with slope 1/2, extending downward and to the left with a vertical tangent at $k_1/k_3 = 2[(1 + 2\gamma)^{1/2} - 1]/\gamma - 1$, $k_2/k_3 = (1 + 2\gamma)^{-1/2}$, and then turning back to the right, crossing the line $k_1/k_3 = 1$ at $k_2/k_3 = 1/(1 + 2\gamma)$ with slope $-1/(4\gamma + 2)$. The bistable region lies to the left of the boundary loop [25]. It is apparent that a high γ greatly restricts the bistable region according to Scheffer's sufficient condition, and that the region is enlarged by lowering γ .

Figure 9 suggests that any nematic material having $k_2/k_3 < (1 + k_1/k_3)/2$ and $k_1/k_3 < 1$ would lie in the bistable region if only γ could be made sufficiently small. For example, for the elastic constants reported for E7, ($k_1/k_3 = 0.6$, $k_2/k_3 = 0.451$) [12], (4.3) is satisfied for $\gamma < 0.956$. Now if the dielectric relaxation of the material is such that the dielectric anisotropy

changes from positive to negative as the frequency is raised, there will certainly be some frequency range in which (4.3) is satisfied.

Scheffer's condition (4.3) is only a sufficient condition for bistability. It does not tell whether bistability may exist under even less stringent conditions, nor does it tell whether the second stable state is another 180° twisted state or the planar vertical state. For better contrast, we would prefer the second state to be the planar vertical state.

A sufficient condition for twist-vertical bistability with $\alpha = 0$ is simply that the holding voltage V_P needed to stabilize the planar vertical state be less than V_F .

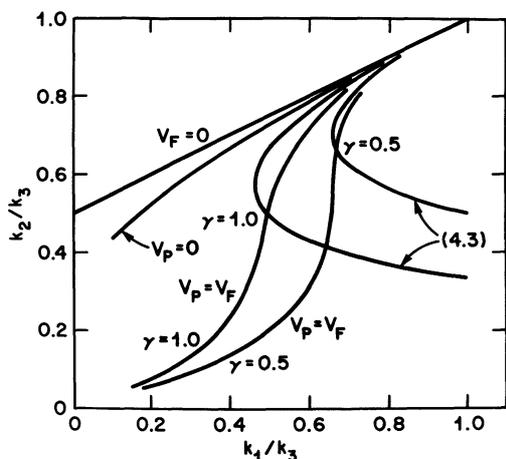


Fig. 10. — Loci of elastic constants for $V_P = 0$ and for $V_P = V_F$, and comparison with Scheffer's sufficient condition (4.3) for boundary tilt bias $\alpha = 0$, dielectric anisotropy $\gamma = 1.0$, and $\gamma = 0.5$. The « V » state is stable at zero voltage for elastic constants above the γ independent line $V_P = 0$. Twist-vertical bistability ($V_P < V_F$) exists to the left of the lines labelled $V_P = V_F$.

Figure 10 compares Scheffer's sufficient condition (4.3) for $\gamma = 1.0$ and $\gamma = 0.5$ with the loci for $V_P = V_F$. Twist-vertical bistability ($V_P < V_F$) exists to the left of the lines labelled $V_P = V_F$. Also shown is the γ independent line $V_P = 0$. This is the same as the locus of elastic constants for the critical angle α_c to be zero. If the elastic constants lie above this line and below the line labelled $V_F = 0$, then both the horizontally twisted state and the planar vertical state with $\alpha = 0$ are stable over the entire voltage range from 0 to V_F where the stability of the twisted state is lost.

According to the curves in figure 10, there are regions of twist-vertical bistability ($V_P < V_F$) that do not satisfy Scheffer's sufficient condition. This means that the voltage *versus* τ curve eventually swings leftward to $V/V_F < 1$, even though its initial trend, given by (4.1), is to the right. Curve 1 in figure 11 illustrates this behaviour for the case $\gamma = 1.0$, $k_1/k_3 = 0.5$, $k_2/k_3 = 0.6$. For these parameters, $d(V/V_F)/d(\tau^2)$ from (4.1) is $+1/36$, yet the voltage

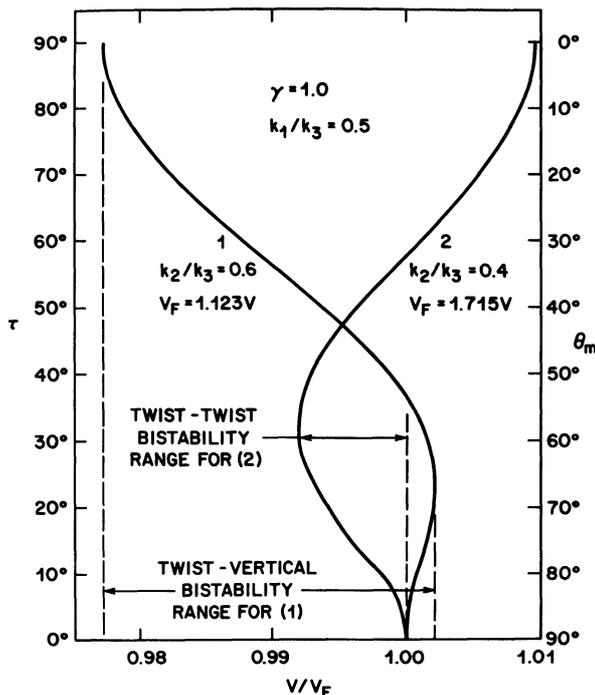


Fig. 11. — V/V_F versus τ curves ($\tau = 90^\circ - \theta_m$) illustrating : (1) twist-vertical bistability for a case that satisfies $V_P < V_F$ but not (4.3), and (2) twist-twist bistability for a case that satisfies (4.3) but has $V_P > V_F$.

versus τ curve reveals twist-vertical bistability with $V_P/V_F = 0.9772$.

When the elastic constants lie to the right of the curves labelled $V_P = V_F$ in figure 10 and below the loop (4.3), there is a bistability between two different 180° twisted states. This is illustrated by the other curve in figure 11, which is for $\gamma = 1.0$, $k_1/k_3 = 0.5$, $k_2/k_3 = 0.4$. Here $d(V/V_F)/d(\tau^2)$ from (4.1) is $-1/14$ so that the initial trend of the voltage *versus* τ curve is to the left, but the curvature reverses to enable the curve to reach $V_P/V_F = 1.0096$ at $\tau = 90^\circ$.

Of greater importance than the mere existence of bistability is the range of holding voltage over which bistability is obtained. One would like V_P/V_F to be as low as possible in order to increase the fractional voltage width $(V_F - V_P)/V_F$ of the bistability. We have already learned from figure 5 that for fixed elastic constants, the voltage range is widened by lowering the dielectric anisotropy γ . Figure 12 shows, for $\gamma = 0.05$, the locus of elastic constants for various constant values of V_P/V_F . With this value of γ , the elastic constants reported for E7 fall on the curve for $V_P = 0.9274 V_F$. These curves make apparent the direction in which the elastic constants should be shifted in order to widen the voltage range of the bistability.

Figure 12 indicates the desirability of lowering k_1/k_3 . The systematic experiments of Leenhouts, Dekker, and de Jeu [26] and of de Jeu and Claassen [27] show that k_3/k_1 varies directly with the length to width ratio for rigid molecules, but decreases with increasing

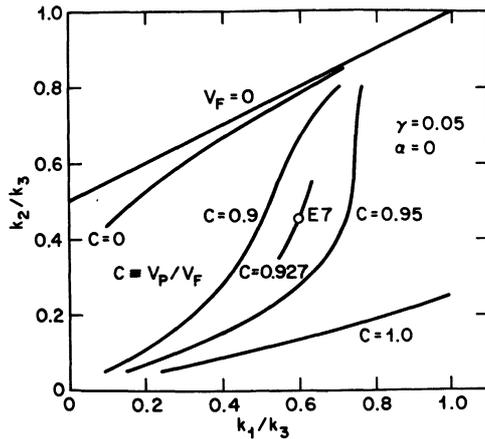


Fig. 12. — Lines of constant $C = V_p/V_F$ in elastic constant space for $\gamma = 0.05$ and $\alpha = 0$.

chain length when a flexible alkyl or alkoxy chain is added to the aromatic core of a molecule [28].

5. The barrier states. — The idea of a barrier state comes from imagining the Gibbs energy at constant voltage plotted against a configuration coordinate such as τ . With $G(\tau, V)$ defined to be the lowest Gibbs function for each (τ, V) , as in section 3, the states of stable equilibrium are at the minima of such a plot. When there is more than one minimum, the minima are obviously separated by maxima, which represent unstable equilibrium barrier states.

With boundary tilt bias $\alpha = 0$ and $\Phi = 180^\circ$, there are for every voltage, at least four topologically equivalent equilibrium states : two oppositely tilted planar vertical states (V) and the horizontal left and right 180° twisted states (HT). For voltage on the interval $0 < V < V_p$, the V states are unstable states representing an energy barrier between left and right twisted states. On $V > V_F$, the HT states are unstable barrier states. For V between V_p and V_F , there is a continuum of equilibrium twisted states (T) with midplane tilt τ ranging from 0 at V_F to 90° at V_p , as in figures 3, 4, 8 and 11 [29].

We proved in section 3 that points on the equilibrium voltage *versus* τ curve represent stable equilibrium states where the curve has positive slope, and unstable equilibrium states where the slope is negative. States associated with negative slope are barrier states between states of higher and lower τ that are in stable equilibrium at the same voltage.

For example, consider the case $\alpha = 0$, $\gamma = 0.05$, and elastic constants of E7, for which the voltage *versus* τ curve is plotted in figure 8 (2nd curve from right). Modified Gibbs energies for these states are shown plotted against V/V_F in figure 13. The quantity plotted is

$$W = (2Gd + \varepsilon_\perp V^2)/\pi^2 k_2 \quad (5.1)$$

where $G \equiv p_0 d$ is the Gibbs function per unit area,

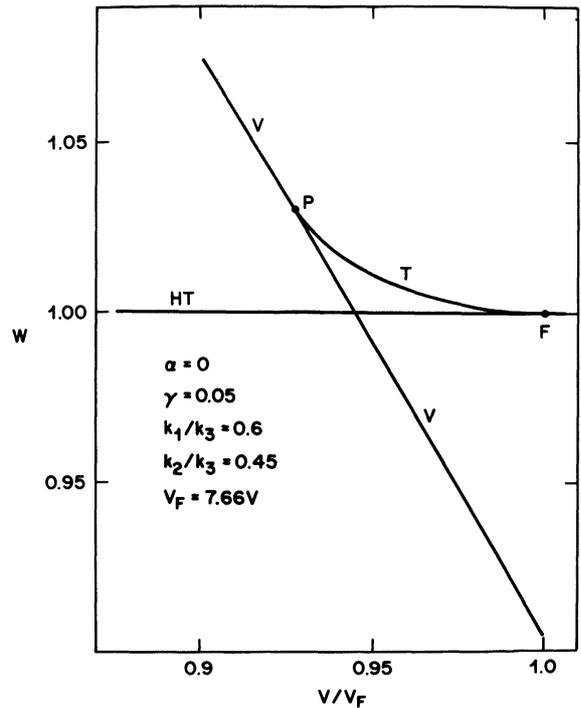


Fig. 13. — Modified Gibbs function, equation (5.1), of the V, T, and HT states for boundary tilt bias $\alpha = 0$, dielectric anisotropy $\gamma = 0.05$, and elastic constants of E7. For voltage $V < V_p$, the planar vertical V state is a barrier state, and the horizontally twisted HT state is stable. For $V_p < V < V_F$, both the V and HT states are stable, separated by a barrier T state. For voltage $V > V_F$, HT becomes a barrier state and the V state remains stable. On the scale shown, the topologically distinct uniform planar horizontal state would appear as a horizontal line at $W = 0$.

$p_0 = f_k - f_\varepsilon$ as in (2.18), and d is the cell depth. The addition of $\varepsilon_\perp V^2$ compensates for the decrease in $2Gd$ with V for horizontal states, and does not affect the difference in the level of states that are compared at the same voltage V . The normalization then makes W exactly 1.0 for the HT state, independent of voltage. For $V < V_p$, HT is stable, there is no equilibrium T state with $\tau \neq 0$, and V is an unstable barrier state. For $V_p < V < V_F$, the unstable barrier T state emerges with Gibbs function above that of both the V and HT states, which are both stable. The barrier is highest at the voltage $V/V_F = 0.945$ where the V and HT states have equal Gibbs functions. At $V = V_F$, the T state disappears, while HT loses its stability and becomes the barrier state for $V > V_F$. The T state merges with the V state at V_p where $\tau = 90^\circ$ and with the HT state at V_F where $\tau = 0$.

6. Elimination of reverse twist. — Boundaries between domains of left and right 180° twist, equally likely to form in a pure nematic with perfectly aligned boundaries, would appear as transmitting lines in a field that is desired to be absorbing. One handedness can be favoured either by adding a small amount of cholesteric or by misalignment to make the total twist Φ less than 180° . In order to determine how much

misalignment can be tolerated before the bistability is lost or its range of holding voltage becomes impractically narrow, one can examine voltage *versus* τ curves for $\Phi \neq 180^\circ$. Figure 14 shows voltage *versus* τ for $\gamma = 0.05$, $\alpha = 1^\circ$, and $\Phi = 180^\circ$, 178° and 175° . With $\Phi = 180^\circ$, the bistable range extends from $V/V_F = 0.908$ to 0.947 , a width of about 4.2 %. We see from figure 14 that with $\Phi = 178^\circ$ the bistable range extends from $V/V_F = 0.924$ to 0.948 , or 2.5 %. With $\Phi = 175^\circ$, the width is reduced to about 1.3 %, and with $\Phi = 170^\circ$ (curve not shown), the voltage width of the bistability is only 0.22 %.

Besides a narrowing of the voltage range of the bistability, misalignment also introduces loss of con-

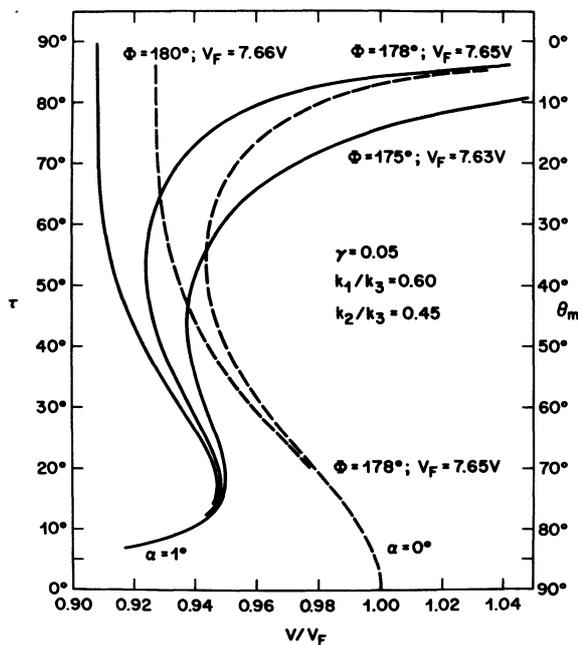


Fig. 14. — Reduced voltage *versus* τ for total twist $\Phi = 180^\circ$, $\Phi = 178^\circ$ and $\Phi = 175^\circ$. $\tau = 90^\circ - \theta_m$. $\alpha = 1^\circ$, $\gamma = 0.05$, and elastic constants of E7. Curves for $\alpha = 0^\circ$, and $\Phi = 178^\circ$ and 180° are included for comparison. Bistability is obtained in the voltage range in which the curve has negative slope.

trast, for, as can be seen from figure 14, the stable high- τ state no longer has $\tau = 90^\circ$, but a maximum τ on the bistable range of about 76° with $\Phi = 178^\circ$, and of only 60° with $\Phi = 175^\circ$.

7. Conclusion. — We have demonstrated analytically a bistability between 180° twisted states and the planar vertical state in pure nematics in the presence of a holding voltage. The effect of varying elastic constants and dielectric anisotropy has been studied with the result that the holding voltage range of the bistability is widened by a lowering of k_1/k_3 and the anisotropy. With reasonable elastic constants (Merck E7) and dielectric anisotropy parameter $(\epsilon_{\parallel} - \epsilon_{\perp})/\epsilon_{\perp} = 0.05$, the theoretical holding voltage range with zero tilt and perfect alignment is 7.3 %.

A low dielectric anisotropy can be obtained by mixing materials of positive and negative anisotropy.

The introduction of boundary tilt bias and misalignment in order to eliminate oppositely tilted and oppositely twisted domains drastically reduces the holding voltage range. For the above-cited elastic constants and anisotropy, a boundary tilt bias of 1° lowers the holding voltage range to 4.2 % with perfect alignment, while the combination of 1° tilt bias and 2° misalignment cuts the holding voltage range of the bistability to 2.5 %.

The studied bistability could probably be demonstrated experimentally, but practical considerations cast doubt on its viability for use in displays.

Acknowledgments. — The numerical integration and root-finding needed in the calculations were accomplished with the programs QUAD and ZERO from the *PORT Mathematical Subroutine Library* [30].

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- [18] T_{φ} given here by
- $$T_{\varphi} = \frac{\gamma D^2 \sin \theta \cos \theta}{\epsilon_{\perp}(1 + \gamma \cos^2 \theta)^2},$$
- is the φ component of the torque per unit volume exerted by the field on the material. As expected from classical concepts, it can be obtained not only from (2.16), but also as the φ component of $\mathbf{P} \times \mathbf{E}$ ($\mathbf{P} = \mathbf{D} - \epsilon_0 \mathbf{E}$, $\epsilon_0 = 8.855 \times 10^{-12}$ farad/meter).
- [19] To identify the elastic torque per unit area, consider an infinitesimally thin layer of thickness dz at the level in the cell where $\theta_z = 0$. (This would be at the midplane of a cell with symmetrical boundary conditions.) If the change in φ across this layer is Φ , then $\varphi_z = \Phi/dz$, and the elastic energy per unit area of the layer of thickness dz is $\frac{1}{2} k_3 g(\theta) \Phi^2/dz$. The torque per unit area is the derivative of this expression with respect to Φ , namely $k_3 g(\theta) \varphi_z = mk_3$.
- [20] The planar states ($\varphi_z \equiv 0$) have $m = 0$, and there is not necessarily any real value of θ at which $\theta_z = 0$. A planar vertical state (planar state that contains $\theta = 0$) may be characterized by p_0 or by the value of θ_z at $\theta = 0$, say $\theta_z(0)$. In this case,
- $$p_0 = \frac{1}{2} k_3 [\theta_z(0)]^2 - \frac{D^2}{2 \epsilon_{\perp}(1 + \gamma)}, \quad m = 0.$$
- If (2.20) is to yield this same p_0 in the limit as $\theta_m \rightarrow 0$, then
- $$\lim_{\theta_m \rightarrow 0} \frac{m^2}{g(\theta_m)} = [\theta_z(0)]^2.$$
- [21] For example, DEULING, H. J., « Elasticity of Nematic Liquid Crystals », p. 77-107 in *Liquid Crystals*, guest editor L. Liebert, Supplement 14 of *Solid State Physics, Advances in Research and Applications*, H. Ehrenreich, F. Seitz, and D. Turnbull, editors. Equation (27), p. 89. Deuling's U_{c1} is given in cgs units on p. 85. Equation (2.39) is in mks units. Equation (2.39) is easily derived by analysis of (2.22)-(2.25) for θ very close to $\pi/2$. With $\tau \equiv \pi/2 - \theta$, one finds, to appropriate order in τ , $\theta_z^2 = \kappa A (\tau_m^2 - \tau^2)$ where
- $$A = m^2 \xi(2 - \xi) + \gamma D^2 / k_3 \epsilon_{\perp}.$$
- Then (2.23) yields $d = \pi/(\kappa A)^{1/2}$. The substitutions $m = \Phi/\xi d$ and $D = \epsilon_{\perp} V_F/d$, which are appropriate in the limit as $\tau_m \rightarrow 0$, then give (2.39).
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- [24] THURSTON, R. N., « Stability of nematic liquid crystal configurations », *J. Physique* **42** (1981) 419-425.
- [25] In the $(k_1/k_3, k_3/k_2)$ plane, the boundary loop (4.3) is a hyperbola with centre at $k_1/k_3 = -(\gamma + 2)/\gamma$, $k_3/k_2 = 0$, and axes rotated by an angle θ where $\tan 2\theta = \gamma$.
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- [28] DE JEU, W. H., *Physical Properties of Liquid Crystalline Materials* (Gordon and Breach, New York) 1980, p. 88-95.
- [29] Exception : if the elastic constants lie between the lines $V_p = 0$ and $V_f = 0$ in figure 10, the V state is stable for any voltage (with $\gamma > 0$), and the voltage versus τ curve never attains 90°, but swings upward and to the left from the Freedericksz voltage at $\tau = 0$ to an intersection with the line $V = 0$ at some $\tau < 90^\circ$.
- [30] P. A. Fox, editor, *PORT Mathematical Subroutine Library*, 2nd edition, 1977 (a Bell Telephone Laboratories proprietary product obtainable from Western Electric Company, Patent Licensing, P.O. Box 20046, Greensboro, NC 27420).