LE JOURNAL DE PHYSIQUE

Classification
Physics Abstracts

1.110 — 7.160 — 7.222

THE CROSSING OF DEFECTS IN ORDERED MEDIA AND THE TOPOLOGY OF 3-MANIFOLDS

V. POENARU

Université Paris-Sud, Département de Mathématiques, 91405 Orsay, France

and

G. TOULOUSE

Ecole Normale Supérieure, Laboratoire de Physique, 24, rue Lhomond, 75231 Paris, France

(Reçu le 15 février 1977, accepté le 26 avril 1977)

Résumé. — Ce travail est un premier pas dans l'étude des obstructions topologiques pour déformer les défauts des milieux ordonnés. Il s'inscrit dans la ligne des théories physiques récentes qui classifient les défauts en termes de groupes d'homotopie d'une certaine variété V , caractéristique pour l'ordre en question.

On montre ici que les seules obstructions pour le croisement des lignes de défaut, dans un échantillon 3-dimensionnel, sont les commutateurs dans le groupe fondamental de V . Ceci est un phénomène qualitativement nouveau pour une certaine classe de matériaux, à $\pi_1 V$ non commutatif, qu'on espère voir synthétisés bientôt. Il en résulte aussi la nécessité d'une révision de certains concepts traditionnels dans la physique de la matière condensée.

Le présent travail contient un cadre mathématique rigoureux pour la description des défauts non commutatifs, une discussion des applications physiques, ainsi que quelques problèmes ouverts.

Abstract. — This investigation is a first step in the study of the topological obstructions involved in deforming the defects of ordered media; it belongs to the framework of the recent physical theories which classify stable defects in terms of the homotopy groups of a certain manifold V , characteristic for the given type of order.

In particular, it is shown here that the only obstruction for having two defect lines in a 3-dimensional sample cross through each other, without getting entangled, is a certain commutator in the fundamental group of V . This represents a qualitatively new phenomenon as far as the behaviour of certain materials (with non-commutative $\pi_1 V$), some still to be synthesized, is concerned; it also asks for a revision of certain traditional concepts in the physical theory of condensed matter.

The present paper contains a rigorous mathematical framework for the description of non-commutative defects, a discussion of some physical applications and some open problems.

1. Introduction. — The origin of this paper is a physical theory which classifies the defects of ordered media in topological terms (see [7, 8]). We recall that purely topological properties can imply the existence of defects in an ordered medium. For example, if the *order parameter* is a 3-dimensional vector, and if along a certain spherical surface the medium has no defects, but the index of the corresponding vector field is non zero, we can be sure that in the region of space enclosed by the surface there are defects. In a similar vein, assume that the isotropy group of the order parameter is a finite group $G \subset \text{SO}(3)$, and consider a closed circuit C which does not contain defects. The variation of the order parameter along C leads to a continuous path $[0, 1] \xrightarrow{\psi} \text{SO}(3)$, starting with the identity. If, by

lifting this path in $\text{SU}(2)$, we get for $t = 1$ a non-trivial spinor, then we can be sure that C is linked with the defect lines; in particular any (singular, self-intersecting) 2-disk whose boundary is C is crossed by a line of defects.

There is today a very similar trend of ideas in the physics of elementary particles, these being possibly like defects in some ordered medium. Their existence and some of their characteristics (quantum numbers...) could have a topological interpretation.

Defects of dimension d' in an ordered medium of dimension d are classified by the homotopy group of dimension $(d-d'-1)$ of a certain *manifold of internal states*, V , characteristic for the order under consideration. The higher homotopy groups (i.e. those of

dimension ≥ 2) are always commutative while the first group (the fundamental group) is not necessarily so. From a physical standpoint, this group leads to punctual defects in dimension 2 and linear defects in dimension 3. The case of non-commutative punctual defects is briefly discussed in the appendix to this paper. The case of linear defects in dimension 3 is of greater interest in physics.

In point of fact, in this paper, we study precisely the problem of deformation of defects in a 3-dimensional medium. Our canonical example will be a liquid crystal of biaxial nematic type, where the order parameter is an ellipsoid with three unequal axes. In this case, the isotropy group G is finite; it is $D_2 \subset SO(3)$. Its lift in $SU(2)$ will be denoted by $\tilde{G}$. The defects are linear in this case [8, 9]. The same holds for any *finite* $G \subset SO(3)$. (See also [10].)

We start by setting up a topological model for the « elementary operations » by which the defects can be physically deformed. This is done in section 2 below. For example, the obstructions for cutting defect lines will be elements in $\tilde{G}$ (as they should be, of course). Generally speaking, the topological obstructions connected to the existence of linear defects, live in the fundamental group of the *manifold of internal states*; in the case of biaxial nematics, this group is, of course, $\tilde{G}$.

Let us assume now that the intrinsic topology of the set of defects is fixed, but that one tries to change, or deform, the position of the set of defects inside the physical space; hence one wants to have lines of defect cross each other. Our main result is that *for such deformations, all the obstructions live in the commutator subgroup of $\tilde{G}$, denoted by $[\tilde{G}, \tilde{G}]$.*

This somewhat vague statement will be given a precise form in section 4; it will be shown that in order to have one particular defect line cross another, it is necessary and sufficient that a specific pair or elements in $\tilde{G}$ should commute. In particular, *if $\tilde{G}$ is abelian, any conceivable deformation is actually possible, while if $\tilde{G}$ is non-abelian the structure of the set of defects is much more rigid.*

Physicists used to classical mechanics had some problems in adjusting to the non-commutativity of quantum mechanics, and similarly, in the physics of condensed matter there is a certain difficulty involved in grasping the non-commutativity of defects. On the other hand, it looks as if topology has vast possibilities of applications in all these matters. So, one way or another, a close intercommunication between the physicists working with condensed matter and topologists, should be fruitful for both. We hope that this paper will be a good incentive for such ties and we have tried to write it in such a way that neither of the two groups of researchers whose subject it touches, should find it unreadable. This is the reason why our account is somewhat long.

The authors thank Claudine Williams who had the idea of putting them into contact.

2. The topological model. — We will give an abstract mathematical model of an ordered medium and its defects. This model is suggested by [7] and [8]. (See also [10].)

The following ingredients will be given :

I. A (topological) space V , called the *manifold of internal states*. In this paper, it will be assumed throughout that V is connected, has a non-trivial fundamental group ($\pi_1 V \neq 1$) and has its second homotopy group trivial ($\pi_2 V = 0$).

The fundamental group which is generally speaking non-abelian will be written multiplicatively, while the higher π_i 's ($i > 1$), which are commutative, are written additively.

In our canonical example, the biaxial nematics, V is constructed as follows. We consider an order parameter whose isotropy group is a *finite* subgroup $G \subset SO(3)$; V is the corresponding homogeneous space $V = SO(3)/G$. The natural map $SO(3) \rightarrow V$ is a Galois covering with fiber (structural group) G . Let $SU(2) \rightarrow SO(3)$ be the universal covering of $SO(3)$ and $\tilde{G} \subset SU(2)$ the subgroup of $SU(2)$ obtained by lifting G . It is clear that $SU(2)$ is the universal covering space of V and that, moreover : $\pi_0 V = 1$, $\pi_1 V = \tilde{G}$, $\pi_2 V = 0$.

[*Remark* : The extension $1 \rightarrow \mathbb{Z}/2 \rightarrow \tilde{G} \rightarrow G \rightarrow 1$, is, in general, non-trivial. It can very well happen that G is abelian and $\tilde{G}$ not. In the physical application we have in mind (see [9]), G is the group with 4 elements whose non-trivial elements are the rotation of angle 180° around the x , y , z -axes in $\mathbb{R}^3$. In this case, $\tilde{G}$ is the group with 8 elements $e_1, e_2, e_3, -e_1, -e_2, -e_3, \tilde{\gamma}, 1$ defined by the relations :

$$e_1^2 = e_2^2 = e_3^2 = \tilde{\gamma}, \quad \tilde{\gamma}^2 = 1,$$

$$\tilde{\gamma}e_i = e_i\tilde{\gamma} = -e_i, \quad e_1e_2e_3 = \tilde{\gamma}.]$$

II. We also consider a *physical space* which, for our purpose, will be a smooth, 3-dimensional manifold M^3 . We assume M^3 to be connected, orientable and compact with boundary ∂M^3 possibly $\neq \emptyset$. (The reader might as well think of M^3 as being a room in $\mathbb{R}^3$ and ∂M^3 its walls.)

III. In M^3 , we have a subset $\Sigma \subset M^3$, which will be called the *set of defects*, and outside Σ a continuous map Φ , from M^3 to V , is given :

$$M^3 - \Sigma \xrightarrow{\Phi} V.$$

Φ associates to every $p \in M^3 - \Sigma$ an *order parameter* $\Phi(p) \in V$. For topological reasons similar to those which make that a vector field on a closed manifold possesses, in general, singularities; we cannot usually expect a Φ as above to be always defined everywhere on M^3 if boundary conditions are imposed, or without locally nonremovable singularities; this leads to the necessity of considering defects (singularities).

[*Remark* : Our whole theory can be extended to bundles with fiber V and cross-sections, instead of maps.]

Generically speaking, we have the following numerical relation (see [7] and [8]) :

$$\dim \Sigma + \{ \text{the lowest } i \text{ such that } \pi_i V \text{ is non trivial} \} + 1 = \{ \text{the dimension of the physical space} \}.$$

Hence, in our context, Σ will be (generically) of dimension one; in principle, Σ will be a finite graph contained in M^3 , touching ∂M^3 with some of its endpoints. The whole theory below could be stated in terms of such graphs, but it will be more convenient for us to work with a set of defects Σ which is a 3-dimensional submanifold of M^3 , of a very special type : we start with a graph $\Gamma \subset M^3$ and then we thicken it into a 3-dimensional object Σ , as in figure 1. Note that Γ determines Σ uniquely (up to *isotopy*), but the converse is not true.

FIG. 1.

Remark : The standard mathematical terminology is to say that Σ *collapses* to Γ ; the collapsing is the operation inverse to thickening.

The data $\Sigma \subset M^3$ and $M^3 - \Sigma \cong V$ are our mathematical model of an ordered medium (and its defects).

Remark : Sometimes it will be convenient to think of Φ as being also defined on $\partial \Sigma - \partial M^3 \cap \Sigma$.

Our Σ which is *essentially* one-dimensional cannot disconnect M^3 and we have a homomorphism :

$$\Phi_* : \pi_1(M^3 - \Sigma) \rightarrow \pi_1 V$$

defined only up to an inner automorphism, as long as we have not chosen our base point $x_0 \in M^3 - \Sigma$.

This is the principal ingredient as far as the topological study of defects is concerned. We will make some remarks on Φ_* at the end of the paragraph.

Let us remark, finally, that since M^3 is orientable, Σ is made out of connected components which are solid tori with p holes ($p \geq 0$).

3. The elementary operations. — We will define several elementary moves which allow us to pass from a (Σ, Φ) to a (Σ', Φ') ; M^3 and V are, of course, given once for all. These elementary moves correspond to (obvious) physical transformations which have been observed in the deformation of defects in ordered media.

The operation 0-1. Let $\Psi_t : M^3 \rightarrow M^3$ be a family of diffeomorphisms of M^3 depending continuously on the parameter $t \in [0, 1]$, such that $\Psi_0 = \text{identity}$, $\Psi_t|_{\partial M^3} = \text{identity}$. [Such a family is called an *isotopy*. If $M^3 = D^3$ it is known that any diffeomorphism equal to the identity when restricted to the boundary, is actually isotopic to the identity; this is a difficult theorem of J. Cerf [1].]

If $M^3 - \Sigma \cong V$ is given, we can transform it by isotopy into (Σ', Φ') , where $\Sigma' = \Psi_1 \Sigma$, $\Phi' = \Phi \circ \Psi_1$. This operation could be physically quite costly, as far as energy is concerned, but from a purely topological viewpoint, does not cost anything.

The operation 0-2. We assume that Σ collapses to a graph Γ such that Γ contains an isolated connected component I which is a closed interval having at most one end point on the boundary ∂M^3 . Let $\Gamma' = \Gamma - I$ and $\Sigma' \subset \Sigma$ be the part of Σ corresponding to Γ' (the thickening of Γ').

Φ can be extended to a continuous map $\Phi' :$

$$M^3 - \Sigma' \rightarrow V.$$

[If $I \subset \text{int } M^3$, we have to use here the fact that $\pi_2 V = 0$. If $I \cap \partial M^3 \neq \emptyset$, the extension is always possible and essentially unique.]

From a physical stand-point, this operation decreases the « core energy » of the system (proportional to the length of Σ). Hence it will tend to be spontaneously realized; and there are anyway no topological obstructions to doing so.

The operation 0-3. Suppose $\text{int } M^3 - \Sigma \cap \text{int } M^3$ contains one embedded 2-sphere S^2 separating M^3 into two components, one of which contains all of ∂M^3 , the other being diffeomorphic to a ball; call it B^3 . Assume also that Σ collapses onto a graph Γ such that $B^3 \cap \Gamma$ is a simple closed curve $C \subset \text{int } B^3$.

Let $\Gamma' = \Gamma - C$ and $\Sigma' \subset \Sigma$ be the thickening of Γ' . Because of the condition $\pi_2 V = 0$, Φ can be extended to a continuous map $\Phi' : M^3 - \Sigma' \rightarrow V$. (It is the restriction of Φ to the complement of $\text{int } B^3$ which is extended to Φ' .) We can make the same remarks as for operation 0-2.

The operation 0-4. Let γ be a simple closed curve contained in $\partial \Sigma - \partial \Sigma \cap \partial M^3$. To γ , we can attach two elements (defined up to inner automorphism) :

$$g(\gamma) \in \pi_1 \Sigma, \quad p(\gamma) \in \pi_1 V,$$

defined as follows :

$g(\gamma)$ is the image of γ via the natural homomorphism $\pi_1(\partial \Sigma) \rightarrow \pi_1 \Sigma$.

$p(\gamma)$ is the homotopy class of $\Phi(\gamma)$.

At this point, we need to state a lemma (the proof will be given in the next paragraph).

Lemma 1. 1° If $g(\gamma) = 1 \in \pi_1 \Sigma$, then γ is the boundary of an embedded 2-disk $D^2 \subset \Sigma$, such that $D^2 \cap \partial \Sigma = \partial D^2 = \gamma$, $D^2 \cap \partial M^3 = \emptyset$, meeting $\partial \Sigma$ transversally. This disk is unique (up to isotopy). If we cut Σ along D^2 , we obtain a 3-manifold $\Sigma' \subset M^3$ which can also be collapsed to some graph $\Gamma' \subset M^3$. (Hence it is of the type which is allowed, *a priori*, to be a defect of M^3 .)

2° If, moreover, $p(\gamma) = 1 \in \pi_1 V$, then Φ can be extended to a continuous mapping :

$$\Phi' : M^3 \rightarrow \Sigma' \rightarrow V. \quad \square$$

By definition, the passage from (Σ, Φ) to (Σ', Φ') is our operation 0-4; and this makes sense because of lemma 1.

Like 0-2, 0-3, this operation decreases the energy. But in order for it to be possible, a topological obstruction, made out of a *geometrical constraint* $g(\gamma)$, and *physical constraint* $p(\gamma)$, has to vanish.

The operation 0-5. We start with (Σ, Φ) where Σ collapses onto some graph Γ . Let $\Gamma' \subset M^3$ be another graph, containing Γ , and such that $\Gamma' - \Gamma$ is one of the three objects listed below :

i) An isolated component I , diffeomorphic to an arc such that $I \cap \Gamma = \emptyset$, and $\partial I \cap \partial M^3$ consists of at most one end-point.

ii) An isolated component which is a simple closed curve contained in $M^3 - \Gamma - \partial M^3$, not linked with Γ .

iii) A not necessarily isolated component I , such that $\text{int } I \cap (\partial M^3 \cup \Gamma) = \emptyset$ and $\partial I \subset \partial M^3 \cup \Gamma$.

We define $\Sigma' =$ the thickening of Γ' ($\Sigma' \supset \Sigma$) and $\Phi' =$ the restriction of Φ to $M^3 - \Sigma'$.

This is the inverse of 0-2, or 0-3, or 0-4.

From a physical standpoint, 0-5 is costly in energy, but from a topological viewpoint, there is no obstruction.

This is our list of elementary moves.

Now some final remarks about the homomorphism Φ_* . If $\pi_1 V$ is abelian, there is a simpler homomorphism Ψ , as below, which completely determines Φ_* :

$$\begin{array}{ccc} \pi_1(M^3 - \Sigma) & \xrightarrow{\Phi_*} & \pi_1 V = H_1 V \\ & \searrow \Psi & \nearrow \\ & H_1(M^3 - \Sigma) = \pi_1(M^3 - \Sigma)/[\pi_1, \pi_1] \end{array}$$

Unlike Φ_* , Ψ which is a homomorphism of abelian groups, can be described very neatly. It is better to think of Σ now as being a graph with edges $a_1, a_2, \dots, a_n$. For each a_i , we consider a small simple closed loop, γ_i , oriented, contained in $M^3 - \Sigma$ and going once around a_i . [If $M^3 = D^3$ the images of γ_i ,

$$[\gamma_i] \in H_1(M^3 - \Sigma)$$

generate $H_1(M^3 - \Sigma)$.] For each vertex $v \in \Sigma \cap \text{int } M^3$, where $a_{i_1}, a_{i_2}, \dots, a_{i_n}$ meet, we have a relation of the form :

$$\varepsilon_{i_1}[\gamma_{i_1}] + \dots + \varepsilon_{i_n}[\gamma_{i_n}] = 0,$$

in $H_1(M^3 - \Sigma)$ with $\varepsilon_{i_j} = \pm 1$. There is, of course, one more such relation for each connected component of ∂M^3 , using exactly the edges which meet this component. [If $M^3 = D^3$, this is a complete system of relations for $H_1(M^3 - \Sigma)$.]

We can consider $\Psi[\gamma_i] = [\Phi\gamma_i] \in H_1 V$ and since Ψ is a homomorphism one gets the familiar Kirchhoff-type rules (*règles des nœuds*), well known in the theory of translation dislocations of crystals :

$$\varepsilon_{i_1}[\Phi\gamma_{i_1}] + \dots + \varepsilon_{i_n}[\Phi\gamma_{i_n}] = 0 \in H_1 V = \pi_1 V.$$

Hence, if $\pi_1 V$ is abelian, everything is completely described in terms of the $[\Phi\gamma_i]$'s, which in this case can be thought of as homotopy or homology classes.

But if $\pi_1 V$ fails to be abelian, the homotopy classes of the $\Phi\gamma_i$'s are elements of $\pi_1 V$ defined only up to some inner automorphism, and the computations from before are meaningless in $\pi_1 V$. Describing Φ_* in this case is a much more painful task. *A posteriori*, one might say that the amount by which the *règles des nœuds* fail to be true, is exactly what forbids deformations of defects.

Proof of lemma 1 (1) : Granted point 1° (of lemma 1), point 2° is obvious.

The existence of the disk D^2 follows from the so-called *Dehn's lemma* which we state below (this was stated by Max Dehn in the early twenties and proved by Papakyriakopoulos in 1957; the reader can find the proof in [3, 4] or [6]).

Dehn's lemma. Let X^3 be a smooth 3-manifold with $\partial X^3 \neq \emptyset$ and $\gamma \subset \partial X^3$ a simple closed curve such that the inclusion $\gamma \subset X^3$ is null-homotopic. Then there is a smoothly embedded 2-disk $D^2 \subset X^3$ such that $D^2 \cap \partial X^3 = \partial D^2 = \gamma$. $\square$

For the uniqueness of our D^2 , we need another very classical result (which is intuitively obvious, but not so easy to prove; see [1]) :

Alexander's theorem. If the 2-sphere S^2 is smoothly embedded in R^3 , there is a diffeomorphism of R^3 mapping its image into the unit sphere. $\square$

Consider now two copies of D^2 , smoothly embedded in Σ , D' and D'' , such that $\partial D' = \partial D'' = \gamma \subset \partial \Sigma$. Since M^3 is orientable, Σ is a solid torus, and hence it can be embedded (smoothly) into R^3 . If $\text{int } D' \cap \text{int } D'' = \emptyset$, it is easy to find an isotopy connecting them, using

(1) This section is relatively technical and some readers might prefer to skip it and accept lemma 1° (which is certainly not intuitively obvious) on faith. But then, for some other readers the arguments of this section might be useful as an introduction to 3-manifolds.

Alexander's theorem. [It is perhaps necessary to explain our terminology. If $D^2 \rightarrow \Sigma$ is a smooth embedding, such that $iD^2 \cap \partial\Sigma = i(\partial D^2)$, an isotopy of i is, by definition, a continuous family of smooth embeddings $D^2 \rightarrow \Sigma$ ($t \in [0, 1]$) such that $\varphi_0 = i$, $\varphi_t | \partial D^2 = i | \partial D^2$. By an easy general theorem, in this situation there exists a lifting of φ_t to an isotopy of diffeomorphisms of Σ .]

Now, in the general case, $\text{int } D' \cap \text{int } D'' \neq \emptyset$, but we can assume without any loss of generality that D' and D'' meet transversally along a finite set of simple closed curves; the whole problem is to reduce their number by successive isotopies, which will eventually bring us back to the easy case. We give an idea of how this can be done, without entering too much into details. If $C \subset \text{int } D' \cap \text{int } D''$ is one of the curves of intersection, C bounds two smaller disks $\delta' \subset \text{int } D'$, $\delta'' \subset \text{int } D''$. Without difficulty, we can always find a « minimal » C , such that $\text{int } \delta' \cap D'' = \emptyset$. Then $\delta' \cup \delta'' \subset \Sigma$ is an embedded 2-sphere and the embedding can easily be made smooth by rounding off the edge $C \subset \delta' \cup \delta''$. Hence, by Alexander's theorem, $\delta' \cup \delta''$ bounds a 3-ball $B^3 \subset \Sigma$. This ball can be used to produce an isotopy of D'' , in such a way that C (and perhaps some other curves of intersection) disappear. Figure 2 below gives an idea of what is to be done.

FIG. 2.

We prove now the last part of statement 1° (in lemma 1). Let us denote by T_p the solid torus with $p \geq 0$ holes (T_0 = the 3-ball B^3), and by M_p its boundary (M_p = the orientable closed surface with p holes). What we have to show is the following :

Lemma 2. Let $D^2 \subset T_p$ be a smoothly embedded 2-disk (such that $\partial D^2 = \partial T_p \cap D^2$) meeting ∂T_p transversally, and X be the 3-manifold obtained by cutting T_p along D^2 . If D^2 separates T_p (into two isolated components), then X has two isolated components, diffeomorphic respectively to T_{p-q} and T_q (for some $0 \leq q \leq p$).

If D^2 does not separate T_p , then X is diffeomorphic to T_{p-1} . □

Proof of lemma 2 : The proof is based on a number of classical results which we recall here :

I. The *loop theorem* of Papakyriakopoulos (see [2], [4], [6]) : Let Z^3 be a 3-manifold with $\partial Z^3 \neq \emptyset$. If the natural homomorphism $\pi_1 \partial Z^3 \rightarrow \pi_1 Z^3$ possesses a

non-trivial kernel, then some element of this kernel, different from 1, can be realized by a *simple* closed loop in ∂Z^3 .

II. A corollary of *Grushko's theorem* (see [4], [6]) : Let G_1 and G_2 be two groups and $G_1 * G_2$ their free product. We consider the rank of G_i , i.e. the minimum number of generators for G_i , denoted $rk G_i$. One has the following relation : $rk(G_1 * G_2) = rk G_1 + rk G_2$.

In contrast to I and II which are quite hard, the next results are easy :

III. The *theorem of Nielsen-Schreier* (see [4]) : any subgroup of a free group is free.

IV. The following easy remark (see [5]) : Let P^3 be a compact, simply-connected 3-manifold, with $\partial P^3 \neq \emptyset$. Every connected component of ∂P^3 is diffeomorphic to S^2 . □

We introduce the following notations : if X is any compact manifold, we consider the following number attached to X :

$$R(X) = \sum_{X_i} rk \pi_1 X_i,$$

where X_i is a connected component of X . For M_p , we consider the *genus* $g(M_p) = p$, and for a not necessarily connected, compact, closed, orientable 2-manifold Y , we define the following number :

$$g(Y) = \sum_{Y_i} g(Y_i)$$

where Y_i is a connected component of Y .

Now, as everybody knows, $\pi_1 T_p = F_p$ (= the free group with p generators) and according to whether D^2 separates T_p or not, we have, as a consequence of II, III above :

$X = X' \cup X''$ (two isolated connected components), with

$$\pi_1 X' = F_{p_1}, \quad \pi_1 X'' = F_{p_2},$$

$$p_1 + p_2 = p, \quad 0 \leq p_i \leq p,$$

or

$$X \text{ is connected and } \pi_1 X = F_{p-1}.$$

From an easy analysis of what happens to the boundary in the passage from T_p to X , we can deduce the following inequalities :

$$p \geq g(\partial X) \geq R(X).$$

[In fact, *a posteriori*, the argument which follows will show that, for every component of X , the corresponding g and R are equal, whence $g(\partial X) = R(X)$.]

Since every component of X has free fundamental group, while at the same time π_1 of a closed surface is never free, we can use the *loop theorem* and *Dehn's lemma* to prove the existence of a smoothly embedded 2-disk $D_1^2 \subset X$ (with $D_1^2 \cap \partial X = \partial D_1^2$), such that the homotopy class of ∂D_1^2 in ∂X is non-trivial.

We can cut X along D_1^2 in the same way as we have cut T_p along D^2 , and get another manifold, X_1 . The number of isolated connected components of X_1 can be 1, 2 or 3, and the fundamental group of each component is free.

As before :

$$g(\partial X) \geq g(\partial X_1) \geq R(X_1).$$

Because $[\partial D_1^2]$ is non-trivial in ∂X for any isolated connected component of X_1 , let's call it Z , we have $R(Z) < R(Z')$, where Z' is that component of the preceding level (i.e. X) to which Z belongs (use II and IV).

In the same way as we passed from X to X_1 , we can use X_1 to construct a new manifold X_2 , and hence a whole sequence $X_2, X_3, \dots$. But, if Z is a component of X_{n+1} , then $R(Z) < R(Z')$, and this implies that the process we just described cannot go on indefinitely. Let us call X_n the last X_i . We have necessarily $g(\partial X_n) = 0$, otherwise we could continue; this also implies that $R(X_n) = 0$, and since

$$X_n \subset X_{n-1} \subset \dots \subset X \subset T_p \subset \mathbb{R}^3,$$

by Alexander's theorem, every isolated component of X_n is a 3-ball. From these balls we can reconstruct X in an obvious way, and our lemma is now proved.

4. The topological obstructions for crossing the lines of defect. — We consider an ordered medium $M^3 - \Sigma \xrightarrow{\phi} V$, where Σ can be collapsed to some graph Γ .

Inside Γ , we consider two small arcs a and b , contained in the interior of the edges. At the level of Σ , a and b are thickened and they become two cylindrical solid tubes, which we call A and B . We also consider a simple arc $l \subset M^3$ joining the middle of a to the middle of b , and not meeting Γ (nor ∂M^3) otherwise.

FIG. 3. — The orientations are chosen arbitrarily.

Let x_0 be the middle point of l and α_1 (or β_1 respectively) the closed loop of $M^3 - \Sigma$, based at x_0 , defined as follows : we go from x_0 to a (or b) along l , we go once around a (or b), and then we come back to x_0 , along l (see Fig. 3).

Let $[\alpha_1], [\beta_1] \in \pi_1(M^3 - \Sigma, x_0)$ be the corresponding based homotopy classes. We have a commutator :

$$[\alpha_1] \cdot [\beta_1] \cdot [\alpha_1]^{-1} \cdot [\beta_1]^{-1} \in \pi_1(M^3 - \Sigma, x_0),$$

and its image under Φ_* :

$$\begin{aligned} [\Phi_*[\alpha_1], \Phi_*[\beta_1]] &= \\ &= \Phi_*[\alpha_1] \cdot \Phi_*[\beta_1] \cdot \Phi_*[\alpha_1]^{-1} \cdot \Phi_*[\beta_1]^{-1} \in \pi_1 V_0. \end{aligned}$$

For what follows next it is immaterial whether α_1 (or β_1) is replaced by α_1^{-1} (or β_1^{-1}), or whether $[[\alpha_1], [\beta_1]]$ is replaced by $[[\beta_1], [\alpha_1]]$; the reader should remember that if x, y , elements of some group G , commute, then x^{-1} and y commute too.

We can use l to make b cross through a . To be more precise, we start by replacing l with a long rectangle L ; L goes along l and its short sides are resting on a and b , as in figure 4. 1. Afterwards, we move b along l until it crosses a , staying as much as possible inside L ; there are exactly two ways of doing this, which are shown in figures 4. 2 and 4. 3 below.

FIG. 4. 1.

Of course L is not unique. Given one such rectangle, we can produce a double infinite sequence of similar objects by twisting as in figure 5. [We cut L from figure 4. 1 along Δ and then we re-glue together the two parts after having twisted, let's say the lower one, with an angle $\varphi = \dots - 360^\circ, -180^\circ, 180^\circ, 360^\circ, \dots$]

Each new L can be used to perform something like figures 4. 2, 4. 3. In this way we have described all the possible ways of letting b cross a , along l . Generally these ways are topologically distinct.

FIG. 5.

Remark : There is no topological difference between letting b cross a along l and letting a cross b along l .

If l and φ are given, we have produced a new graph $\Gamma' = \Gamma_{l,\varphi}$ and by thickening it we get $\Sigma' = \Sigma_{l,\varphi}$. In what follows, l is fixed but the precise way of letting b cross a (along l), parametered by the angle φ , is not. [The origin of the angle φ is obtained from L , not from l alone. Strictly speaking read $\Gamma' = \Gamma_{L,\varphi}$]

Theorem. Assume that for some fixed φ_0 , we can pass from the ordered medium

$$M^3 - \Sigma \xrightarrow{\Phi} V$$

to an ordered medium with defects Σ'_{1,φ_0} :

$$M^3 - \Sigma'_{1,\varphi_0} \xrightarrow{\Phi'} V,$$

by a sequence of operations 0-1, 0-2, 0-3 not touching 1, 0-4, 0-5.

Then the same thing is possible for any other $\Sigma'_{1,\varphi}$.

Moreover, the necessary and sufficient condition for these passages to be possible is that the elements $\Phi_*[\alpha_i]$ and $\Phi_*[\beta_i]$, of $\pi_1 V$, COMMUTE, i.e.:

$$[\Phi_*[\alpha_i], \Phi_*[\beta_i]] = 1 \in \pi_1 V. \quad \square$$

Proof of the theorem: We shall show that the operation described by figure 4.2 (or 4.3) can be realized by the operations 0-*i* if and only if $\Phi_*[\alpha_i]$ and $\Phi_*[\beta_i]$ commute. Once this is proved, everything else follows (in particular the first part of the statement, since α_i and β_i are defined only in terms of *l* and not of *L*).

It will be more convenient now to return to our 3-dimensional model for the defects. So, we consider $A, B \subset \Sigma$ with

$$A = \Delta' \times [-1, +1], \quad B = \Delta'' \times [-1, +1],$$

where Δ', Δ'' are 2-disks, and using *L* we can produce (in a well-defined manner, up to isotopy) a box $B^3 \subset M^3$, diffeomorphic to the 3-ball, such that

$$B^3 \supset L \cup A \cup B, \quad B^3 \cap \Sigma = A \cup B.$$

Hence:

$$\begin{aligned} \partial B^3 \cap \Sigma &= \Delta' \times (-1) \cup \Delta' \times \\ &\quad \times (+1) \cup \Delta'' \times (-1) \cup \Delta'' \times (+1). \end{aligned}$$

We consider $X^2 = \partial B^3 - \partial B^3 \cap \Sigma$ (as in figure 6).

FIG. 6.

It is understood that the four small disks which belong to $\partial B^3 \cap \Sigma$ are in no way touched by the various isotopies and deformations which will be described next; one will think of them as being rigid, while all the rest of $\Sigma \cap B^3$ will be flexible.

Inside X^2 we consider three simple closed loops $\lambda_0, \lambda_1, \lambda_2$, as in figures 6 and 7.

FIG. 7.

The result of the operations described in figure 4.2 is the subset $A \cup B'$ contained inside B^3 (Fig. 7). This operation happens in the interior of B^3 while λ_1 and λ_2 are on the boundary.

In $B^3 - (A \cup B)$ the commutators of the form $[\alpha_i]^{\pm 1} [\beta_i]^{\pm 1} [\alpha_i]^{\mp 1} [\beta_i]^{\mp 1}$ are *freely* homotopic to $[\lambda_1]^{\pm 1}$ or $[\lambda_2]^{\pm 1}$ (as the case might be).

On the other hand, from our figure 7, it is obvious that in $B^3 - (A \cup B')$, λ_1 is null-homotopic.

Let Σ' be Σ with $A \cup B$ replaced by $A \cup B'$, and assume that one can go from $M^3 - \Sigma \xrightarrow{\Phi} V$ to an ordered medium of the form $M^3 - \Sigma' \xrightarrow{\Phi'} V$ via a sequence of elementary moves 0-*i* (*i* = 1, ..., 5). Without any loss of generality, we can assume that these moves do not touch λ_1 and that $\Phi|_{\lambda_1} = \Phi'|_{\lambda_1}$. Since $\Phi'[\lambda_1] = 1 \in V$ one also has $\Phi_*[\lambda_1] = 1$ and this means that $\Phi_*[\alpha_i]$ and $\Phi_*[\beta_i]$ commute. A similar argument works for the operation described in figure 4.3, using λ_2 instead of λ_1 . So the commutativity condition is necessary.

We shall prove now that if $\Phi_*[\alpha_i]$ and $\Phi_*[\beta_i]$ commute, then one can pass from $M^3 - \Sigma \xrightarrow{\Phi} V$ to an ordered medium of the form $M^3 - \Sigma' \xrightarrow{\Phi'} V$ by one operation 0-5, followed by one operation 0-4.

The operation 0-5 consists of joining *A* and *B* by a solid tube along *l*. Then Σ is replaced by Σ'' such that $\Sigma'' \supset \Sigma$, $\Sigma'' - B^3 = \Sigma - B^3$, $\Sigma'' \cap B^3$ being as in figure 8. We go from $M^3 - \Sigma \xrightarrow{\Phi} V$ to

$$M^3 - \Sigma'' \xrightarrow{\Phi'' = \Phi|_{M^3 - \Sigma''}} V.$$

Note that $\Sigma'' \cap \partial B^3 = \Sigma \cap \partial B^3$ and that there is an obvious diffeomorphism $h : \Sigma'' \cap B^3 \rightarrow B^3$, which is the identity on $\Sigma'' \cap \partial B^3$ and homotopic to the natural inclusion $\Sigma'' \cap B^3 \subset B^3$, rel $\Sigma'' \cap \partial B^3$. [h is unique, up to an isotopy rel $\Sigma'' \cap \partial B^3$. We can realize it by inflating $\Sigma'' \cap B^3$, keeping $\Sigma'' \cap \partial B^3$ fixed.]

FIG. 8.

Let $\gamma = h^{-1}(\lambda_1)$. If we consider $g(\gamma)$ and $p(\gamma)$ defined as in section 1, then clearly $g(\gamma) = 1$; moreover, $[\Phi_*[\alpha_i], \Phi_*[\beta_i]] = 1$ is equivalent to $p(\gamma) = 1$. Hence we can use γ in order to perform an operation 0-4. It is not hard to see that the new set of defects is exactly our Σ' . This finishes the proof.

Remark : One could also go back from Σ'' to Σ via an operation 0-4 with $\gamma_0 = h^{-1}(\lambda_0)$.

5. Physical applications; some open problems. —

We start by recalling the foundations of the topological classification of defects in ordered media : at those points where the order parameter is zero (the core of the defect), the medium is in a higher energy state than at the points where the order parameter is non-zero ; in this way, we have energy proportional to the core volume. Because of this, the system will have a natural tendency to reduce the core volume as much as possible. In particular the dimension of the defects will be the lowest possible ; the stable defects are such that there is a topological obstruction to reducing the core volume. The ubiquity of this phenomenon fits well with a very general topological analysis.

But, of course, as soon as a particular material is being investigated, a detailed energetical analysis becomes necessary in order to account for the observable effects. As an example : for crystals, the dislocations by translation or rotation are both stable from a topological standpoint ; nevertheless the former are much more usual because of much lower energy. That is why, when physical applications are considered, one should always remember that the purely topo-

logical analysis has always to be supplemented by specific energetic considerations, taking into account the particular material under investigation.

The defects play an important role for the way in which the medium responds to constraints from outside. The existence of a long range order gives a certain « rigidity » to the medium ; thus in a crystal, unlike a liquid, moving one end draws along the other end. What happens if opposing constraints are applied at the two ends simultaneously ? The properties of the medium in this situation will be governed by the displacement of the defects. This is the subject matter of the theories of plasticity in solids (metallurgy), magnetic hysteresis, superfluid flows, and so on. Generally speaking, the intrinsic rigidity of the ordered medium is to a certain extent (and in a certain sense) compensated by the mobility of the defects. It is by acting on this mobility that one usually changes the response properties of the medium. (For example, one introduces certain impurities, in order to « hook » or « pin » the defects.)

Now, it should be noted that *the topological rigidity, discussed in this paper, introduces qualitatively new phenomena as far as the mobility of defects is concerned.* We expect these phenomena for a special class of materials (materials with non-commutative $\pi_1 V$), whose experimental investigation appears now to be important.

In the present terminology of the theory of elementary particles, one might say, with tongue in cheek, that the topological rigidity which forbids two defect lines to cross clearly, without staying entangled, is a *confinement* (or *slavery*) mechanism.

Many media with periodic order have a non-commutative $\pi_1 V$ (because of the non-commutativity between translations and rotations in the isotropy group). But because of the large difference in energy between dislocations of translation and rotation already mentioned, these kinds of materials are not the most suitable ones for our study. The biaxial nematics do not have such drawbacks and would appear to be very well adapted to this type of investigation ; but they have yet to be prepared [9].

While still waiting for this necessary experimental study, we end the paper with some open questions related to our model :

I. What happens if we change our conditions on $\pi_i V$ or the dimension of M ? This last question could be of some interest in elementary particle physics (quark confinement). Our commutator $[,]$ would have to be replaced by a Whitehead product (paper in preparation).

II. If M^3 and V are fixed, the various ordered media (Σ, Φ) fall into large *classes* $K_1, K_2, \dots$ which are defined by the fact that one can pass from one medium to another by elementary moves if and only if they belong to the same class. One should try to understand these classes.

Moreover one should try to find some natural probability-measure on the set of ordered media which will then give a sense to the *measure* of K_i . The general idea would be that the (Σ, Φ) 's falling into K_i 's with larger mass are more likely to occur than the others (at least from a purely topological viewpoint).

III. Develop a theory as before, for the case where the operations 0-3 touch 1.

Appendix : punctual defects in ordered media of dimension 2. — We will illustrate, by a simple, physically realizable example, the effects of non-commutativity for the junction of punctual defects. In this example, the order is a density wave (this could be, for example, a wave of density-of-matter in a liquid crystal, or a wave of density-of-electric charge in a metal).

The isotropy group of the order parameter is the semi-direct product of Z (= translations) with $Z/2$ (= rotations of angle π). The manifold of internal states V is the Klein bottle, and hence $\pi_1 V = Z \tilde{\times} Z$, a classical result.

A configuration of the ordered medium is represented by drawing the lines of equal phase ($\Phi = 2\pi n, n \in \mathbb{Z}$). Figure 9 represents three such configurations, which we will discuss.

In 9.1, we have a configuration with defects A, B. If A and B are brought closer to each other, by a continuous movement, they will eventually annihilate each other.

In 9.2, we have again two defects A and B, and the local situation in their neighbourhoods is the same as before. But the intermediary lines are glued together differently and, if we bring these A and B close together, they do not cancel out; we get, instead, another punctual defect, namely a double dislocation.

In 9.3, we have essentially something like figure 9.1, but also, in the region between A and B, a new defect C (a simple dislocation). If A and B are brought closer, on a path going below C, they cancel each other out.

But, if A and B are brought closer on a path going above C, then they will give a double dislocation.

FIG. 9.1.

FIG. 9.2.

FIG. 9.3.

Hence the interactions of two defects can be completely modified by the passage of a third defect between them.

This example shows that the local topology around the defects is not enough to predict the interactions. The whole situation of the medium between comes into play, and the result of a junction depends on the path chosen in order to realize it.

These kinds of effects depend on the non-commutativity of $\pi_1 V$ and disappear in the commutative case.

References

- [1] CERF, J., *Sur les difféomorphismes de la sphère de dimension trois* ($\Gamma_4 = 0$), (Springer L.N.) 53 1968.
- [2] PAPA KYRIAKOPOULOS, C. D., *On solid tori*, *Proc. London Math. Soc.* **317** (1957) 281.
- [3] PAPA KYRIAKOPOULOS, C. D., *On Dehn's lemma and the asphericity of knots*, *Ann. Math.* **66** (1957) 1.
- [4] POÉNARU, V., *Groupes discrets* (Springer L.N.) 421 (1974).
- [5] SEIFERT, H., THREL FALL, W., *Lehrbuch der Topologie*, Teubner, Leipzig (1934).
- [6] STALLINGS, J., *Group theory and three-dimensional manifolds*, *Yale Math. Monographs* 4 (1971).
- [7] TOULOUSE, G. and KLEMAN, M., *J. Physique-Lett.* **37** (1976) L-149.
- [8] TOULOUSE, G., *Pourquoi et comment classer les défauts dans les milieux ordonnés ?* *Bull. Soc. Fr. Phys.*, octobre 1976.
- [9] TOULOUSE, G., *Pour les nématiques biaxes*, *J. Physique-Lett.* **38** (1977) 67.
- [10] KLEMAN, M., MICHEL, L. and TOULOUSE, G., *Classification of topologically stable defects in ordered media* (to appear in *J. Physique-Lettres*).