# A polynomial parametrization of torus knots

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**Abstract.** For every odd integer $N$ we give an explicit construction of a polynomial curve $\mathcal{C}(t) = (x(t), y(t))$, where $\deg x = 3$, $\deg y = N + 1 + 2\left[\frac{N}{4}\right]$ that has exactly $N$ crossing points $\mathcal{C}(t_i) = \mathcal{C}(s_i)$ whose parameters satisfy $s_1 < \cdots < s_N < t_1 < \cdots < t_N$. Our proof makes use of the theory of Stieltjes series and Padé approximants. This allows us an explicit polynomial parametrization of the torus knot $K_{2,N}$.

**keywords:** Polynomial curves, Stieltjes series, Padé approximant, torus knots

## 1 Introduction

Let $N$ be an odd integer. We look for a parametrized curve $\mathcal{C}(t) = (x(t), y(t))$ of minimal lexicographic degree such that $\mathcal{C}$ has exactly $N$ crossing points, corresponding to parameters $(s_i, t_i)$ such that

$$\mathcal{C}(s_i) = \mathcal{C}(t_i),\ s_1 < \cdots < s_N < t_1 < \cdots < t_N. \tag{1}$$

Here we look for curves with $\deg x = 3$. As a consequence of Bézout theorem, we have $\deg y \geq N + 1$. We have translated this problem into a problem on real roots of certain real polynomials in one variable. In [KP] we proved that if $N > 3$, there is no solution with $\deg y = N + 1$. We have computed the first examples and we have shown that the minimal degrees are $\deg y = N+1+2\left[\frac{N}{4}\right]$ for $N = 3, 5, 7$.

The purpose of this paper is to give an explicit construction at any order of such curves with $\deg y = N + 1 + 2\left[\frac{N}{4}\right]$.

In section **2.**, we first recall some properties of the Chebyshev polynomials. Our construction is based on certain relations in the space spanned by some of these polynomials.

The explicit construction is given in section **3**. It involves some particular polynomial basis whose existence is proved in section **6.**, using Stieltjes series theory and Padé approximation theory (see [BG]).

In section **5.**, we will show that the algebraic relation between $\cos 2\theta$ and $\cos 6\theta$ may be seen as a Stieltjes series, namely some algebraic hypergeometric function. We will recall some properties of these functions and their approximations by rational functions in section **6.**, the so-called Padé approximants.

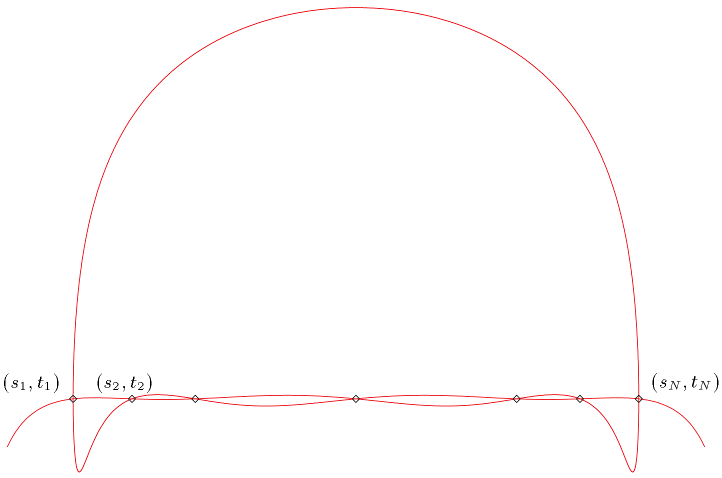



**Fig. 1.** Curve of degree $(3, 19)$ in logarithmic scale in $y$

The polynomial curves whose existence are proved are of interest for an explicit polynomial parametrizations of the $(2, N)$-type torus knot $K_{2,N}$ (see [Ad,KP,Mu,RS]). In section **4.**, we give an explicit parametrizations for the knots $K_{2,N}$. They are symmetric with respect to the $y$-axis and of smaller degrees than those already known.

## 2 Some properties of the Chebyshev polynomials

**Definition 1 (Monic Chebyshev polynomials).**
*If* $t = 2\cos\theta$, *let* $T_n(t) = 2\cos(n\theta)$ *and* $V_n(t) = \dfrac{\sin((n+1)\theta)}{\sin\theta}$.

$T_n$ and $V_n$ are both monic and have degree $n$. It is convenient for our problem to consider them as basis of $\mathbf{R}[t]$.

Looking for a polynomial curve $\mathcal{C}(t) = (x(t), y(t))$ where $\deg x = 3$, one can suppose that

$$x(t) = T_3(t),\ \ y(t) = T_m(t) + a_{m-1}T_{m-1}(t) + \cdots + a_1T_1(t).$$

In [KP] (Lemma **A**) we have shown that if $s \neq t$ are real numbers such that $T_3(s) = T_3(t)$, then for any integer $k$ we have

$$\frac{T_k(t) - T_k(s)}{t - s} = \frac{2}{\sqrt{3}}\sin\frac{k\pi}{3}\,V_{k-1}(s+t) = \varepsilon_k V_{k-1}(s+t). \tag{2}$$

We proved the following:

**Proposition 2.** *Let* $\varepsilon_k = \dfrac{2}{\sqrt{3}} \sin \dfrac{k\pi}{3} = V_{k-1}(1)$ *and*

$$R_m = \varepsilon_m V_{m-1} + \varepsilon_{m-1} a_{m-1} V_{m-2} + \cdots + \varepsilon_1 a_1 V_0. \tag{3}$$

*— If $R_m$ has exactly $N$ distinct roots $-1 < u_1 < \cdots < u_N < 1$ and no other in $[-2, 2]$, then*

$$\mathcal{C}(t) = (T_3(t),\ T_m(t) + a_{m-1}T_{m-1}(t) + \cdots + a_1 T_1(t))$$

*has exactly $N$ crossing points.*
*— Let $u_i = 2\cos\alpha_i$, then*

$$s_i = 2\cos(\alpha_i + \pi/3),\, t_i = 2\cos(\alpha_i - \pi/3) \tag{4}$$

*are the parameters of the crossing points and satisfy*

$$s_1 < \cdots < s_N < t_1 < \cdots < t_N.$$

We look for polynomials $R_m$ in $\mathbf{R}[t]$ having $N$ roots that are linear combinations of the $V_k$, where $k$ is not equal to 2 [mod 3]. We will consider separately $E \subset \mathbf{R}[t]$ spanned by $V_{6k+1}$ and $V_{6k+3}$ and $\tilde{E}$ spanned by the $V_{6k}$ and $V_{6k+4}$. We first describe these vectorial spaces as direct sums:

**Lemma 3.** $E = T_1 \cdot \mathbf{R}[T_6] \oplus T_1 \cdot T_2 \cdot \mathbf{R}[T_6]$, $\quad \tilde{E} = 1 \oplus T_3 \cdot E$.

*Proof.* — From $\sin(x+y) - \sin(x-y) = 2\cos(x)\sin(y)$ we deduce that for every integers $n$ and $p$, we have

$$V_{n+p} - V_{n-p} = V_{p-1}T_{n+1}.$$

We thus deduce that $V_1 = T_1 = t$, $V_3 = T_1 \cdot T_2$ and

$$V_{6k+1} - V_{6k-3} = T_1 \cdot T_{6k},\ V_{6k+3} - V_{6k-5} = V_3 \cdot T_{6k} = T_1 \cdot T_2 \cdot T_{6k}.$$

From $T_{6k} = T_k(T_6)$, we deduce by induction that

$$E = T_1 \cdot \mathbf{R}[T_6] \oplus T_1 \cdot T_2 \cdot \mathbf{R}[T_6].$$

— From $\sin(x+y) + \sin(x-y) = 2\cos(y)\sin(x)$, we get

$$V_{n+3} + V_{n-3} = T_3 V_n$$

so

$$V_{6k+6} + V_{6k} = T_3 V_{6k+3},\ V_{6k+4} + V_{6k-2} = T_3 V_{6k+1}.$$

As $V_0 = 1$ and $V_{-2} = -1$, we thus deduce by induction that $\tilde{E} = 1 \oplus T_3 \cdot E$. $\square$

**Definition 4.** *Let us define for $k \geq 0$,*

$$\tilde{W}_{2k} = V_{6k},\ W_{2k} = V_{6k+1},\ W_{2k+1} = V_{6k+3},\ \tilde{W}_{2k+1} = V_{6k+4}.$$

We have $\deg W_n = 2n + 2\left[\frac{n}{2}\right] + 1$ and $E = \text{vect } (W_k,\, k \geq 0)$.
We have $\deg \tilde{W}_n = 2n + 2\left[\frac{n+1}{2}\right]$ and $\tilde{E} = \text{vect } (\tilde{W}_k,\ k \geq 0)$.

Using the Padé approximation theory, we will prove in section **6** (p. 15)

**Theorem 5.** *There exists a sequence of odd polynomials $C_n$ in $E$ such that*

$$\text{vect } (W_0, \ldots, W_n) = \text{vect } (C_0, \ldots, C_n),\ C_n = t^{2n+1} F_n,\ F_n(0) = 1.$$

*Furthermore $F_n(t) > 0$ when $t \in [-2, 2]$.*

We find, up to some multiplicative constant,

$C_0 = t = W_0$,
$C_1 = t^3 = W_1 + 2\,W_0$,
$C_2 = t^5\left(t^2 - 6\right) = W_2 - 10\,W_1 - 16\,W_0$,
$C_3 = t^7\left(t^2 - 9/2\right) = W_3 + 7/2\,W_2 - 15\,W_1 - 21\,W_0$,
$C_4 = t^9\left(t^4 - 12\,t^2 + 33\right) = W_4 - 22\,W_3 - 56\,W_2 + 176\,W_1 + 231\,W_0$,
$C_5 = t^{11}\left(t^4 - \dfrac{102}{11}t^2 + \dfrac{234}{11}\right) = W_5 + \dfrac{52}{11}W_4 - 40\,W_3 - \dfrac{910}{11}W_2 + 208\,W_1 + 260\,W_0$.

We deduce from theorem **5** and lemma **3** the following useful result for the construction of the height function $z(t)$ of our knots (see section **4.**).

**Corollary 6.** *The sequence $\tilde{C}_0 = 1$, $\tilde{C}_n = -\dfrac{1}{3}T_3 C_{n-1}$ of even polynomials in $\tilde{E}$ satisfies:*

$$\text{vect } (\tilde{W}_0, \ldots, \tilde{W}_n) = \text{vect } (\tilde{C}_0, \ldots, \tilde{C}_n),\ \tilde{C}_n = t^{2n}\tilde{F}_n,\ \tilde{F}_n(0) = 1.$$

*Furthermore $\tilde{F}_n(t) > 0$ when $t \in [-2, 2]$.*

## 3 Construction of the prescribed curves

We will construct polynomials $R_m$ in $E$ with $N = 2n + 1$ real roots in $[-1, 1]$ and no other roots in $[-2, 2]$. They will be chosen as a slight deformation of $C_n$. Let us first show properties of the polynomials $C_n$.

**Lemma 7.** *Let $0 < u_1 < \cdots < u_n < 1$ be real numbers. For $\varepsilon$ being small enough,*

1. *there exists a unique $(a_0, \ldots, a_{n-1})$ such that $\{0, \pm\varepsilon u_1, \ldots, \pm\varepsilon u_n\}$ are roots in $[-1, 1]$ of*
$$A_n(\varepsilon) = C_n + a_{n-1}C_{n-1} + \cdots + a_0 C_0.$$
2. *$\{0, \pm\varepsilon u_1, \ldots, \pm\varepsilon u_n\}$ are the only real roots in $[-2, 2]$ of $A_n(\varepsilon)$.*

*Proof.* Looking for $A_n(\varepsilon)$ with roots 0 and $\pm\varepsilon u_i$ is equivalent to the linear system

$$\begin{pmatrix} C_0(\varepsilon u_1) & C_1(\varepsilon u_1) & \cdots & C_{n-1}(\varepsilon u_1) \\ C_0(\varepsilon u_2) & & \cdots & C_{n-1}(\varepsilon u_2) \\ \vdots & & & \vdots \\ C_0(\varepsilon u_n) & C_1(\varepsilon u_n) & \cdots & C_{n-1}(\varepsilon u_n) \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_{n-1} \end{pmatrix} = - \begin{pmatrix} C_n(\varepsilon u_1) \\ C_n(\varepsilon u_2) \\ \vdots \\ C_n(\varepsilon u_n) \end{pmatrix}$$

whose determinant is $\varepsilon^{(1+3+\cdots+2n-1)} \begin{vmatrix} u_1 F_0(\varepsilon u_1) & u_1^3 F_1(\varepsilon u_1) & \cdots & u_1^{2n-1} F_{n-1}(\varepsilon u_1) \\ u_2 F_0(\varepsilon u_2) & & \cdots & u_2^{2n-1} F_{n-1}(\varepsilon u_2) \\ \vdots & & & \vdots \\ u_n F_0(\varepsilon u_n) & u_n^3 F_1(\varepsilon u_n) & \cdots & u_n^{2n-1} F_{n-1}(\varepsilon u_n) \end{vmatrix}$.
It is equivalent to the classical Vandermonde-type determinant when $\varepsilon \to 0$:

$$\varepsilon^{n^2} \begin{vmatrix} u_1 & u_1^3 & \cdots & u_1^{2n-1} \\ u_2 & & \cdots & u_2^{2n-1} \\ \vdots & & & \vdots \\ u_n & u_n^3 & \cdots & u_n^{2n-1} \end{vmatrix} = \varepsilon^{n^2} u_1 \cdots u_n \prod_{1 \le i < j \le n} (u_j^2 - u_i^2) \neq 0.$$

Therefore, this system has a unique solution.

— Using Cramer formulas, we get $a_k(\varepsilon) = \dfrac{G_k(\varepsilon)}{G_n(\varepsilon)}$ where

$$G_k(\varepsilon) = \det(C_{i_j}(\varepsilon u_i))_{1 \le i,j \le n}$$

and $i_1 < i_2 < \cdots < i_n \in \{0, \ldots, n\} - \{k\}$. We get

$$\begin{aligned} G_k(\varepsilon) &= \varepsilon^{2(i_1+\cdots+i_n)+n} \begin{vmatrix} u_1^{2i_1+1} F_{i_1}(\varepsilon u_1) & u_1^{2i_2+1} F_{i_2}(\varepsilon u_1) & \cdots & u_1^{2i_n+1} F_{i_n}(\varepsilon u_1) \\ u_2^{2i_1+1} F_{i_1}(\varepsilon u_2) & & \cdots & u_2^{2i_n+1} F_{i_n}(\varepsilon u_2) \\ \vdots & & & \vdots \\ u_n^{2i_1+1} F_{i_1}(\varepsilon u_n) & u_n^{2i_2+1} F_{i_2}(\varepsilon u_n) & \cdots & u_n^{2i_n+1} F_{i_n}(\varepsilon u_n) \end{vmatrix} \\ &\underset{\varepsilon \to 0}{\simeq} \varepsilon^{(n+1)^2-(2k+1)} \begin{vmatrix} u_1^{2i_1+1} & u_1^{2i_2+1} & \cdots & u_1^{2i_n+1} \\ u_2^{2i_1+1} & & \cdots & u_2^{2i_n+1} \\ \vdots & & & \vdots \\ u_n^{2i_1+1} & u_n^{2i_2+1} & \cdots & u_n^{2i_n+1} \end{vmatrix}. \end{aligned}$$

— We thus deduce that $a_k(\varepsilon) = \mathcal{O}(\varepsilon^{2(n-k)})$ and therefore

$$\lim_{\varepsilon \to 0} A_n(\varepsilon) = C_n = t^{2n+1} F_n.$$

Let $A_n(\varepsilon) = t \prod_{i=1}^n (t^2 - \varepsilon^2 u_i^2) D_n(\varepsilon)$. We deduce that $\lim_{\varepsilon \to 0} D_n = F_n$. Let $\varepsilon$ be small enough, we get $D_n(t) > 0$ for $t \in [-2, 2]$ because of the compactness of $[-2, 2]$. □

**Proposition 8.** *Let $N = 2n+1$ be an odd integer. There exists a curve $\mathcal{C}(t) = (x(t), y(t))$, where $\deg x = 3$ and $\deg y = N + 2\left[\frac{N}{4}\right] + 1$, such that $\mathcal{C}$ has exactly $N$ crossing points corresponding to parameters $(s_i, t_i)$ such that*

$$\mathcal{C}(s_i) = \mathcal{C}(t_i), \ s_1 < \cdots < s_N < t_1 < \cdots < t_N. \tag{5}$$

*Proof.* Let $N = 2n+1$. Let us choose $\varepsilon$ and $0 < u_1 < \cdots < u_n$, such that there exists a polynomial $A_n(\varepsilon) \in \mathrm{vect}\ (W_0, \ldots, W_n)$ having exactly $N$ distinct roots $\{0, \pm\varepsilon u_1, \ldots, \pm\varepsilon u_n\}$ in $[-2, 2]$. It has degree $\deg W_n = 2n + 2\left[\frac{n}{2}\right] + 1 = N + 2\left[\frac{N}{4}\right] = m - 1$. We have

$$A_n = W_n + \cdots + a_0 = V_{m-1} + \varepsilon_{m-1} a_{m-1} V_{m-2} + \cdots + a_2 V_1.$$

Using proposition **2**, the curve

$$x(t) = T_3(t), \ y(t) = \varepsilon_m T_m(t) + a_{m-1} T_{m-1}(t) + \cdots + a_2 T_2(t)$$

has the required properties. □

*Example 9 ($N = 9$).* We chose

$$y(t) = -\frac{27}{10} T_{14} + 10\, T_{12} - 23\, T_{10} + 42\, T_8 - 64\, T_6 + 85\, T_4 - 100\, T_2 + 112.$$

The roots of $Q$ are $u_0 = 0, \pm u_1 = .355, \pm u_2 = .584, \pm u_3 = .785, \pm u_4 = 1.073$. We obtain a polynomial parametrization of degree $(3, 14)$. Note that we choose $u_4 > 1$ for a nicer picture (see fig. 2). Note that the parameters of the crossing points satisfy $s_1 < \cdots < s_9 < t_1 < \cdots < t_9$.

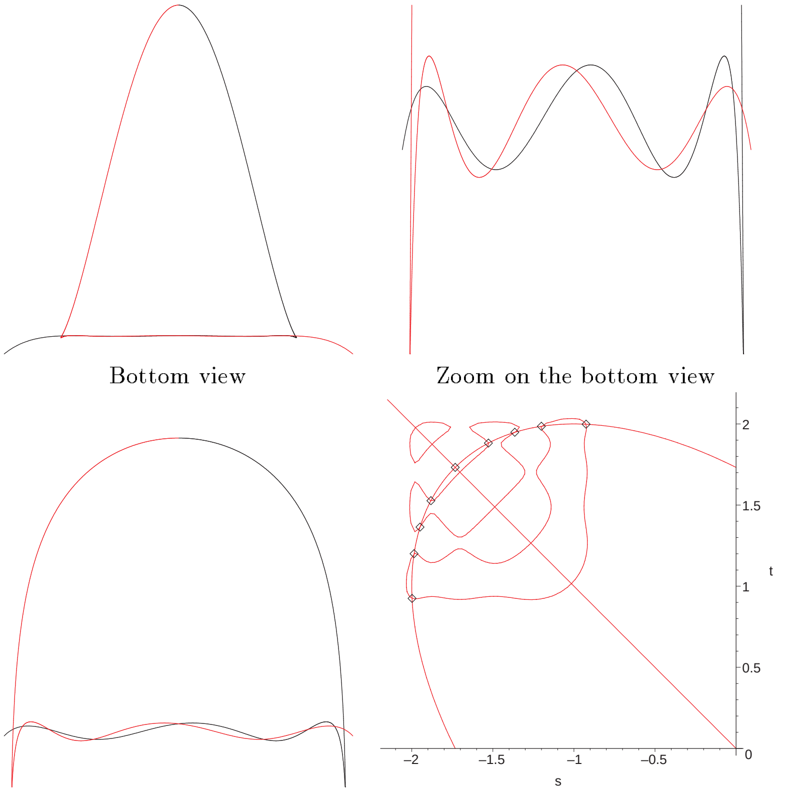
## 4 Construction of the torus knots

If $N = 2n+1$ is odd, the torus knot $K_{2,N}$ of type $(2, N)$ is the boundary of a Moebius band twisted $N$ times (see [Ad,Mu] and fig. (3)). The purpose of this section is to give an explicit construction of a polynomial curve $\mathcal{C}(t) = (x(t), y(t), z(t))$ that is equivalent (in the one point compactification $\mathbf{S}^3$ of the space $\mathbf{R}^3$) to the torus knot $K_{2,N}$.

Vassiliev (see [Va]) proved that any non-compact knot type can be obtained from a polynomial embedding $t \mapsto (f(t), g(t), h(t)), t \in \mathbf{R}$, using the Weierstrass approximation theorem.

Shastri [Sh] gave a detailed proof of this theorem, and a simple polynomial parametrizations of the trefoil and of the figure eight knot. A. Ranjan and R. Shukla [RS] have found small degree parametrizations for $K_{2,N}$, $N$ odd. They proved that these knots can be attained from polynomials of degrees $(3, 2N-2, 2N-1)$.

In [KP], we proved that it is not possible to attain the torus knot $K_{2,N}$ with polynomial of degrees $(3, N+1, m)$ when $N > 3$. We gave explicit parametrization of degrees $(3, N+2\left[\frac{N}{4}\right]+1, N+2\left[\frac{N+1}{4}\right])$ for $N = 3, 5, 7, 9$ and showed that they were of minimal lexicographic degree for $N \leq 7$.




Bottom view — Zoom on the bottom view

logarithmic scale — $\dfrac{x(s)-x(t)}{s-t}=0,\ \dfrac{y(s)-y(t)}{s-t}=0, s<t$

**Fig. 2.** $N = 9$. Curve of degree $(3, 14)$

A sufficient condition is to construct a parametrized curve $\mathcal{C}(t) = (x(t), y(t), z(t))$ such that $(x(t), y(t))$ has exactly $N = 2n + 1$ crossing points corresponding to parameters

$$s_1 < \cdots < s_N < t_1 < \cdots < t_N$$

and such that

$$x(t_i) = x(s_i),\ \ y(t_i) = y(s_i),\ (-1)^i(z(t_i) - z(s_i)) > 0,\ \ i = 1, \ldots, N.$$

We look first for minimal degree in $x$. $x(t)$ must be nonmonotonic and therefore has degree at least 2. In case when $\deg x = 2$, we would have constant $t_i + s_i$ and not the condition (1). We will give a construction for $\deg x = 3$.

**Proposition 10.** *For any odd integer* $N = 2n + 1$*, there exists a curve* $\mathcal{C}(t) = (x(t), y(t), z(t))$ *of degree* $(3, N + 2[\frac{N}{4}] + 1, N + 2[\frac{N+1}{4}])$ *such that the curve*

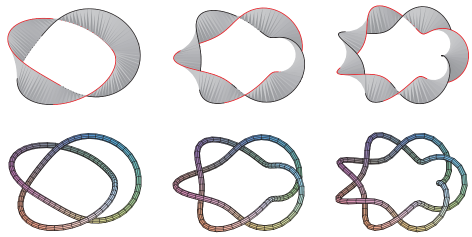


**Fig. 3.** $K_{2,N}$, $N = 3, 5, 7$.

*$(x(t), y(t))$ has exactly $N$ crossing points*

$$x(t_i) = x(s_i),\ \ y(t_i) = y(s_i),\ s_1 < \cdots < s_N < t_1 < \cdots < t_N$$

*and*

$$(-1)^i (z(t_i) - z(s_i)) > 0,\ \ i = 1, \ldots, N.$$

*Proof.* — Following the construction of section **3.**, we first choose $\varepsilon$ to be small enough and $0 < c_1 < \cdots < c_n < \varepsilon < 1/2$, such that

$$\begin{vmatrix} C_0(c_1) & C_1(c_1) & \cdots & C_{n-1}(c_1) \\ C_0(c_2) & & \cdots & C_{n-1}(c_2) \\ \vdots & & & \vdots \\ C_0(c_n) & C_1(c_n) & \cdots & C_{n-1}(c_n) \end{vmatrix} \neq 0.$$

Consider

$$u_{n+1} = 0,\ u_i = 2\cos\alpha_i = -c_{n+1-i},\ u_{n+1+i} = 2\cos\alpha_{n+1+i} = c_i,\ i = 1, \ldots, n.$$

We thus have $-1 < u_1 < \ldots < u_N < 1$. Let

$$s_i = 2\cos(\alpha_i + \pi/3),\ t_i = 2\cos(\alpha_i - \pi/3),\ i = 1, \ldots N.$$

— Using the proposition (8), there is a polynomial

$$y(t) = T_m(t) + a_{m-1}T_{m-1}(t) + \cdots + a_1 T_1(t),$$

of degree $m = N + 2\left[\frac{N}{4}\right] + 1$ such that

$$x(t_i) = x(s_i),\ \ y(t_i) = y(s_i),\ i = 1, \ldots N.$$

— As for lemma **7**, there exists a unique $(b_0, \ldots, b_n)$ such that

$$B_n = b_n\tilde{C}_n + b_{n-1}\tilde{C}_{n-1} + \cdots + b_0\tilde{C}_0$$

satisfies $B_n(u_i) = (-1)^i$, $i = 1, \ldots N$. Namely, $(b_0, \ldots, b_n)$ is the solution of the system

$$b_n\tilde{C}_n(u_i) + b_{n-1}\tilde{C}_{n-1}(u_i) + \cdots + b_0\tilde{C}_0(u_i) = (-1)^i,\ i = 1, \ldots, N.$$

Because $u_i = -u_{N+1-i}$ and $\tilde{C}_k$ are even polynomials, the system is equivalent to

$$\begin{pmatrix} \tilde{C}_0(u_{n+1}) & \tilde{C}_1(u_{n+1}) & \cdots & \tilde{C}_n(u_{n+1}) \\ \tilde{C}_0(u_{n+2}) & & \cdots & \tilde{C}_n(u_{n+2}) \\ \vdots & & & \vdots \\ \tilde{C}_0(u_N) & \tilde{C}_1(u_N) & \cdots & \tilde{C}_n(u_N) \end{pmatrix} \begin{pmatrix} b_0 \\ b_1 \\ \vdots \\ b_n \end{pmatrix} = \begin{pmatrix} (-1)^{n+1} \\ (-1)^{n+2} \\ \vdots \\ (-1)^N \end{pmatrix}.$$

From $\tilde{C}_0 = 1$, $\tilde{C}_n = -\dfrac{1}{3}T_3 C_{n-1}$ and $C_k(u_{n+1}) = 0$, we deduce that the determinant of the previous system is

$$\pm \frac{1}{3^n} T_3(c_1) \cdots T_3(c_n) \begin{vmatrix} C_0(c_1) & C_1(c_1) & \cdots & C_{n-1}(c_1) \\ C_0(c_2) & & \cdots & C_{n-1}(c_2) \\ \vdots & & & \vdots \\ C_0(c_n) & C_1(c_n) & \cdots & C_{n-1}(c_n) \end{vmatrix} \neq 0.$$

$B_n$ is a linear combination of $(\tilde{W}_n, \ldots, \tilde{W}_0)$ and it has degree $m' = N + 2\left[\frac{N+1}{4}\right]$:

$$B_n = b'_{m'} V_{m'} + \cdots + b'_0 V_0.$$

Consider now $z(t) = \varepsilon_{m'+1} b'_{m'} T_{m'+1} + \cdots + \varepsilon_1 b'_0 T_1$, we have using eq. (2):

$$\frac{z(t_i) - z(s_i)}{t_i - s_i} = B_n(u_i) = (-1)^i.$$

Because $t_i > s_i$ we deduce that $z(t_i) - z(s_i)$ has alternate signs. □

## 5 $T_2$ as a power series of $T_6 + 2$

Looking for identities in the vectorial space $\mathbf{R}[T_6] + T_2 \cdot \mathbf{R}[T_6]$, we show first some relation between $T_2$ and $T_6$.

**Lemma 11.** *For $t \in [-1, 1]$, we have*

$$T_2 + 2 = 4\sin^2\left(\frac{1}{3}\arcsin\sqrt{\frac{T_6 + 2}{4}}\right).$$

*Proof.* Let $t \in [-1, 1]$ and $x \in [\pi/3, 2\pi/3]$ such that $t = 2\cos x$.
We get $3x - \pi \in [0, \pi]$ and $\cos(3x - \pi) = -\dfrac{1}{2}T_3$ so

$$x = \frac{\pi}{3} + \frac{1}{3}\arccos\left(-\frac{T_3}{2}\right) = \frac{\pi}{3} + \frac{1}{3}\left(\frac{\pi}{2} + \arcsin\frac{T_3}{2}\right) = \frac{\pi}{2} + \frac{1}{3}\arcsin\frac{T_3}{2}.$$

We thus have

$$T_1 = 2\cos\left(\frac{\pi}{2} + \frac{1}{3}\arcsin\frac{T_3}{2}\right) = -2\sin\left(\frac{1}{3}\arcsin\frac{T_3}{2}\right).$$

We thus deduce the lemma from $T_2 = T_1^2 - 2$ and $T_6 = T_3^2 - 2$. □

**Lemma 12.** *Let* $\varphi(u) = 4\sin^2\left(\frac{1}{3}\arcsin\sqrt{u}\right)$. *For* $u \in [0,1]$, *we have*

$$\varphi(u) = \sum_{n\geq 1} \varphi_n u^n \text{ where } \varphi_1 = \frac{4}{9},\ \varphi_{n+1} = \frac{2}{9}\frac{(3n+1)(3n-1)}{(n+1)(2n+1)}\varphi_n.$$

*Proof.* We have $\varphi(u) = 2 - 2\cos\left(\frac{2}{3}\arcsin\sqrt{u}\right)$. We deduce that

$$\begin{cases} \varphi(u) = -2A + 2 \\ \frac{d}{du}\varphi(u) = \frac{2}{3}B \\ \frac{d^2}{du^2}\varphi(u) = -\frac{2}{9}\frac{1}{(u-u^2)}A + \frac{1}{3}\frac{(2u-1)}{(u-u^2)}B \end{cases} \tag{6}$$

where $A = \cos\left(\frac{2}{3}\arcsin\sqrt{u}\right)$ and $B = \dfrac{\sin\left(\frac{2}{3}\arcsin\sqrt{u}\right)}{\sqrt{u-u^2}}$.

Eliminating $A$ and $B$ from system (6), we find that

$$-4 + 2\,\varphi(u) + 9\,(1-2\,u)\,\frac{d}{du}\varphi(u) + 18\,(u-u^2)\,\frac{d^2}{du^2}\varphi(u) = 0. \tag{7}$$

$\varphi$ has a power series expansion and we get from (7)

$$\varphi_0 = 0,\ \varphi_1 = \frac{4}{9},\ \varphi_{n+1} = \frac{2}{9}\frac{(3n+1)(3n-1)}{(n+1)(2n+1)}\varphi_n.$$

□

*Remark 13.* There is no need to know explicitely $\varphi$ with the lemma **11**. One can see from $4u = v(v-3)^2$ that $\varphi$ is an algebraic function. It is therefore the solution of a differential equation we can find using Euclid algorithm. Recursion formula for the $\varphi_n$ and the differential equation can be easely obtained using the Maple package `gfun` (see [SZ]).

**Definition 14.** *Let* $\Delta$ *defined by* $\Delta f_n = f_{n+1} - f_n$. *We say that* $f_n$ *is totally monotone when for every integer* $k$ *and every* $n \geq 1$, *we have*

$$(-1)^k \Delta^k f_n > 0.$$

*Example 15.* — Let $f_n = \exp(-n)$. We get $(-1)^k\Delta^k f_n = f_n(1-1/e)^k$.
— Let $f_n = \frac{1}{n}$. We get $(-1)^k\Delta^k f_n = f_n\frac{1}{\binom{n+k}{k}}$.

They are both totally monotone.

**Proposition 16.** $\varphi_n$ *is totally monotone.*

*Proof.* We will show that

$$(-1)^k \Delta^k \varphi_n = \varphi_n \frac{P_k(n)}{(n+1)\cdots(n+k)\cdot(2n+1)\cdots(2n+2k-1)} > 0.$$

— We get

$$\Delta\varphi_n = \varphi_{n+1} - \varphi_n = \varphi_n \left(\frac{2(3n-1)(3n+1)}{9(n+1)(2n+1)} - 1\right) = -\varphi_n \frac{3n+11/9}{(n+1)(2n+1)}.$$

Suppose now that $(-1)^k \Delta^k \varphi_n = \varphi_n \dfrac{P_k(n)}{(n+1)\cdots(n+k)\cdot(2n+1)\cdots(2n+2k-1)}$.
We thus deduce

$$\begin{aligned}
(-1)^{k+1}\Delta^{k+1}\varphi_n &= -\Delta\left[\varphi_n \frac{P_k(n)}{(n+1)\cdots(n+k)\cdot(2n+1)\cdots(2n+2k-1)}\right] \\
&= \varphi_n \frac{P_k(n)}{(n+1)\cdots(n+k)\cdot(2n+1)\cdots(2n+2k-1)} - \\
&\qquad \varphi_{n+1} \frac{P_k(n+1)}{(n+2)\cdots(n+k+1)\cdot(2n+3)\cdots(2n+2k+1)} \\
&= \varphi_n \frac{(n+k+1)(2n+2k+1)P_k(n) - 2(n^2-1/9)P_k(n+1)}{(n+1)\cdots(n+k+1)\cdot(2n+1)\cdots(2n+2(k+1)-1)}.
\end{aligned}$$

We thus obtain

$$(-1)^k \Delta^k \varphi_n = \varphi_n \frac{P_k(n)}{(n+1)\cdots(n+k)\cdot(2n+1)\cdots(2n+2k-1)},$$

where $P_0 = 1$ and

$$P_{k+1}(n) = (n+k+1)(2n+2k+1)P_k(n) - 2(n^2-1/9)P_k(n+1).$$

— We will show now by induction that $P_k = a_k X^k + \cdots + a_0$ where $a_k > 0$. Suppose it is true for a given $k$, we thus deduce that

$$\begin{aligned}
P_{k+1} &= (X+k+1)(2X+2k+1)(a_k X^k + a_{k-1}X^{k-1} + \cdots) - \\
&\qquad 2(X^2-1/9)(a_k X^k + (a_{k-1} + k a_k)X^{k-1} + \cdots) \\
&= 2a_k X^{k+2} + ((4k+3)a_k + 2a_{k-1}))\, X^{k+1} + \cdots - \\
&\qquad \left[2a_k X^{k+2} + (2a_{k-1} + 2k a_k)X^{k+1} + \cdots\right] \\
&= (2k+3)a_k X^{k+1} + \cdots. \qquad (8)
\end{aligned}$$

$P_k$ is a polynomial of degree $k$ whose leading coefficient is $1 \cdot 3 \cdots (2k+1)$.

— Let us prove now by induction the following

$$(-1)^i P_k(-i) > 0,\ i = 0, \ldots, k.$$

This is true for $k = 0$.

Suppose now it is true for $P_k$. Intermediate values theorem says that $P_k$ has exactly $k$ real roots in $]-k, 0[$, so $P_k(x) > 0$ when $x \geq 0$ or when $x + k \leq 0$.

Let us compute

$$P_{k+1}(0) = (k+1)(2k+1)P_k(0) + 2/9P_k(1) > 0$$

For $i = 1, \ldots, k$ :

$$(-1)^i P_{k+1}(-i) = (k-i+1)(2(k-i)+1)(-1)^i P_k(i) + 2(i^2-2/9)(-1)^{i-1}P_k(-(i-1)) > 0.$$

For $i = -(k+1)$ we get

$$(-1)^{k+1}P_{k+1}(-(k+1)) = 0 - 2((k+1)^2 - 2/9)(-1)^{k+1}P_k(-k) > 0$$

We thus deduce that $(-1)^i P_{k+1}(-i) > 0$ for $i = 0, \ldots, k+1$.

— We thus deduce that $P_k$ has exactly $k$ roots in $]-k, 0[$ so $P_k(n)$ is nonnegative for any integer $n$. $\square$

**Definition 17.** *$f(z) = \sum_{n\geq 1} f_n z^n$ is a Stieltjes series if for every $n \geq 1$ and $m \geq 0$, one has*

$$\begin{vmatrix} f_n & f_{n+1} & \cdots & f_{n+m} \\ f_{n+1} & f_{n+2} & \cdots & f_{n+m+1} \\ \vdots & & & \vdots \\ f_{n+m} & f_{n+m+1} & \cdots & f_{n+2m} \end{vmatrix} > 0.$$

*Remark 18.* This last condition is related to the problem of Hamburger moments. It is the Stieltjes condition. The totally monotonicity is related to the Hausdorff condition (see [Ha]).

The Hausdorff condition and the Stieltjes condition are equivalent if the series is not a rational function (see [BG], p. 194 and the proof of Schönberg, [Wa], p. 267 or [Sc]). We thus deduce that

**Theorem 19.** *$\varphi(z) = \sum_{n\geq 1} \varphi_n z^n$ is a Stieltjes series.*

*Proof.* $\varphi(u)$ is an algebraic function that satisfies $4u = \varphi(\varphi - 3)^2$. Suppose that $\varphi = p/q$ where $p(u)$ and $q(u)$ are relatively prime polynomials in $u$, then we would have $4uq^3 - p^3 + 6p^2q - 9pq^2 = 0$ and $p$ would divide $u$ and $q$ would divide 1. We would have $\varphi(u) = \lambda u$ and it is not the case. Thus $\varphi$ is not a rational function and is therefore a Stieltjes function. $\square$

*Remark 20.* In example (15), the sequence $\exp(-n)$ is totally monotonic. But $\sum_n \exp(-n)z^n = \dfrac{1}{1 - e \cdot z}$ is a rational function and the condition (17) does not hold.

*Remark 21.* $\varphi(u) = 2 - 2F(1/3, -1/3, 1/2; u)$ where $F(a, b, c; z)$ is the hypergeometric function. It results from eq. (7) that is known as the hypergeometric equation ([BG,Wa])

$$\left(u - u^2\right) \frac{d^2}{du^2} f(u) + (c - (1 + a + b)\, u) \frac{d}{du} f(u) - ab\, f(u) = 0.$$

for $\varphi - 2 = -2f$, $a = -b = \dfrac{1}{3}$, $c = \dfrac{1}{2}$.

## 6 Padé approximation

Rational approximations of Stieltjes series have remarkable properties. Let us remind the following construction of Padé approximants:

**Theorem 22 (Padé approximant).** *Let $f(x) = \sum_{k \geq 1} f_k x^k$ be a Stieltjes series and consider two integers $m \leq n$. There is a unique solution $(P_n, Q_m) \in \mathbf{R}_n[x] \times \mathbf{R}_m[x]$, such that*

$$Q_m(0) = 1,\ P_n - fQ_m = 0\, [\bmod\ x^{n+m+1}]. \tag{9}$$

*Furthermore we have* $\deg P_n = n$ *and* $\deg Q_m = m$.

*Proof.* Let us write

$$P_n = p_0 + \cdots + p_n x^n,\ Q_m = q_0 + q_1 x + \cdots + q_m x^n.$$

Eq. (9) gives

$$\begin{cases} p_0 = f_0, \\ p_1 = f_0 q_1 + f_1 q_0 \\ \quad \vdots \\ p_n = f_{n-m} q_m + f_{n-m+1} q_{m-1} + \cdots + f_n q_0, \end{cases} \tag{10}$$

$$\begin{cases} 0 = f_{n-m+1} q_m + f_{n-m+2} q_{m-1} + \cdots + f_{n+1} q_0 \\ 0 = f_{n-m+2} q_m + f_{n-m+3} q_{m-1} + \cdots + f_{n+2} q_0 \\ \ \vdots \\ 0 = f_n q_m + f_{n+1} q_{m-1} + \cdots + f_{m+n} q_0. \end{cases} \tag{11}$$

The last $m \times m$ system (11) is

$$\begin{pmatrix} f_{n-m+1} & f_{n-m+2} & \cdots & f_n \\ f_{n-m+2} & f_{n-m+3} & \cdots & f_{n+1} \\ \vdots & & & \vdots \\ f_n & f_{n+1} & \cdots & f_{m+n-1} \end{pmatrix} \begin{pmatrix} q_m \\ q_{m-1} \\ \vdots \\ q_1 \end{pmatrix} = -q_0 \begin{pmatrix} f_{n+1} \\ f_{n+2} \\ \vdots \\ f_{m+n} \end{pmatrix} \tag{12}$$

and therefore has a unique solution because $f$ is a Stieltjes series and $q_0 = 1$. The first system (10) is then solved for $p_0, \ldots, p_n$.

— System (11) may be also written

$$\begin{pmatrix} f_{n-m+2} & f_{n-m+3} & \cdots & f_{n+1} \\ f_{n-m+3} & f_{n-m+4} & \cdots & f_{n+2} \\ \vdots & & & \vdots \\ f_{n+1} & f_{n+2} & \ldots & f_{m+n} \end{pmatrix} \begin{pmatrix} q_{m-1} \\ q_{m-2} \\ \vdots \\ q_0 \end{pmatrix} = -q_m \begin{pmatrix} f_{n-m+1} \\ f_{n-m+2} \\ \vdots \\ f_n \end{pmatrix}.$$

We thus deduce that if $q_m = 0$ then $Q_m = 0$ and $Q_m(0) = 0$.

— With the last equation of (10) and (11), we have

$$\begin{pmatrix} f_{n-m} & f_{n-m+1} & \cdots & f_n \\ f_{n-m+1} & f_{n-m+2} & \cdots & f_{n+1} \\ \vdots & & & \vdots \\ f_n & f_{n+1} & \ldots & f_{m+n} \end{pmatrix} \begin{pmatrix} q_m \\ q_{m-1} \\ \vdots \\ q_0 \end{pmatrix} = \begin{pmatrix} p_n \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

We thus deduce that $p_n \neq 0$. □

*Remark 23.* The system (12) shows that if $Q_m(0) = 0$, then $Q_m = 0$.

**Definition 24.** *We say that $f^{[n/m]} = P_n/Q_m$ is the Padé approximant of order $(n, m)$ of $f$.*

We will make use of a very useful theorem concerning Stieltjes series.

**Theorem 25.** *Let $f(x)$ be a Stieltjes series with radius of convergence $R$ and let us denote by $f^{[n/m]}$ its Padé approximant $P_n/Q_m$. Then*

1. *$Q_m$ has exactly $m$ real roots in $]R, +\infty[$.*
2. *Let $f^{[n/m]}(x) = \sum_{k\geq 1} f_k^{[n/m]} x^k$. We have*
   (a) *for $1 \leq k \leq n+m$, $0 < f_k^{[n/m]} = f_k$.*
   (b) *$0 \leq f_{n+m+1}^{[n/m]} < f_{n+m+1}$.*
   (c) *for $k \geq n+m+1$, $0 \leq f_k^{[n/m]} \leq f_k$.*

*Proof.* The assertion (1) is proved in [BG], p. 220. Note that the authors use the function $f(-z)$. Assertion (2a) is a consequence of the Padé approximation definition. Assertion (2c) is proved in [BG], p. 212. Note that the authors have shown that $0 \leq f_k^{[n/m]} \leq f_k$. Suppose now that $f_{n+m+1}^{[n/m]} = f_{n+m+1}$. From theorem **22**, we would have $\deg P_n + \deg Q_m = n + m + 1$ and this is not the case. We thus have $f_{n+m+1}^{[n/m]} < f_{n+m+1}$. □

We thus deduce

**Corollary 26.** *Let $m \leq n$. There are polynomials $P_n \in \mathbf{R}_n[u]$, $Q_m \in \mathbf{R}_m[u]$ and $F_{n,m} \in \mathbf{R}[v]$ such that*

$$Q_m(u)v - P_n(u) = v^{n+m+1} F_{n,m}(v),$$

*where $F_{n,m}(0) = 1$. Furthermore, we have $F_{n,m}(v) > 0$ when $v \in [0, 1]$, $\deg P_n = n$, $\deg Q_m = m$ and $Q_m(u) > 0$ for $u \in [0, 1]$.*

*Proof.* $\varphi$ is a Stieltjes series and because $\dfrac{\varphi_{n+1}}{\varphi_n} \underset{n\to\infty}{\simeq} 1 - \dfrac{3}{2n}$, we deduce that its radius of convergence $R$ is 1 and that $\sum_{n\geq 1}\varphi_n = \varphi(1) = 1$. Let $\varphi^{[n/m]} = P_n/Q_m$ be the Padé approximant of $\varphi$, we deduce that

$$\varphi(u) - \varphi^{[n/m]}(u) = \sum_{k\geq n+m+1} (\varphi_k - \varphi_k^{[n/m]})u^k = u^{n+m+1}\psi_{n,m}(u),\ 0 \leq u \leq 1.$$

We have $\psi_{n,m}(u) > 0$ for $u \in [0,1]$), from theorem **25**, (2b). From $Q_m(0) = 1$ and theorem **25**, (1) we get $Q_m(u) > 0$ for $u \in [0,1]$, and

$$vQ_m(u) - P_n(u) = u^{n+m+1}\psi_{n,m}(u)Q_m(u) > 0.$$

On the other hand, as $4u = v(v-3)^2 \underset{v\to 0}{\simeq} 9v$, we deduce that $vQ_m(u) - P_n(u)$ is a polynomial in $v$ with 0 as root of order $n+m+1$. We deduce that

$$vQ_m(u) - P_n(u) = v^{n+m+1}F_{n,m}(v),$$

where $F_{n,m}$ is a polynomial. □

We deduce

**Proposition 27.** *There exists a family $C_n$ in* vect $(W_0, \ldots, W_n)$*, such that*

$$C_n = t^{2n+1}F_n,\ F_n(0) = 1.$$

*Furthermore,* $\deg C_n = 2n + 2\left[\frac{n}{2}\right] + 1$ *and $F_n(t) > 0$ for $t \in [-2,2]$.*

*Proof.* Let us consider

$$C_{k,l}(t) = vQ_l(u) - P_k(u) = v^{k+l+1}F_{k,l}(v)$$

given by corollary **26**. Note that $Q_l(u) > 0$ for $u \in [0,1]$.

— If $t \in [-1,1]$, we have $u, v \in [0,1]$ and the announced result by corollary **26**.

— We have $u([1,2]) = u([-2,-1]) = u([0,1]) = [0,1]$. Let $|t| \in [1,2]$. There exists $t_1 \in ]0,1]$, such that $u = u(t) = u(t_1) = u_1$ and we have $v = v(t) = t^2 \geq t_1^2 = v(t_1) = v_1$. We deduce

$$\begin{aligned} C_{k,l}(t) = vQ_l(u) - P_k(u) &= vQ_l(u_1) - P_k(u_1) \\ &\geq v_1Q_l(u_1) - P_k(u_1) > 0. \end{aligned}$$

In conclusion, for $t \in [-2,2]$, we have $F_{k,l}(t) > 0$.

— $Q_l(u) \in (T_2+2)\mathbf{R}_l[T_6+2]$ and $P_k \in \mathbf{R}_k[T_6+2]$. We thus deduce that $C_{k,l} \in \mathbf{R}[T_6] \oplus T_2\mathbf{R}[T_6]$. Note that $\deg C_{k,l} = \max(6k+2, 6l)$.

— If $n = 2k+1$, let $C_n = t \cdot C_{k,k}$. If $n = 2k$, let $C_n = t \cdot C_{k,k-1}$. $C_n$ has degree $2n + 2\left[\frac{n}{2}\right] + 1$ and therefore $C_n \in$ vect $(W_0, \ldots, W_n)$. □

*Remark 28.* We have proved the existence of $C_n$. This is an upper-triangular basis of $E$ with respect to the $W_i$. It is unique and it can be computed by simple LU-decomposition of the matrix whose lines are the $W_i$.

## 7 Conclusion

We have shown in this paper the existence of plane polynomial curves of degree $(3, N + 2[\frac{N}{4}] + 1)$ having the required properties. We think that they are of minimal lexicographic degrees (it is true for $N = 3, 5, 7, 9$). This question is related to the following question: where are the real zeros of polynomials in vect $(V_k, k \neq 2 \,[\mathrm{mod}\ 3])$? We guess that such polynomials cannot have too many zeroes in $[-1, 1]$. It would give a lower bound for the degrees of the torus knots approximation by polynomial curves.

We have not given explicit formulas for our polynomials. We have just shown that they can be found by solving some explicit linear system. In a near future, we hope we will be able to give explicit function of the degree $N$.

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