

# On the Chow ring of certain algebraic hyper-Kähler manifolds

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*Pour Fedya Bogomolov, en l'honneur de ses 60 ans*

## 0 Introduction

This paper proposes and studies a generalization of a conjecture made by Beauville in [3]. Recall that Beauville and the author proved the following result in [5]:

**Theorem 0.1** *Let  $S$  be an algebraic K3 surface. Then there exists a degree 1 0-cycle  $o$  on  $S$  satisfying the property that for any line bundle  $L$  on  $S$ , one has*

$$c_1(L)^2 = [c_1(L)]^2 o \text{ in } CH_0(S).$$

*Furthermore, we have  $c_2(T_S) = 24o$ .*

(In this paper, Chern classes will be Chern classes in the Chow ring tensored by  $\mathbb{Q}$ , and we will denote by  $[c_i]$  the corresponding rational cohomology classes.)

This result can be rephrased by saying that any polynomial relation

$$P([c_1(L_i)]) = 0 \text{ in } H^*(S, \mathbb{Q}), L_i \in \text{Pic } S,$$

already holds in  $CH(S)$ .

In [3], Beauville conjectured that a similar result holds for algebraic hyper-Kähler varieties:

**Conjecture 0.2** *(Beauville) Let  $Y$  be an algebraic hyper-Kähler variety. Then any polynomial cohomological relation*

$$P([c_1(L_i)]) = 0 \text{ in } H^*(Y, \mathbb{Q}), L_i \in \text{Pic } Y$$

*already holds at the level of Chow groups :*

$$P(c_1(L_i)) = 0 \text{ in } CH(Y).$$

He proved in [3] this conjecture in the case of the second and third punctual Hilbert scheme of an algebraic K3 surface.

In this paper, we observe that the results of [5] can lead to a more general conjecture concerning the Chow ring of an algebraic hyper-Kähler variety. Namely, the full statement of Theorem 0.1 can be interpreted by saying that any polynomial relation between  $[c_2(T_S)], [c_1(L_i)]$  in  $H^*(S, \mathbb{Q})$ , already holds between  $c_2(T_S), c_1(L_i)$  in  $CH(S)$ . The purpose of this paper is to study the following conjecture:

**Conjecture 0.3** *Let  $Y$  be an algebraic hyper-Kähler variety. Then any polynomial cohomological relation*

$$P([c_1(L_j)], [c_i(T_Y)]) = 0 \text{ in } H^{2k}(Y, \mathbb{Q}), L_j \in \text{Pic } Y$$

*already holds at the level of Chow groups :*

$$P(c_1(L_j), c_i(T_Y)) = 0 \text{ in } CH^k(Y).$$

We shall prove the following results:

**Theorem 0.4** *1) Conjecture 0.3 holds for  $Y = S^{[n]}$ , for  $n \leq 2b_2(S)_{tr} + 4$  and any  $k$ , where  $S^{[n]}$  is the Hilbert scheme of length  $n$  subschemes of an algebraic K3 surface  $S$ .*

*2) Conjecture 0.3 is true for any  $k$  when  $Y$  is the Fano variety of lines of a cubic fourfold.*

In 1),  $b_2(S)_{tr} = b_2(S) - \rho$  is the rank of the transcendental lattice of  $S$ .

Concerning point 2), recall from [4] that the variety of lines  $F$  of a cubic fourfold  $X$  is a deformation of  $S^{[2]}$ , for  $S$  an algebraic K3 surface, but that for general  $X$ , it has  $\text{Pic } F = \mathbb{Z}$  and thus it is not a Hilbert scheme. Even when  $\rho(F) \geq 2$  it is not necessarily the case that  $F$  is a  $S^{[2]}$ . In [3], Beauville asked whether his conjecture 0.2 holds true for the variety of lines of a cubic fourfold.

Finally, we also prove the following.

**Theorem 0.5** *Conjecture 0.3 holds for  $Y = S^{[n]}$ , and  $k = 2n - 2, 2n - 1, 2n$ , for any  $S$  as above and any  $n$ .*

The cohomology ring of the Hilbert scheme of a K3 surface has been computed in [14], [16]. For the subring generated by  $H^2$ , one can use the result of Verbitsky [18], [9]. The question of understanding more precisely the Chow ring is rather delicate and we are dealing here only with a small part of it.

We prove in section 1 part 1) of Theorem 0.4 and Theorem 0.5 . The proof involves particular cases of the following statement :

**Conjecture 0.6** *Let  $S$  be an algebraic K3 surface. For any integer  $m$ , let  $P \in CH(S^m)$  be a polynomial expression in*

$$pr_i^* c_1(L_s), L_s \in \text{Pic } S, pr_j^* o, pr_{kl}^* \Delta_S.$$

*Then if  $[P] = 0$ , we have  $P = 0$ .*

We also prove that Conjecture 0.6 for  $S$  and any  $m' \leq m$  implies Conjecture 0.3 for  $Y = S^{[m]}$ .

In section 2 we deal with the case of the variety of lines of the cubic fourfold (Theorem 0.4, 2)).

It is a pleasure to dedicate this paper to Fedya Bogomolov, who greatly contributed in the papers [7], [8], [9] to the study of hyper-Kähler manifolds.

# 1 Case of the Hilbert scheme of a $K3$ surface

Let  $S$  be an algebraic  $K3$  surface, and  $S^{[n]}$  be the Hilbert scheme of length  $n$  subschemes of  $S$ . For any line bundle  $L$  on  $S$ , there is an induced line bundle, which we still denote by  $L$  on  $S^{[n]}$ , which is the pull-back via the Hilbert-Chow morphism of the line bundle on  $S^{(n)}$  corresponding to the  $\mathfrak{S}_n$ -invariant line bundle  $L \boxtimes \dots \boxtimes L$  on  $S^n$ .

There are furthermore two natural vector bundles on  $S^{[n]}$ , namely  $\mathcal{O}_{[n]}$ , which is defined as  $R^0 p_* \mathcal{O}_{\Sigma_n}$ , where

$$\Sigma_n \subset S^{[n]} \times S, p = pr_1 : \Sigma_n \rightarrow S^{[n]}$$

is the incidence scheme, and the tangent bundle  $T_n$ . It is not clear that the Chern classes of  $\mathcal{O}_{[n]}$  can be expressed as polynomials in  $c_1(\mathcal{O}_{[n]})$  and the Chern classes of  $T_n$ . The following result may thus be stronger than Theorem 0.4, 1):

**Theorem 1.1** *Let  $n \leq 2b_2(S)_{tr} + 4$ , and let  $P \in CH(S^{[n]})$  be any polynomial expression in the variables*

$$c_1(L), L \in Pic S \subset Pic S^{[n]}, c_i(\mathcal{O}_{[n]}), c_j(T_n) \in CH(S^{[n]}).$$

*Then if  $P$  is cohomologous to 0, we have  $P = 0$  in  $CH(S^{[n]})$ .*

This implies Theorem 0.4 for the  $n$ -th Hilbert scheme of  $K3$  surface  $S$  with  $n \leq 2b_2(S)_{tr} + 4$ , because we have  $c_1(\mathcal{O}_{[n]}) = -\delta$ , where  $2\delta \equiv E$  is the class of the exceptional divisor of the resolution  $S^{[n]} \rightarrow S^{(n)}$ , and it is well-known that  $Pic S^{[n]}$  is generated by  $Pic S$  and  $\delta$ .

To start the proof of this theorem, we establish first the following Proposition 1.2, which gives particular cases of Conjecture 0.6. Let  $o \in CH^2(S)$  be the cycle introduced in the introduction. Let  $m$  be an integer.

**Proposition 1.2** *Let  $P \in CH(S^m)$  be a polynomial expression in the variables*

$$pr_i^* \left( \frac{1}{24} c_2(T) \right) = pr_i^* o, pr_j^* c_1(L_s), L_s \in Pic S, pr_{kl}^* \Delta_S, k \neq l,$$

*where  $\Delta_S \subset S \times S$  is the diagonal. Assume that one of the following assumptions is satisfied:*

1.  $m \leq 2b_2(S)_{tr} + 1$ .
2.  $P$  is invariant under the action of the symmetric group  $\mathfrak{S}_{m-2}$  acting on the  $m - 2$  first indices.

*Then if  $P$  is cohomologous to 0, it is equal to 0 in  $CH(S^m)$ .*

Using the results of [5], this proposition is a consequence of the following lemma:

**Lemma 1.3** *The polynomial relations  $[P] = 0$  in the cohomology ring  $H^*(S^m)$ , satisfying one of the above assumptions on  $m, P$ , are all generated (as elements of the ring of all polynomial expressions in the variables above) by the following polynomial relations, the list of which will be denoted by (\*) :*

1.  $[pr_i^*(c_1(L)) \cdot pr_i^*o] = 0, L \in Pic S, [pr_i^*(o) \cdot pr_i^*(o)] = 0.$
2.  $[pr_i^*(c_1(L)^2 - [c_1(L)]^2o)] = 0, L \in Pic S.$
3.  $[pr_{ij}^*(\Delta_S \cdot p_1^*o - (o, o))] = 0,$  where  $p_1$  here is the first projection of  $S \times S$  to  $S,$  and  $(o, o) = p_1^*o \cdot p_2^*o.$
4.  $[pr_{ij}^*(\Delta_S \cdot p_1^*c_1(L) - c_1(L) \times o - o \times c_1(L))] = 0, L \in Pic S,$  where  $p_1$  here is the first projection of  $S \times S$  to  $S,$  and  $c_1(L) \times o = p_1^*c_1(L) \cdot p_2^*o.$
5.  $[pr_{ijk}^*(\Delta_3 - p_{12}^*\Delta_S \cdot p_3^*o - p_1^*o \cdot p_{23}^*\Delta_S - p_{13}^*\Delta_S \cdot p_2^*o + p_{12}^*(o, o) + p_{23}^*(o, o) + p_{13}^*(o, o))] = 0.$
6.  $[pr_{ij}^*\Delta_S]^2 = 24pr_{ij}^*(o, o) = 24pr_i^*o \cdot pr_j^*o.$

In 5,  $\Delta_3$  is the small diagonal of  $S^3$  and the  $p_i, p_{ij}$  are the various projections from  $S^3$  to  $S, S \times S$  respectively. Note that  $\Delta_3$  can be expressed as  $p_{12}^*\Delta_S \cdot p_{23}^*\Delta_S.$  Furthermore we have

$$pr_{ij}^* \circ p_1^* = pr_i^*, pr_{ijk}^* \circ p_{12}^* = pr_{ij}^*, pr_{ijk}^* \circ p_i^* = pr_i^*.$$

Thus all the relations in (\*) are actually polynomial expressions in the variables

$$[pr_i^*o], [pr_j^*c_1(L)], L \in Pic S, [pr_{kl}^*\Delta_S], k \neq l.$$

Assuming this Lemma, we conclude that for  $m \leq 2b_2(S)_{tr} + 1,$  all polynomial relations  $[P] = 0$  in the variables  $pr_i^*o, pr_j^*c_1(L), L \in Pic S, pr_{kl}^*\Delta_S, k \neq l$  which hold in  $H^*(S^m)$  also hold in  $CH(S^m),$  because we know from [5] that the relations listed in (\*) hold in  $CH(S^m).$  In fact, (apart from the relations 1 and 3 which obviously hold in  $CH(S^m),$  these relations are pulled-back, via the maps  $pr_i,$  resp.  $pr_{ij},$  resp.  $pr_{ijk},$  from relations in  $CH(S),$  resp.  $CH(S^2),$  resp.  $CH(S^3),$  which are established in [5].

Similarly, for any  $m,$  the same conclusion holds for polynomial relations invariant under  $\mathfrak{S}_{m-2}.$

This concludes the proof of Proposition 1.2. ■

**Proof of Lemma 1.3.** Let  $\mathcal{B}$  be a basis of  $Pic S.$  It is clear that modulo the relations generated by (\*), any polynomial in the variables

$$[pr_{ij}^*\Delta_S], [pr_k^*c_1(L)], L \in \mathcal{B}, [pr_l^*o], \tag{1.1}$$

can be written as a combination of monomials having the property that an index  $i \in \{1, \dots, n\}$  appears only once. Indeed, these relations express any product with a repeated index as a combination of monomials with no repeated index. Furthermore, if we start from a polynomial which is invariant under the action of  $\mathfrak{S}_{m-2},$  as the set of relations (\*) is stable under this action, it is clear that replacing systematically each repeated index by the corresponding combination with no repeated index using (\*), we will end with a polynomial expression invariant under the action of  $\mathfrak{S}_{m-2}.$

We claim now that if  $m \leq 2b_2(S)_{tr} + 1,$  no non zero combination of monomials with no repeated index vanishes in  $H^*(S^m).$  Furthermore, for any  $m,$  no non zero combination of monomials with no repeated index which is invariant under  $\mathfrak{S}_{m-2}$  vanishes in  $H^*(S^m).$

To prove the claim, consider the transcendental part of  $H^2(S, \mathbb{Q})$ ,

$$H^2(S, \mathbb{Q})_{tr} := NS(S)^\perp.$$

We have the direct sum decomposition

$$H^*(S, \mathbb{Q}) = H^2(S, \mathbb{Q})_{tr} \oplus H^*(S, \mathbb{Q})_{alg}, \quad (1.2)$$

where  $H^*(S, \mathbb{Q})_{alg}$  is generated by

$$H^0(S, \mathbb{Q}), NS(S)_\mathbb{Q}, H^4(S, \mathbb{Q}).$$

The decomposition (1.2) induces for any  $m$  a direct sum decomposition of

$$H^*(S^m, \mathbb{Q}) = H^*(S, \mathbb{Q})^{\otimes m} = pr_1^* H^*(S, \mathbb{Q}) \otimes \dots \otimes pr_m^* H^*(S, \mathbb{Q}).$$

Let  $[\Delta_S]_{tr}$  be the projection of  $[\Delta_S]$  in the direct summand

$$H^2(S, \mathbb{Q})_{tr} \otimes H^2(S, \mathbb{Q})_{tr}$$

of  $H^4(S \times S, \mathbb{Q})$ . Then we have  $[\Delta_S]_{tr} \neq 0$  and

$$[\Delta_S] = [\Delta_S]_{tr} + p_1^*[o] + p_2^*[o] + \sum_{i,j \in \mathcal{B}} \alpha_{ij} p_1^*[c_1(L_i)] \cdot p_2^*[c_1(L_j)].$$

It is then clear that it suffices to prove the claim with  $pr_{ij}^*[\Delta_S]$  replaced by  $pr_{ij}^*[\Delta_S]_{tr}$  in the set of variables (1.1).

Let  $M$  be a monomial of the form above, and let  $I_M \subset \{1, \dots, m\}$  be the set of indices  $i$  appearing in  $M$  via a diagonal, i.e. for some  $l \neq i$ , the variable  $pr_{il}^*[\Delta_S]_{tr}$  appears in  $M$ . Then  $I_M$  is also the unique index set for which the projection of  $M$  in

$$\bigotimes_{i \notin I_M} H^*(S, \mathbb{Q})_{alg} \otimes \bigotimes_{i \in I_M} H^2(S, \mathbb{Q})_{tr}$$

is non zero. Hence it follows that a relation

$$\sum_M \alpha_M M = 0$$

implies for each fixed  $I \subset \{1, \dots, m\}$  (of even cardinality), by projection onto

$$\bigotimes_{i \notin I} pr_i^* H^*(S, \mathbb{Q})_{alg} \otimes \bigotimes_{i \in I} pr_i^* H^2(S, \mathbb{Q})_{tr},$$

a relation of the form

$$\sum_{M, I_M=I} \alpha_M M = 0. \quad (1.3)$$

Now note that we can further decompose each term  $\bigotimes_{i \notin I} pr_i^* H^*(S, \mathbb{Q})_{alg}$  using the basis of  $H^*(S, \mathbb{Q})_{alg}$  given by  $H^4(S, \mathbb{Q})$ ,  $H^0(S, \mathbb{Q})$  and the basis  $\mathcal{B}$ . Then the relation (1.3) decomposes into a sum of relations of the form

$$(\bigotimes_{j \notin I} pr_j^* \alpha_j) \otimes \left( \sum_{I_{M'}=I} \alpha'_{M'} M' \right), \quad (1.4)$$

for any given  $I$  and given set  $(\alpha_j)_{j \notin I}$  of chosen elements in the basis

$$H^4(S, \mathbb{Q}), H^0(S, \mathbb{Q}), \mathcal{B}$$

of  $H^*(S, \mathbb{Q})_{alg}$ . Here each  $M'$  is a monomial in the  $pr_{ij}^*[\Delta_S]_{tr}$ , and  $I_{M'} = I$  means that only indices  $i, j \in I$  appear in the monomial  $M'$ , and each index  $l \in I$  appears exactly once.

Of course, (1.4) is equivalent to the relation

$$\sum_{I_{M'}=I} \alpha'_{M'} M' = 0, \quad (1.5)$$

which has to hold in  $H^{2s}(S^m, \mathbb{Q})$  or equivalently in  $H^{2s}(S^I, \mathbb{Q})$ . (Here  $2s$  is the cardinality of  $I$ , and  $S^I$  is the product of the copies of  $S$  indexed by  $I$ . Thus clearly (1.5) is pulled-back via the projection  $S^m \rightarrow S^I$  from the corresponding relation in  $S^I$ .)

Now observe that if we started with a polynomial relation invariant under the action of  $\mathfrak{S}_{m-2}$ , each relation we get in (1.5) is invariant under the symmetric group  $\mathfrak{S}'_I$  permuting the elements of  $I$  which are  $\leq m-2$ .

In conclusion, we are reduced to prove that for each  $I \subset \{1, \dots, m\}$  of cardinality  $2s$ , and thus satisfying  $2s \leq m \leq 2b_2(S)_{tr} + 1$ , there are no relations in  $H^{2s}(S_I)$  between monomials of the form

$$\prod pr_{ij}^*[\Delta_S]_{tr}, \quad (1.6)$$

where each index  $i, j \in I$  appears exactly once. Furthermore, for any  $m$ , there are no  $\mathfrak{S}'_I$ -invariant relations in  $H^{2s}(S_I)$  between monomials of the form above.

For the statement concerning  $\mathfrak{S}'_I$ -invariant relations, this is obvious, as the symmetric group  $\mathfrak{S}'_I$  acts with at most two distinct orbits on the set of monomials of degree  $s$  in the variables  $pr_{ij}^*[\Delta_S]_{tr}$ ,  $i, j \in I$ , with no repeated indices, namely, in the case where  $m-1, m \in I$ , those monomials containing  $p_{m-1, m}^*[\Delta_S]_{tr}$  and those not containing it. Assume that  $m-1, m \in I$ , as otherwise the action is transitive and the result is still simpler.

Then the only possible non zero  $\mathfrak{S}'_I$ -invariant relation would be of the form:

$$\beta \sum_{M \in \mathcal{E}} M = \alpha \sum_{M \in \mathcal{F}} M, \quad (1.7)$$

where  $\mathcal{E}$  is the set of monomials not containing  $p_{m-1, m}^*[\Delta_S]_{tr}$ , and  $\mathcal{F}$  is the set of monomials containing  $p_{m-1, m}^*[\Delta_S]_{tr}$ . We identify  $I$  with  $\{1, \dots, 2s\}$  in such a way that  $m-1$  is identified with  $2s-1$  and  $m$  with  $2s$ . Then this gives an identification of  $S^I$  with  $S^s \times S^s$  and we can consider each  $M$  as above as a self-correspondence of  $S^s$ . Let  $p_1, p_2$  be the two projections of  $S^{2s}$  to  $S^s$ . Then each monomial  $M$  as above induces a map

$$\gamma_M : H^{2s,0}(S^s) \rightarrow H^{2s,0}(S^s),$$

given by

$$\gamma_M(\eta) = p_{1*}(M \cdot p_2^*\eta). \quad (1.8)$$

Now we have  $\gamma_M = 0$  when the monomial  $M$  has the property that for two indices  $i, j \leq s$ ,  $pr_{ij}^*[\Delta_S]_{tr}$  divides  $M$ , because  $\omega^2 = 0$  on  $S$ . In particular, we have  $\gamma_M = 0$  for  $M \in \mathcal{F}$ . Thus (1.7) gives

$$\beta \sum_{M \in \mathcal{E}} \gamma_M = 0. \quad (1.9)$$

On the other hand, when the monomial  $M$  has the property that for any indices  $i \neq j \leq s$ ,  $pr_{ij}^*[\Delta_S]_{tr}$  does not divide  $M$ ,  $M$  is of the form

$$\prod_{i \leq s} pr_{i, s+\sigma(i)}^*[\Delta_S]_{tr}$$

for some permutation  $\sigma$  of  $\{1, \dots, s\}$ . In that case, we have

$$\gamma_M(pr_1^*\omega \cdot \dots \cdot pr_s^*\omega) = pr_{\sigma(1)}^*\omega \cdot \dots \cdot pr_{\sigma(s)}^*\omega = p_1^*\omega \cdot \dots \cdot p_s^*\omega. \quad (1.10)$$

Thus we get by (1.9) and (1.10)

$$\beta \sum_{M \in \mathcal{E}} \gamma_M = \beta s! Id_{H^{2s,0}(S^s)} = 0,$$

which implies that  $\beta = 0$ , and that if  $\alpha \neq 0$ , the relation reduces to  $\sum_{M \in \mathcal{F}} M = 0$ . But elements of  $\mathcal{F}$  are of the form

$$p_{m-1, m}^*[\Delta_S]_{tr} \cdot M',$$

where  $M' \in \mathcal{F}'$  are the monomials with no repeated indices in the  $pr_{ij}^*[\Delta_S]_{tr}$ ,  $i, j < m-1$ ,  $i, j \in I$ . The relation  $\sum_{M \in \mathcal{F}} M = 0$  thus provides  $\sum_{M' \in \mathcal{F}'} M' = 0$ , which has just been proved to be impossible.

In the case where  $2s \leq m \leq 2b_2(S)_{tr} + 1$ , we have  $s \leq b_2(S)_{tr}$ . The intersection form on  $H^2(S, \mathbb{C})_{tr}$  is non degenerate. Thus we can choose an orthonormal basis  $\alpha_i$ ,  $1 \leq i \leq b_2(S)_{tr}$ , of  $H^2(S, \mathbb{C})_{tr}$ . Let

$$\eta := pr_1^*\alpha_1 \cdot \dots \cdot pr_s^*\alpha_s.$$

For each monomial  $M$ , consider now  $\gamma_M : H^{2s}(S^s, \mathbb{C}) \rightarrow H^{2s}(S^s, \mathbb{C})$ , defined as in (1.8). As we have  $\langle \alpha_i, \alpha_j \rangle = 0$  for  $i \neq j$ , we find that if there are two indices  $i, j > s$  such that  $p_{ij}^*[\Delta_S]_{tr}$  appears in  $M$ , we have  $\gamma_M(\eta) = 0$ , and that the remaining  $M$ 's are in one-to-one correspondence with permutations  $\sigma$  of  $\{1, \dots, s\}$ . Then, as before, we have for such  $M$ :

$$\gamma^*(\eta) = p_{\sigma(1)}^*\alpha_1 \cdot \dots \cdot p_{\sigma(s)}^*\alpha_s = p_1^*\alpha_{\sigma^{-1}(1)} \cdot \dots \cdot p_s^*\alpha_{\sigma^{-1}(s)}.$$

As the tensors  $pr_1^*\alpha_{\sigma^{-1}(1)} \cdot \dots \cdot pr_s^*\alpha_{\sigma^{-1}(s)}$ ,  $\sigma \in \mathfrak{S}_s$  are linearly independent in  $H^2(S, \mathbb{C})_{tr}$ , we conclude from these two facts that a relation  $\sum_M \alpha_M M = 0$  implies  $\alpha_M = 0$  for all those  $M$  such that for no indices  $i, j > s$ ,  $pr_{ij}^*[\Delta_S]_{tr}$  appears in  $M$ .

To show that the other coefficients  $\alpha_M$  must be also 0, we introduce maps similar to the  $\gamma_M$ , defined by choosing any subset  $I_1 = \{i_1, \dots, i_s\}$ ,  $i_1 < \dots < i_s$  of  $I$ . Denoting by  $I_2$  the complementary set, we define  $\gamma_M^{I_1}$  as  $p_{I_1^*} \circ (M \cdot p_{I_2}^*)$ , where  $p_{I_1}$  (resp.  $p_{I_2}$ ) is the projection from  $S_I \cong S^{2s}$  to  $S^s$  determined by the (ordered) set  $I_1$  (resp.  $I_2$ ). For any  $M$ , there is a choice of  $I_1$  such that for no indices  $i, j \in I_2$ ,  $pr_{ij}^*[\Delta_S]_{tr}$  appears in  $M$ , and then we conclude as before that  $\alpha_M$  must also be 0.

Thus Lemma 1.3 and also Proposition 1.2 are proved. ■

We come now to the geometry of  $S^{[n]}$ . Let us introduce the following notation: let

$$\mu = \{\mu_1, \dots, \mu_m\}, m = m(\mu), \sum_i |\mu_i| = n$$

be a partition of  $\{1, \dots, n\}$ . Such a partition determines a partial diagonal

$$S_\mu \cong S^m \subset S^n,$$

defined by the conditions

$$x = (x_1, \dots, x_n) \in S_\mu \Leftrightarrow x_i = x_j \text{ if } i, j \in \mu_l, \text{ for some } l.$$

Consider the quotient map

$$q_\mu : S^m \cong S_\mu \rightarrow S^{(n)},$$

and denote by  $E_\mu$  the following fibered product:

$$E_\mu := S_\mu \times_{S^{(n)}} S^{[n]} \subset S^m \times S^{[n]}.$$

We view  $E_\mu$  as a correspondence between  $S^m$  and  $S^{[n]}$  and we will denote as usual by  $E_\mu^* : CH(S^{[n]}) \rightarrow CH(S^m)$  the map

$$\alpha \mapsto pr_{1*}(pr_2^*(\alpha) \cdot E_\mu).$$

Let us denote by  $\mathfrak{S}_\mu$  the subgroup of  $\mathfrak{S}_m$  permuting only the indices  $i, j$  for which the cardinalities of  $\mu_i, \mu_j$  are equal. The group  $\mathfrak{S}_\mu$  can be seen as the quotient of the global stabilizer of  $S_\mu$  in  $S^n$  by its pointwise stabilizer. In this way the action of  $\mathfrak{S}_\mu$  on  $S_\mu \cong S^m$  is induced by the action of  $\mathfrak{S}_n$  on  $S^n$ .

We have the following result:

**Proposition 1.4** *Let  $P \in CH(S^{[n]})$  be a polynomial expression in  $c_i(\mathcal{O}_{[n]})$ ,  $c_j(T_n)$ . Then for any  $\mu$  as above,  $E_\mu^*(P) \in CH(S^m)$  is a polynomial expression in  $pr_s^*o$ ,  $pr_{lk}^*\Delta_S$ . Furthermore,  $E_\mu^*(P)$  is invariant under the group  $\mathfrak{S}_\mu$ .*

Note that the last statement is obvious, since  $\mathfrak{S}_\mu$  leaves invariant the correspondence  $E_\mu \subset S_\mu \times S^{[n]}$ .

We postpone the proof of this proposition and conclude the proofs of the theorems.

**Proof of Theorem 1.1.** From the work of De Cataldo-Migliorini [11], it follows that the map

$$(E_\mu^*)_{\mu \in Part(\{1, \dots, n\})} : CH(S^{[n]}) \rightarrow \Pi_\mu CH(S^{m(\mu)})$$

is injective. Let now  $P \in CH(S^{[n]})$  be a polynomial expression in  $c_1(L)$ ,  $L \in Pic S \subset Pic S^{[n]}$ ,  $c_i(\mathcal{O}_{[n]})$ ,  $c_j(T_n) \in CH(S^{[n]})$ . Note first that for  $L \in Pic S$ , and for each  $\mu$ , the restriction of  $pr_2^*L$  to  $E_\mu \subset S_\mu \times S^{[n]}$  is a pull-back  $pr_1^*L_{\mu|E_\mu}$ , where  $L_\mu \in Pic S_\mu = Pic S^m$  is equal to  $L^{\otimes |\mu_1|} \boxtimes \dots \boxtimes L^{\otimes |\mu_m|}$ . This follows from the fact that  $L$  is the pull-back of a line bundle on  $S^{(n)}$ . Note that  $L_\mu$  is invariant under  $\mathfrak{S}_\mu$ .

Thus it follows from Proposition 1.4 and the projection formula that for each partition  $\mu$ ,  $E_\mu^*(P)$  is a polynomial expression in  $pr_i^*c_1(L)$ ,  $pr_k^*o$ ,  $pr_{lm}^*\Delta$  which is invariant under the group  $\mathfrak{S}_\mu$ .

Now, if  $P$  is cohomologous to 0, each  $E_\mu^*(P)$  is cohomologous to 0. Let us now verify that the assumptions of Proposition 1.2 are satisfied. Recall that we assume  $n \leq 2b_2(S)_{tr} + 4$ . If  $m(\mu) \leq 2b_2(S) + 1$ , Proposition 1.2 applies. Otherwise,  $m(\mu) \geq 2b_2(S) + 2$  and, as  $n \leq 2b_2(S)_{tr} + 4$ , it follows that the partition  $\mu$  contains at most two sets of cardinality  $\geq 2$ . Thus the group  $\mathfrak{S}_\mu$  contains in this case a group conjugate to  $\mathfrak{S}_{m(\mu)-2}$ . Proposition 1.2 thus applies, and gives  $E_\mu^*(P) = 0$  in  $CH(S_\mu)$ , for all  $\mu$ .

It follows that  $P = 0$  by the result of De Cataldo-Migliorini. This concludes the proof of Theorem 1.1.  $\blacksquare$

**Proof of Theorem 0.5.** Let  $P \in CH^k(S^{[n]})$ , with  $k \geq 2n - 2$  be a polynomial expression in  $c_1(L)$ ,  $L \in Pic S \subset Pic S^{[n]}$ ,  $c_i(\mathcal{O}_{[n]})$ ,  $c_j(T_n) \in CH(S^{[n]})$ , and assume that  $[P] = 0$ . Notice that because  $k \geq 2n - 2$ , we have  $E_\mu^*P = 0$  if the image of  $E_\mu$  in  $S^{[n]}$  has codimension  $> 2$ . This is the case once  $m(\mu) < n - 2$ . On the other hand, if  $m(\mu) \geq m - 2$ , the partition  $\mu$  has at most two sets  $\mu_i$  of cardinality  $\geq 2$ . Hence for  $m(\mu) \geq 2$ , the group  $\mathfrak{S}_\mu$  contains a group conjugate to  $\mathfrak{S}_{m(\mu)-2}$ . As  $[E_\mu^*P] = 0$ , and  $E_\mu^*P$  is a  $\mathfrak{S}_\mu$ -invariant polynomial expression in  $pr_i^*c_1(L)$ ,  $pr_j^*o$ ,  $pr_{ij}^*\Delta_S$ , Proposition 1.2 thus applies, and gives  $E_\mu^*(P) = 0$  in  $CH(S_\mu)$  for  $m(\mu) \geq n - 2$ . As we also have  $E_\mu^*(P) = 0$  in  $CH(S_\mu)$  for  $m(\mu) < n - 2$ , the theorem of De Cataldo-Migliorini shows that  $P = 0$ .  $\blacksquare$

To conclude, let us notice that Proposition 1.4 and the end of the proof of Theorem 1.1 prove the following:

**Proposition 1.5** *Conjecture 0.6 for  $S$ , and any  $m \leq n$ , implies Conjecture 0.3 for  $S^{[n]}$ .*

It remains to prove Proposition 1.4. For the proof, we use the formulas proved in [13], which allow induction on  $n$ . As in [13], in order to get the result by induction, we will need to introduce a more general induction statement, which is the following: For each integer  $l$ , we can also consider the correspondence

$$E_{\mu,l} := E_\mu \times \Delta_{S^l}$$

between  $S_\mu \times S^l$  and  $S^{[n]} \times S^l$ , where  $\Delta_{S^l}$  is the diagonal of  $S^l$ . On  $S^{[n]} \times S^l$ , we have the natural classes  $pr_{0i}^*c_s(\mathcal{I}_n)$ , where  $\mathcal{I}_n$  is the ideal sheaf of the universal subscheme  $\Sigma_n \subset S^{[n]} \times S$ , and  $pr_{0i}$  is the projection onto the product of the first factor  $S^{[n]}$  and the  $i$ -th factor of  $S^l$ . We shall denote  $pr_0$  the projection onto the first factor  $S^{[n]}$ , and  $pr_i$  the projection onto the  $i$ -th factor of  $S^l$ .

The induction statement, which will be proved by induction on  $n$ , is the following generalization of Proposition 1.4 (which is the  $l = 0$  case):

**Proposition 1.6** *Let  $P \in CH(S^{[n]} \times S^l)$  be a polynomial expression in*

$$pr_0^*c_r(\mathcal{O}_{[n]}), pr_0^*c_s(T_n), pr_{0i}^*c_s(\mathcal{I}_n), 1 \leq i \leq l.$$

*Then for any  $\mu$  as above,*

$$E_{\mu,l}^*(P) \in CH(S^m \times S^l) = CH(S^{m+l})$$

*is a polynomial expression in the  $pr_j^*o$ ,  $pr_{ik}^*\Delta$ ,  $i, j, k \leq m + l$ .*

**Proof.** Consider the smooth variety  $S^{[n,n-1]}$  parameterizing pairs  $(z, z')$  of subschemes of  $S$ , of length  $n$  and  $n-1$  respectively, such that  $z' \subset z$ .

$S^{[n,n-1]}$  admits a natural map  $\rho$  to  $S$ , which to  $(z, z')$  associates the residual point of  $z'$  in  $z$ . Together with the two natural projections  $\psi$  to  $S^{[n]}$  and  $\phi$  to  $S^{[n-1]}$  respectively, this gives two maps:

$$\psi : S^{[n,n-1]} \rightarrow S^{[n]}, \quad \sigma = (\phi, \rho) : S^{[n,n-1]} \rightarrow S^{[n-1]} \times S.$$

$\sigma$  is birational; in fact it is the blow-up of  $S^{[n-1]} \times S$  along the incidence subscheme  $\Sigma_{n-1} \subset S^{[n-1]} \times S$ . We shall denote by  $\mathcal{L}$  the line bundle  $\mathcal{O}(-E)$  on  $S^{[n,n-1]}$ , where  $E$  is the exceptional divisor of  $\sigma$ . Thus we have

$$Im(\sigma^* \mathcal{I}_{n-1} \rightarrow \mathcal{O}_{S^{[n,n-1]}}) = \mathcal{O}(-E).$$

The map  $\psi$  has degree  $n$ , and  $(\psi, \rho)$  is a birational map from  $S^{[n,n-1]}$  to the incidence subscheme  $\Sigma_n \subset S^{[n]} \times S$ .

Let now  $\mu = \{\mu_1, \dots, \mu_m\}$  be a partition of  $\{1, \dots, n\}$ , and  $S_\mu \cong S^m \subset S^n$  be as above. Consider the fibered product

$$S_\mu \times_{S^{(n)}} S^{[n,n-1]},$$

which is also equal to

$$E_\mu \times_{S^{[n]}} S^{[n,n-1]}.$$

It obviously has exactly  $m$  components dominating  $S_\mu$ , according to the choice of the residual point. Let us choose one component, say the one where over the generic point  $(x_1, \dots, x_n) \in S_\mu$ , the residual point is  $x_n$ . Let  $\mu'$  be the partition of  $\{1, \dots, n-1\}$  deduced from  $\mu$  by putting

$$\mu'_i = \mu_i, \text{ if } n \notin \mu_i, \quad \mu'_i = \mu_i \setminus \{n\}, \text{ if } n \in \mu_i.$$

Let us denote by  $E_{\mu, \mu'} \subset S_\mu \times S^{[n,n-1]}$  the underlying reduced variety of the component defined above, and note that via the projection  $\pi$  from  $S_\mu$  to  $S_{\mu'}$  (forgetting the  $n$ -th factor), and the map  $\sigma$ , we get a natural map

$$\chi_{\mu'} = (\pi, \sigma)|_{E_{\mu, \mu'}} : E_{\mu, \mu'} \rightarrow E_{\mu'} \times S.$$

On the other hand, we have the natural map

$$\chi_\mu := (Id_{S_\mu}, \psi)|_{E_{\mu, \mu'}} : E_{\mu, \mu'} \rightarrow E_\mu.$$

Now, observe that the following diagram is commutative:

$$\begin{array}{ccccc} E_\mu & \xleftarrow{\chi_\mu} & E_{\mu, \mu'} & \xrightarrow{\chi_{\mu'}} & E_{\mu'} \times S \\ p_\mu \downarrow & & & & \downarrow (p_{\mu'}, Id), \\ S_\mu & & \xrightarrow{\pi'} & & S_{\mu'} \times S \end{array}$$

where  $p_\mu$  is the restriction to  $E_\mu \subset S_\mu \times S^{[n]}$  of the first projection, and similarly for  $p_{\mu'}$ , and where  $\pi' : S_\mu \rightarrow S_{\mu'} \times S$  is given by  $(\pi, pr_n|_{S_\mu})$ . Note also that both  $\chi_\mu$  and  $\chi_{\mu'}$  are generically finite of degree 1. Thus we have the following equalities :

$$E_{\mu, \mu'}^* \circ \psi^* = E_\mu^* : CH(S^{[n]}) \rightarrow CH(S_\mu),$$

$$\pi'_* \circ E_{\mu, \mu'}^* \circ \sigma^* = (E_{\mu'} \times \Delta_S)^* : CH(S^{[n-1]} \times S) \rightarrow CH(S_{\mu'} \times S).$$

Similarly, for any integer  $l$ , we can consider the induced correspondence

$$E_{\mu, \mu', l} := E_{\mu, \mu'} \times \Delta_{S^l}$$

between  $S_\mu \times S^l$  and  $S^{[n, n-1]} \times S^l$ . Then we have the formulas

$$E_{\mu, l}^* = E_{\mu, \mu', l}^* \circ (\psi, Id_l)^* : CH(S^{[n]} \times S^l) \rightarrow CH(S_\mu \times S^l), \quad (1.11)$$

$$E_{\mu', l+1}^* = \pi'_{l*} \circ E_{\mu, \mu', l}^* \circ (\sigma, Id_l)^* : CH(S^{[n-1]} \times S^{l+1}) \rightarrow CH(S_{\mu'} \times S^{l+1}). \quad (1.12)$$

Here,  $Id_l$  denotes the identity of  $S^l$ , and  $\pi'_l$  is defined by

$$\pi'_l = (\pi', Id_l) : S_\mu \times S^l \rightarrow S_{\mu'} \times S^{l+1}.$$

Furthermore, for any  $\gamma \in CH(S^{[n, n-1]} \times S^l)$ , one has

$$\pi'_{l*} \circ E_{\mu, \mu', l}^*(\gamma) = E_{\mu', l+1}^*((\sigma, Id_l)_* \gamma). \quad (1.13)$$

Indeed, this follows from the fact that the correspondences  $E_{\mu, \mu'} \subset S_\mu \times S^{[n, n-1]}$  and  $E_{\mu'} \times \Delta_S \subset S_{\mu'} \times S \times S^{[n-1]} \times S$  satisfy the relation:

$$(\pi', Id_{S^{[n, n-1]}})_*(E_{\mu, \mu'}) = (Id_{S_{\mu'}}, \sigma, Id_S)^*(E_{\mu'} \times \Delta_S) \quad (1.14)$$

in  $CH(S_{\mu'} \times S \times S^{[n, n-1]})$  and similarly with  $l > 0$ . From (1.14), we deduce that for  $\gamma \in CH(S^{[n, n-1]})$ , one has

$$\begin{aligned} \pi'_{l*} \circ E_{\mu, \mu'}^*(\gamma) &= (p_{S_{\mu'} \times S})_*((\pi', Id_{S^{[n, n-1]}})_*(E_{\mu, \mu'})) \cdot p_{S^{[n, n-1]}}^*(\gamma) \\ &= (p_{S_{\mu'} \times S})_*((Id_{S_{\mu'}}, \sigma, Id_S)^*(E_{\mu'} \times \Delta_S)) \cdot p_{S^{[n, n-1]}}^*(\gamma) \\ &= (p_{S_{\mu'} \times S})_* \circ (Id_{S_{\mu'}}, \sigma, Id_S)_* ((Id_{S_{\mu'}}, \sigma, Id_S)^*(E_{\mu'} \times \Delta_S)) \cdot p_{S^{[n, n-1]}}^*(\gamma) \\ &= (p_{S_{\mu'} \times S})_*((E_{\mu'} \times \Delta_S)) \cdot (Id_{S_{\mu'}}, \sigma, Id_S)_*(p_{S^{[n, n-1]}}^*(\gamma)) \\ &= (p_{S_{\mu'} \times S})_*((E_{\mu'} \times \Delta_S)) \cdot p_{S^{[n-1]} \times S}^*(\sigma_* \gamma) \\ &= E_{\mu', 1}^*(\sigma_* \gamma), \end{aligned}$$

which proves (1.13) for  $l = 0$ . One argues similarly for  $l > 0$ .

From (1.13), using the projection formula, one deduces that for any

$$\alpha \in CH(S^{[n, n-1]} \times S^l), \quad \beta \in CH(S^{[n-1]} \times S^{l+1}),$$

one has :

$$\pi'_{l*} \circ E_{\mu, \mu', l}^*(\alpha \cdot (\sigma, Id_l)^* \beta) = E_{\mu', l+1}^*((\sigma, Id_l)_* \alpha \cdot \beta). \quad (1.15)$$

The key point is now the following formulas proved by Ellingsrud, Göttsche, Lehn in [13]: here we work on the  $K_0$  groups (the varieties considered are smooth and projective). The morphism  $\phi^! : K_0(Y) \rightarrow K_0(X)$  for a morphism  $\phi : X \rightarrow Y$  between smooth varieties is induced by the morphism  $\phi^*$  on vector bundles. The morphism  $M \mapsto M^\vee$  is induced by the morphism  $E \mapsto E^*$  on vector bundles, and the product  $\cdot$  is induced by the tensor product between vector bundles. Then we have (here we use for simplicity the fact that  $K_S$  is trivial) :

**Theorem 1.7** ([13], Lemma 2.1 and Proposition 2.3) We have in  $K_0(S^{[n,n-1]})$  :

$$\psi^!T_n = \phi^!T_{n-1} + \mathcal{L} \cdot \sigma^!\mathcal{I}_{n-1}^\vee - \rho^!(1 - T_S). \quad (1.16)$$

$$\psi^!\mathcal{O}_{[n]} = \phi^!\mathcal{O}_{[n-1]} + \mathcal{L}. \quad (1.17)$$

Furthermore, we have in  $K_0(S^{[n,n-1]} \times S)$ :

$$(\psi, Id_S)^!T_n = (\phi, Id_S)^!T_{n-1} - pr_0^!(\mathcal{L}) \otimes (\rho, Id_S)^!\mathcal{O}_{\Delta_S}. \quad (1.18)$$

Another very important property is

**Lemma 1.8** ([13], Lemma 1.1) In  $CH(S^{[n-1]} \times S)$ , we have the relation

$$\sigma_*(c_1(\mathcal{L})^i) = (-1)^i c_i(-\mathcal{I}_{n-1}).$$

Theorem 1.7 can be translated into statements concerning the Chern classes of the considered sheaves (or elements of the  $K_0$  groups). Namely we conclude from (1.16) that the Chern classes  $c_i(T_n)$  satisfy the property that  $\psi^*c_i(T_n)$  can be expressed as polynomials in

$$\phi^*c_j(T_{n-1}), c_1(\mathcal{L}), \sigma^*c_s(\mathcal{I}_{n-1}), \rho^*c_2(T_S) = 24\rho^*o.$$

Similarly, we get from (1.17) that the Chern classes  $c_i(\mathcal{O}_{[n]})$  satisfy the property that  $\psi^*c_i(\mathcal{O}_{[n]})$  can be expressed as polynomials in

$$\phi^*c_j(\mathcal{O}_{[n-1]}), c_1(\mathcal{L}).$$

Finally, from (1.18) we conclude that the Chern classes of  $\mathcal{I}_n$  satisfy the property that  $(\psi, Id_S)^*c_i(\mathcal{I}_n) \in CH(S^{[n,n-1]} \times S)$  can be expressed as polynomials in

$$(\phi, Id_S)^*c_j(\mathcal{I}_{n-1}), pr_0^*c_1(\mathcal{L}), (\rho, Id_S)^*c_s(\mathcal{O}_{\Delta_S}).$$

Note that because  $K_S$  is trivial, the Chern classes of  $\mathcal{O}_{\Delta_S}$  reduce to

$$c_2(\mathcal{O}_{\Delta_S}) = -\Delta_S \in CH^2(S \times S)$$

and  $c_4(\mathcal{O}_{\Delta_S})$ , which is proportional to  $(o, o)$  as  $c_2(T_S)$  is proportional to  $o$ .

Let now  $P \in CH(S^{[n]} \times S^l)$  be a polynomial expression in

$$pr_0^*c_r(\mathcal{O}_{[n]}), pr_0^*c_s(T_n), pr_{0i}^*c_t(\mathcal{I}_n), 1 \leq i \leq l$$

as in Proposition 1.6. Applying (1.11), we get

$$E_{\mu,l}^*(P) = E_{\mu,\mu',l}^* \circ (\psi, Id_l)^*(P). \quad (1.19)$$

As just explained above,  $(\psi, Id_l)^*(P) \in CH(S^{[n,n-1]} \times S^l)$  can be expressed as a polynomial in

$$\begin{aligned} &(\phi, pr_i)^*c_t(\mathcal{I}_{n-1}), pr_0^*c_1(\mathcal{L}), (\phi \circ pr_0)^*c_r(\mathcal{O}_{[n-1]}), (\phi \circ pr_0)^*c_s(T_{n-1}), \\ &(pr_{1,i} \circ (\sigma, Id_l))^*\Delta_S, (pr_i \circ (\sigma, Id_l))^*o, 1 \leq i \leq l+1. \end{aligned}$$

Observing that

$$\phi \circ pr_0 : S^{[n,n-1]} \times S^l \rightarrow S^{[n-1]}$$

is equal to  $pr_0 \circ (\sigma, Id_l)$ , the variables above can all be expressed as pull-back via  $(\sigma, Id_l)$  of the following variables in  $CH(S^{[n-1]} \times S^{l+1})$ :

$$\begin{aligned} & pr_{1,i}^* \Delta_S, pr_i^* o, 1 \leq i \leq l+1, \\ & pr_{0,i}^* c_t(\mathcal{I}_{n-1}), pr_{0,r}^* c_r(\mathcal{O}_{[n-1]}), pr_{0,s}^* c_s(T_{n-1}), \end{aligned} \quad (1.20)$$

except for  $pr_0^* c_1(\mathcal{L})$ . Thus we have in  $CH(S^{[n,n-1]} \times S^l)$ :

$$(\psi, Id_l)^*(P) = \sum_i pr_0^* c_1(\mathcal{L})^i (\sigma, Id_l)^* Q_i, \quad (1.21)$$

where  $Q_i \in CH(S^{[n-1]} \times S^{l+1})$  is a polynomial expression in the variables (1.20).

From (1.21) and (1.19), applying (1.15), we deduce that

$$\begin{aligned} \pi'_{l*}(E_{\mu,l}^*(P) &= \pi'_{l*}(E_{\mu,\mu',l}^* \circ (\psi, Id_l)^*(P)) = \\ \pi'_{l*}(E_{\mu,\mu',l})^* & \left( \sum_i pr_0^* c_1(\mathcal{L})^i (\sigma, Id_l)^* Q_i \right) = E_{\mu',l+1}^* \left( \sum_i Q_i \cdot (\sigma, Id_l)_*(pr_0^* c_1(\mathcal{L})^i) \right). \end{aligned} \quad (1.22)$$

Using Lemma 1.8, we find that  $(\sigma, Id_l)_*(pr_0^* c_1(\mathcal{L})^i)$  is a polynomial expression in the  $pr_{0j}^* c_s(\mathcal{I}_{n-1})$ , and thus

$$\sum_i Q_i \cdot (\sigma, Id_l)_*(pr_0^* c_1(\mathcal{L})^i)$$

is a polynomial expression in the variables (1.20). Applying induction on  $n$  and the projection formula to the right hand side, we conclude that  $\pi'_{l*}(E_{\mu,l}^*(P))$  is a polynomial expression in the variables

$$pr_j^* o, pr_{ik}^* \Delta_S, i, j, k, \leq l+m.$$

There are finally two cases to consider here, according to whether  $|\mu(n)| = 1$  or  $|\mu(n)| \geq 2$ , where  $\mu(n)$  is the element of the partition  $\mu$  to which  $n$  belongs (so  $|\mu(n)|$  is the multiplicity of  $n$  in the diagonal  $S_\mu$ ). In the first case, we have

$$\pi' : S_\mu \cong S_{\mu'} \times S,$$

while in the second case, we have  $\pi : S_\mu \cong S_{\mu'}$  and  $\pi'$  is the embedding of  $S_\mu \cong S^m$  in  $S_{\mu'} \times S \cong S^{m+1}$  which is given by the diagonal on the last factor. In the first case,  $\pi'$  being an isomorphism, we proved that  $E_{\mu,l}^*(P)$  is a polynomial in the variables  $pr_{ij}^* \Delta_S, pr_k^* o$ . In the second case, we get that  $pr_{\mu'} \circ \pi'$  is an isomorphism from  $S_\mu$  to  $S_{\mu'}$ , and applying  $(pr_{\mu'}, Id_l)$  to both sides of (1.22), we get the same conclusion.

This proves Proposition 1.6, and thus also Proposition 1.4.  $\blacksquare$

**Remark 1.9** *It is presumably the case that Proposition 1.4 could be obtained as a consequence of the Bridgeland-King-Reid-Haiman equivalence of categories between the derived category of  $S^{[n]}$  and the derived category of  $\mathfrak{S}_n$ -equivariant coherent sheaves on  $S^n$  (see [10], [15]), combined with results on equivariant K-theory of Vistoli [20], and Riemann-Roch type theorems by Toen [19].*

*However, the explicit computation of the equivariant complex associated to a given sheaf on  $S^{[n]}$  is rather complicated. It is done in [17] for  $\mathcal{O}_{[n]}$ , but not for  $T_n$ , and the computation is more difficult than the method of [13], that we have been using here.*

## 2 Case of the variety of lines of a cubic fourfold

We shall use the following notations: the cubic fourfold will be denoted by  $X$  and its Fano variety of lines by  $F$ .  $F$  is contained in the Grassmannian  $G := G(2, 6)$  of lines in  $\mathbb{P}^5$ , and we shall denote by

$$l \in CH^1(F, \mathbb{Z}), c \in CH^2(F, \mathbb{Z})$$

the Chern classes of the rank 2 quotient bundle  $\mathcal{E}$  induced on  $F$ . Thus if

$$\begin{array}{ccc} P & \xrightarrow{q} & X \\ p \downarrow & & \\ F & & \end{array} \quad (2.23)$$

is the incidence diagram,  $P$  is a  $\mathbb{P}^1$ -bundle over  $F$ , and  $\mathcal{E} = R^0 p_* q^*(\mathcal{O}_X(1))$ .

We shall denote by  $H \in CH^1(X)$  the class  $c_1(\mathcal{O}_X(1))$  and by  $h$  its pull-back to  $P$ ,  $h = q^*H$ .

Let  $I \subset F \times F$  be the incidence subvariety, which is the codimension 2 subset of  $F \times F$  defined as

$$I = (p, p)(q, q)^{-1}(\Delta_X), \quad (2.24)$$

where  $\Delta_X$  is the diagonal of  $X$ . Thus  $I$  is the set of pairs  $(\delta, \delta')$  of intersecting lines. We shall denote by the same letter  $I$  the class of  $I$  in  $CH^2(F \times F)$ .

We start the proof with a few remarks concerning the Chern classes of  $F$ . As it is known that  $F$  is symplectic holomorphic, one has  $T_F \cong \Omega_F$ , and thus only the even Chern classes of  $F$  can be non zero. We shall denote them by  $c_2, c_4$ . It is immediate to compute that  $c_2$  and  $c_4$  can be written as polynomials in  $c$  and  $l$ . Indeed  $F \subset G$  is defined as the zero set of a section of the vector bundle  $S^3 \mathcal{E}_G$  on  $G$ , and thus the normal bundle of  $F$  in  $G$  is isomorphic to  $S^3 \mathcal{E}$ . The normal bundle exact sequence then shows that the Chern classes of  $F$  are polynomials in  $l, c$  and in the Chern classes of  $G$  restricted to  $F$ . But the later are also polynomials in  $c$  and  $l$ , as are the restrictions of all cycles on the Grassmannian.

Thus, in this case, Theorem 0.4, 2) is equivalent to the following :

**Theorem 2.1** *Any polynomial expression in  $D \in CH^1(F)$  and  $c \in CH^2(F)$  which vanishes in cohomology, vanishes in  $CH(F)$ .*

We observe first that there is no cohomological relation in degree 4 of the form above. Indeed, as  $F$  is a deformation of a  $S^{[2]}$ , one knows that

$$H^4(F, \mathbb{Q}) \cong S^2 H^2(F, \mathbb{Q}).$$

Thus there is only one cohomological relation of the form

$$[c_2(F)] = P,$$

where  $P \in S^2 H^2(F, \mathbb{Q})$ . But this  $P$  is non degenerate because its kernel is a sub-Hodge structure of  $H^2(F, \mathbb{Q})^*$ , which must be trivial because it is stable under deformation of  $F$ , and in particular under a deformation for which  $NS(F)$  becomes trivial. Thus there cannot be any relation of the form

$$[c_2(F)] = Q,$$

where  $Q \in S^2(NS(F))$ , because  $NS(F)$  never generates  $H^2(F, \mathbb{Q})$ .

Thus we only have to study relations in  $H^6$  and  $H^8$ . We first deal with the relations between  $l$  and  $c$  in degree 8. There are obviously two such relations, as  $l^4, c^2, l^2c$  are all proportional in  $H^8(F, \mathbb{Q})$ . Let us prove:

**Lemma 2.2** *There exists a 0-cycle  $o \in CH^4(F)$ , which is of degree 1, and such that*

$$l^4, c^2, l^2c$$

*are multiples of  $o$  in  $CH^4(F)$ .*

**Proof.** We observe first that for generic  $X$ , there is one surface  $\Sigma$  of class  $c$  which is a singular rational surface (namely, its desingularization is rational). Indeed, surfaces in the class  $c$  are surfaces of lines of hyperplane sections of  $X$ . When an hyperplane section  $Y$  acquires a node  $x$ , its surface of lines becomes birationally equivalent to a symmetric product  $S^2E_x$ , where  $E_x$  is the curve of lines in  $Y$  (or  $X$ ) passing through  $x$  (see [12]). This curve of lines has genus 4, and imposing four “independent” supplementary nodes to  $Y$  creates four nodes on the curve  $E_x$ , which remains irreducible, so that the normalization of  $E_x$  becomes rational. In that case, the desingularization of the surface of lines of  $Y$  is rational. Now, for generic  $X$  it is easy to see that there exists such an hyperplane section  $Y$  with five independent nodes (which means that the associated vanishing cycles are independent).

Of course, all points of  $\Sigma$  are rationally equivalent in  $F$ . For some particular  $X$ , it might be that the surface  $\Sigma$  degenerates to a non rational surface, but it still will remain true that all the points of the degenerate surface  $\Sigma$  are rationally equivalent in  $F$ .

We shall denote by  $o \in CH^4(F)$  this degree 1 0-cycle. As  $c^2$  is supported on  $\Sigma$ ,  $c^2$  is a multiple of  $o$  in  $CH^4(F)$ . Similarly  $c \cdot l^2$  is supported on  $\Sigma$ , hence it has to be a multiple of  $o$  in  $CH^4(F)$ .

Next, with the same notations as above, we note that the curve  $E_x$  is contained in  $\Sigma$ . Thus we have a relation in  $CH^4(X)$ :

$$l \cdot E_x = \mu o, \tag{2.25}$$

for some coefficient  $\mu$  equal to the degree of  $l \cdot E_x$ . The class of  $E_x$  is computed as follows: As  $CH_0(X) = \mathbb{Z}$ , this class does not depend on  $x$ , and in fact we have :

$$3E_x = p_* h^4,$$

because  $3x$  is rationally equivalent to  $H^4$  in  $X$ . Now we have the relation defining Chern classes:

$$(p^*l - h)h = p^*c$$

in  $CH^2(P)$ , which gives

$$h^2 = p^*l \cdot h - p^*c, \quad h^3 = p^*l \cdot (p^*l \cdot h - p^*c) - h \cdot p^*c = p^*(l^2 - c) \cdot h - p^*(l \cdot c),$$

$$h^4 = p^*(l^2 - c) \cdot (p^*l \cdot h - p^*c) - p^*(l \cdot c) \cdot h = p^*(l^3 - 2lc) \cdot h - p^*((l^2 - c)c).$$

Thus we have

$$3E_x = l^3 - 2lc \text{ in } CH^3(F). \tag{2.26}$$

Equation (2.25) thus gives a relation

$$3l(l^3 - 2lc) = \mu o,$$

and thus  $l^4$  is also a multiple of  $o$ . ■

We now introduce a relation in the Chow ring of  $F \times F$  which generalizes the results obtained in [22] (which concerned the Chow ring of the surface of conics of a Fano threefold). This relation will be essential to understand the group  $CH_1(F)$ .

**Proposition 2.3** *There is a quadratic relation in  $CH^4(F \times F)$*

$$I^2 = \alpha \Delta_F + \Gamma \cdot I + \Gamma', \quad (2.27)$$

where  $\alpha \neq 0$ , and  $\Gamma$  is a codimension 2 cycle of  $F \times F$  which is a degree 2 polynomial in

$$l_1 := p_1^* l, l_2 = p_2^* l,$$

and  $\Gamma'$  is a codimension 4 cycle which is a degree 2 weighted polynomial in  $l_1, l_2, p_1^* c, p_2^* c$ .

**Proof.** We first prove the existence of a relation of the above form, and we will show later on that the coefficient  $\alpha$  is not 0.

To get such a relation, it suffices to show the existence of a relation

$$I_0^2 = \Gamma \cdot I_0 + \Gamma' \text{ in } CH^4(F \times F \setminus \Delta_F), \quad (2.28)$$

where  $\Gamma, \Gamma'$  are as above and  $I_0$  is the restriction of  $I$  to  $F \times F \setminus \Delta_F$ .

Note that  $I$  is the image in  $F \times F$  via the map  $(p, p)$  of

$$\tilde{I} := (q, q)^{-1}(\Delta_X).$$

Furthermore, over a point  $(\delta, \delta') \in F \times F$ , the fiber of the map

$$p' := (p, p)|_{\tilde{I}} : \tilde{I} \rightarrow F \times F$$

identifies schematically to the intersection of the corresponding lines  $L, L'$  in  $X$ . Thus, away from the diagonal, this fibre is a reduced point, and the restriction  $p'_0$  of  $p'$  to  $\tilde{I}_0 := \tilde{I} \setminus (p')^{-1}(\Delta_F)$  is an isomorphism onto  $I_0$ .

Furthermore, as  $\tilde{I}_0$  is a local complete intersection, and  $(p, p)$  is a submersion,  $I_0$  is also a local complete intersection, and thus  $I_0^2$  is equal to  $j_*(c_2(N_{I_0}))$ , where  $N_{I_0}$  is the normal bundle of  $I_0$  in  $F \times F \setminus \Delta_F$  and  $j$  is the inclusion of  $I_0$  in  $F \times F \setminus \Delta_F$ . On the other hand, as  $p'_0$  is an isomorphism onto  $I_0$ , the normal bundle of  $\tilde{I}_0$  in  $P \times P$  fits into a normal sequence

$$0 \rightarrow T_{P \times P / F \times F}|_{\tilde{I}_0} \rightarrow N_{\tilde{I}_0 / P \times P} \rightarrow (p'_0)^* N_{I_0 / F \times F} \rightarrow 0. \quad (2.29)$$

We deduce from this that  $p'_0{}^* c_2(N_{I_0 / F \times F})$  can be expressed as a polynomial in the Chern classes  $c_1, c_2$  of the normal bundle  $N_{\tilde{I}_0 / P \times P}$  and in the Chern classes of  $T_{P \times P / F \times F}|_{\tilde{I}_0}$ .

The later ones are polynomials in  $h_1, l'_1, h_2, l'_2$ , where

$$h_i = pr_i^* h, l'_i = pr_i^*(p^* l), i = 1, 2,$$

and  $pr_i$  are the two projections of  $P \times P$  onto  $P$ . Next we observe that, as  $\tilde{I} = (q, q)^{-1}(\Delta_X)$ , we have the equalities

$$c_i(N_{\tilde{I}_0/P \times P}) = q_0^* c_i(T_X),$$

where  $q_0 : \tilde{I}_0 \rightarrow X$  is the restriction of  $(q, q)$  to  $\tilde{I}_0$ . But  $c_i(T_X)$  are polynomials in  $H$ . Thus we conclude that we have a relation:

$$\begin{aligned} I_0^2 &= p'_{0*}(p'^*_0 c_2(N_{I_0/F \times F})) \\ &= p'_{0*}(P(h_i, l'_i)), \end{aligned}$$

for some degree 2 polynomial  $P$  in  $h_i, l'_i|_{\tilde{I}_0}$  (in fact  $h_1 = h_2$  on  $\tilde{I}_0$ ). This can also be written as

$$I_0^2 = (p, p)_*(P(h_i, l'_i) \cdot \tilde{I})|_{F \times F \setminus \Delta_F}.$$

Let us now write the quadratic polynomial  $P$  as

$$P = h_1 A + h_2 B + Q,$$

where  $A, B$  are linear in  $h_i, l'_i$ , and  $Q$  is quadratic in  $l'_1, l'_2$ . We have by the projection formula, noting that  $l'_i = (p, p)^* l_i$ ,

$$(p, p)_*(Q(l'_i) \cdot \tilde{I}) = Q(l_i) \cdot I,$$

which is of the form  $\Gamma' \cdot I$ .

At this point we proved

$$I_0^2 = \Gamma' \cdot I_0 + (p, p)_*((h_1 A + h_2 B) \cdot \tilde{I})|_{F \times F \setminus \Delta_F}. \quad (2.30)$$

Finally, we observe that the diagonal of  $X$  admits a Künneth type decomposition:

$$\Delta_X = \Delta_1 + \Delta_0,$$

where  $\Delta_1$  can be written as a sum

$$\Delta_1 = \sum_i \alpha_i H_1^i \cdot H_2^{4-i}$$

and  $\Delta_0$  has the property that

$$H_1 \cdot \Delta_0 = 0, H_2 \cdot \Delta_0 = 0 \text{ in } CH^6(X \times X). \quad (2.31)$$

Here  $H_i = pr_i^* H$ ,  $i = 1, 2$ , and  $pr_i$  are the two projections on  $X \times X$ . We obtain this decomposition as follows: we choose the  $\alpha_i$  in such a way that we have the following equalities between intersection numbers:

$$\Delta_X \cdot H_1^i \cdot H_2^{4-i} = \Delta_1 \cdot H_1^i \cdot H_2^{4-i}, \text{ for } i = 0, \dots, 4.$$

Then the cycle  $\Delta_0 = \Delta_X - \Delta_1$  is such that its image under each inclusion

$$j_1 : X \times X \hookrightarrow \mathbb{P}^5 \times X, j_2 : X \times X \hookrightarrow X \times \mathbb{P}^5$$

is rationally equivalent to 0, because  $j_{1*}\Delta_X = \Delta_{\mathbb{P}^5|\mathbb{P}^5 \times X}$ . This implies (2.31) because

$$j_1^* \circ j_{1*} = 3H_1, \quad j_2^* \circ j_{2*} = 3H_2.$$

From the decomposition above, and recalling that

$$\tilde{I} = (q, q)^{-1}(\Delta_X) = (q, q)^*\Delta_X, \quad h_i = (q, q)^*H_i$$

we conclude that

$$h_1 A \cdot \tilde{I} = A \cdot (q, q)^*(H_1 \cdot \Delta_X) = A \cdot (q, q)^*(H_1 \Delta_1).$$

But as  $H_1 \Delta_1$  is a polynomial in  $H_1, H_2$ , it is then clear that  $(p, p)_*(h_1 A \cdot \tilde{I})$  is a cycle of the form  $\Gamma''$  as in the Proposition. Similarly for  $(p, p)_*(h_2 B \cdot \tilde{I})$ . Thus, using (2.30), the existence of a quadratic relation (2.27) is proven.

We now show that  $\alpha \neq 0$ . Mimicking the arguments in [22], one sees that there exist an hypersurface  $W \subset F$  and a non zero coefficient  $\gamma \in \mathbb{Z}$  such that for each  $\delta \in F$ , there is a relation

$$\gamma\delta = S_\delta^2 + z,$$

where  $z$  is a 0-cycle supported on  $W$ . Here  $S_\delta$  is the surface of lines of  $X$  meeting  $\delta$ , so that  $S_\delta = I^*\delta$  in  $CH^2(F)$  and

$$S_\delta^2 = \gamma\delta - z = (I^2)^*\delta \text{ in } CH^4(F). \quad (2.32)$$

We have an equality

$$I^2 = \alpha\Delta_F + \Gamma \cdot I + \Gamma' \text{ in } CH^4(F \times F),$$

from which we deduce that  $(I^2)^*$  acts as multiplication by  $\alpha$  on  $H^{4,0}(F) \neq 0$ . On the other hand, (2.32) together with the generalized Mumford theorem (cf [23], Proposition 10.24), shows that  $(I^2)^*$  acts as multiplication by  $\gamma$  on  $H^{4,0}(F)$ . Thus  $\alpha = \gamma \neq 0$ . ■

We have the following corollary of Proposition 2.3.

**Corollary 2.4** *Let  $z \in CH_1(F) = CH^3(F)$  be a 1-cycle. Assume that  $z$  is rationally equivalent to a combination of rational curves  $C_i \subset F$ ,*

$$z = \sum_i n_i C_i,$$

*that  $z$  is cohomologous to 0, and that one (or equivalently any) point  $x_i$  of  $C_i$  is rationally equivalent to  $o$  in  $F$ . Then  $z = 0$  in  $CH^3(F)$ .*

**Proof.** Indeed, observe that since

$$z = \sum_i n_i C_i,$$

with  $C_i$  rational, we have

$$\Delta_{F*} z = \sum_i n_i (x_i \times C_i + C_i \times x_i) \text{ in } CH(F \times F), \quad (2.33)$$

where  $x_i$  is any point of  $C_i$ . Now  $I^2$  is the restriction of  $I \times I$  to the diagonal  $\Delta_{F \times F}$  of  $F \times F$ . Thus we have

$$(I^2)^*z = ((I \times I)^*(\Delta_{F^*}z))|_{\Delta_F}.$$

From (2.33), we conclude that

$$(I^2)^*z = 2 \sum_i n_i I^* C_i \cdot I^* x_i. \quad (2.34)$$

By assumption, we have  $I^* x_i = I^* o$  in  $CH^2(F)$ , thus (2.34) is equal to

$$2I^* o \cdot \sum_i n_i I^* C_i = 2I^* o \cdot I^* z. \quad (2.35)$$

But  $z$  is homologous to 0, so  $I^* z \in CH^1(F)$  is also homologous to 0, hence it is rationally equivalent to 0. Thus  $(I^2)^*z = 0$  in  $CH^3(F)$ .

Now we apply Proposition 2.3 which gives a relation

$$\alpha z = (I^2)^*z - (\Gamma \cdot I)^*z - \Gamma'^*z.$$

As  $(I^2)^*z = 0$ , the right hand side is equal to

$$-(\Gamma \cdot I)^*z - \Gamma'^*z.$$

But we know that both  $I^*z$  and  $l \cdot z$  are rationally equivalent to 0 : for the first, this was noticed just before, and for the second, this is because it is a multiple of  $o$  and homologous to 0. Hence it follows that  $-(\Gamma \cdot I)^*z - \Gamma'^*z = 0$  and, as  $\alpha \neq 0$ , we conclude that  $z = 0$ . ■

As a consequence, we can start the computation of relations in  $CH^3(F)$  by showing the following Lemma 2.5: Notice that  $[l^3]$  and  $[lc]$  are proportional in  $H^6(F, \mathbb{Q})$ . Let this relation be

$$[\mu cl - \nu l^3] = 0 \text{ in } H^6(F, \mathbb{Q}), \mu \neq 0, \nu \neq 0.$$

**Lemma 2.5** *We have the equality*

$$\mu cl - \nu l^3 = 0$$

*in  $CH^3(F)$ .*

**Proof.** Indeed, it suffices to prove this relation for generic  $X$ . In that case, we proved that the cycles  $l^3$  and  $lc$  are supported on a rational surface of class  $c$ , all points of which are rationally equivalent to  $o$  in  $F$ . Thus the cycle  $z = \mu cl - \nu l^3$  satisfies the assumptions of Corollary 2.4. ■

In conclusion, we proved in Lemma 2.2 and Lemma 2.5 that all polynomial cohomological relations between  $l$  and  $c$  hold in  $CH(F)$ .

Let us decompose now  $CH^1(F)$  as

$$CH^1(F) = \langle l \rangle \oplus CH^1(F)_0,$$

where  $CH^1(F)_0 = p_*q^*CH^2(X)_{prim}$ . Recall the following from [23], 9.3.4. Let  $Z \in CH^2(X)_{prim} := \{Z \in CH^2(F), [Z] \in H^4(X, \mathbb{Q})_{prim}\}$ . Write

$$q^*Z = hp^*D + p^*Z'.$$

Then from

$$H \cdot Z = 0 \text{ in } CH^3(X),$$

(see [23], 9.3.4), we get, using  $h^2 = hp^*l - p^*c$ ,

$$h^2p^*D + hp^*Z' = 0 = hp^*(lD + Z') - p^*(cD).$$

Thus we have  $D = p_*q^*Z$ , and

$$Z' = -lD, \quad cD = 0 \text{ in } CH(F). \quad (2.36)$$

In particular

$$q^*Z = (h - p^*l)p^*D. \quad (2.37)$$

Let us deduce from this the following:

**Lemma 2.6** *For any  $D \in CH^1(F)_0$ , we have the relations:*

$$l^2D^2 = Cq([D])o,$$

$$lD^2 = C'q([D])E_x,$$

where  $q$  is the Beauville-Bogomolov quadratic form on  $H^2(F)$ ,  $C, C'$  are constants, and  $E_x = p_*q^*x$  was already introduced and shown to be proportional to  $l^3$  and  $cl$  in  $CH^3(F)$ .

**Proof.** Note that since  $X$  is Fano, we have  $CH^4(X) = \mathbb{Q}$  and thus

$$Z^2 = \langle Z, Z \rangle x \quad (2.38)$$

for any  $x \in X$ . Using (2.37), we get

$$q^*(Z^2) = (h - p^*l)^2p^*(D^2). \quad (2.39)$$

Next we use the relations  $cD = 0$ ,  $h^2 = hp^*l - p^*c$ , and (2.38) to rewrite (2.39) as

$$\begin{aligned} \langle Z, Z \rangle q^*x &= hp^*lp^*D^2 - 2hp^*lp^*D^2 + p^*(l^2D^2) \\ &= -hp^*(lD^2) + p^*(l^2D^2). \end{aligned}$$

Note now that  $\langle Z, Z \rangle = -C'q([D])$  for some constant  $C'$ , as proved in [4], so that pushing forward via  $p$  the above expression, we get

$$C'q([D])E_x = lD^2.$$

Finally, applying  $l$  to this, we get

$$l^2D^2 = C'q(D)l \cdot E_x = Cq(D)o,$$

with  $C = C' \deg(l \cdot E_x)$ . (We use (2.26) and Lemma 2.5 to get the last equality.) ■

Summing-up what we have done up to now, we get:

**Proposition 2.7** *Any polynomial relation*

$$[P] = 0 \text{ in } H^6(F, \mathbb{Q}) \text{ or in } H^8(F, \mathbb{Q}),$$

in the variables  $l, c, D \in CH^1(F)_0$ , which is of degree  $\leq 2$  in  $D$ , is already satisfied in  $CH^3(F)$ , resp.  $CH^4(F)$ .

**Proof.** Indeed, consider first the case of  $H^8$ . The polynomial expression  $P$  is then of the form

$$P = cQ + l^2Q' + clA + l^3A' + \alpha c^2 + \beta cl^2 + \gamma l^4,$$

where  $Q, Q' \in S^2CH^1(F)_0$ ,  $A, A' \in CH^1(F)_0$  and  $\alpha, \beta, \gamma$  are constants. But we know (cf (2.36)) that

$$cQ = 0, cA = 0,$$

and that  $l^2Q', c^2, cl^2, \gamma l^4$  are all multiples of  $o$  (cf Lemma 2.2, Lemma 2.6). On the other hand, as we proved that the cycle  $l^3$  is rationally equivalent to a cycle supported on a rational surface in the class  $c$ , and all points of  $\Sigma$  are rationally equivalent to  $o$ , it follows that  $l^3A'$  is also a multiple of  $o$ . Thus  $P$  is a multiple of  $o$  in  $CH^4(F)$ , and as it is cohomologous to 0, it must be 0.

Next we consider the case of degree 6. Then  $P$  can be written as

$$P = lQ + cA + l^2A' + \alpha l^3 + \beta cl,$$

where  $Q \in S^2CH^1(F)_0$ ,  $A, A' \in CH^1(F)_0$  and  $\alpha, \beta$  are constants.

We know that  $cA = 0$  and we proved already that the cycles

$$lQ, l^3, cl$$

are all proportional in  $CH^3(F)$  (cf Lemma 2.5, Lemma 2.6). Using these proportionality relations, we get an equality in  $CH(F)$ :

$$P = l^2(A' + \gamma l),$$

where the number  $\gamma$  depends on  $Q, \alpha, \beta$  and involves the constants  $\mu, \nu, C'$  of Lemma 2.5, Lemma 2.6. But we know that  $[P] = 0$ , and thus the hard Lefschetz theorem implies that  $[A' + \gamma l] = 0$ . Thus, as we are in  $CH^1(F) \subset H^2(F, \mathbb{Q})$ , we have  $A' + \gamma l = 0$  and  $P = 0$ . ■

We now turn to polynomials of degree at most 3 in  $D$ . Let us first consider the case of polynomials of degree 4, that is  $P \in CH^4(F)$ .

**Lemma 2.8** *Any polynomial expression  $P \in CH^4(F)$  in  $l, c, D \in CH^1(F)_0$  which is of degree at most 3 in  $D$  is a multiple of  $o$ . Thus, if  $[P] = 0$  in  $H^8(F, \mathbb{Q})$ , then  $P = 0$ .*

**Proof.** Indeed this was already proved for polynomial expressions of degree at most 2 in  $D$  (cf Proposition 2.7), and thus, we only have to consider expressions of the form

$$P = lT,$$

where  $T \in S^3CH^1(F)_0$ . Now Lemma 2.6 says that for  $D \in CH^1(F)_0$ ,  $lD^2$  is proportional to  $l^3$  in  $CH^3(F)$ . Hence  $lD^3$  is proportional to  $l^3D$  in  $CH^4(F)$ . But by Proposition 2.7, we know that  $l^3D$  is a multiple of  $o$  in  $CH^4(F)$ , as is any polynomial expression of degree  $\leq 2$  in  $D$ . ■

We turn now to the cubic polynomial relations in  $CH^3(F)$ . First of all we have the following lemma:

**Lemma 2.9** *For any  $D \in CH^1(F)_0$ , one has*

$$[D^3] = \frac{3}{q([l])}q([D])[l^2D]. \quad (2.40)$$

**Proof.** Recall from [18], [9] that, in the complex cohomology algebra  $H^*(F, \mathbb{C})$ , one has the relations

$$d'^3 = 0,$$

for  $d' \in H^2(F, \mathbb{C})$  such that  $q(d') = 0$ .

It follows that we have more generally a relation of the form

$$d'^3 = q(d')A(d'),$$

where  $A(d') \in H^6(F, \mathbb{C})$  is a linear function of  $d'$ . We apply this to  $d' = d + \lambda[l]$ , where  $\lambda \in \mathbb{C}$ ,  $d = [D]$ ,  $D \in CH^1(F)_0$ . Then we get, recalling that  $q(d, [l]) = 0$ ,

$$d^3 + 3\lambda d^2[l] + 3\lambda^2 d[l]^2 + \lambda^3 [l]^3 = (q(d) + \lambda^2 q([l]))A(d') \text{ in } H^6(F, \mathbb{C}). \quad (2.41)$$

Write  $A(d') = a(d)N + \lambda M$ . Then we get by taking the 0-th order term in  $\lambda$ :

$$d^3 = q(d)a(d)N.$$

The order 2 term in  $\lambda$  gives now

$$3d[l]^2 = q([l])a(d)N,$$

from which we conclude that

$$d^3 = \frac{3}{q([l])}q(d)l^2d.$$

■

We will show the following proposition.

**Proposition 2.10** *For any  $D \in CH^1(F)_0$ , we have the relation*

$$D^3 = \frac{3}{q([l])}q([D])l^2D \text{ in } CH^3(F). \quad (2.42)$$

Postponing the proof of Proposition 2.10, we conclude now the proof of Theorem 2.1, or equivalently of Theorem 0.4, 2).

**Proof of Theorem 2.1.** Let us first treat the case of a polynomial expression  $P \in CH^3(F)$ , which has to be of degree at most 3 in  $Pic F_0$ . So assume  $[P] = 0$ , where  $P = T + lQ + l^2L + cL' + C$ , is the decomposition of  $P$  into elements of  $Sym CH^1(F)_0$  of degree 3, 2, 1 and 0 respectively, whose coefficients are polynomials in  $c, l$ . We know from (2.36) that  $cL' = 0$ . We also know from Lemma 2.6 and Lemma 2.5 that  $lQ$  and  $C$  are proportional to  $l^3$  in  $CH^3(F)$ . Thus we have

$$lQ + C = \gamma l^3 \text{ in } CH^3(F).$$

Finally, it follows from Proposition 2.10 that  $T$  is equal in  $CH^3(F)$  to  $l^2D$  for some  $D \in Pic F_0$ .

Thus we have  $P = l^2(D + L) + \gamma l^3$  in  $CH^3(F)$  and the relation  $[P] = 0$  implies

$$[l^2][D + L + \gamma l] = 0 \text{ in } H^6(F, \mathbb{Q}).$$

But the hard Lefschetz theorem implies then that  $[D + L + \gamma l] = 0$ . Thus  $D + L + \gamma l = 0$  and  $P = 0$ .

To conclude the proof of the theorem, we now have to consider the case of a polynomial  $P \in CH^4(F)$  of degree 4 in  $D \in CH^1(F)_0$ . But Proposition 2.10 shows that, for any  $D \in CH^1(F)_0$ , we have the relation

$$D^4 = \frac{3}{q([l])} q([D]) l^2 D^2 \text{ in } CH^4(F).$$

We proved in Lemma 2.6 that  $l^2 D^2$  is proportional to  $o$  in  $CH^4(F)$ . Thus  $D^4$  is a multiple of  $o$  and so is any quartic homogeneous polynomial expression in  $D \in Pic F_0$ .

By Lemma 2.8, the same is true of any polynomial expression of degree  $\leq 3$  in  $D$ , with coefficients which are polynomials in  $l, c$ . Thus any polynomial expression  $P$  of degree 4 in  $D$ , with coefficients in  $l, c$  is a multiple of  $o$  in  $CH^4(F)$ . In particular, if  $[P] = 0$ , we have  $P = 0$ . ■

**Proof of Proposition 2.10.** We first prove the result under the assumption that  $X$  contains no plane. We will show later on how to deduce the result when  $X$  contains planes.

Let us introduce the following object:

$$\tilde{F} = \{(\delta_1, \delta_2) \in F \times F, \exists P \cong \mathbb{P}^2 \subset \mathbb{P}^5, P \cap X = 2\delta_1 + \delta_2\}.$$

Because we made the assumption that  $X$  does not contain any plane,  $\tilde{F}$  is irreducible, and is the graph of the rational map  $\phi : F \dashrightarrow F$  described in [21]. We shall denote by

$$\tau : \tilde{F} \rightarrow F, \tilde{\phi} : \tilde{F} \rightarrow F,$$

the restrictions to  $\tilde{F}$  of the two projections. Thus  $\tau$  is birational and  $\phi = \tilde{\phi} \circ \tau^{-1}$ .

Note that  $\tilde{F}$  may be singular, which may imply that the groups  $CH_i(\tilde{F})$  and  $CH^{4-i}(\tilde{F})$  differ, and cause troubles because on one hand we compute relations in  $CH_*(\tilde{F})$ , and on the other hand, we use intersection product on  $CH(\tilde{F})$ . However, there is a desingularization of  $\tilde{F}$  which is obtained by a sequence of blow-ups starting from  $F$ . We leave to the reader to adapt the following arguments using this smooth model, and in the sequel, we do as if  $\tilde{F}$  were smooth.

We will prove the following two Lemmas:

**Lemma 2.11** *For  $D \in CH^1(F)_0$ , we have  $\tilde{\phi}^* D = -2\tau^* D$  in  $CH^1(\tilde{F})$ .*

**Lemma 2.12** *Let  $I \subset F \times F$  be the incidence subscheme defined in 2.24. Then*

$$(\tilde{\phi}, Id)^* I = -2(\tau, Id)^* I + Z \tag{2.43}$$

in  $CH^2(\tilde{F} \times F)$ , where  $Z$  is a cycle of the form

$$Z = Z_1 \times F + D' \times l + \tilde{F} \times Z_2, \quad (2.44)$$

with  $Z_1 \subset \tilde{F}$  a codimension 2 cycle,  $D' \subset \tilde{F}$  a codimension 1 cycle,  $Z_2 \subset F$  a codimension 2 cycle.

Assuming these lemmas, let us show how to conclude the proof: First of all, from Lemma 2.11, we deduce that for  $D \in Pic F_0$ , we have

$$\tilde{\phi}^* D^3 = -8\tau^* D^3 \text{ in } CH^3(\tilde{F}). \quad (2.45)$$

Next, from lemma 2.12, we deduce that

$$(\tilde{\phi}, Id)^* I^2 = 4(\tau, Id)^* I^2 - 4Z \cdot (\tau, Id)^* I + Z^2. \quad (2.46)$$

Note now that by definition of  $\phi^*$  :

$$\tau_* \circ \tilde{\phi}^* = \phi^*,$$

acting on  $CH(F)$ . Furthermore we have, applying  $\tau_*$ ,  $(\tau, Id)_*$  to (2.45), (2.46):

$$\phi^* D^3 = -8D^3, \quad (2.47)$$

$$(\phi, Id)^* I^2 = 4I^2 - 4I \cdot Z' + Z'', \quad (2.48)$$

where

$$Z' := (\tau, Id)_* Z, \quad Z'' = (\tau, Id)_* Z^2.$$

Observe now that

$$\phi^*((I^2)^*(z)) = ((\phi, Id)^*(I^2))^*(z), \quad \forall z \in CH_1(F).$$

Combining this with (2.48) and the quadratic relation (2.27) given in Proposition 2.3, we get, for any  $z \in CH_1(F)$ :

$$\begin{aligned} \phi^*(\alpha z + (\Gamma \cdot I)^* z + (\Gamma')^* z) &= 4(I^2)^* z - 4(I \cdot Z')^* z + (Z'')^* z \\ &= 4(\alpha z + (\Gamma \cdot I)^* z + (\Gamma')^* z) - 4(I \cdot Z')^* z + (Z'')^* z. \end{aligned} \quad (2.49)$$

Applying this to  $z = D^3$  and using (2.47), we finally get

$$\begin{aligned} &-8\alpha D^3 + \phi^*((\Gamma \cdot I)^* D^3 + (\Gamma')^* D^3) \\ &= 4\alpha D^3 + 4((\Gamma \cdot I)^* D^3 + (\Gamma')^* D^3 - (I \cdot Z')^* D^3) + (Z'')^* D^3. \end{aligned} \quad (2.50)$$

In conclusion, we proved that

$$12\alpha D^3 = \phi^*((\Gamma \cdot I)^* D^3 + (\Gamma')^* D^3) - 4((\Gamma \cdot I)^* D^3 + (\Gamma')^* D^3) - (I \cdot Z')^* D^3 - (Z'')^* D^3.$$

We claim now that  $(\Gamma')^* D^3$ ,  $\phi^*((\Gamma')^* D^3)$  and  $(Z'')^* D^3$  are all multiples of  $l^3$  (or equivalently  $cl$ ).

In the case of  $(\Gamma')^* D^3$ , this is a consequence of the fact that  $\Gamma' \in CH^4(F \times F)$  is a polynomial in  $pr_1^* l$ ,  $pr_2^* l$ ,  $pr_1^* c$ ,  $pr_2^* c$ , and of lemma 2.5. This implies also the claim

for  $\phi^*((\Gamma')^*D^3)$ , as one shows easily (using Lemma 2.5) that  $\phi^*l^3$  is a multiple of  $l^3$ . As for  $(Z'')^*D^3$ , we observe that we have for any  $z \in CH_1(F)$ ,

$$(Z'')^*z = \tau_*((Z^2)^*z),$$

and using formula (2.44) for  $Z$ , this gives

$$(Z'')^*z = 2\tau_*(Z_1D')deg(l \cdot z). \quad (2.51)$$

Hence it suffices to show that  $\tau_*(Z_1D')$  is a multiple of  $l^3$ . Now we have by (2.51) applied to  $l^3$ :

$$(Z'')^*l^3 = 2\tau_*(Z_1D')deg l^4.$$

Thus it suffices to show that  $(Z'')^*l^3$  is a multiple of  $l^3$ . This follows now from (2.49) applied to  $z = l^3$ , and from the fact that

$$\phi^*l^3, (\Gamma \cdot I)^*l^3, (\Gamma')^*l^3, (I \cdot Z')^*l^3$$

are all multiples of  $l^3$ . For the first three, this follows easily from the definition of  $\phi$  and from the form of  $\Gamma, \Gamma'$ ; for the last one, this follows from the fact that, for any  $z \in CH_1(F)$ ,  $(I \cdot Z')^*z$  is a linear combination of  $\tau_*(Z_1) \cdot I^*(z)$  and  $\tau_*D' \cdot I^*(lz)$ . Then the result is a consequence of the fact that

$$I^*l^3, I^*l^4, \tau_*(Z_1), \tau_*D'$$

are polynomial expressions in  $l$  and  $c$ , which is proved using (2.43) and the definitions of  $\tilde{F}$  and  $I$ .

Next recall that the codimension 2-cycle  $\Gamma$  is a linear combination of  $l_1^2, l_2^2, l_1l_2$  on  $F \times F$ . Thus  $(\Gamma \cdot I)^*D^3$  is a combination of  $l^2I^*(D^3)$  and of  $lI^*(lD^3)$ . Next, for the same reason,  $(I \cdot Z')^*D^3$  is a linear combination of  $\tau_*(Z_1) \cdot I^*(D^3)$  and  $\tau_*D' \cdot I^*(lD^3)$ , that is of

$$l^2 \cdot I^*(D^3), c \cdot I^*(D^3), l \cdot I^*(lD^3).$$

Thus our relation (2.50) becomes:

$$\begin{aligned} 12\alpha D^3 = \phi^*(\mu l^2 I^*(D^3) + \nu l I^*(lD^3)) \\ + \mu' l^2 I^*(D^3) + \nu' l I^*(lD^3) + \mu'' c I^*(D^3) + \mu''' l^3. \end{aligned} \quad (2.52)$$

Recall from Lemma 2.8 that  $lD^3$  is proportional to  $o$ . Thus  $lI^*(lD^3)$  is proportional to  $lI_o$  which is a multiple of  $l^3$  and  $cl$  in  $CH^3(F)$ . Furthermore, we mentioned already that  $\phi^*(l^3)$  is also proportional to  $l^3$ .

Next we have

**Lemma 2.13** *For some constant  $\beta$ , and for any  $D \in CH^1(F)_0$ , one has*

$$I^*(D^3) = \beta q([D])D.$$

**Proof.** Indeed, as we are in  $CH^1(F)$ , it suffices to show this in  $H^2(F, \mathbb{Q})$ . But we know that  $[D^3] = \frac{3}{q(l)}q(D)[l^2D]$ . Thus it suffices to show that for some constant  $\beta'$ , and for any  $[D] \in H^2(F, \mathbb{Q})_0$ ,

$$I^*([l]^2[D]) = \beta'[D].$$

This is immediate because the left hand side is a morphism of Hodge structure from  $H^2(F, \mathbb{Q})_0$  to  $H^2(F, \mathbb{Q})$  which is defined for general  $X$ , hence has to be a multiple of the identity, because the Hodge structure on  $H^2(F, \mathbb{Q})_0$  for general  $X$  is simple with  $h^{2,0} = 1$ , while  $H^2(F, \mathbb{Q}) = H^2(F, \mathbb{Q})_0 + \mathbb{Q}[l]$ .  $\blacksquare$

From this lemma, we get in particular that  $cI^*(D^3) = 0$ , and we deduce from (2.52) a relation:

$$12\alpha D^3 = \phi^*(\mu q([D])l^2 D) + \mu'_1 q([D])l^2 D + \nu' l^3.$$

Furthermore, we recall that by Lemma 2.11

$$\tilde{\phi}^* D = -2\tau^* D.$$

Hence it follows that

$$\phi^*(l^2 D) = \tau_*(-2\tau^* D \tilde{\phi}^* l^2) = -2D\phi^* l^2.$$

It is easy to verify that  $\phi^* l^2$  is a combination of  $l^2$  and  $c$ . As  $cD = 0$ , we conclude that  $\phi^*(l^2 D)$  is a multiple of  $l^2 D$ . Thus we finally proved that we have a relation

$$12\alpha D^3 = \mu'' q([D])l^2 D + \nu' l^3. \quad (2.53)$$

On the other hand, we know that we have the cohomological relation

$$[D^3] = \frac{3}{q(l)} q([D])[l^2 D].$$

Using the hard Lefschetz theorem, and comparing with the cohomological relation

$$12\alpha [D^3] = \mu'' q([D])[l^2 D] + \nu'' [l^3]$$

deduced from (2.53), we conclude that  $\nu' = 0$ , and that

$$\frac{\mu''}{12\alpha} = \frac{3}{q(l)}.$$

This concludes the proof of Proposition 2.10 when  $X$  contains no plane.

It remains to see how to do when  $X$  contains a plane. Let  $D := D_Z$  for some primitive class  $[Z] \in H^4(X, \mathbb{Q})$ . In that case, either  $[Z]$  is a multiple of the primitive component  $[H]^2 - 3[\mathbb{P}]$  of the cohomology class of a plane  $\mathbb{P} \subset X$ , or it is not. In the later case, one can show by deformation theory that a generic deformation of  $X$  preserving the class  $Z$  does not preserve any plane contained in  $X$ . Then we know that (2.42) is satisfied by  $D_t \in \text{Pic } F_t$  for the generic member of a family of deformations of the pair  $(D, F)$ . Thus it is also satisfied by  $(D, F)$ .

Thus it remains only to consider the case where  $D = D_Z$ ,  $[Z] = [H]^2 - 3[\mathbb{P}]$ . Thus  $D = l - 3D_{\mathbb{P}}$ , where  $D_{\mathbb{P}}$  is the divisor of lines meeting  $P$ . But this case is easy because away from the dual plane  $\mathbb{P}^* \subset F$ ,  $D_{\mathbb{P}}$  is isomorphic via  $p$  to  $\tilde{D} := q^{-1}(\mathbb{P}) \subset P$ . It follows that the restriction  $(D_{\mathbb{P}})|_{D_{\mathbb{P}}}$  identifies away from  $\mathbb{P}^*$  as  $\det q^* N_{\mathbb{P}/X} - T_{P/F}|_{\tilde{D}}$ , that is to the restriction of a combination of  $h$  and  $l$  to  $\tilde{D}$ . From this, one deduces easily that (2.42) is satisfied in  $F \setminus \mathbb{P}^*$ , and as it is satisfied in cohomology, while

$$CH_1(\mathbb{P}^*) = H_2(\mathbb{P}^*, \mathbb{Z}) = \mathbb{Z} \subset H^3(F, \mathbb{Q}),$$

it follows that it is satisfied as well on  $F$ .

Thus Proposition 2.10 is proved, modulo Lemmas 2.11 and 2.12. ■

**Proof of Lemma 2.11.** Note that  $\tau : \tilde{F} \rightarrow F$  is the contraction of a ruled divisor  $E$  to the surface  $T$  of points  $l \in F$  having the property that there is a  $\mathbb{P}_l^3 \subset \mathbb{P}^5$  which is everywhere tangent to the corresponding line  $\Delta_l \subset X$ . (One verifies that  $T$  is always a surface, and the fiber of  $\tau$  over  $l \in T$  identifies to the  $\mathbb{P}^1$  parameterizing planes  $\mathbb{P}^2$  contained in  $\mathbb{P}_l^3$  and containing  $\Delta_l$ , because  $X$  contains no plane.)

Thus for any divisor  $D \in CH^1(F)$ , there must be a relation

$$\tilde{\phi}^*D = \tau^*D' + \sum_i \alpha_i E_i \text{ in } CH_3(\tilde{F}),$$

where the  $E_i$  are the irreducible components of  $E$ . Here the  $\alpha_i$  are computed as  $D \cdot \tilde{\phi}(E_{i,l})$ , where  $E_{i,l}$  is the fiber of  $E_i$  over  $l \in T_i$ . (Here  $T_i$  is the irreducible component of  $T$  corresponding to  $E_i$ .) However, the curve  $\tilde{\phi}(E_{i,l})$  is the family of lines contained in a cubic surface  $S$  in  $X$  which is singular along the line  $\Delta_l$ . Thus the surface in  $X$  swept out by the lines parameterized by  $\tilde{\phi}(E_{i,l})$  is the cubic surface  $S$ , and for  $D = D_Z$ , with  $Z \subset X$  a cycle with primitive cohomology class, one has

$$\alpha_i = -D \cdot \tilde{\phi}(E_{i,l}) = \langle Z, S \rangle = 0.$$

Thus we have

$$\tilde{\phi}^*D = \tau^*D' \text{ in } CH_3(\tilde{F}),$$

and clearly  $D' = \phi^*D \in CH^1(F)$ . But the action of  $\phi^*$  on  $CH^1(F)_0$  is the restriction of the action of  $\phi^*$  on  $H^2(F, \mathbb{Q})_0 := p_*q^*H^4(X, \mathbb{Q})_{prim}$ . This action is multiplication by  $-2$ , because it is multiplication by  $-2$  on  $H^{2,0}(F)$  (cf [21]), and for general  $X$  the Hodge structure on  $H^2(F, \mathbb{Q})_0$  is simple. Thus  $D' = -2D$  and the lemma is proven.  $\blacksquare$

**Proof of Lemma 2.12.** We observe first that it suffices to prove the lemma for generic  $F$ , because the family of  $\tilde{F}$  parameterized by the set  $U \subset \mathbb{P}(H^0(\mathcal{O}_{\mathbb{P}^5}(3)))$  corresponding to smooth cubic hypersurfaces which do not contain a plane is flat.

Next we note that because  $Pic^0 F = 0$ , (which implies that divisors on any product  $K \times F$  are rationally equivalent to sum of pull-backs of divisors on each factor), and  $Pic F = \mathbb{Z}$ , which implies that divisors on  $F$  are rationally equivalent to a multiple of  $l$ , any codimension 2 cycle in  $\tilde{F} \times F$  which is supported on  $D \times F$  is of the form

$$Z_1 \times F + D' \times l,$$

where  $Z_1, D'$  have respectively codimension 2 and 1 in  $\tilde{F}$ .

We use now the fact that for  $L \in F$ , the points  $L$  and  $\phi(L)$  of  $F$  parameterize lines

$$\Delta_L, \Delta_{\phi(L)}$$

in  $X$  which satisfy the property

$$2\Delta_L + \Delta_{\phi(L)} = H^3 \text{ in } CH^3(X).$$

Thus we also have

$$2I_L + I_{\phi(L)} = C \text{ in } CH^2(F),$$

where  $C = p_*q^*H^3$  is a constant. We then apply the Bloch-Srinivas argument [6] ([23],10.3.1), to conclude that  $2(\tau, Id)^*I + (\tilde{\phi}, Id)^*I$  is rationally equivalent to the

sum of a cycle of the form  $\tilde{F} \times C$  and of a cycle  $W$  supported (via the first projection) on a divisor of  $\tilde{F}$ . We can thus apply the remark above, which gives

$$2(\tau, Id)^* I + (\tilde{\phi}, Id)^* I = \tilde{F} \times C + Z_1 \times F + D' \times l,$$

that is formula (2.44) with  $Z_2 = C$ . ■

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