Artificial Intelligence and Mathematics Education : Expectations and Questions

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Abstract: The main contribution of Artificial Intelligence (A.I.) to mathematics education is to provide concepts, methods and tools for the design of flexible and relevant computer-based systems for teaching and learning purposes. Such systems convey *great expectations* as : direct manipulation of abstract objects, customized explanations, intelligent microworlds allowing learning by discovery. Many *questions* are associated to these expectations, as in the first place the question of knowing what can be learned and what is learned by means of interaction with such AI-systems ? But, other important questions must be asked concerning the consequences of the knowledge reification implied by AI modelling and the design of friendly interfaces or the ways such systems can cooperate with teachers in the mathematics classroom. We will consider these questions in the context of the Cabri-géomètre project.

1. Expectations

Artificial intelligence (A.I.) has the practical objective of designing and implementing systems which behaviors appear intelligent to the eyes of human observers : looking at the system, one can legitimately conjecture that its behavior is due to some kind of reasoning. A related theoretical objective is the modeling of knowledge in an operational way. That implies a clear identification of what knowledge consists of and of the ways in which it can be represented. For this reason any research and development in A.I. implies epistemology, explicitly or in action.

Thus, intelligence in the expression "Artificial Intelligence" means essentially that implemented models enable a machine to solve problems, in the sense that solutions of these problems have not been a priori encoded, but that they are constructed originally by the machine.

A brief report on the historical development of the relationships between A.I. and Education might help the reader to understand the context of this presentation.

The first significant projects of A.I. in the field of educational technology for mathematics education appeared in the early seventies. One can for example mention the Integration Tutor from Kimball (1973). But the emblematic projects of this period were not in the domain of mathematics, they were in geography with SCHOLAR (Carbonnell 1970), and medicine with GUIDON (Clancey 1979). Actually it is better to see projects of this period as being A.I. projects taking Education as a field of application than to see them as projects in the domain of Educational Technology as such. At the same period started the LOGO project (Papert 1973) which is one of the first significant A.I. having some specificities with respect to Mathematics Education. It is in the eighties that we can identify projects specific to education and A.I. and concerning mathematics. Among them one can mention BUGGY (Brown and Burton 1978), PIXIE (Sleeman 1982) , ALGEBRALAND (Brown 1983), GEOMETRY-tutor (Anderson et *al*. 1985), WEST (Burton and Brown 1979), etc.. A new step has been made in the early nineties with the growth of the number of A.I. projects claiming their specificity to Education. In this trend a community is grown, gathering researchers from A.I., Education and Psychology, organizing itself. Even a specific journal exists since 1990 : "Journal of Artificial Intelligence and Education".

I will not here cover all the field, even its restriction to mathematics. My purpose will be to consider it from the point of view of *didactique* , trying to clarify some questions related to our expectations, which might leads us to new research.

By *didactique* I mean the theoretical study of the characteristics and of the properties of situations specifically organized to allow the understanding and the acquisition of a given piece of knowledge. This theoretical approach, initiated in France by Brousseau (1972), is original by its emphasis on a modelization of teaching/learning situations with respect to the specificity of the knowledge they intend to allow one to learn. Didactique also studies phenomena related to the implementation of didactical situations in real teaching practice (Brousseau 1992).

I will here present briefly two of the essential concepts of didactique (for a more detailed presentation see Brousseau 1986, Balacheff 1990):

- The specification and the design of a didactical situation rests on an analysis of the specific "ecology" of the knowledge concerned. It means an analysis of the circumstances which can favor the learners' construction of a correct meaning for this piece of knowledge. But such a situation cannot exit by itself, it must be introduced to the learners by the teacher. This is done through a process by which the teacher and learners negotiate the meaning of the situation in the context of the current classroom activity. The outcome of this negotiation is what we call a *didactical contract*, it defines the nature of relationships teacher and students have with respect to the situation concerned, and thus the meaning of the knowledge to be learned. But, what this didactical contract consists of remains largely implicit and open to research questions.

- To become teachable, any piece of knowledge needs to be adapted in order to fulfill the specific constraints of teaching and learning. Among these constraints one can point out : time constraints on teaching, previous learner knowledge, nature of the means at disposal, organization of the classroom society, etc. It is very tempting to see this adaptation as a mere elementarization. Unfortunately what happens is quite more radical. We can speak of a transformation of knowledge by its transposition to a didactical context. The process of *didactical transposition* (Chevallard 1985) is very complex, it can not be precisely located. Instead it is the concrete outcome of interactions among a society which involves teachers, teacher educators, members of the political and economical world, academics, parents and others. The more evident outcome of the didactical transposition can be observed in the form of official texts describing curricula or making recommendations on the way these curricula must be implemented. But it consists also of all the evolution of teacher education, and of teaching materials proposed to teachers.

Expectations concerning the usefulness and efficiency of intelligent teaching/learning environment covers many aspects among which the main ones are : making knowledge more accessible, allowing more autonomy to learners, helping or eventually replacing teachers in some tasks. I will try to enlighten these issues and raise some of the questions I think the community of research in mathematics education will have to consider in the near future. These questions are related to knowledge modelization, learner modeling and understanding of errors, design and management of learner/machine interaction.

2. Reification of mathematical knowledge

2.1. Drawing, figures and geometry

The most important outcome of the evolution of person/machine interaction of the last decade, as demonstrated by the Macintosh interface, is interfaces allowing sophisticated dynamic graphical display and their direct manipulation. In the case of education, the consequences are numerous and very important.

Such interfaces open the possibility to learner activities in conceptual domains which otherwise would require a higher level of symbolic manipulation. The basic idea of reification of knowledge is "to transform implicit and unobservable phenomena into objects that can be visualized and studied" (Wenger 1987, p.317).

In mathematics graphical displays and direct manipulation have led to the development of softwares dedicated to the learning of geometry (Laborde 1992). The power of these softwares is such that they are likely to be at the origin of a renewal of geometry teaching. For example, Cabri-géomètre (Laborde 1986, Baulac 1990, Bellemain 1992) whose design is based on the principle of direct manipulation, provides users with a model of geometry allowing geometrical experience in a very friendly and flexible way.

Such an environment does not only provide students with a tool which help them to escape the tedious task of making correct geometrical drawings, it changes the meaning of this task. The object constructed at the interface of Cabri-géomètre, starting from few basic objects (point, line, segment, etc.) and construction tools (like the drawing of a perpendicular line) is correct only if the geometrical properties specified by the construction are preserved by the manipulation. Thus a learner can produce a "correct" drawing at the interface, that is a drawing which is perceptively correct, which does not correspond to a correct geometrical figure (c.f. the case of the symmetric of a line-segment in section 3.2.).

In a paper-and-pencil environment only the result of the construction is accessible, not the process which led to it. So, "to construct" in a paper-and-pencil environment means to construct one drawing, whereas in Cabri-géomètre it means to construct a class of drawings. The geometrical drawing can then be defined as the class itself, each specific drawing at the interface is a representative of this class.

The computer-based environment provides learners with a sophisticated milieu for the learning of geometry. In this context, geometry becomes a good theory to understand and to explain the behavior of this milieu. Experiments, in relation to problem-solving, which were not possible before — in a paper-and-pencil context — become possible in this milieu. Let us take the example of the following problem:

> Let ABC be a right triangle with A vertex of the right angle. P is a point on BC. Let be I and J the bases of the perpendicular lines drawn from P to AB and AC. How can P be chosen to minimize the length of IJ ?

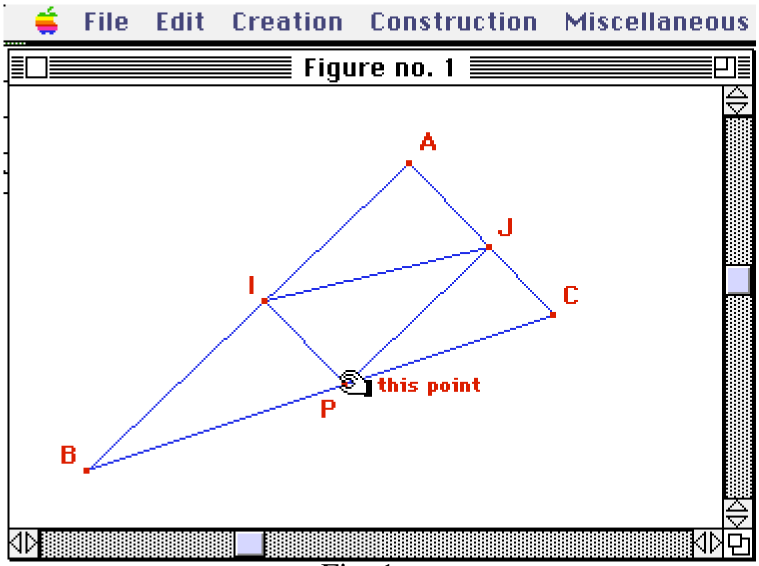

Fig. 1

The direct manipulation of the point P on BC (Fig. 1) allows learners to observe the behavior of IJ and the variation of its length in the Cabri-géomètre calculation window. They are likely then to question the common false answer "P must be the mid point of BC".

Since the meaning students attach to geometry will be the result of their interaction with this system, it is important to explore to which extend this meaning could depend on its characteristics. For instance, in the case of the preceding example, the complexity of the task is quite different depending on the software environment used (Laborde 1992a pp.131-132), and thus the nature of a solution and the related learning outcomes. I invite the reader to think of an other example: the problem of drawing three circles externally tangent two by two, in the context of paper-and-pencil, LOGO and CABRI-géomètre.

Actually, the specificity of the programme which underlies the software used, and to some extend the characteristics of the machine itself, are likely to play an important rôle the learners' construction of meaning since they have already an effects on the knowledge to be taught. I will consider this point in the following section.

2.2. The computational transposition

Two classical effects of representation technics and tools on a given knowledge, widely recognized by A.I. researchers, are *granularity* and *compilation* :

- Granularity: The description of a model of an "object" X requires the choice of a grain for the decomposition of X in elementary components which provide its basic elements (axioms of a formal system, primitive in language generation, basic principles of a simulation, etc.). Then, the implementation of the model fixes up this choice and hence some limits to the behavior of the software produced. In the case of educational systems it raises the question of the level of decomposition of the content to be taught. But such decisions must not be confused with decisions concerning prerequisites which state conditions on learners knowledge. Granularity decisions do interact with decisions about prerequisites, but they have a broader scope of concern. They bind the coherence and the generality of the implemented model (Psotka et al. p.282).

- Knowledge compilation: it refers to a process which transforms a piece of declarative knowledge into a piece of procedural knowledge available for action Anderson (1983). An essential consequence of this process is that the rule by itself doesn't give account of its origin, its validity or of its relationships with other pieces of knowledge. The phenomena of knowledge compilation plays an important rôle as soon as one want the system to explain its own behavior.

Knowledge compilation and granularity bind on the inspectability of systems and their capacity for the generation of explanations . Obviously, the later is of a special importance in an educational perspective (Delozanne 1992).

These aspects of implementation are quite classical and often mentioned, I would like to point other ones which could be of importance in our evaluation of computer-based learning environments.

Let us take the example of Cabri-géomètre. This software allows to draw a point on a segment (Fig. 2) without any other constraint than being one of the points of the segment, what I will call it "any-point" to keep the idea of a random point carried by the French expression "*point quelconque*".

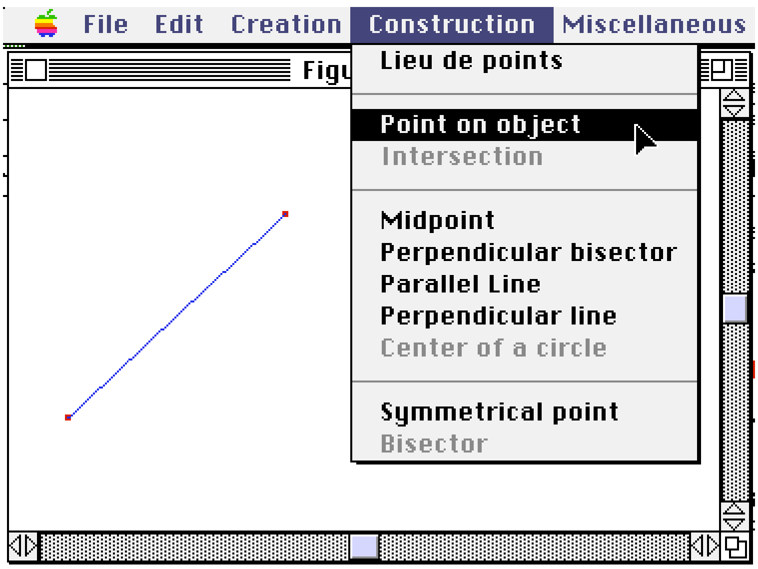

Fig. 2

When one extremity of the segment is dragged on the screen, the any-point must move. So a decision is taken about the behavior of this point. One can follow what might happen with paper and pencil, choosing randomly a new any-point for each new position of the extremities of the segment . But in this case the any-point might "jump" from place to place, badly surprising learners. Instead, they probably expect the drawing to evolve smoothly : the any-point following a continuous trajectory like the other points. This is obtained in the case of Cabri-géomètre by constraining the any-point to always divides the segment according to the same ratio. The consequence is that, from a Cabri-géomètre point of view, so to say, this point is no longer an any-point : when one extremity of the segment moves while staying on a given line, the trajectory of the any-point is an homothetic line. Analogous decisions are taken for the other objects. Any-points on a straight line or on a circle have a behavior which fulfill the WYSIWYE principle : "what you see is what you expect" (Bresenham 1988, p.348).

Important decisions of implementation are related to time management, they imply the introduction of explicit order where usually users do not pay much attention, or even don't matter. Let's take an other example in geometry : an equilateral triangle is completely defined once one of its sides is known. So, it can be defined for Cabri-géomètre as a macro-construction following the classical construction using two circles of the same radius |AB| centered on the extremities of a given segment AB. The arguments of this macro-construction will be the extremities of a segment. The figure (Fig. 3), shows such a triangle constructed on AB, but if the user gives the argument in the order B then A (Fig.4), the triangle drawn has a different orientation. This demonstrate that the introduction of time has as a consequence the introduction of order, and thus of an orientation of the plan (Payan 1992).

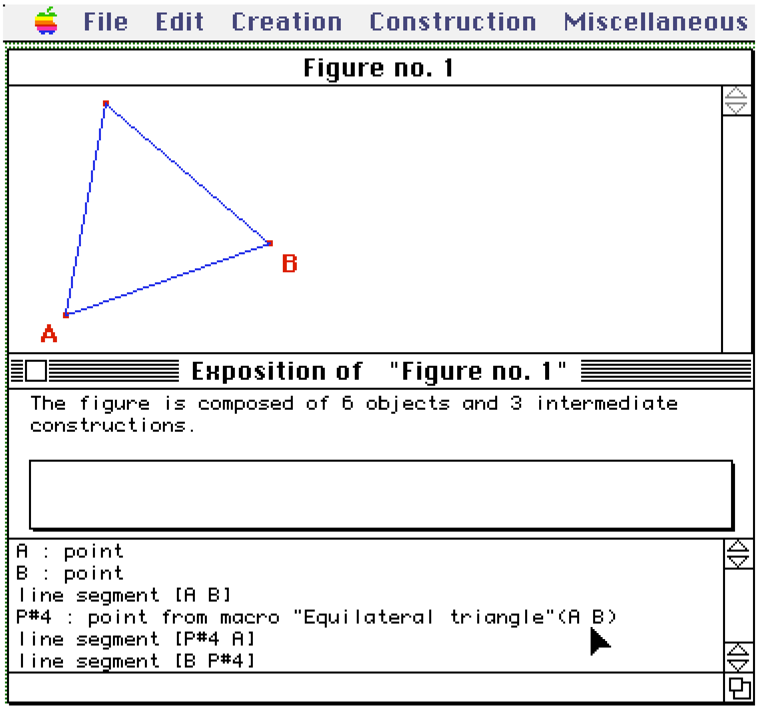

Fig.3

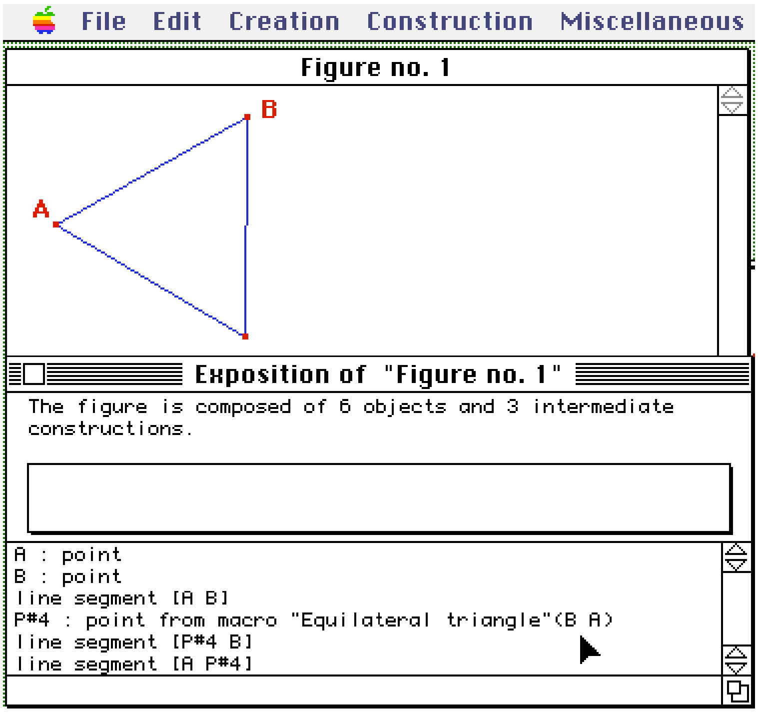

Fig.4

To illustrate the problems raised by implementation choices, I will briefly give an other example since one might consider the preceding ones as too specific because of their relationships to graphical representation. Let's consider algebraic expressions like the one we manipulate in elementary algebra. These expression can be seen as strings of characters like the following : 5x+2x(x-3) or as a tree structure. But the choice of a knowledge structure does fix the kind of manipulation possible at the interface of the system. If the tree structure is chosen, what is the case of APLUSIX (Nicaud 1992), then some learners manipulations are not possible. For example : 5x+2x cannot be extracted from 5x+2x(x-3) or the transformation 5+3x —> 8x has no interpretation (one cannot find any sub-tree manipulation which "explain it"). On the other hand, the choice of a list structure, what is the case for PIXIE (Sleeman 1982), allows such manipulations and thus gives the possibility to take into account successes and failures of a learner at the level of basic algebraic manipulations. Actually, APLUSIX and PIXIE have not the same teaching objectives.

Reacting to what might be seen as technological limitations, people might suggest other implementations so that the effects mentioned no longer exist. But such suggestions miss the fact that other implementations would give rise to other "side" effects, or unintended effects. The question is not to suppress them, since any representation has productive effects, but to be able to say in details what they are. Multiple representations are very often suggested as possible solutions for this kind of problem, but it supposes that one can exhaustively enumerate and describe representations related to a given piece of knowledge. A quick look, at the history of human knowledge, even restricted to science, or at the various representations punctuating the development of learners, can convince any one that the enterprise is condemned to failure. An other solution which has not been explored yet is to characterize the domain of validity of the chosen representations, and thus the domain of validity of the educational software itself. Such analysis needs a strong pluridisciplinary competence.

These problems and phenomena are consequences of general constraints of computerization and its physical implementation, they are the indication of the work made to adapt knowledge representation to requirements of symbolic representation and computation. They reach beyond education, they are general problems as well in knowledge engineering and computer graphic, especially when person/machine interaction is concerned. For these reasons I call *computational transposition* (Balacheff 1991a) the process which leads to the implementation of a knowledge model.

2.3. Expectations and questions

The progress of direct manipulation and reification of knowledge let us expect powerful environments enabling learners to experiment in domains up to now not accessible without a high level of competence. The computer effects, which are already well-known in numerical domains, will extend their significant influence to formal calculus and graphic manipulation.

Since students knowledge is an emergent property of a dialectic relationship between perception and conceptualization in the context of the interaction at the interface of the system, which itself depends on the nature of the internal representation, the choice of knowledge representation and treatment plays an essential role. For this reason, specification, implementation and didactical evaluation of educational technology requires the analysis of the way knowledge is defined, represented and finally implemented in systems.

When specifying a computer-based learning environment, authors — actually, it is in general a team of authors — the content to be taught must be instanciated in a way that makes it operational for teaching purposes. Such a work which is usually done by teachers in the intimacy of their classroom, is made public by designers. The exigencies of knowledge computation require to solve a lot of problems which until then remained hidden.

Let us take a simple example in geometry. The micro-world Cabri-géomètre allows one to draw geometrical figures by assembling elementary objects like points, segments, straight lines, triangles or circles. Then, dragging on the screen the basic points of the figure, one can observe deformation and thus some invariant properties. For example, if one draws the three altitudes of a triangle and then moves any of the vertices of this triangle, then one observes that the three altitudes intersect at a common point. But, for some of the possible triangles, they no longer intersect (Fig. 5). The reason is that altitudes were drawn as segments, whereas in some cases they have to be drawn as straight lines. In their classrooms, teachers ask students to "extend" the altitude and then they try to negotiate the conflict between the drawing and the "formal definition". Designers of geometrical drawing tools have to explicitly make a decision to consider the altitudes as segments or straight lines. Some softwares, like Geometric supposer (Chazan D. & Houde R. 1989, pp.52-55), solve this problem like what is shown by figure (Fig. 5), Cabri-géomètre allows an other solution closer to what teachers expect from students: when the triangle is accute the altitudes are drawn as segments, when not they are drawn as straight-lines (Fig. 6).

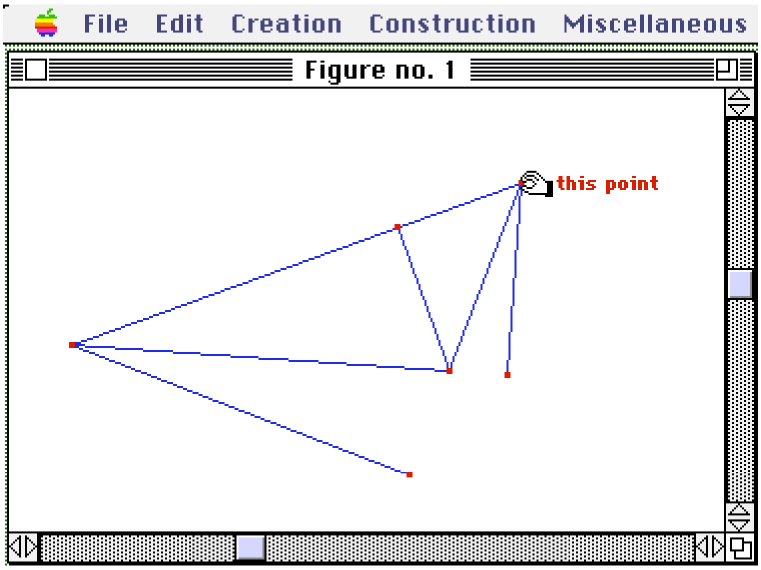

Fig. 5

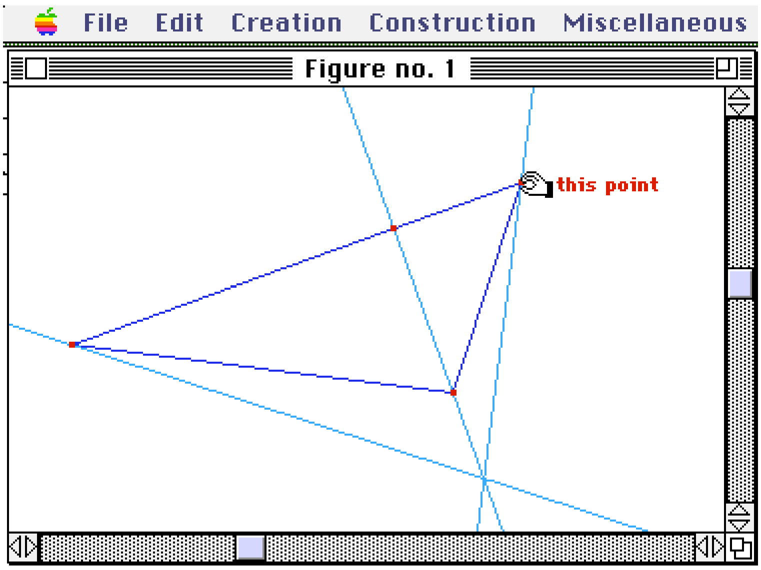

Fig. 6

By this type of activity, mainly having to take decisions about teaching facts or phenomena usually implicit or ill defined, designers and developers participate to the didactical transposition.

At the cross point of didactical transposition and computational transposition stands the problem of the relationships of the knowledge taught as it results from the behavior of the system with the knowledge it intends to teach. This problem is that of the characterization of *the domain of epistemological validity* of the software.

3. Understanding learners' understanding

3.1. The issue of learner modeling

Insofar as complex knowledge is concerned, the constructivist hypothesis appears as the most fruitful foundation for research on mathematics learning. This hypothesis states that learners construct their own knowledge, they don't receive it as being ready made. The fact that previous knowledge is questioned, the disequilibration in the Piagetian sense, results in the construction of new knowledge as a necessary response to the learners' environment.

For a given piece of knowledge, let us name *conception* the model of the related learners' mental construct. This designation emphasis the fact that conceptions are related to individuals, but also it refers to the fact that they are models of learners' mental construct which are real piece of knowledge since they allow them to solve problems, to make decisions, to perform actions. They are not beliefs more or less stable or efficient, more or less arbitrary.

Learners' conceptions do have a domain of validity and of efficiency.

In this theoretical context, problems play an essential role (Vergnaud 1992), first they are the source of the meaning of knowledge, but also intellectual productions can turn into knowledge only if they prove to be efficient and reliable in solving problems that have been identified as being important practically (they need to be solved frequently and thus economically) or theoretically (their solution allows a new understanding of the related conceptual domain). But, because they are the product of adaptations to problem situations, learners' conceptions are likely to be local or context dependent.

Then, in the case of learning supported by a computer based environment, the characteristics of the interactions between the learner and the machine play an essential role in the construction of meaning. Learners make sense of the feedback of the machine, and as a result their conception depend on their understanding of these feedback. The gap between the meaning they have constructed and the intended meaning is evidenced by their dysfunctionning in some situations. Actually these gaps can be of very different nature, I will here use the word *error* to refers to gaps which can be considered as symptoms of the underlying conception of the learner.

So it is essential that the machine can diagnose such gaps and that it can provide adequate feedback to students. One solution to this problem consist of giving to the machine the capacity to understand students' understanding. This problem is referred to by the AI community as student modeling.

Several solutions are explored concerning this problem, the main ones are :

- the implementation of a catalogue of errors : the machine try to match the gap it observes at the interface to errors *a priori* described in a catalogue. It then provides some ad hoc feedback .

- error generation : a model is implemented which allows the reconstruction of conceptions which can be the source of the errors.

- error reconstruction : using some machine learning algorithms, the machine attempts to automatically deduce mal-rules which might "explain" the observed gaps.

One of the important difficulty of student modeling is related to the characteristics of the interfaces which on the one hand constitute filters for the observation of students' behaviors, and on the other hand are likely to produce noises. These problems are classical and studied in quite an intensive way in computer science and A.I., as well as questions like stability of errors, or error migration.

I will here focus on the problem of coping with implicit meaning of actions carried on by learners at the interface. This problem is related to the identification of learners' intentions and thus plays an essential role in student modeling. The development of graphical interface and direct manipulation pushes this question on the fore front of the educational scene.

3.2. Coping with implicitness

In this section I will just give an illustration of the problems raised by coping with the implicit intentions related to actions performed by learners at the interface of systems.

Let's consider the problem of the construction of the image of a line-segment by a reflection of a given axis, using Cabri-géomètre. The figure (Fig. 7) is a screen image of the solution of a learner A*. This drawing seems satisfactory, but when one moves the axis of symmetry (Fig. 8) it appears that the behavior of the geometrical figure no longer fulfill the constraints of symmetry.

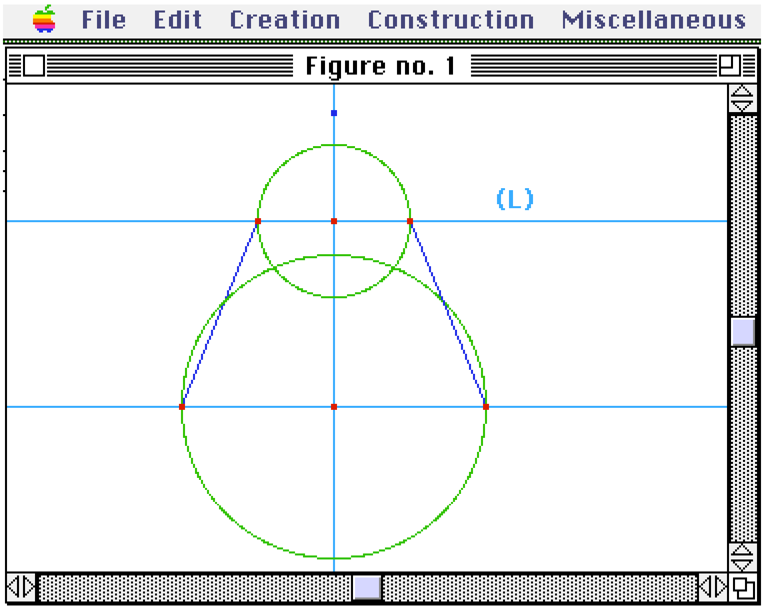

Fig. 7

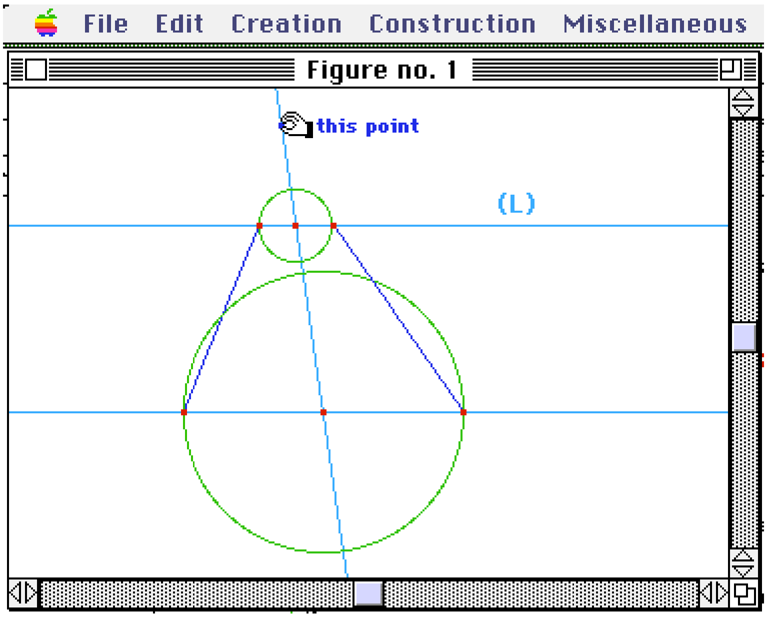

Fig. 8

Actually, this learner A* has not constructed the line (L) as a perpendicular line to the axis but as a line perceptively horizontal (the pixels are aligned) while the axis is vertical. For Cabri-géomètre (L) is a line with no specific constraints, so when one of the basic point are dragged, the perpendicularity is not kept. Cabri-géomètre can identify the wrong feature of the drawing if it is asked whether (L) and the axis are perpendicular, it answers that it seems true on the current drawing (Fig. 9), but that it is indeed false. Cabri-géomètre then proposes a counter-example.

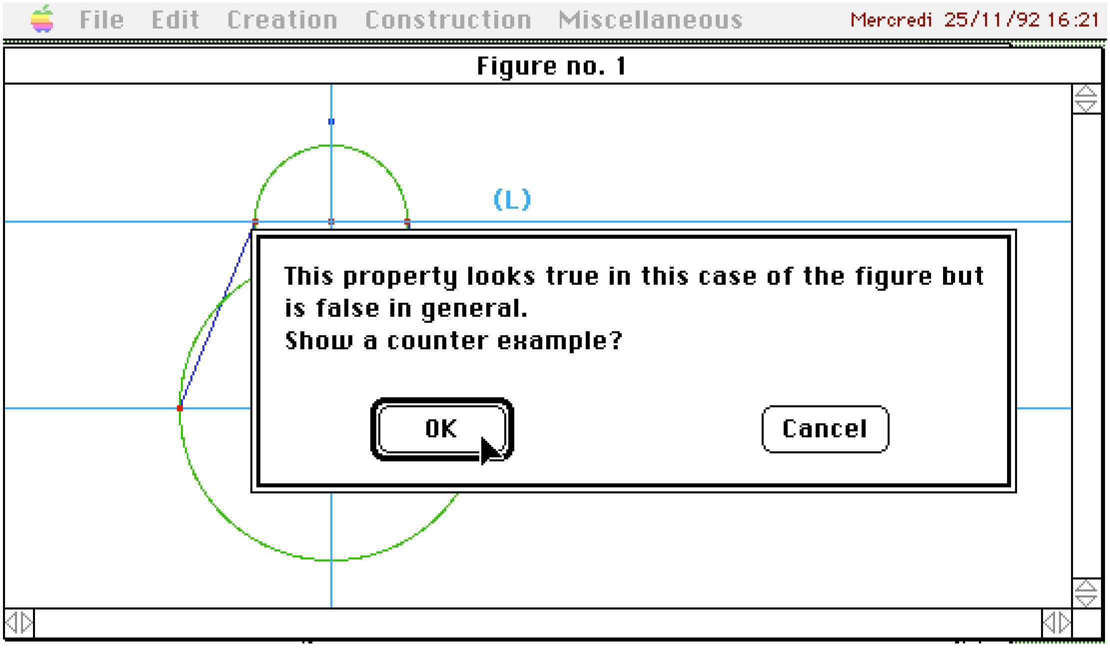

Fig. 9

The erroneous structure of the learner construction can be identified on its internal model, the graph which can be associated to the construction is not connected. If we interpret this graph as an "image" of the behavior of the learner, then it is an illustration of the fact that it is possible to identify such errors at the behavioral level. Here, Cabri-géomètre can make sense of it since it can recognize that the two lines are "perceptively" perpendicular.

This example shows that the problem of learner modeling starts right from the specification of the data to be retained as inputs for the diagnosis function. I will develop this point as one of the essential questions related to learner modeling.

3.2. Expectations and questions

The issue of student modeling conveys great expectations since it promises systems with more relevant behavior, allowing individualization and more learner autonomy. The progress in learner modeling will lead to systems which can adapt themselves to the specific cognitive characteristics of individuals they interact with. Making sense of the understanding of the learner, systems could decide on a better basis on the more relevant explanations or challenging activity or questions to provide to them.

At the research level, it raises difficult questions for the researchers in mathematics education as well as for researchers in artificial intelligence. One of the main question is that of the evaluation of the validity of the model implemented in the machine, or of the one it constructs itself as a result of some machine learning algorithms. To clarify this problem, let us distinguish the behavioral level and the epistemological level of modeling and look at the way the learner can be "understood" at each of these levels. This distinction is not by itself original, but the way I will define it differs from what can be in general found in the literature (Wenger 1987).

The preceding section demonstrates that right from the first decision to consider some events at the interface of the system, among several others which might have been considered as well, as being events which could help to characterize a behavior, implies an interpretation at an epistemological level. Thus, the fact that there is, or that there is not, interpretations or decisions is not a way to distinguish among these two levels.

Let us think of a computer-based system as being a device splitting the Universe into an external and an internal universe. Then consider the following schema (Fig. 10).

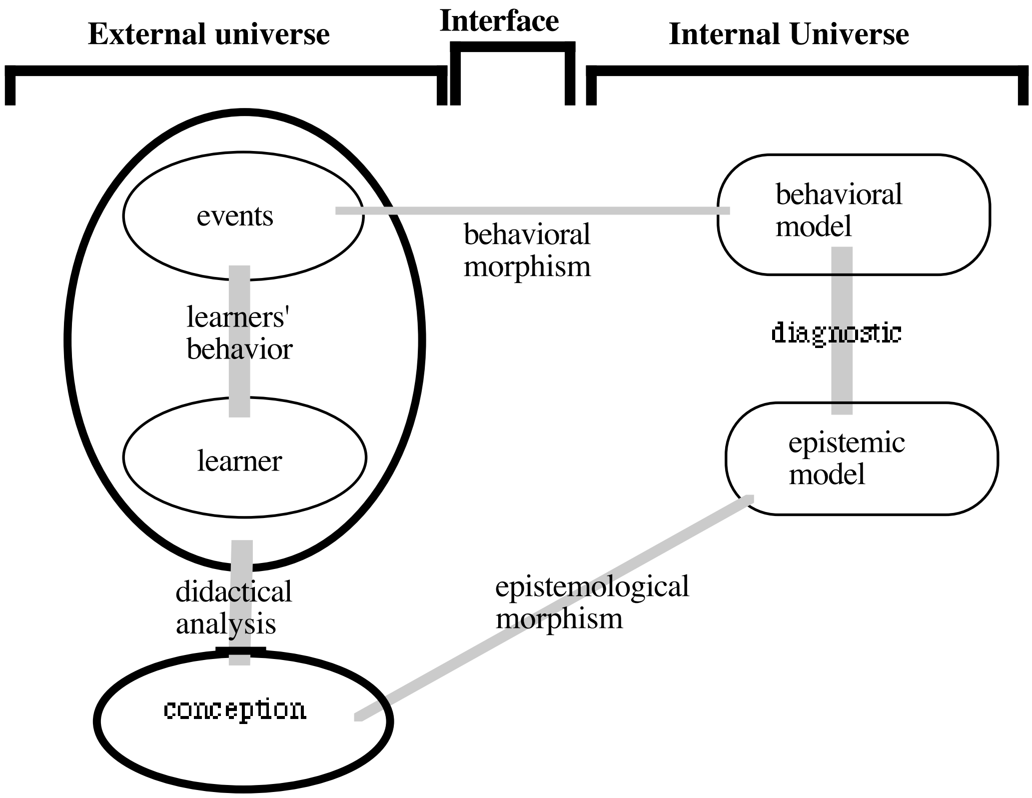

Fig. 10

The identification of "facts" is the result of a cutting out and of an organization of "reality" under the control of a theory and of a methodology. When researchers in mathematics education create a corpus of observed "facts", itsrelevance depends on the theory and the methodology they use. Some of these facts are organized in order to describe what we call *behaviors*. In their turn, these behaviors are used as data to elaborate models : the *learners' conceptions*. It is clear that such models are a construction of researchers and not at all what is "in" the students' heads".

In the case of a computer-based system as an observer of the external "reality", the events which are likely to constitute the observed "facts" are physical events captured at the interface. Physics can provide us with a description, but even this level of description cannot be considered as "objective": it is a kind of organization of some "reality".

Then, I define as *behavioral model* the model obtained from the analysis of the events occurring at the interface of the system. This analysis consists essentially of ignoring "non relevant" events and using high level language to describe some sets of events. The criterion I propose for the validation of the behavioral model is the following : when runned in the same environment, it creates at the interface the same observable "facts" as those observers have identified with reference to a precise didactical analysis. I call *behavioral morphism* this mapping between the behavioral model and the description of the student behavior that researchers may produce as the result of their observations in the external universe. Behavioral mapping is more than just the history of students actions at the interface, it gives account of the structure of this action seen through the lenses of the system.

The *epistemic model* is constructed using data which are produced by the behavioral model. The construction of the epistemic model is the result of what is usually called a *diagnostic*. I would consider that such a model is valid if it exists an *epistemological morphism* between it and the students' conception as elaborated by research, that is : a mapping which preserve the epistemological structures.

The theoretical framework and the methodology for the evaluation of such level of modeling and for the construction of such morphism is one of the more important to be investigated in the field of AI and Education. Our progress would be directly related to the progress made on learner modeling in the external universe; in other words it depends on the progress we will make in research in didactique.

4. Managing didactical interaction

4.1. Micro-worlds versus tutoring systems

Micro-worlds and tutoring systems stands at two extreme points of a continuum of computer-based learning environments which can be organized on the basis of their directiveness. On the one hand, micro-worlds offer to learners open worlds in which they can freely explore problem situations, on the other hand tutoring systems provide students with strong guiding feedback.

LOGO is a prototypical example of a micro-world. Starting with few simple primitives, the learner can construct more and more sophisticated objects and define more and more complex tools for further investigation. In some sense the micro-world evolve as the learner's knowledge grows. This is a crucial feature of micro-worlds. It is a significant difference between micro-worlds and most of the simulation systems. In general simulations consist of models whose parameters can be instanciated by learners, then they can observe the effect of the chosen inputs. But the initial models do not evolve as the learner progress, instead some environments change the model in order to obtain this adaptive behavior. The criterion for the evaluation of these simulation is the quality of their evocation of a model of reference or of some "reality", this evaluation includes that of the visual feedback which are in general in the iconic or graphical register. Some micro-worlds can be considered as simulations, they have then to satisfy epistemological criteria with respect to some model of reference. Cabri-géomètre is a good example of such a micro-world, it is a simulation of geometry which, by means of macro-constructions, enables the learner to built into the micro-world new objects or new tools with as starting points few elementary primitives.

Geometry Tutor (Anderson *et al.* 1985) is a prototypical example of an intelligent tutoring system. It guides learners in the construction of a mathematical proof to the solution of a problem in geometry, providing immediate feedback, clear hints and help when the learner fails or gets lost. A friendly interface allows a reification of the learner progress by a tree which is dynamically built, at the end of the process this tree represents the structure of the proof. Geometry Tutors just accepts learners explorations which are likely to lead to a correct proof (Guin 1991). Other tutors exist which are less restriction to learners freedom. For example APLUSIX (Nicaud 1992) in algebra leaves to learners the possibility to explore different strategies, efficient or not, in order to find the solution to a problem of factorization of an elementary algebraic expression. But, as a tutor it does not accept errors in the application of algebraic rules and in case of bad inputs it provides immediate feedback. A feature of APLUSIX which is quite classical to tutors is the capacity to provide learners with explanations of the current actions even with reference to a strategy. In geometry MENTHONIEZH (Py and Nicolas 1990) provides learners with an environment for quite a flexible investigation of mathematical proof, the construction of the proof is not necessarily "linear", but still keeping them on the correct track.

The evaluation of such environments leads to two remarks :

- First, a free exploration of a micro-world offers to learners a rich experience but does not guaranty that a specific learning occurs. For this purpose the micro-worlds must be embedded in a milieu organized by a teacher. It is the didactical characteristics of this milieu which could ensure the expected learning outcome.

- Second, the close interaction of a tutor does guaranty performances but not the nature of the underlying meaning. One reason is that the learner cannot express his or her view about the knowledge concerned, so the tutor feedback focuses more on the knowledge of reference than on the learner's knowledge and its evolution. An other reason is that learning in such an environment could means learning how to obtain the best hints and help from the tutor so that the problem at hand can be solved. In other words, the learner can learn how to optimize the use of the tutor feedback instead of the knowledge the task is supposed to convey.

A trend of research now look for environments with a better balance between these two extreme points. I will present it briefly in the following section.

4.2. Guided discovery learning

Guided discovery learning (Elsom-Cook 1990) refers to environments which adapt their level of directiveness to the current state of the learners' knowledge. They could provide very open environment, like micro-worlds, for some purpose and shift to a tutor-like behavior if the situation of learner is such that it seems better to do it, or if the teaching target appears to be better reached in this way.

HyperCabri (Laborde J.-M. & Strässer 1990) is a prototype of such an environment in the case of geometry. It is the implementation, with HyperCard on Macintosh, of a didactical process organized around the problem of the construction of a square. The learning target is essentially the concept of circle as a set of equidistant points to a given point. The main feature of this environment is that learners are free to explore the problem situation in the framework of the micro-world. When they have finished, their solution is evaluated by the tutoring module of HyperCabri. If a learner has not succeeded then the feedback can be either a new task to allow the study of a relevant and helpful sub-problem, or direct hints or helps. The design of this environment is based on the *a priori* analysis of the possible solutions to the problem and the related student's conceptions.

DEFI-CABRI (Baulac & Giorgiutti 1991) is an implementation of a guided discovery architecture based on the interaction between the micro-world Cabri-géomètre and the tutor DEFI. The original aim of DEFI is to provide learners with an environment for the learning of mathematical proof in geometry. For the investigation of a given problem, learners can construct the related geometrical figure in Cabri-géomètre and freely explore its properties, then DEFI makes an evaluation of the figure drawn and coach the learners' construction of a mathematical proof to the solution they have found. At the present stage of development, DEFI does not allow learners to engage in a wrong direction, but the feedback it provides does much rely on learners mathematical commitment. During the exploration of the figure properties, DEFI asks questions like "do you know how to prove such property ?", if the learner does then he or she can continue the exploration. If he or she does not, then DEFI provides some helps like question inviting to try new exploration of the figure. This strategy allows to escape the rigid approach top/down or bottom/up usually imposed by tutors for the search of mathematical proof.

In such environments the more difficult task is the evaluation of the learner activity in the micro-world and the related productions. In the case of geometry a crucial question is that of the genericity (Laborde 1992b) of the drawing produced and the fact that it correctly fulfill the initial problem specification (Allen, Desmoulins & Trilling 1992).

4.3. Expectations and questions

The progress towards Guided Discovery Learning Environments let us expect to have at disposal in the future educational softwares closer to a constructivist conception of construction of meaning, and so escaping the programmed teaching and behavioral learning paradigm.

But the effects we emphasized concerning tutoring systems might not disappear that easily, the Guided Discovery Learning conception of design does not guaranty by itself the correctness of the learning outcomes. Since these systems have an explicit objective of teaching they cannot escape the emergence of a didactical contract and its related consequences. So, learners could still construct knowledge in order to satisfy some machine expectation instead of a construction specific to the intended problem-situation.

In this context, the problem of coping with learners errors and misunderstanding occupies a central position. The didactical problem then is not to avoid or to make disappear such learning phenomena as inopportune facts, but to understand their origin and to specify the conditions needed for the evolution of the underlying conceptions. Many errors are due to conceptions which existence are necessary for the construction of meaning, learners must both construct these conceptions and then reject them to have access to the meaning that teaching intends to convey (Brousseau 1986). This problem is very difficult because the evolution of these conceptions can happen in unintended directions. One cannot know *a priori* if the overcoming of a difficulty or any obstacle is a progress, what is quite coherent with the fundamental hypothesis of constructivism (Balacheff 1991b). In particular the adaptation might be ad hoc or local, being the cheapest way to adapt to the situation. It might be good to remember that this behavior is not only that of young learners : even scientists confronted to anomaly could "devise numerous articulations and *ad hoc* modifications of their theory in order to eliminate any apparent conflict" (Kuhn 1962 p.78).

These questions evidence the difficulty we could encounter in considering learning environments as autonomous systems. To overcome such difficulties we must consider the computer-based environment not as an isolated system but as part of a larger system which includes the teacher. As a conclusion, I will consider this point in the last section of this paper.

5. Questions

The introduction of computer-based systems of whatever types they are, complexify the teaching/learning situation from a didactical point of view because such systems are first of all materializations of a symbolic technology. This particularity plays a role in two ways :

- on one hand, by modifying the object of teaching as a result of the process of the computational transposition ;
- on the other hand, because the student, as well as the teacher, might consider the machine as a new partner.

In such a context, to negotiate a learning situation with students needs to take into account the amount of didactical responsibility which can be passed from the teacher to the machine, and not only the characteristics of the task and its meaning in relation to a given piece of knowledge. Indeed, the relevance of learner/machine interactions allowing the emergence of knowledge depends on the quality of this aspect of the negotiation. The study of the characteristics of these situations and of their functioning are at their very beginning.

Teachers will not be able to insert the new technology in their daily practice, if they cannot be well informed on all the aspects which could determine its place and its precise role in a didactical process. To some extent, we can say that they must know the computer based system from a didactical point of view, as they would like to know a colleague with who they might have to share in turn the responsibility of the class.

Indeed that raises the question of the specification and the information about the knowledge to be "encapsulated" in the machine and the ways it functions with respect to a given learning/teaching target. But it raises also the problem of the communication between the teacher and the machine concerning the learning process to occur. Many research in artificial intelligence have dealt with explanation for teaching purposes, all of them are centered on learners, none on teachers and the explanation they might expect about a didactical interaction. Giving back the trace of the learner/machine interaction will not be efficient nor significant since it will consist of giving too much information at a too low level. Thus, the machine must be able to handle and produce relevant didactical information about the teaching process, in order to be able to interact and cooperate properly with the teacher. That is nowadays an open problem for both researchers in mathematics education and computer scientists, but it is one of the conditions for tomorrow cohabitation of artificial intelligence and real teaching.

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