A Logo-based Task for Arithmetical Activity

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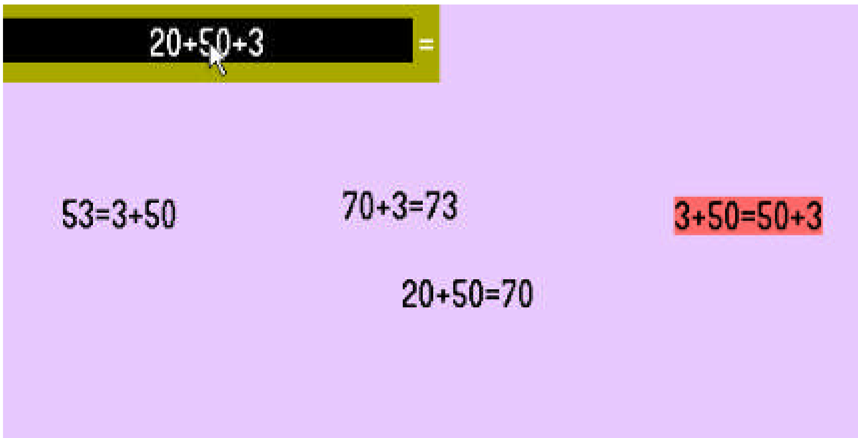
Abstract

Young children attend to answer-getting readings of arithmetical notation. This is evidenced by many children's exclusive acceptance of $a+b=c$ syntaxes that lend themselves to computational readings (e.g. Behr et al., 1976; Carpenter & Levi, 2000; Knuth, Stephens, McNeil & Alibali 2006). Even those children who do accept a wider variety of syntaxes, such as $a+b=b+a$ and $c=a+b$, adhere to a computational view involving getting answers to both sides of the equals sign and checking they are the same (Jones, 2006).

I report here on the trialling of a Logo-based task (see screenshot below) designed to foster meaningful engagement with the structural properties of arithmetical equality statements. The design rationale is that of *diagrammatic reasoning* where arithmetic inscriptions onscreen are observed and manipulated according to precise operational rules. The dual functionality of selecting equality statements and substituting terms is intended to afford the specific diagrammatic activities of *iconic matching* and *transformation making* towards the construction of purposeful meanings for differing statement syntaxes.



The data show that the children's readings of arithmetic notation were broadened from computation to iconic matching during trialling. This pattern-based observation of notation combined with the activity of transformation making resulted in commutative and partitional meaning-making for $a+b=b+a$ and $c=a+b$ syntaxes respectively. The data also suggest that various factors impacted on the diagrammatic strategies developed by the children to complete the task. These factors included blind experimentation, systematic testing, existing computational readings, and emergent commutative and partitional readings.

Keywords

arithmetic, equals sign, meaning-making, task design

INTRODUCTION

Arithmetic is about performing computations in order to get a result (Hewitt, 1998). Accordingly, written arithmetic commonly appeals to answer-getting readings in the form of a string of numerals and operation signs followed by an equals sign and then the result, a sequence familiar to anyone who has used a simple calculator. There is rarely any need to appeal to commutative or partitional readings, as in $32+45=45+32$ or $77=45+32$, because all that is usually required is a final answer. Various studies have demonstrated that as a consequence many school children exclusively favour $a+b=c$ syntaxes. (e.g. Behr et al., 1976; Carpenter & Levi, 2000; Knuth, Stephens, McNeil & Alibali 2006). Even those children who do accept $a+b=b+a$ and $c=a+b$ syntaxes adhere to a computational view involving getting answers to both sides of the equals sign and checking they are the same (Jones, 2006). This closed, computational reading of arithmetic notation is of concern to mathematics educators on two levels. First, it is bereft of the operationalised relational thinking that Piaget (1952) reported present in young children in non-notational contexts. Second, many children encounter significant difficulties in later schooling when formal algebraic notating is required by the curriculum (McNeil & Alibali, 2005).

My focus here is on the first of these issues: the apparent gulf between young children's powerful mathematical thinking reported by Piaget and their narrow answer-getting readings of formal notation. I am not explicitly exploring learners' difficulties with the curriculum transition from arithmetic to algebra at the start of secondary schooling, although this issue is of implicit interest. My specific aim is to render structural readings of notation noticeable and "useful" (Ainley, Pratt, & Hansen 2006) by providing tools for manipulating onscreen arithmetic symbols towards a specified task goal.

DESIGN RATIONALE

Children's observations and manipulations of physical objects led Piaget (1952) to theorise an "operational plane" (p.220) of reversible equivalence and non-equivalence relations that is integral to the very notion of number itself. Quantitative and qualitative relationships were investigated by pouring liquids between differently shaped containers, arranging lines and piles of beads, matching eggs with egg-cups and so on. My research question might be framed as: How can we provide opportunities for children to observe and manipulate arithmetical symbols as they do physical objects? A way forward might be to provide learners with opportunities to engage in "diagrammatic activity" (Dörfler, 2006). Diagrammatic activity is the emergence of mathematical meaning-making through the systematic observation and transformation of "inscriptions" according to precise "diagrammatic operation rules" (p.105). Such activity is empirical and observation based, involving hypothesising and experimenting with inscriptions on a page or computer screen. Diagrammatic activities make no appeal to real-world metaphor or concrete referents. Instead, meaning-making is a social phenomenon requiring "a discursive context which offers a rich language to speak about the diagrams and their transformations" such that students "learn this language simultaneously with their development of the diagrammatic practice" (p.108).

In the remainder of this section I will describe a diagrammatic microworld I developed using *Imagine Logo*[1] (I chose Logo because it lends itself to the fast prototyping of microworlds). An example screenshot of the software, called *Sum Puzzles*, is shown in Figure 1. The software supports two basic functionalities: selecting equality statements and clicking terms. In Figure 1, $5+6=11$ has been selected by clicking on the equals sign. In this sense the inscription = might be considered as a handle for taking hold of the equality statement. A term can be substituted by

[1] Imagine (Kalas & Blaho, 2003) has been developed by a team at Comenius University, Bratislava, Slovakia and is published by Logotron.

clicking on it and the resulting substitution (if any) is specified by the currently selected equality statement. For example, in Figure 1, if the inscription $5+6$ in the black box were to be clicked, it would change to 11 as shown in Figure 2. Likewise, if the inscription $5+6$ in the statement $5+6=6+5$ were clicked the statement would become $11=6+5$; if the inscription 11 in the statement $11+3=14$ were clicked the statement would become $5+6+3=14$. Note that substitutions are reversible: if the inscription 11 in the black box in Figure 2 were clicked while the statement $5+6=11$ is still selected the situation would return to that shown in Figure 1.

Figure 1: Sum Puzzles software

Figure 2: Substituting a term

The central goal is to transform the term in the black box into its answer by using the statements to make substitutions. The software presents a series of puzzles of increasing complexity (e.g. Figure 3). It is predicted that the embedded functionalities of selecting equality statements and clicking terms will afford the diagrammatic activity of *iconic matching* (Conjecture 1). Iconic matching is the process of looking for visually identical inscriptions. The emergence of such activity during trials would provide evidence that children's readings of arithmetical notation can broaden from computation to include pattern awareness through resource and task design.

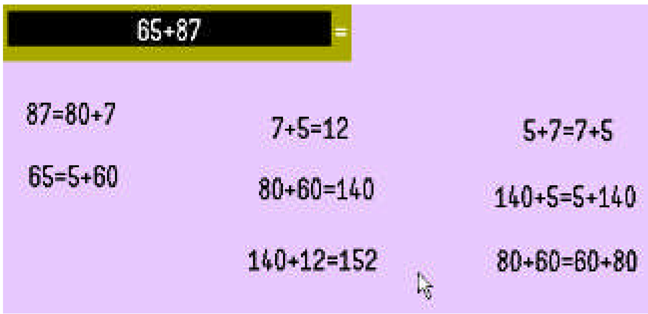


Figure 3: Puzzle containing eight statements

Gray and Tall (1994) have described operationalised readings as involving the ability to make reversible interchanges of equivalent arithmetic terms. In *Sum Puzzles* such reversible interchanges are carried out explicitly onscreen through *transformation making*, i.e. substituting a term for a second, equivalent term. Such transformation making potentialises a powerful design principle, “using before knowing” (Papert, 1996). Evidence for emerging operationalisation would include children making commutative and partitional meanings for $a+b=b+a$ and $c=a+b$ statements respectively (Conjecture 2) when engaging in diagrammatic activity.

According to diSessa and Sherin (1998), existing and emerging ideas co-exist in fragmentary, often conflicting, patterns. Their “co-ordination classes” account predicts that children’s empirical experimentations with arithmetic turtles will be no simple story of diagrammatic activities replacing computational readings. Rather, diagrammatic activity and computation will impact upon one another and the trial data should be expected to show evidence of interference between the two when solving arithmetic puzzles (Conjecture 3).

METHODS

The two trials reported in this paper constitute one iterative step of my wider study. This iteration provides a test of the above conjectures, which were informed by previous pilot trials of the software (see Jones, in press).

Each trial involved two children working collaboratively through a sequence of 11 onscreen puzzles and lasted almost 40 minutes. The role of collaboration in task trialling is to foster a shared dialogue about diagrams in order to enable meaningfulness to emerge. The resulting composite data of onscreen manipulations and naturalistic discourse offers a window onto deep mathematical thinking (Noss and Hoyles, 1996). This method of data capture tends to be most insightful where the task is designed to challenge and reshape existing notions thereby rendering key shifts in mathematical thinking readily identifiable.

During the trials, each pair of children was first shown how to operate the software. For each puzzle I challenged the children to make the “answer” to the “sum” appear in the black box (Challenge 1). Twice in each trial I then set another challenge to get the original sum back again (Challenge 2). Challenge 2 was intended to emphasise reversibility but I dropped it soon into each trial as it proved ineffectual for generating eventful data. On other occasions when the data became uneventful I set a more difficult challenge: make it so that the term in the black box can be changed into its answer with a single click (Challenge 3). The distinction between Challenge 1 and Challenge 3 can be illustrated for the case of Puzzle 10 (Figure 3): Challenge 1 would be to transform $65+87$ in the black box into 152; Challenge 3 would be to transform the statement $140+12=152$ into $65+87=152$, and then use it to transform the term in the black box into 152 in a single click. Other than demonstrating the software and setting challenges, my role as researcher was to prompt for verbal elaborations from the children, mainly by asking “why” questions of the type “Why do you think that didn’t work?”

The children in the trial were aged 9 and 10 years (Year 5). The first pair of children, T and A, were boys deemed by their class teacher to be above average in mathematical ability. The second pair, K and Z, were girls deemed average in mathematical ability.

DATA AND DISCUSSION

In this section I will present a brief overview of the two trials followed by data and analysis relating to the three conjectures set out above. All excerpts are referenced by the time at which they began, in minutes and seconds.

Overview

Trial 1: T and A

T and A were confident with atypical arithmetical syntaxes at the start of the trial, as illustrated by their reaction the first time a statement of the form $a+b=b+a$ appeared onscreen (Puzzle 3):

08:37 T: 1 add 9 equals 9 add 1.
A: Yeah it does.
T: Yeah I know.

T and A got to grips quickly with the software and were able to develop strategies for solving puzzles (Challenge 1) confidently and efficiently. The most telling data was generated when I set Challenge 3 as this caused significant perturbations to their emerging (and successful) puzzle solving strategies. T and A's level of engagement was high throughout the trial, and characterised by cycles of (amused) frustration and triumph. The boys reflected on this at the end of the trial:

35:50 A: It's quite challenging [T: Yeah] It's kind of addictive in a way. You don't want to stop. [T: Yeah] Kind of like, really determined.
T: Bit fiddly. And it can sometimes get really annoying but it's still fun.

Trial 2: K and Z

K and Z were hesitantly accepting of $c = a + b$ syntaxes and unfamiliar with $a + b = b + a$ syntaxes as illustrated at the end of the trial when I asked them whether they had encountered such statements before. Only K claimed to remember encountering $c = a + b$ syntaxes previously ("I remember doing one of them in Year 4"), and neither girl remembered encountering $a + b = b + a$ syntaxes.

K and Z were somewhat reserved at the beginning of the trial but gradually opened up over the first 15 to 20 minutes. They solved some of the puzzles with apparent effortlessness and others caused them significant problems. Their inconsistent performance reflects a trial-and-error approach in contrast to T and A's strategic approach. Challenge 1 proved adequate for generating eventful data and the girls were not set Challenge 3 during the trial. Their engagement over the duration of the trial might be described as cautious experimentation increasingly punctuated by confident strategising during the final few puzzles.

Conjecture 1: Evidence for iconic matching

Conjecture 1 predicts that the functionalities of the software will broaden the children's readings of arithmetical notation to include iconic matching.

Initially during the T and A trial, readings were mainly computational, as in T's explanation of Puzzle 2 (Figure 4):

04:20 T: I think it might be this one. Yeah, [A: Yeah] 'cause 7 add 1 equals 8, 8 add 8 is 16.

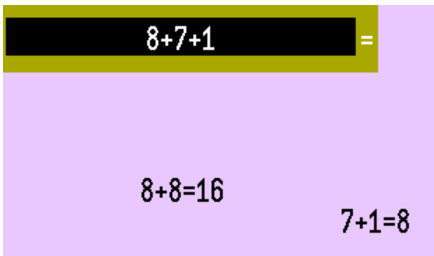


Figure 4: Puzzle 2

Evidence for a shift to iconic matching first occurred during Puzzle 4 (Figure 5):

10:46 A: 1 add 7 add 1 add 1.
T: 1 add 7 equals 7 add 1. Try that.
A: 7 add 1 add 1 add 1.

T: Okay. Let's try this one [1+1+1=3]. You want to do this one now [7+3=10]. Yeah. Now you do this one. It did it!

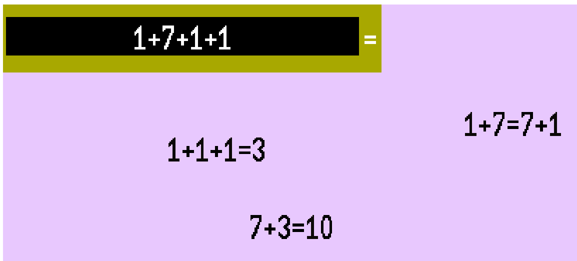


Figure 5: Puzzle 4

This approach, guided by iconic matches, came to dominate T and A's activity most of the time as can be seen in many of the data snippets in the following subsections. However, on occasion, when the boys became stuck, they attempted non-iconic substitutions. For example, at *28:35*, T attempted a transformation of $65+87$ into 152 using the iconic mismatch $80+60+5+7=152$. At other moments of impasse they ignored iconic matching entirely and resorted to selecting a statement and systematically clicking every other term onscreen, though not with much expectation:

30:02 A: Try 12 equals 7 plus 5. … Try all the numbers.

T: [laughs] Yeah, that's what I'm doing.

A: They're not going to work though.

T: Yeah, don't think it will.

K and Z were somewhat reticent during the first three puzzles and appeared to engage in arbitrary statement selecting and term clicking. The first explicit evidence for a computational reading occurred during Puzzle 4 (Figure 5):

07:05 K: 8 add 2 are 10.

Z: Yeah.

K and Z's grasp of iconic matching emerged gradually, and was preceded by attendance to non-iconic perceptual properties of equality statements. For example, when I asked why $1+7=7+1$ had made a substitution in $1+7+1+1$ (Figure 5) but $7+3=10$ had not, K answered (*08:50*) "is it because that's only got two numbers to add to 10 but that one's got four", and Z answered (*09:15*) "Is it because that one's got the equals sign in the middle? And that one hasn't."

Evidence for an emerging iconic reading first occurred during Puzzle 7. The statement $53=50+3$ had been selected and I asked why it made a substitution in $20+50+3$ when the inscription $50+3$ was clicked but not when $20+$ was clicked. Z answered (*17:00*) "Because you have to click on the 50 and there's more 50s there." From then on the girls became iconically guided in their decisions of which statement to try next. However, throughout their trial they also attended to iconic near-matches, such as attempting to make a substitution in $20+3+50$ using $20+50=70$ (at *17:20*), much more frequently than did T and A.

Conjecture 2: Evidence for commutative and partitional readings

Conjecture 2 predicts the emergence of commutative and partitional readings of $a+b=b+c$ and $c=a+b$ statements respectively during diagrammatic activity.

Commutation

The children's awareness of commutation appeared to arise from the need to make implicit knowledge explicit in order to operate the software. For example, early on in the T and A trial, T attempted to transform $9+1$ into 10 using $1+9=10$ (R is researcher):

09:21 R: Why do you think that wasn't working?

T: [selects $1+9=9+1$] Maybe because... 1 and 9 is...

A: Oh, because it hasn't got that sum in it.

R: What do you mean?

A: Well 'cause that's got 1 add 9 but then the end of that's got 9 add 1.

T soon found a use for commutation when wanting to transform $6+5+3$ into $11+3$ using $5+6=11$:

12:45 T: I think we try that one [$5+6=6+5$]... try that and yeah [$6+5+3$ becomes $5+6+3$]. It's changed it. That makes it 5 add 6 so you can do this one now.

Around halfway through the trial T began to express commutation more consistently. Initially he used the phrase "changed around", then "turned round", and then "swap it round". From then on he consistently used the verb "swap" when suggesting and discussing commutative transformations.

K and Z's explication of commutation occurred more gradually. Early in their trial, the girls offered a computational equivalence (rather than a commutative) explanation for $5+6=11$ failing to transform $6+5+3$:

11:50 K: Is it because… um, that one won't work because it doesn't equal the same as the top.

R: What do you mean?

Z: That one's got 3 more.

K: That's a different answer to that one.

Just over halfway through the trial the girls began to express commutation using the verb "swap", as had T. This first occurred when K commented "that's just swapped it" after transforming $40+30+4$ into $30+40+4$ using $40+30=30+40$. However, across the two trials there was a notable contrast in the role of commutation. T commonly made strategic use of commutative transformations to produce statements required for a subsequent step towards solving a puzzle. Two examples from the transcript illustrate how commutation was a means not an end:

24:20 T: No. Oh you need to move 'em, you need to swap 'em round.

A: Hm.

T: No. Those need to be swapped round as well. ... There.

27:50 T: I'm thinking. 140 add 5, if you swap those over somehow. Try that.

In contrast, K and Z's use of commutation tended to be as part of a step-by-step iconic matching approach. For K, commutative transformations "just swap" numbers; but for Z, at least they do *something*:

31:15 Z: Try that on the other one.

K: No, it's just swapped them.

Z: Shall we try swapping and then we can try… [does not finish]

33:45 K: No, that's just swapping them.

Z: Yeah but, we could do it if it was swapped or something.

Only towards the end of the trial did Z come to notice the role of commutation in a sequence of transformations:

35:20 Z: I get this now 'cause like that one swapped those two, and there was two of those which means they've swapped, and like when you clicked on that one I thought they would have added to make 15 so it would be 300 and, add 15 and then click on that one 300 add 15 equals 315. That make sense?

Following this insight K utilised commutation strategically, as a means not an end:

36:20 K: So if we try and click on that one again [Z: Can we try on that one?] then that one would swap and then that would swap as well.

The data in this subsection suggest transformation making afforded T using-before-knowing; he made strategic use of commutation early and became increasingly adept at expressing it. In contrast, transformation making afforded K and Z knowing-before-using; they noticed they could "swap" things early and found a strategic use for doing so later. It should be noted that commutation was expressed exclusively as an action; for example, none of the children used "swap" as a noun.

Partition

The first $c = a + b$ form appears in Puzzle 6. After some initial discussion about which statement in the puzzle comes last (see snippet *19:40*, next subsection) T commented on $41 = 40 + 1$:

20:00 T: Oh! That's the one that you do first! It has to be.

R: Why?

T: Because it's splitting up the 40 and the 1.

T's reaction here demonstrates his ability to infer a partitioning use for $c = a + b$ syntaxes from the diagrammatic rules supported by the software. From then on, making partitional transformations became the starting strategy for both T and A. For example:

22:50 A: Which one's most split up? That one.

T: Forty... 34 add... splits it up there.

A: Yeah.

T: Right. That's the only one that splits up so you just try the rest...

In one tantalising excerpt T appears almost to use "split" (or a derivative) as a noun to describe $87 = 80 + 7$:

24:00 T: Okay! 65 add 87. Any spli...? 8... Yeah. That one. That splits it up.

During Puzzle 9, K suggested using $34 = 30 + 4$ to transform $40 + 34$ in terms of the visual size of the resulting statement:

27:10 K: Try that one. Make it bigger. A bit bigger, then the answer.

However the data provide no other evidence for K and Z noticing or employing partition during the trial. There was no notable reflection on $c = a + b$ syntaxes and no specific vocabulary, such as "split", employed when considering them (bar snippet *27:10*, above). At the end of the trial I asked the girls what they thought the different types of statements on the screen meant. In reference to $109 = 100 + 9$ and $206 = 200 + 6$ they offered computational readings:

38:43 Z: Those mean, like, those two will equal that when you click on them.

K: That... the answer is 109, but like separate that would be 100 add 9 so it's just the... sum is like swapped round.

38:55 K: ...109 would normally be where the 100 and... the 100 plus the 9 would normally be where the 109 is so it's just that they're swapped.

It should be noted for contrast that, immediately prior to this, Z had described $a + b = b + a$ statements in terms of commutating:

38:41 Z: Erm, I think they mean that they swap.

The data in this subsection suggest that when T first encountered an $c = a + b$ inscription (snippet *20:00*, above) he inferred a partitional meaning from his emerging grasp of the diagrammatic rules. However, Conjecture 2 was refuted for partitioning in the K and Z trial. When prompted, K and Z read $c = a + b$ inscriptions as computations written backwards as the wider literature would predict. This contrast in meanings for $c = a + b$ statements appears to be key to the contrast in puzzle solving performance across the two trials, discussed in the following subsection.

Conjecture 3: Evidence for interaction of diagrammatic activity and computation

T and A's puzzle solving strategy throughout much of the trial was sophisticated and efficient. Their emergent "splitting" meaning for $c = a + b$ statements enabled them to make sense of increasingly complex puzzles, and their strategic use of "swapping" to generate statements needed for subsequent steps enabled them to progress without difficulty. The impact of computational readings on their diagrammatic strategies appeared to be minimal and consisted merely of identifying which statement in a given puzzle needed to be left until last. For example, A correctly eliminated $70 + 1 = 71$ from his choice of two statements on the grounds that it is computationally equivalent to the term in the black box ($30 + 41$):

19:40 A: That one goes with that one, because that one's the sum you do at the end isn't it?
R: Which one's the sum you do at the end?
A: That one. Because 70 plus 1, because you get 71... that must be the same.

K and Z did not attend to the partitioning effects of $c = a + b$ statements and were less able to make sense of increasingly sophisticated puzzles. However, when starting a puzzle, they commonly identified the last statement as the one that contained the answer to the sum in the black box (e.g. snippet *07:05*, above) as had T and A. Their general approach was that of experimenting with step-by-step iconic matches. This experimentation was guided by systematic testing of each and every statement rather than computational readings:

34:49 Z: I think we tried everything.

35:11 Z: I think we still haven't done that one yet. Yeah. So it will be 300 add 15 if it works.

Overall, the data suggest that Conjecture 3 is only weakly supported. Computational readings of equality statements had less impact on diagrammatic strategies than the previous pilot trials had led me to expect (see Jones, in press). Rather, partitional and commutative readings, along with looking for iconic matches one step ahead ("iconic strategising") and systematic testing of every statement and every term, appear to be the dominant features of the children's puzzle solving strategies.

CONCLUSION

Conjecture 1 states that the software and task design privilege pattern finding readings over computational readings. This is supported because all the children looked for iconic matches throughout the trials. Conjecture 2 states that the diagrammatic activities of iconic matching and transformation making afford meaning-making for the structure of equality statements. This is supported across the two trials for the case of commutation. It is also supported in the T and A trial, but refuted in the K and Z trial, for the case of partition. Conjecture 3 states that computational readings of equality statements impact on diagrammatic strategies and is only weakly supported across the two trials. It seems instead that commutative and partitional readings, iconic strategising and systematic testing were the more significant attributes of the children's diagrammatic experimentation and puzzle solving.

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