Regularity, Local and Microlocal Analysis in Theories of Generalized Functions

Jean-André Marti
Equipe Analyse Algébrique Non Linéaire-*Laboratoire G T S I*
Université Antilles-Guyane

November 23, 2007

Abstract

We introduce a general context involving a presheaf $\mathcal{A}$ and a subpresheaf $\mathcal{B}$ of $\mathcal{A}$. We show that all previously considered cases of local analysis of generalized functions (defined from duality or algebraic techniques) can be interpretated as the $\mathcal{B}$-local analysis of sections of $\mathcal{A}$.

But the microlocal analysis of the sections of sheaves or presheaves under consideration is dissociated into a "frequential microlocal analysis " and into a "microlocal asymptotic analysis". The frequential microlocal analysis based on the Fourier transform leads to the study of propagation of singularities under only linear (including pseudodifferential) operators in the theories described here, but has been extended to some non linear cases in classical theories involving Sobolev techniques. The microlocal asymptotic analysis can inherit from the algebraic structure of $\mathcal{B}$ some good properties with respect to nonlinear operations.

Contents

1 Introduction

The notion of regularity in algebras or spaces of generalized functions can be formulated in a general way with the help of sheaf theory. In section 2, when $\mathcal{A}$ is a presheaf of algebras or vector spaces on a topological space X, and $\mathcal{B}$ a subpresheaf of $\mathcal{A}$, for each open set Ω in X, we consider $\mathcal{B}(\Omega)$ as the space or algebra of some regular elements of $\mathcal{A}(\Omega)$. This leads to the notion of $\mathcal{B}$-singular support which refines the notion of support of a section $u \in \mathcal{A}(\Omega)$ provided the localization principle (F_1) holds: if u and v are global sections of $\mathcal{A}$ which agree on each open set of a family $(\Omega_i)_{i\in I}$ of open set in X, they agree on the union $\underset{i\in I}{\cup}\Omega_i$.

We can give many examples of this situation in the framework of theories of generalized functions: distributions [38] or Colombeau-type algebras [1, 7, 8, 9, 10]. To illustrate this, let us consider the following sequence of sheaf embeddings, defined for each Ω open set in $\mathbb{R}^n$ by

$$\mathrm{C}^\infty(\Omega) \to \mathcal{L}(\mathrm{C}_c^\infty(\Omega),\mathbb{C}) = \mathcal{D}'(\Omega) \to \mathcal{G}(\Omega) \to \mathcal{L}\left(\mathcal{G}_c(\Omega),\widetilde{\mathbb{C}}\right)$$

$\mathcal{G}(\Omega)$ being the Colombeau algebra, $\widetilde{\mathbb{C}}$ the ring of Colombeau's generalized numbers, $\mathcal{G}_c(\Omega)$ the set of elements in $\mathcal{G}(\Omega)$ with compact support and $\mathcal{L}\left(\mathcal{G}_c(\Omega),\widetilde{\mathbb{C}}\right)$ defined in [17] the space of all continuous and $\widetilde{\mathbb{C}}$ linear functionals on $\mathcal{G}_c(\Omega)$. Each term of the sequence can be considered as the $\mathcal{B}(\Omega)$ regular space or algebra of the following algebra or space $\mathcal{A}(\Omega)$. It is the basis for a local analysis of the elements in $\mathcal{A}(\Omega)$. Some results on propagation of singularities under $\mathcal{B}$-compatible operators permit to explain and summarize the classical results involving differential or pseudo-differential ones. But if we want to define a more precise "microlocal" analysis which gives some informations not only on the locus, but on the causis of the singularities described as a fibered space above that locus, we have first to give a precise local characterization of the singularities under consideration.

A review on the ideas, technics and results on microlocalization is given in section 3. The first step was to follow the Hörmander ideas about the wave front set $WF(u)$ of a distribution u, whose construction is deduced from the classical Fourier characterization of smoothness of distributions with compact support. For a general $v \in \mathcal{E}'(\mathbb{R}^n)$ Hörmander introduces the cone $\Sigma(v)$ of all $\eta \in \mathbb{R}^n \backslash 0$ having no conic neighbourhood V such that the Fourier transform $\widehat{v}$ is rapidly decreasing in V. Lemma 8.1.1. in [22] proves that *if* $\Phi \in \mathcal{D}(\mathbb{R}^n)$ and $v \in \mathcal{E}'(\mathbb{R}^n)$ then $\Sigma(\Phi v) \subset \Sigma(v)$. It follows that if Ω is an open set in $\mathbb{R}^n$ and $u \in \mathcal{D}'(\Omega)$, setting: $\Sigma_x(u) = \underset{\Phi}{\cap}\Sigma(\Phi u)\,;\ \Phi \in \mathcal{D}(\Omega)$, $\Phi(x) \neq 0$, one can define the wave front set of u as $WF(u) = \{(x,\xi) \in \Omega \times (\mathbb{R}^n \backslash 0)\,; \xi \in \Sigma_x(u)\}$.

This way was led in the sheaf $\mathcal{A} = \mathcal{G}$ of Colombeau simplified algebras by Nedeljkov, Pilipovic and Scarpalezos [31] by taking $\mathcal{B} = \mathcal{G}^\infty$ as regular subsheaf of $\mathcal{G}$. This subsheaf, introduced by Oberguggenberger [32], generalizes in a natural way in $\mathcal{G}$ the regular properties of C^∞ in $\mathcal{D}'$. We can find in the literature a description of the main properties of Fourier transform of compacted supported elements in $\mathcal{G}^\infty(\Omega)$, which leads to a frequential microlocal analysis similar to the Hörmander's one (see [37] for instance). The crucial point was the conservation of the power of the lemma 8.1.1. leading to the definition of the generalized wave front set of $u \in \mathcal{G}(\Omega)$ denoted $WF_g(u)$.

Recently, A. Delcroix has extended in [12] the $\mathcal{G}^\infty$ regularity to a so called $\mathcal{G}^{\mathcal{R}}$ regularity, which still preserves the statements of the above quoted lemma, and gives a $\mathcal{G}^{\mathcal{R}}$ frequential microanalysis. We can chose $\mathcal{R}$ such that $\mathcal{G}^{\mathcal{R}}$ countains an embedding of $\mathcal{D}'$ into $\mathcal{G}$, which is not the case for $\mathcal{G}^\infty$ ($\mathcal{G}^\infty \cap \mathcal{D}' = \mathrm{C}^\infty$ is a result of [32]). Then it becomes possible to investigate the frequential $\mathcal{D}'$-singularities of $u \in \mathcal{G}(\Omega)$.

Inspired by the classical theories, many results on propagation of singularities and pseudodifferential techniques have been obtained during the last years by De Hoop, Garetto, Hörmann, Gramchev, Grosser, Kunzinger, Steinbauer and others (see [15, 18, 19, 25, 26, 27]). For example, when $u \in \mathcal{G}(\Omega)$, Hörmann and Garetto [18] obtain characterisations of $WF_g(u)$ in terms of intersections of some domains corresponding to pseudodifferential operators similarly to Hörmander's characterizations of $WF(u)$ for $u \in \mathcal{D}'(\Omega)$ [23]. Following the ideas and technics of [18] and making use of the theory of pseudodifferential operators with generalized symbols ([15, 18]), Garetto [17] has recently extended the definition of $WF_g(u)$ when $u \in \mathcal{G}(\Omega)$ to the definitions of $WF_{\mathcal{G}}(T)$ and $WF_{\mathcal{G}^\infty}(T)$ when $T \in \mathcal{L}\left(\mathcal{G}_c(\Omega), \widetilde{\mathbb{C}}\right)$. She can also give a Fourier-transform characterization of these wave front sets when T is a basic functional. Nevertheless, these very interesting and deep results are still mainly limited to linear cases, at least in the framework developed above. More precisely, even when $\mathcal{A}$ is a sheaf of factor algebras, we don't know any study on the microlocal behaviour of singularities under nonlinear operations by means of frequential methods based on the Fourier transform.

However, such studies exist in a classical framework involving some spaces of Sobolev type. In section 8 of [24], Hörmander uses the results on microlocal $H^{loc}_{(s)}$-regularity of nonlinear operations for tempered distributions in $\mathcal{S}'(\mathbb{R}^n)$ to discuss semi-linear equations, following Rauch [35]. By means of paradifferential techniques some general results for quasilinear equations are given in Bony [4]. Then, from a general result on propagation of singularities for pseudo-differential operators and a Bony's linearization theorem, Hörmander ([24], section 11) can discuss fully nonlinear equations and obtain precise propagation results for hyperbolic second order semi-linear equations. Extensions of the previous results can be found in works of Beals [2, 3] and

Bony[5].

The Fourier transform is still the main tool involved in other generalized cases, where the $\mathcal{G}^\infty$-regularity is subordinated to an additional condition (such as an estimate on the growth of derivatives) characterizing a special property such as to belong to an analytic, Gevrey or C^L class in Hörmander sense ([22], section 8.4). It is the case of "analytic" algebra: $\mathcal{G}^A$ studied by Pilipovic, Scarpalezos and Valmorin [34], of "C^L class"algebra: $\mathcal{G}^L$ introduced by Marti [30], which are subalgebras of $\mathcal{G}$, of "regular Gevrey ultradistributions" algebra: $\mathcal{G}^{\sigma,\infty}$ of Bouzar and Benmeriem [6] which is a subalgebra of $\mathcal{G}^\sigma$, the "generalized Gevrey ultradistributions". In these examples, the aim is always to perform the $\mathcal{B}$-frequential microlocalization of generalized functions from the starting algebra $\mathcal{A}(\Omega)$, when $\mathcal{B}$ is $\mathcal{G}^A$, $\mathcal{G}^L$or $\mathcal{G}^{s,\infty}$ and $\mathcal{A}$ is $\mathcal{G}$ or $\mathcal{G}^\sigma$. All these cases are special cases of the more general one obtained when taking $\mathcal{A} = \mathcal{G}^r$ which extends the Colombeau algebra $\mathcal{G}$ and $\mathcal{B} = \mathcal{G}^{r,\mathcal{R},L}$ which generalizes all the previous regularity cases. The Fourier transform is still used to characterize the $\mathcal{B}$-regularity with the corresponding constraints. But this is not so easy or natural. For instance, in [30] one starts by giving a characterization of local $\mathcal{G}^L$-regularity by means of some sequence u_k of generalized functions with compact support whose the Fourier transform $\widehat{u_k}$ verifies an estimate involving a special sequence $(L_k)_{k\in\mathbb{N}}$. Indeed u_k is constructed as product of $u \in \mathcal{G}(\Omega)$ and a suitable cutoff sequence $\mathcal{X}_k$ whose derivatives are controled up to the order k. This leads to define the $\mathcal{G}^L$-wave front set of a generalized function: $WF_g^L(u) \subset \Omega \times (\mathbb{R}^n \setminus 0)$ and prove, by refining the cutoff sequence $\mathcal{X}_k$, that its projection on Ω is the $\mathcal{G}^L$-singular support of u. Then, $WF_g^L(u)$ gives a spectral decomposition of $\operatorname{sing\,supp}^L u$. A generalization of these results to the case of local $\mathcal{G}^{r,\mathcal{R},L}$-regularity of elements in $\mathcal{G}^r(\Omega)$ is given in [13]. They lead to define the $\mathcal{G}^{r,\mathcal{R},L}$-wave front set of $\mathcal{G}^r$-generalized functions.

These sophisticated constructions give a synthetic description of the frequential microanalysis but the proofs seem to tell that the Fourier transform is not really the good tool to perform this description in the above cases. Perhaps the Fourier-Bros-Iagolnitzer transform would permit to give a better approach of the problem in the future. For the analytic generalized wave front set, we also can think of refering to boundary values techniques which embed distributions into hyperfunctions. But we don't expect results about nonlinear cases in these ways.

We recall here that generalized functions in the initial definition (sections of the sheaf $\mathcal{G}$) are classes of families $(u_\varepsilon)_\varepsilon$ of classical functions. But in the definitions of generalized (frequential) wave front set considered above (when $\mathcal{B}$ is $\mathcal{G}^\infty$, $\mathcal{G}^\mathcal{R}$, $\mathcal{G}^A$ or $\mathcal{G}^L$), the parameter ε does not play a specific role. It has only to ensure the correct use of some notions as regularity, rapid decrease, analyticity, and so on in the definition of the different algebras under consideration. For example the generalized wave front set of any Dirac-delta function Δ in the generalized framework is exactly the same as the classical wave front set of the distribution δ. It is still the same as the generalized wave front set of any power Δ^m of Δ without possibility to compare them. The main reason lies in the structure of Fourier transform. A paradigmatic alternative can be found in the concept of asymptotic analysis.

The idea of an "asymptotic" analysis [14, 28, 29] of $u = [u_\varepsilon] \in \mathcal{A}(\Omega) = \mathcal{G}(\Omega)$ is the following. Let $\mathcal{F}$ be a subsheaf of vector spaces (or algebras) of $\mathcal{G}$. One defines first the sheaf $\mathcal{B}$ such that, for any open set V in $\mathbb{R}^n$, $\mathcal{B}(V)$ is the space of elements $u = [u_\varepsilon] \in \mathcal{A}(V)$ such that u_ε has a limit in $\mathcal{F}(V)$ when ε tends to 0. Then $\mathcal{O}_\mathcal{G}^\mathcal{F}(u)$ is the set of all $x \in \Omega$ such that u agrees with a section of $\mathcal{B}$ above some neighbourhood of x. The $\mathcal{F}$-singular (or $\mathcal{B}$-singular) support of u is $\Omega \backslash \mathcal{O}_\mathcal{G}^\mathcal{F}(u)$. For fixed x and u, $N_x(u)$ is the set of all $r \in \mathbb{R}_+$ such that $\varepsilon^r u_\varepsilon$ tends to a section of $\mathcal{F}$ above some neighbourhood of x. The $\mathcal{F}$-singular spectrum of u is the set of all $(x,r) \in \Omega \times \mathbb{R}_+$ such that $r \in \mathbb{R}_+ \backslash N_x(u)$. It gives a spectral decomposition of the $\mathcal{F}$-singular support of u. As example, take $\delta_\varepsilon(x) = \dfrac{1}{\varepsilon}\varphi\left(\dfrac{x}{\varepsilon}\right)$ where $\varphi \in \mathcal{D}(\mathbb{R}), \varphi \geq 0$ and $\int \varphi(x)\,dx = 1$. Then, for $m \geq 1$, $\Delta^m = [\delta_\varepsilon^m]$ is a generalized function in $\mathcal{G}(\mathbb{R})$. Except for $m = 1$, for which Δ is associated with $\delta \in \mathcal{D}'(\mathbb{R})$, Δ^m is not locally associated with an element of $\mathcal{D}'(V)$ in any neighbourhood V of 0.

But, for $r \geq m-1$, $[\varepsilon^r]\,\Delta^m$ is locally associated with such an element. It follows that for all m, the $\mathcal{D}'$-singular support of Δ^m is $\{0\}$ but its $\mathcal{D}'$-singular spectrum is the set $\{(0,\emptyset)\}$ if $m=1$ or $\{(0,[0,m-1))\}$ if $m>1$. It gives a more precise description of the singularities of Δ^m that its $\mathcal{D}'$-singular support and even that its frequential generalized wave front set $\{(0,\mathbb{R}\backslash 0)\}$ which doesn't depend upon m.

This asymptotic analysis is extended to $(\mathcal{C},\mathcal{E},\mathcal{P})$ algebras. This gives the general asymptotic framework, in which the net $(\varepsilon^r)_\varepsilon$ is replaced by any net a satisfying some technical conditions, leading to the concept of the $(a,\mathcal{F})$-singular parametric spectrum. The main advantage is that this asymptotic analysis is compatible with the algebraic structure of the presheaf $\mathcal{F}$ asymptotically associated to $(\mathcal{C},\mathcal{E},\mathcal{P})$ algebras. Thus the $(a,\mathcal{F})$-singular asymptotic spectrum inherits good properties with respect to nonlinear operations when $\mathcal{F}$ is a presheaf of topological algebras. Moreover, even when $\mathcal{F}$ is a presheaf (or sheaf) of vector spaces (for instance $\mathcal{F}=\mathcal{D}'$), some results on microlocal analysis are still obtained for nonlinear operations (see paragraph 4.3.1) on $(a,\mathcal{D}')$-singular asymptotic spectrum of powers of δ functions. In [14], various examples of propagation of singularities through nonlinear differential operators are given, connected to some results of Oberguggenberger, Rauch, Reeds and Travers ([33, 36, 39]).

The paper is organized as follows. In section 2 we introduce the local analysis of generalized functions. Subsections of 2 give the basic ingredients, some examples in algebraic or duality theories, and define $\mathcal{G}^{r,\mathcal{R},L}$-local analysis. $\mathcal{G}$-local analysis of functional sections of $\mathcal{L}\left(\mathcal{G}_c,\widetilde{\mathbb{C}}\right)$ and $\mathcal{F}$-local analysis for sections of some $(\mathcal{C},\mathcal{E},\mathcal{P})$ algebra are recalled. Section 3 is devoted to the frequential microlocal analysis, with characterization of $\mathcal{G}^{r,\mathcal{R}}$ and $\mathcal{G}^{r,\mathcal{R},L}$-local regularities and corresponding wave front sets. We also give the result proved in [17] on the Fourier transform characterisation of $WF_{\mathcal{G}}(T)$ when T is a basic functional. The asymptotic microlocal analysis studied in [14] is detailed in section 4, with examples and applications to nonlinear partial differential equations.

2 The local analysis of generalized functions

The purpose of this section is to localize the singularities of some generalized functions. We refer the reader to [20] for more details on the sheaf theory involved in the sequel.

2.1 The basis ingredients

The basis ingredients of such an analysis are very simple and general ; even in this subsection no algebraic condition is required.

• $\mathcal{A}$ is a given sheaf of sets (or presheaf with localization principle (F_1) in addition) over a topological space X.

• $\mathcal{B}$ is a given subsheaf (or subpresheaf) of $\mathcal{A}$.

Definition 1 : ***$\mathcal{B}$-global regularity***

For any open set Ω in X, the elements in $\mathcal{B}(\Omega)$ are considered as regular, and called $\mathcal{B}$-regular elements of $\mathcal{A}(\Omega)$.

Definition 2 : ***$\mathcal{B}$-local regularity***

An element $u\in\mathcal{A}(\Omega)$, where Ω is any open set in X, is called $\mathcal{B}$-regular at $x\in\Omega$ if there exists an open neighbourhood V of x such that the restriction $u\mid_V$ is in $\mathcal{B}(V)$.

Definition 3 : ***$\mathcal{B}$-regular open set***

We denote by $\mathcal{O}_{\mathcal{A}}^{\mathcal{B}}(u)$ the set of all $x \in \Omega$ such that u is $\mathcal{B}$-regular at x. We also can write

$$\mathcal{O}_{\mathcal{A}}^{\mathcal{B}}(u) = \{x \in \Omega, \exists V \in \mathcal{V}_x, u\,|_V \in \mathcal{B}(V)\}$$

$\mathcal{V}_x$ being the family of all open neighbourhoods of x.

This very simple framework suffices to state the following

Definition 4 : ***$\mathcal{B}$-singular support***

For any section $u \in \mathcal{A}(\Omega)$, Ω any open set in X, the $\mathcal{B}$-singular support of u is

$$\mathcal{S}_{\mathcal{A}}^{\mathcal{B}}(u) = \Omega \setminus \mathcal{O}_{\mathcal{A}}^{\mathcal{B}}(u).$$

Remark 1 *(i) The gluing principle (F_2) is not needed to get the notion of $\mathcal{B}$-singular support of a section $u \in \mathcal{A}(\Omega)$. More precisely, when $\{b\}$ is the constant presheaf defined by a gobal section of $\mathcal{B}$, the localization principle (F_1) is sufficient to prove the following: the set*

$$\mathcal{O}_{\mathcal{A}}^{\{b\}}(u) = \{x \in \Omega \;\; \exists V \in \mathcal{V}_x,\; u\,|_V = b|_V\}$$

is exactly the union $\Omega_{\mathcal{A}}(u)$ of the open subsets of Ω on which u agrees with b.
Indeed, (F_1) allows to show that u agrees with b on an open subset $\mathcal{O}$ of Ω if, and only if, it agrees with b on an open neighborhood of every point of $\mathcal{O}$. This leads immediately to the required assertion.
Moreover, $\Omega_{\mathcal{A}}(u) = \mathcal{O}_{\mathcal{A}}^{\{b\}}(u)$ is the largest open set on which u agrees with b, and the $\mathcal{B}$-singular support of u is a closed subset of its $\{b\}$-singular support $\mathcal{S}_{\mathcal{A}}^{\{b\}}(u) = \Omega \setminus \mathcal{O}_{\mathcal{A}}^{\{b\}}(u)$.

(ii) When the embedding $\mathcal{B} \to \mathcal{A}$ is a sheaf morphism of abelian groups where 0 denote the null global section, $\mathcal{S}_{\mathcal{A}}^{\{0\}}(u) = \Omega \setminus \mathcal{O}_{\mathcal{A}}^{\{0\}}(u)$ is exactly the support of u in its classical definition.

(iii) In contrast to the situation described above for the support or the $\{b\}$-singular support, we need the gluing principle (F_2) if we want to prove that the restriction of u to $\mathcal{O}_{\mathcal{A}}^{\mathcal{B}}(u)$ belongs to $\mathcal{B}(\mathcal{O}_{\mathcal{A}}^{\mathcal{B}}(u))$. We make this precise in the following

Proposition 1 *Let $u \in \mathcal{A}(\Omega)$. Set $\Omega_{\mathcal{A}}^{\mathcal{B}}(u) = \cup_{i \in I}\Omega_i$, $(\Omega_i)_{i \in I}$ denoting the collection of all open subsets of Ω such that $u\,|_{\Omega_i} \in \mathcal{B}(\Omega_i)$. Then, if $\mathcal{B}$ is a sheaf (even if $\mathcal{A}$ is only a prehesaf),*

(i) $\Omega_{\mathcal{A}}^{\mathcal{B}}(u)$ is the largest open subset $\mathcal{O}$ of Ω such that $u\,|_{\mathcal{O}}$ is in $\mathcal{B}(\mathcal{O})$;

(ii) $\Omega_{\mathcal{A}}^{\mathcal{B}}(u) = \mathcal{O}_{\mathcal{A}}^{\mathcal{B}}(u)$ and $\mathcal{S}_{\mathcal{A}}^{\mathcal{B}}(u) = \Omega \setminus \Omega_{\mathcal{A}}^{\mathcal{B}}(u)$.

Proof. (i) For $i \in I$, set $u\,|_{\Omega_i} = f_i \in \mathcal{B}(\Omega_i)$. The family $(f_i)_{i \in I}$ is coherent by assumption: From (F_2), there exists $f \in \mathcal{B}(\Omega_{\mathcal{A}}^{\mathcal{B}}(u))$ such that $f\,|_{\Omega_i} = f_i$. But from (F_1), we have $f = u$ on $\cup_{i \in I}\Omega_i = \Omega_{\mathcal{A}}^{\mathcal{B}}(u)$. Thus $u\,|_{\Omega_{\mathcal{A}}^{\mathcal{B}}(u)} \in \mathcal{B}(\Omega_{\mathcal{A}}^{\mathcal{B}}(u))$, and $\Omega_{\mathcal{A}}^{\mathcal{B}}(u)$ is clearly the largest open subset of Ω having this property.

(ii) First, $\mathcal{O}_{\mathcal{A}}^{\mathcal{B}}(u)$ is clearly an open subset of Ω. For $x \in \mathcal{O}_{\mathcal{A}}^{\mathcal{B}}(u)$, set $u\,|_{V_x} = f_x \in \mathcal{B}(V_x)$ for some suitable neighborhood V_x. The open set $\mathcal{O}_{\mathcal{A}}^{\mathcal{B}}(u)$ can be covered by the family $(V_x)_{x \in \mathcal{O}_{\mathcal{A}}^{\mathcal{F}}(u)}$. As the family (f_x) is coherent, we get from (F_2) that there exists $f \in \mathcal{B}\left(\cup_{x \in \mathcal{O}_{\mathcal{A}}^{\mathcal{B}}(u)} V_x\right)$ such that $f\,|_{V_x} = f_x$. From (F_1), we have $u = f$ on $\cup_{x \in \mathcal{O}_{\mathcal{A}}^{\mathcal{B}}(u)} V_x$ and, therefore, $u\,|_{\mathcal{O}_{\mathcal{A}}^{\mathcal{B}}(u)} \in \mathcal{B}(\mathcal{O}_{\mathcal{A}}^{\mathcal{B}}(u))$. Thus $\mathcal{O}_{\mathcal{A}}^{\mathcal{B}}(u)$ is contained in $\Omega_{\mathcal{A}}^{\mathcal{B}}(u)$. Conversely, if $x \in \Omega_{\mathcal{A}}^{\mathcal{B}}(u)$, there exists an open neighborhood V_x of x such that $u\,|_{V_x} \in \mathcal{B}(V_x)$. Thus $x \in \mathcal{O}_{\mathcal{A}}^{\mathcal{B}}(u)$ and the assertion (ii) holds. ■

Remark 2 *When $\mathcal{A}$ is a sheaf and $\mathcal{B}$ a subpresheaf of $\mathcal{A}$, we can associate to $\mathcal{B}$ a subsheaf of $\mathcal{A}$ as follows. When Ω is an open set of X, we note $u \in \mathcal{B}(x)$ if $u \in \mathcal{A}(\Omega)$ is $\mathcal{B}$-regular at x according to Definition 2. Set*

$$\mathcal{B}_*(\Omega) = \{u \in \mathcal{A}(\Omega) \mid \forall x \in \Omega \ u \in \mathcal{B}(x)\}.$$

Let $\mathcal{B}_*$ be the functor $\Omega \to \mathcal{B}_*(\Omega)$. We intend to prove that $\mathcal{B}_*$ is a subsheaf of $\mathcal{A}$. The presheaf structure of $\mathcal{A}$ induces immediately the same one for $\mathcal{B}_*$ and principle (F_1) is also fullfiled. To prove that the gluing principle (F_2) holds, we consider a collection $(\Omega_i)_{i\in I}$ of open subset Ω_i of Ω such that $\Omega = \cup_{i\in I}\Omega_i$ and a coherent family $(u_i)_{i\in I}$ of elements $u_i \in \mathcal{B}_*(\Omega_i)$. First, we can glue the u_i into $u \in \mathcal{A}(\Omega)$. Now, we have to prove that $u \in \mathcal{B}_*(\Omega)$. For any $x \in \Omega$ choose i such that $x \in \Omega_i$. Then, we have

$$\mathcal{B}_*(\Omega_i) \ni u_i = u\,|_{\Omega_i} \in \mathcal{A}(\Omega_i).$$

Therefore there exists $V_i \subset \Omega_i$, $V_i \in \mathcal{V}_x$ such that $u_i\,|_{V_i} \in \mathcal{B}(V_i)$, from what we deduce

$$u\,|_{V_i} = (u\,|_{\Omega_i})\,|_{V_i} = u_i\,|_{V_i} \in \mathcal{B}(V_i)$$

which proves that $u \in \mathcal{B}(x)$ for each $x \in \Omega$. Then $\mathcal{B}_*$ is a sheaf. In fact it is the sheaf associated to $\mathcal{B}$ in the sense of [20]. Its construction is simplified by using the sheaf structure of $\mathcal{A}$. Roughly speaking, $\mathcal{B}_*$ is constructed thanks to a local procedure which adds many sections to the $\mathcal{B}$ ones. Then, one wishes to compare the corresponding singular supports of the same $u \in \mathcal{A}(\Omega)$. The answer is given by the following proposition which shows that the singularities of sections of $\mathcal{A}$ don't decrease when replacing $\mathcal{B}$ by $\mathcal{B}_*$.

Proposition 2 *Suppose that $\mathcal{A}$ is a sheaf and $\mathcal{B}$ a subpresheaf of $\mathcal{A}$. Let $\mathcal{B}_*$ the subsheaf of $\mathcal{A}$ associated to $\mathcal{B}$. Then, for any section u of $\mathcal{A}$ over the open set Ω of X we have*

$$\mathcal{S}_{\mathcal{A}}^{\mathcal{B}}(u) = \mathcal{S}_{\mathcal{A}}^{\mathcal{B}_*}(u).$$

Proof. For $u \in \mathcal{A}(\Omega)$, the presheaf mapping: $\mathcal{B} \to \mathcal{B}_*$ leads immediately to the set inclusion: $\mathcal{S}_{\mathcal{A}}^{\mathcal{B}}(u) \supset \mathcal{S}_{\mathcal{A}}^{\mathcal{B}_*}(u)$. Conversely, let be $x \in \mathcal{O}_{\mathcal{A}}^{\mathcal{B}_*}(u)$. There exists $V \in \mathcal{V}_x$ such that $u\,|_V \in \mathcal{B}_*(V)$. But as $x \in V$, there exists $W \in \mathcal{V}_x \cap V$ such that

$$u\,|_W \in \mathcal{B}(W).$$

Then $x \in \mathcal{O}_{\mathcal{A}}^{\mathcal{B}}(u)$. We have proved the inclusion $\mathcal{O}_{\mathcal{A}}^{\mathcal{B}_*}(u) \subset \mathcal{O}_{\mathcal{A}}^{\mathcal{B}}(u)$ which gives the converse one for the respective singular supports and leads to the required equality. ■

2.2 Some properties of $\mathcal{B}$-singular support

2.2.1 Elementary algebraic properties

Proposition 3 *We suppose that $\mathcal{B}$ and $\mathcal{A}$ are presheaves of $\mathbb{K}$-vector spaces, (resp. algebras). Let $(u_j)_{1\leq j\leq p}$ be any finite family of elements in $\mathcal{A}(\Omega)$ and $(\lambda_j)_{1\leq j\leq p}$ any finite family of elements in $\mathbb{K}$. We have:*

$$\mathcal{S}_{\mathcal{A}}^{\mathcal{B}}\Big(\sum_{1\leq j\leq p} \lambda_j u_j\Big) \subset \bigcup_{1\leq j\leq p} \mathcal{S}_{\mathcal{A}}^{\mathcal{B}}(u_j).$$

In the resp. case, we have in addition:

$$\mathcal{S}_{\mathcal{A}}^{\mathcal{B}}\Big(\prod_{1\leq j\leq p} u_j\Big) \subset \bigcup_{1\leq j\leq p} \mathcal{S}_{\mathcal{A}}^{B}(u_j).$$

In particular, if $u_j = u$ for $1 \leq j \leq p$, we have $\mathcal{S}_{\mathcal{A}}^{\mathcal{B}}(u^p) \subset \mathcal{S}_{\mathcal{A}}^{B}(u)$.

Proof. If $x \in \Omega$ is in $\underset{1\leq j\leq p}{\cap} \mathcal{O}_{\mathcal{A}}^{\mathcal{B}}(u_j)$, there exists V_j in $\mathcal{V}_x$ such that $u_j \left|_{V_j}\right. \in \mathcal{B}(V_j)$. Thus $\left(\sum_{1\leq j\leq p} \lambda_j u_j\right)\Big|_{\underset{1\leq j\leq p}{\cap} V_j} \in \mathcal{B}(\underset{1\leq j\leq p}{\cap} V_j)$ (resp. $\left(\prod_{1\leq j\leq p} u_j\right)\Big|_{\underset{1\leq j\leq p}{\cap} V_j} \in \mathcal{B}(\underset{1\leq j\leq p}{\cap} V_j)$, which implies

$$\underset{1\leq j\leq p}{\cap} \mathcal{O}_{\mathcal{A}}^{\mathcal{B}}(u_j) \subset \mathcal{O}_{\mathcal{A}}^{\mathcal{B}}(\sum_{1\leq j\leq p} \lambda_j u_j) \qquad (\text{resp. } \underset{1\leq j\leq p}{\cap} \mathcal{O}_{\mathcal{A}}^{\mathcal{B}}(u_j) \subset \mathcal{O}_{\mathcal{A}}^{\mathcal{B}}\left(\prod_{1\leq j\leq p} u_j\right).$$

The result follows by taking the complementary sets in Ω. ■

2.2.2 $\mathcal{B}$-compatible operators and propagation of singularities

We begin by a general result which doesn't need algebraic assumptions. Let Ω be a given open subset of X. A presheaf operator A in $\mathcal{A}(\Omega)$ is defined as a presheaf morphism $\mathcal{A}(\Omega) \to \mathcal{A}(\Omega)$ compatible with restrictions. More precisely if $\mathcal{O}_\Omega$ denote the category of all open sets in Ω, A may be given by a collection $(A_V)_{V\in\mathcal{O}_\Omega}$ of mappings $A_V : \mathcal{A}(V) \to \mathcal{A}(V)$ such that for each $V \in \mathcal{O}_\Omega$ and $u \in \mathcal{A}(\Omega)$ we have: $A_\Omega(u)|_V = A_V(u|_V)$. Thus we can simplify the notations and write A instead A_V when acting on sections over V.

Definition 5 *Let A be a presheaf operator in $\mathcal{A}(\Omega)$. We say that A is locally $\mathcal{B}$-compatible if for each triple $(x, V, v) \in \Omega \times \mathcal{V}_x \times \mathcal{B}(V)$ there exists $W \in \mathcal{V}_x$, $W \subset V$, such that $A(v)|_W \in \mathcal{B}(W)$.*

Proposition 4 *Suppose that the assumption given in subsection 2.1 are fulfilled. Let A be a presheaf operator in $\mathcal{A}(\Omega)$ locally $\mathcal{B}$-compatible. Then we have*

$$\mathcal{S}_{\mathcal{A}}^{\mathcal{B}}(A(u)) \subset \mathcal{S}_{\mathcal{A}}^{\mathcal{B}}(u).$$

Proof. If $x \in \Omega$ belongs to $\mathcal{O}_{\mathcal{A}}^{\mathcal{B}}(u)$ there exists V in $\mathcal{V}_x$ such that $u|_V \in \mathcal{B}(V)$. Then, there exists $W \in \mathcal{V}_x$, $W \subset V$, such that $A(u|_V)|_W \in \mathcal{B}(W)$. We have

$$A(u|_V)|_W = (A(u)|_V)|_W = A(u)|_{V\cap W} = A(u)|_W .$$

Then x belongs to $\mathcal{O}_{\mathcal{A}}^{\mathcal{B}}(A(u))$, and we have proved that $\mathcal{O}_{\mathcal{A}}^{\mathcal{B}}(u) \subset \mathcal{O}_{\mathcal{A}}^{\mathcal{B}}(A(u))$. The result follows by taking the complementary sets in Ω. ■

The following weakened form of locally $\mathcal{B}$-compatibility may be more practical for applications

Definition 6 *A presheaf operator A in $\mathcal{A}(\Omega)$ is said $\mathcal{B}$-compatible if for each open set V of Ω it maps $\mathcal{B}(V)$ into itself.*

It is easy to see that a $\mathcal{B}$-compatible operator is locally $\mathcal{B}$-compatible but with the above definition we can get some useful results. The simplest one concerns the composition product, with an obvious proof.

Proposition 5 *If a presheaf operator A in $\mathcal{A}(\Omega)$ is $\mathcal{B}$-compatible, then for any $p \in N$, the composition product $A^p = \overbrace{A \circ A \circ ...A}^{p}$ is $\mathcal{B}$-compatible.*

Adding some algebraic hypothesis leads to the following definitions and results as corollaries of propositions 3 and 4. When $\mathcal{B}$ is a presheaf of algebras and $\mathcal{A}$ a presheaf of vector spaces and a $\mathcal{B}$-module we recall that the external sheaf product $\mathcal{B} \times \mathcal{A} \to \mathcal{A}$ extending the usual algebra product $\mathcal{B} \times \mathcal{B} \to \mathcal{B}$ is defined for any $\Omega \in \mathcal{O}_X$ and $(b, u) \in \mathcal{B}(\Omega) \times \mathcal{A}(\Omega)$ by $(b, u) \mapsto bu \in \mathcal{A}(\Omega)$ with $bu|_V = b|_V\, u|_V$ for each open set V in Ω.

Definition 7 *We suppose that $\mathcal{B}$ is a presheaf of algebras and $\mathcal{A}$ a presheaf of vector space and a $\mathcal{B}$-module. Let b be a given element in $\mathcal{B}(\Omega)$. We define the B-operator of multiplication in $\mathcal{A}(\Omega)$ by the map $\mathcal{A}(\Omega) \to \mathcal{A}(\Omega)$ such that $B(u) = bu$.*

Proposition 6 *B is a presheaf operator $\mathcal{B}$-compatible.*

Proof. First, we have for each open set V in Ω

$$B(u)|_V = bu|_V = b|_V \, u|_V = B(u|_V).$$

When v is in $\mathcal{B}(V)$, the external product $b|_V \, v$ agrees with the standard product in the algebra $\mathcal{B}(V)$ and lies in it. Then, for each pair $(V, v) \in \mathcal{O}_\Omega \times \mathcal{B}(V)$, $B(v)$ is in $\mathcal{B}(V)$. ■

Corollary 7 *We keep the same assumption as above and consider a family $(b_\alpha)_{\alpha\in\mathfrak{A}}$ of elements in $\mathcal{B}(\Omega)$ and another family of $\mathcal{B}$-compatible operators $(A_\alpha)_{\alpha\in\mathfrak{A}}$ in $\mathcal{A}(\Omega)$ where $\mathfrak{A}$ is a set of indices. Then, for any finite part $\mathfrak{A}_0$ of $\mathfrak{A}$, $\sum\limits_{\alpha\in\mathfrak{A}_0} b_\alpha A_\alpha$ is a $\mathcal{B}$-compatible operator in $\mathcal{A}(\Omega)$.*

Proof. For each $\alpha \in \mathfrak{A}_0$ and V open set in Ω we have

$$b_\alpha A_\alpha(u)|_V = b_\alpha|_V \, A_\alpha(u)|_V = b_\alpha|_V \, A_\alpha(u|_V) = b_\alpha A_\alpha(u|_V).$$

When v belongs to $\mathcal{B}(V)$, $A_\alpha(v)$ belongs to $\mathcal{B}(V)$ from the hypothesis on A_α. Then, the external product $b_\alpha|_V \, A_\alpha(v)$ agrees with the standard product in the algebra $\mathcal{B}(V)$ and lies in it. Then, for each pair $(V, v) \in \mathcal{O}_\Omega \times \mathcal{B}(V)$, $b_\alpha A_\alpha(v)$ is in $\mathcal{B}(V)$ and it is the same for the finite sum $\sum\limits_{\alpha\in\mathfrak{A}_0} b_\alpha A_\alpha(v)$. ■

Corollary 8 *Let P be the polynomial in $\mathcal{A}(\Omega)$ defined for each $u \in \mathcal{A}(\Omega)$ by $P(u) = \sum\limits_{1\leq j\leq p} b_j u^j$ where $b_j \in \mathcal{B}(\Omega)$. We suppose that $\mathcal{B}$ and $\mathcal{A}$ are presheaves of algebras. Then P is $\mathcal{B}$-compatible.*

Proof. It suffices to remark that for each $j \in \mathbb{N}$ the map $A_j : u \mapsto u^j$ is a $\mathcal{B}$-compatible presheaf operator of $\mathcal{A}(\Omega)$. Putting $\alpha = j$ in the above corollary gives the result. ■

Collecting all these informations we can summarize the previous results in the following:

Proposition 9 *We suppose that $\mathcal{B}$ is a presheaf of algebras and $\mathcal{A}$ a presheaf of vector spaces and a $\mathcal{B}$-module. $\mathfrak{A}$ being a set of indices, let $(b_\alpha)_{\alpha\in\mathfrak{A}}$ be a family of elements in $\mathcal{B}(\Omega)$, $(A_\alpha)_{\alpha\in\mathfrak{A}}$ a family of $\mathcal{B}$-compatible operators in $\mathcal{A}(\Omega)$ and $(p_\alpha)_{\alpha\in\mathfrak{A}}$ a family of positive integers. Then,*

(i) For any finite part $\mathfrak{A}_0$ of $\mathfrak{A}$, $\sum\limits_{\alpha\in\mathfrak{A}_0} b_\alpha A_\alpha^{p_\alpha}$ is a $\mathcal{B}$-compatible operator in $\mathcal{A}(\Omega)$.

(ii) If $\mathcal{A}$ is a presheaf of algebras, $u \mapsto \sum\limits_{\alpha\in\mathfrak{A}_0} b_\alpha (A_\alpha(u))^{p_\alpha}$ is a $\mathcal{B}$-compatible operator in $\mathcal{A}(\Omega)$.

2.3 Examples

Example 1 : C^∞**-local analysis in** $\mathcal{D}'(\Omega)$

Let $\mathcal{A} = \mathcal{D}'$, $\mathcal{B} = \mathrm{C}^\infty$. Then, for any distribution $u \in \mathcal{D}'(\Omega)$ where Ω is an open set of $\mathbb{R}^n$

$$\mathcal{S}_{\mathcal{D}'}^{\mathrm{C}^\infty}(u) = sing\,supp\,(u)$$

where $sing\,supp\,u$ is, in the Hörmander sense, the closet subset aff all $x \in \Omega$ having no neighbourhood in which the distribution u is smooth.

Example 2 : $\mathcal{G}^{\infty}$**-local analysis in** $\mathcal{G}(\Omega)$

Let $\mathcal{A}=\mathcal{G}$ *and* $\mathcal{B}=\mathcal{G}^{\infty}$ *the "regular" subsheaf of* $\mathcal{G}$*, the sheaf of Colombeau's generalized functions.* $\mathcal{G}^{\infty}(\Omega)$ *is defined as the sections of* $\mathcal{G}(\Omega)$ *having a representative verifying*

$$\forall K \Subset \Omega\ \exists p \geq 0\ \forall \alpha \in \mathbb{N}^{n}\ \sup_{x\in K}\left|\partial^{\alpha}u_{\varepsilon}(x)\right| = \mathcal{O}\left(\varepsilon^{-p}\right)\ \text{as } \varepsilon \to 0.$$

Then

$$\mathcal{S}_{\mathcal{G}}^{\mathcal{G}^{\infty}}(u) = sing\,supp_{g}(u)$$

where $sing\,supp_{g}\,u$ *is the generalized singular support of* u *defined in the literature as the set of all* $x \in \Omega$ *having no neighbourhood* V *such that* $u|_{V} \in \mathcal{G}^{\infty}(V)$.

Example 3 : $\mathcal{G}^{\mathcal{R}}$**-local analysis in** $\mathcal{G}(\Omega)$

In [12] the $\mathcal{G}^{\infty}$*-regularity is extended into a* $\mathcal{G}^{\mathcal{R}}$ *one. Starting from a set* $\mathcal{R}$ *of sequences of positive numbers,* $\mathcal{G}^{\mathcal{R}}$ *is defined by the sections* $u \in \mathcal{G}(\Omega)$ *having a representative verifying*

$$\forall K \Subset \Omega\ \exists\left(N_{l}\right)_{l\geq 0} \in \mathcal{R}\ \sup_{x\in K}\left|\partial^{\alpha}u_{\varepsilon}(x)\right| = \mathcal{O}\left(\varepsilon^{-N_{|\alpha|}}\right)\ \text{as } \varepsilon \to 0.$$

Under certain stability conditions on the set $\mathcal{R}$ *exposed in [12],* $\mathcal{G}^{\mathcal{R}}$ *is a subsheaf of differential algebras of* $\mathcal{G}$ *and when* $\mathcal{R}$ *consists of the set of all bounded sequences, then* $\mathcal{G}^{\mathcal{R}}=\mathcal{G}^{\infty}$.
Then

$$\mathcal{S}_{\mathcal{G}}^{\mathcal{G}^{\mathcal{R}}}(u) = sing\,supp^{\mathcal{R}}(u)$$

where $sing\,supp^{\mathcal{R}}\,u$ *is the set of all* $x \in \Omega$ *having no neighbourhood* V *such that* $u|_{V} \in \mathcal{G}^{\mathcal{R}}(V)$.

Example 4 : **Local analysis of** $\mathcal{G}^{L}$**-type**

In [30] one constructs $\mathcal{B}=\mathcal{G}^{L}$ *as a special regular sub(pre)sheaf of* $\mathcal{A}=\mathcal{G}$*, extending in a generalized sense the* C^{L} *classes of Hörmander [22] containing analytic and Gevrey classes and constructed from an increasing sequence* L_{k} *of positive numbers such that* L_{0}=1 *and*

$$k \leq L_{k},\ L_{k+1} \leq CL_{k}$$

for some constant C*. When taking* $L_{k}=k+1$*, we obtain the analytic case* $\mathcal{G}^{A}$ *studied in [34] involving special properties of holomorphic generalized functions which give to* $\mathcal{G}^{A}$ *a sheaf property.*

Example 5 : $\mathcal{G}^{\sigma,\infty}$**-local analysis in** $\mathcal{G}^{\sigma}(\Omega)$

In [6] Bouzar and Benmeriem introduce a sheaf of algebra $\mathcal{G}^{\sigma} \neq \mathcal{G}$ *of Gevrey ultradistributions with another asymptotic scale than the Colombeau one by replacing the estimate* $O\left(\varepsilon^{-m}\right)$ *by* $O\left(e^{\varepsilon^{-\frac{1}{2\sigma-1}}}\right)^{m}$ *(resp.* $O\left(\varepsilon^{p}\right)$ *by* $O\left(e^{-\varepsilon^{-\frac{1}{2\sigma-1}}}\right)^{p}$*)in the definition of moderate (resp. null) elements When taking* $L_{k}=(k+1)^{\sigma}$ *they can construct a subpresheaf* $\mathcal{G}^{\sigma,\infty}$ *of* $\mathcal{G}^{\sigma}$ *and give a study of* $\mathcal{G}^{s,\infty}$*-singularity.*

By choosing $\mathcal{R}$ as a regular subset of $\mathbb{R}_{+}^{\mathbb{N}}$, Delcroix has extended the $\mathcal{G}^{\infty}$-regularity into the $\mathcal{G}^{\mathcal{R}}$ one, and in a work in progress Bouzar replaces the classical regularity by the $\mathcal{R}$-regularity to extend the regular generalized Gevrey ultradistributions. We follow this way in view of constructing a general model containing all the previous examples but we have to add two other parameters: an asymptotic scale $r=\left(r_{\lambda}\right)_{\lambda} \in \left(\mathbb{R}_{+}^{*}\right)^{\Lambda}$ and a sequence L_{k} of positive numbers such that L_{0}=1 and $k \leq L_{k},\ L_{k+1} \leq CL_{k}$ for some constant C.

2.4 $\mathcal{G}^{r,\mathcal{R}}$ and $\mathcal{G}^{r,\mathcal{R},L}$-local analysis in $\mathcal{G}^r(\Omega)$

2.4.1 The $\mathcal{G}^r$ sheaf of algebras

Let us consider

• $\mathcal{E} = \mathrm{C}^\infty$ as starting sheaf of algebras, for each open set Ω in $\mathbb{R}^n$, $\mathrm{C}^\infty(\Omega)$ is endowed by the usual family $p_{K,\alpha}$ of seminorms

• Λ a set of indices left-filtering for the given (partial) order relation $\prec$.

• an asymptotic scale $r = (r_\lambda)_\lambda \in (\mathbb{R}_+^*)^\Lambda$ such that $\lim_\Lambda r_\lambda = 0$, (or $r_\lambda \to 0$), that is to say: for each $\mathbb{R}$-neighbourhood W of 0, there exists $\lambda_0 \in \Lambda$ such that

$$\lambda \prec \lambda_0 \Longrightarrow r_\lambda \in W.$$

Define the functors $\mathcal{X}^r$ (resp. $\mathcal{N}^r$): $\Omega \mapsto \mathcal{X}^r(\Omega)$ (resp. $\mathcal{N}^r(\Omega)$) by

$$\mathcal{X}^r(\Omega) = \left\{(u_\lambda)_\lambda \in [\mathrm{C}^\infty(\Omega)]^\Lambda, \forall K \Subset \Omega, \forall \alpha \in \mathbb{N}^n, \exists N \in \mathbb{N}, p_{K,\alpha}(u_\lambda) = O\left(r_\lambda^{-N}\right) \text{ for } r_\lambda \to 0\right\}$$
$$\mathcal{N}^r(\Omega) = \left\{(u_\lambda)_\lambda \in [\mathrm{C}^\infty(\Omega)]^\Lambda, \forall K \Subset \Omega, \forall \alpha \in \mathbb{N}^n, \forall m \in \mathbb{N}, p_{K,\alpha}(u_\lambda) = O\left(r_\lambda^{m}\right) \text{ for } r_\lambda \to 0\right\}$$

it is not difficult to prove with the same techniques as Colombeau ones [7] that $\mathcal{X}^r$ and $\mathcal{N}^r$ are respectively a sheaf of differential algebras and a subsheaf of ideals of $\mathcal{X}^r$over the ring

$$\mathcal{X}^r(\mathbb{C}) = \left\{(s_\lambda)_\lambda \in \mathbb{C}^\Lambda, \exists N \in \mathbb{N}, |s_\lambda| = O\left(r_\lambda^{-N}\right) \text{ for } r_\lambda \to 0\right\}.$$

Then $\mathcal{X}^r/\mathcal{N}^r = \mathcal{G}^r$ is a priori a factor presheaf of Colombeau type. It is well known that $\mathcal{G}$ is a sheaf and even a fine sheaf. The first assumption (a result from Aragona and Biagioni [1]) is based on the existence of a C^∞ partition of unity associated to any open covering of Ω (due to the fact that $\mathbb{R}^d$ is a locally compact Hausdorff space). On the other hand, we can notice that C^∞ is a fine sheaf because multiplication by a smooth function defines a sheaf homomorphism in a natural way. Hence the usual topology and C^∞ partition of unity define the required sheaf partition of unity according to the definition in sheaf theory. This leads very easily to the second assumption, from the well known result that any sheaf of modules on a fine sheaf is itself a fine sheaf: it is precisely the case of $\mathcal{G}$ which is a sheaf of C^∞ modules. And it is the same for $\mathcal{G}^r$ which is a fine sheaf of C^∞ modules and also a sheaf of differential algebras over the ring $\mathcal{X}^r(\mathbb{C})/\mathcal{N}^r(\mathbb{C})$ with

$$\mathcal{N}^r(\mathbb{C}) = \left\{(s_\lambda)_\lambda \in \mathbb{C}^\Lambda, \forall m \in \mathbb{N}, |s_\lambda| = O\left(r_\lambda^{m}\right) \text{ for } r_\lambda \to 0\right\}.$$

2.4.2 The $\mathcal{G}^{r,\mathcal{R}}$ subsheaf of $\mathcal{G}^r$

The $\mathcal{G}^{r,\mathcal{R}}$ regularity of $\mathcal{G}^r$ generalizes the $\mathcal{G}^\infty$ regularity of $\mathcal{G}$. We begin by defining a regular subspace $\mathcal{R}$ of $\mathbb{R}_+^n$ in the Delcroix sense [12]:

Definition 8 *A subspace $\mathcal{R}$ of $\mathbb{R}_+^{\mathbb{N}}$ is* regular *if $\mathcal{R}$ is non empty and*

(i) $\mathcal{R}$ is "overstable" by translation and by maximum

$$\forall N \in \mathcal{R},\ \forall (k,k') \in \mathbb{N}^2,\ \exists N' \in \mathcal{R},\ \ \forall n \in \mathbb{N}, \quad N(n+k) + k' \leq N'(n),$$
$$\forall N_1 \in \mathcal{R},\ \forall N_2 \in \mathcal{R},\ \exists N \in \mathcal{R},\ \forall n \in \mathbb{N}, \quad \max(N_1(n), N_2(n)) \leq N(n).$$

(ii) For all N_1 and N_2 in $\mathcal{R}$, there exists $N \in \mathcal{R}$ such that

$$\forall (l_1,l_2) \in \mathbb{N}^2, \quad N_1(l_1) + N_2(l_2) \leq N(l_1+l_2).$$

Then, for any regular subset $\mathcal{R}$ of $\mathbb{R}_+^{\mathbb{N}}$ we can set

$$\mathcal{X}^{r,\mathcal{R}}(\Omega) = \left\{ (u_\lambda)_\lambda \in [\mathrm{C}^\infty(\Omega)]^\Lambda, \forall K \Subset \Omega, \exists N \in \mathcal{R}, \forall \alpha \in \mathbb{N}^n, p_{K,\alpha}(u_\lambda) = O\left(r_\lambda^{-N(|\alpha|)}\right) \text{ for } r_\lambda \to 0 \right\}$$
$$\mathcal{N}^{r,\mathcal{R}}(\Omega) = \left\{ (u_\lambda)_\lambda \in [\mathrm{C}^\infty(\Omega)]^\Lambda, \forall K \Subset \Omega, \forall m \in \mathcal{R}, \forall \alpha \in \mathbb{N}^n, p_{K,\alpha}(u_\lambda) = O\left(r_\lambda^{m(|\alpha|)}\right) \text{ for } r_\lambda \to 0 \right\}.$$

Proposition 10

(i) For any regular subspace $\mathcal{R}$ of $\mathbb{R}_+^{\mathbb{N}}$, the functor $\Omega \to \mathcal{X}^{r,\mathcal{R}}(\Omega)$ defines a sheaf of differential algebras over the ring $\mathcal{X}^r(\mathbb{C})$.

(ii) The set $\mathcal{N}^{r,\mathcal{R}}(\Omega)$ is equal to $\mathcal{N}^r(\Omega)$. Thus, the functor $\mathcal{N}^{\mathcal{R}} : \Omega \to \mathcal{N}^{\mathcal{R}}(\Omega)$ defines a sheaf of ideals of the sheaf $\mathcal{X}^{\mathcal{R}}(\cdot)$.

(iii) For any regular subspaces $\mathcal{R}_1$ and $\mathcal{R}_2$ of $\mathbb{R}_+^{\mathbb{N}}$, with $\mathcal{R}_1 \subset \mathcal{R}_2$, the sheaf $\mathcal{X}^{\mathcal{R}_1}(\Omega)$ is a subsheaf of the sheaf $\mathcal{X}^{\mathcal{R}_2}(\Omega)$.

Proof. The proof follows the same lines as in the case of $\mathcal{G}^{\mathcal{R}}$ algebras (see [12], Proposition 1.). We have to verify that our asymptotic scale $(r_\lambda)_\lambda$ involving a more general parametrization doesn't modify the results. We deduce assertion (i) from the assertion (i) in the definition of $\mathcal{R}$. For the equality $\mathcal{N}^{r,\mathcal{R}}(\Omega) = \mathcal{N}^r(\Omega)$, take first $(u_\lambda)_\lambda \in \mathcal{N}^{r,\mathcal{R}}(\Omega)$. For any $K \Subset \Omega$, $\alpha \in \mathbb{N}^n$ and $m \in \mathbb{N}$, choose $N \in \mathcal{R}$. From (i) in definition 8 there exists $N' \in \mathcal{R}$ such that $N + m \leq N'$. Thus $p_{K,\alpha}(u_\lambda) = O\left(r_\lambda^{N'(|\alpha|)}\right) = O(r_\lambda^m)$ and $(u_\lambda)_\lambda \in \mathcal{N}^r(\Omega)$. Conversely for given $(u_\lambda)_\lambda \in \mathcal{N}^r(\Omega)$ and $N \in \mathcal{R}$ we have $p_{K,\alpha}(u_\lambda) = O\left(r_\lambda^{N(|\alpha|)}\right)$ since this estimate holds for all $m \in \mathbb{N}$. For the sheaf properties we have to replace Colombeau's estimates by $\mathcal{X}^{r,\mathcal{R}}$ estimates and consider only a finite number of terms by compactness. Thus, from (ii) in definition 8 we have the results. The inclusion $\mathcal{X}^{\mathcal{R}_1}(\Omega) \subset \mathcal{X}^{\mathcal{R}_2}(\Omega)$ prove (iii). ■

According to same arguments as those used for $\mathcal{G}^r$ the presheaf $\mathcal{G}^{r,\mathcal{R}} = \mathcal{X}^{r,\mathcal{R}}/\mathcal{N}^{r,\mathcal{R}} = \mathcal{X}^{r,\mathcal{R}}/\mathcal{N}^r$ turns to be a sheaf of differentiable algebras on the ring $\mathcal{X}^{r,\mathcal{R}}(\mathbb{C})/\mathcal{N}^r(\mathbb{C})$ with

$$\mathcal{N}^r(\mathbb{K}) = \left\{ (s_\lambda)_\lambda \in \mathbb{C}^\Lambda, \forall m \in \mathbb{N}, |s_\lambda| = O(r_\lambda^m) \text{ for } r_\lambda \to 0 \right\}, \quad \mathbb{K} = \mathbb{R} \text{ or } \mathbb{K} = \mathbb{C}.$$

Moreover, from (ii) in the above proposition, $\mathcal{G}^{r,\mathcal{R}}$ is a subsheaf of $\mathcal{G}^r$.

Definition 9 *For any regular subset $\mathcal{R}$ of $\mathbb{R}_+^{\mathbb{N}}$, the sheaf of algebras (subsheaf of $\mathcal{G}^r$)*

$$\mathcal{G}^{r,\mathcal{R}} = \mathcal{X}^{r,\mathcal{R}}/\mathcal{N}^{r,\mathcal{R}}$$

is called the sheaf of $(r,\mathcal{R})$-regular algebras of (nonlinear) generalized functions.

Example 6 *Taking $\lambda = \varepsilon \in]0,1]$, $r_\varepsilon = \varepsilon$ and $\mathcal{R} = \mathbb{R}_+^{\mathbb{N}}$, we recover the sheaf $\mathcal{G}$ of Colombeau simplified algebras.*

Taking $\lambda = \varepsilon \in]0,1]$, $r_\varepsilon = \varepsilon$ and $\mathcal{R} = \mathcal{B}o$ (the set of bounded sequences), we obtain the sheaf of $\mathcal{G}^\infty$-generalized functions [12].

Taking $\lambda = \varepsilon \in]0,1]$, $r_\varepsilon = e^{\varepsilon^{-\frac{1}{2\sigma-1}}}$ and $\mathcal{R} = \mathbb{R}_+^{\mathbb{N}}$, we obtain the sheaf of so called $\mathcal{G}^s$-generalized functions in [6].

2.4.3 The $\mathcal{G}^{r,\mathcal{R},L}$ subpresheaf of $\mathcal{G}^r$ and $(r,\mathcal{R},L)$-analysis

Let L_k be an increasing sequence of positive numbers such that L_0=1 and

$$k \leq L_k, \; L_{k+1} \leq CL_k$$

for some constant C. According to Hörmander definition given in subsection 8.4 of [22], we shall denote by C^L the sheaf of $\mathbb{K}$-algebras on $\mathbb{R}^n$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$) such that, for any open set $\Omega \subset \mathbb{R}^n$

$$\mathrm{C}^L(\Omega) = \left\{ u = \mathrm{C}^\infty(\Omega) \mid \forall K \Subset \Omega, \exists c > 0, \forall \alpha \in \mathbb{N}^n, \sup_{x \in K} |\mathcal{D}^\alpha u(x)| \leq c \left(cL_{|\alpha|} \right)^{|\alpha|} \right\}$$

When $L_k = k + 1$, C^L is the sheaf A of analytical functions. If $L_k = (k+1)^a$, $a > 1$, C^L is the sheaf G_a of the Gevrey class of order a.

It is possible to enlarge the above definition into a generalized one involving three parameters r, $\mathcal{R}$, L corresponding to a choice of some asymptotic scale $r = (r_\lambda)_\lambda \in \left(\mathbb{R}_+^*\right)^\Lambda$, a regular subset $\mathcal{R}$ of $\mathbb{R}_+^{\mathbb{N}}$, and a sequence $L = (L_k)_k$.

Definition 10 *Let us define the functors $\mathcal{X}^{r,\mathcal{R},L}$ (resp. $\mathcal{N}^{r,\mathcal{R},L}$): $\Omega \mapsto \mathcal{X}^{r,\mathcal{R},L}(\Omega)$ (resp. $\mathcal{N}^{r,\mathcal{R},L}(\Omega)$) by*

$$\begin{cases} \mathcal{X}^{r,\mathcal{R},L}(\Omega) = \Big\{ (u_\lambda)_\lambda \in [\mathrm{C}^\infty(\Omega)]^\Lambda, \forall K \Subset \Omega, \exists N \in \mathcal{R}, \exists c > 0, \exists \lambda_0 \in \Lambda \\ \qquad \forall \alpha \in \mathbb{N}^n, \forall \lambda \prec \lambda_0 : \sup_{x \in K} |D^\alpha u_\lambda(x)| \leq c r_\lambda^{-N(|\alpha|)} \left(cL_{|\alpha|} \right)^{|\alpha|} \Big\}, \\ \mathcal{N}^{r,\mathcal{R},L}(\Omega) = \mathcal{X}^{r,\mathcal{R},L}(\Omega) \cap \mathcal{N}^r(\Omega). \end{cases}$$

Lemma 11 *$\mathcal{X}^{r,\mathcal{R},L}$ is a subsheaf of $\mathcal{X}^{r,\mathcal{R}}$, $\mathcal{N}^{r,\mathcal{R},L}$ is a sheaf of ideals of $\mathcal{X}^{r,\mathcal{R},L}$*

Proof. For each Ω, $\mathcal{X}^{r,\mathcal{R},L}(\Omega)$ is a subalgebra of $\mathcal{X}^{r,\mathcal{R}}(\Omega)$, and the restriction and localization processes are obvious. Let us try to glue together the bits, giving some family $(\Omega_i)_{i \in I}$, with $\Omega = \underset{1 \leq i \leq L}{\cup} \Omega_i$, and $U_i = (u_{i,\lambda})_\varepsilon \in \mathcal{X}^{r,\mathcal{R},L}(\Omega_i)$ with $U_i = U_j$ on $\Omega_i \cap \Omega j$. We begin to define $U(x)$ as $U_i(x)$ when $x \in \Omega$ lies in Ω_i. Clearly U belongs to $[\mathrm{C}^\infty(\Omega)]^\Lambda$. Let $K \Subset \Omega$. We can cover K by a finite number of $\Omega_i : \Omega_1, ...\Omega_p$ such that $K = \underset{1 \leq i \leq L}{\cup} K_i$, with $K_i = K \cap \overline{\Omega_i'} \subset \Omega_i$. This is possible by choosing $\Omega_i' \subsetneqq \Omega_i, d(\Omega_i', \Omega_i) \leq d(K, \Omega)$ for $1 \leq i \leq p$. Then

$$\exists N_i \in \mathcal{R}, \exists c_i > 0, \exists \lambda_{0,i} \in \Lambda, \forall \alpha \in \mathbb{N}^n, \forall \lambda \prec \lambda_{0,i} : \sup_{x \in K_i} |D^\alpha u_{i,\lambda}(x)| \leq c_i r_\lambda^{-N_i(|\alpha|)} \left(c_i L_{|\alpha|} \right)^{|\alpha|}$$

From the assumption there exits some $\lambda_0 \in \Lambda$ such that $\lambda_0 \prec \lambda_{0,i}$ and $N \in \mathcal{R}$ such that $N \geq N_i$ for $1 \leq i \leq p$. Set $c = \max_{1 \leq i \leq L} c_i$. Then for each $\alpha \in \mathbb{N}^n$ and $\lambda \prec \lambda_0$ we have

$$\sup_{x \in K} |D^\alpha u_\lambda(x)| = \sum_{1 \leq i \leq p} c r_\lambda^{-N_i(|\alpha|)} \left(cL_{|\alpha|} \right)^{|\alpha|} \leq p c r_\lambda^{-N(|\alpha|)} \left(cL_{|\alpha|} \right)^{|\alpha|} \leq c' r_\lambda^{-N(|\alpha|)} \left(c' L_{|\alpha|} \right)^{|\alpha|}.$$

Thus U belongs to $\mathcal{X}^L(\Omega)$. It is easy to see that for each Ω, $\mathcal{N}_*^L(\Omega)$ is an ideal of $\mathcal{X}^L(\Omega)$, and the same proof as above leads to the sheaf structure of $\mathcal{N}_*^L$. Then we can define a new factor presheaf of C^L-type algebras which is a subpresheaf of $\mathcal{G}^{r,\mathcal{R}}$ and $\mathcal{G}^r$, according to the definition of the ideal $\mathcal{N}^{r,\mathcal{R},L}(\Omega) = \mathcal{X}^{r,\mathcal{R},L}(\Omega) \cap \mathcal{N}^r(\Omega)$. ■

Definition 11 *For any asymptotic scale $r = (r_\lambda)_\lambda \in \left(\mathbb{R}_+^*\right)^\Lambda$, any regular subset $\mathcal{R}$ of $\mathbb{R}_+^{\mathbb{N}}$, and any sequence $L = (L_k)_k$, the presheaf of algebras (subpresheaf of $\mathcal{G}^r$)*

$$\mathcal{G}^{r,\mathcal{R},L} = \mathcal{X}^{r,\mathcal{R},L} / \mathcal{N}^{r,\mathcal{R},L}$$

is called the sheaf of $(r, \mathcal{R}, L)$-regular algebras of (nonlinear) generalized functions.

Example 7 *Taking* $\lambda = \varepsilon \in \left]0,1\right]$, $r_\varepsilon = \varepsilon$, $\mathcal{R} = \mathcal{B}o$ *(the set of bounded sequences), and some* $L = (L_k)_k$*, we obtain the presheaf of* $\mathcal{G}^L$*-generalized functions (subpresheaf of* $\mathcal{G}$*) [30].*

Taking $\lambda = \varepsilon \in \left]0,1\right]$, $r_\varepsilon = \varepsilon$, $\mathcal{R} = \mathcal{B}o$ *and* $L_k = k+1$*, we obtain the sheaf of* $\mathcal{G}^A$*-generalized functions (subsheaf of* $\mathcal{G}$*) [34].*

Taking $\lambda = \varepsilon \in \left]0,1\right]$, $r_\varepsilon = e^{\varepsilon^{-\frac{1}{2\sigma-1}}}$, $\mathcal{R} = \mathcal{B}o$ *and* $L_k = (k+1)^\sigma$*, we obtain the presheaf of* $\mathcal{G}^{\sigma,\infty}$*-generalized functions (subpresheaf of* $\mathcal{G}^\sigma$*) [6].*

Remark 3 *We proved that* $\mathcal{G}^{r,\mathcal{R},L}$ *is a presheaf ; the localization principle* (F_1) *is not difficult to prove. However, we lack some suitable partition of unity or other argument which preserve the* $\mathcal{X}^{r,\mathcal{R},L}$ *estimates and permit to glue together the bits and get* (F_1)*. But instead of introducing the sheaf associated to* $\mathcal{G}^{r,\mathcal{R},L}$*, one can keep its presheaf structure in the following localization processes.*

Taking $\mathcal{A} = \mathcal{G}^r$ *and* $\mathcal{B} = \mathcal{G}^{r,\mathcal{R},L}$*, the* $\mathcal{G}^{r,\mathcal{R},L}$*-singularities of* $u \in \mathcal{G}^r(\Omega)$ *are localized in*

$$\mathcal{S}_{\mathcal{G}^r}^{\mathcal{G}^{r,\mathcal{R},L}}(u) = sing\,supp^{(r,\mathcal{R},L)}(u)$$

where $sing\,supp^{(r,\mathcal{R},L)}(u)$ *is the set of all* $x \in \Omega$ *having no neighbourhood* V *such that* $u|_V \in \mathcal{G}^{r,\mathcal{R},L}(V)$*. But we cannot prove that* $\Omega \backslash sing\,supp^{(r,\mathcal{R},L)}(u)$ *is the largest open set* $\mathcal{O}$ *such that* $u \in \mathcal{G}^{r,\mathcal{R},L}(\mathcal{O})$*. Due to the lack of the* (F_2) *principe in lemma 2, we don't know how to prove the existence of such an open set.*

Remark 4 *Following the previous definition 2 of* $\mathcal{B}$*-local regularity of an element* $u \in \mathcal{A}(\Omega)$*, we can naturally set that* $u \in \mathcal{G}^r(\Omega)$ *is locally* $\mathcal{G}^{r,\mathcal{R},L}$ *at* $x \in \Omega$ *if for some neighbourhood* V *of* $x \in \Omega$ *the restriction* $u|_V$ *belongs to* $\mathcal{G}^{r,\mathcal{R},L}(V)$*, that is to say if for some neighbourhood* V *of* x *there exists a representative* $(u_\lambda)_\lambda$ *of* u *such that* $(u_\lambda|_V)_\lambda$ *belongs to* $\mathcal{X}^{r,\mathcal{R},L}(V)$*. Let* $\mathcal{H}^{r,\mathcal{R},L}(\Omega)$ *be the set of all* $u \in \mathcal{G}^r(\Omega)$ *which are locally* $\mathcal{G}^{r,\mathcal{R},L}$ *at* $x \in \Omega$*. It is not difficult to prove that the sheaf associated to* $\mathcal{G}^{r,\mathcal{R},L}$ *is the functor* $\Omega \mapsto \mathcal{H}^{r,\mathcal{R},L}(\Omega)$*. It is a subsheaf of algebras of* $\mathcal{G}^r$*. But in the general case, we cannot prove that* $u \in \mathcal{H}^{r,\mathcal{R},L}(\Omega)$ *has a global representative in* $\mathcal{X}^{r,\mathcal{R},L}(\Omega)$*. However this is fulfilled when taking* $r_\varepsilon = \varepsilon$, $\mathcal{R} = \mathcal{B}o$, $L_k = k+1$*, corresponding to the analytic case studied in [34] involving special properties of holomorphic generalized functions. And then* $\mathcal{H}^{r,\mathcal{R},L} = \mathcal{G}^A$*, the subsheaf of generalized analytic functions of the sheaf* $\mathcal{G}$*. But when* $\mathcal{A} = \mathcal{G}$ *and* $\mathcal{B} = \mathcal{G}^A$*, the* $\mathcal{G}^A$*-singularities of* $u \in \mathcal{G}(\Omega)$ *are always localized in*

$$\mathcal{S}_{\mathcal{G}}^{\mathcal{G}^A}(u) = sing\,supp^A(u)$$

where $sing\,supp^A\,u$ *is the set of all* $x \in \Omega$ *having no neighbourhood* V *such that* $u|_V \in \mathcal{G}^A(V)$*. Here, the sheaf structure of* $\mathcal{G}^A$ *provides the following precision: from lemma 2 involving the* (F_2) *principe, we can prove that* $\Omega \backslash sing\,supp^A\,u$ *is also the largest open set* $\mathcal{O}$ *such that* $u \in \mathcal{G}^A(\mathcal{O})$*.*

2.4.4 Canonical embeddings

Lemma 12 *We have the following commutative diagram in which the arrows are canonical embeddings*

$$\begin{array}{ccc} \mathrm{C}^L & \rightarrow & \mathrm{C}^\infty \\ \downarrow & & \downarrow \\ \mathcal{G}^{r,\mathcal{R},L} & \rightarrow & \mathcal{G}^{r,\mathcal{R}} \end{array}$$

Proof. The canonical (pre)sheaf embedding of C^L into $\mathcal{G}^{r,\mathcal{R},L}$ (resp. of C^∞ into $\mathcal{G}^{r,\mathcal{R}}$) is defined for each open set $\Omega \subset \mathbb{R}^n$ in by the canonical map

$$\mathrm{C}^L(\Omega) \rightarrow \mathcal{G}^{r,\mathcal{R},L}(\Omega) \text{ (resp. } \mathrm{C}^\infty \rightarrow \mathcal{G}^{r,\mathcal{R}}\text{) : } u \mapsto [u_\lambda]\text{, with } u_\lambda = u \text{ for } \lambda \in \Lambda$$

which is an injective homomorphism of algebras, $[u_\lambda]$ being the class of $u \in \mathrm{C}^L(\Omega)$ (resp. $\mathrm{C}^\infty(\Omega)$) in the factor algebra $\mathcal{G}^{r,\mathcal{R},L}(\Omega)$ (resp.$\mathcal{G}^{r,\mathcal{R}}(\Omega)$). In order to construct the sheaf embedding : $\mathcal{G}^{r,\mathcal{R},L} \to \mathcal{G}^{r,\mathcal{R}}$, we recall that

$$\mathcal{X}^{r,\mathcal{R},L}(\Omega) \subset \mathcal{X}^{r,\mathcal{R}}(\Omega)\,; \mathcal{N}^{r,\mathcal{R},L}(\Omega) = \mathcal{X}^{r,\mathcal{R},L}(\Omega) \cap \mathcal{N}^{r}(\Omega)$$

is a necessary and sufficient condition to embed $\mathcal{G}^{r,\mathcal{R},L}(\Omega)$ into $\mathcal{G}^{r,\mathcal{R}}(\Omega)$, from which we deduce the required sheaf embedding. ∎

Remark 5 *When $\lambda = \varepsilon$ and $r_\varepsilon = \varepsilon$, we can suppress the symbol r in the previous formulation. When taking $\mathcal{R} = \mathcal{B}o$, the symbol $\mathcal{R}$ becomes ∞. For example, when we do that simultaneously, we have*

$$\mathcal{G}^r = \mathcal{G}\ ; \mathcal{G}^{r,\mathcal{R}} = \mathcal{G}^\infty\ ; \mathcal{G}^{r,\mathcal{R},L} = \mathcal{G}^L.$$

2.5 $\mathcal{G}$-local analysis in $\mathcal{L}\left(\mathcal{G}_c(\Omega), \widetilde{\mathbb{C}}\right)$

2.5.1 Duality in the Colombeau context

Starting from the usual family of semi norms $(p_i)_{i\in I}$ defining the topology of $\mathrm{C}^\infty(\Omega)$ by

$$p_i(f) = p_{K,l}(f) = \sup_{x\in K, |\alpha|\le l} |\partial^\alpha f(x)|$$

the so-called sharp topology of $\mathcal{G}(\Omega)$ [31]is defined by the family of ultra-pseudo-seminorms $(\mathcal{P}_i)_{i\in I}$ such that $\mathcal{P}_i(u) = e^{-v_{p_i}(u)}$ where v_{p_i} is the valuation defined for $u = [u_\varepsilon]$ by

$$v_{p_i}(u) = v_{p_i}\left((u_\varepsilon)_\varepsilon\right) = \sup\left\{b \in \mathbb{R} : p_i(u_\varepsilon) = O\left(\varepsilon^b\right) \text{ as } \varepsilon \to 0\right\}.$$

The valuation on the ring $\widetilde{\mathbb{C}}$ of generalized numbers given for each $r = [r_\varepsilon]$ by

$$v(r) = v\left((r_\varepsilon)_\varepsilon\right) = \sup\left\{b \in \mathbb{R} : |u_\varepsilon| = O\left(\varepsilon^b\right) \text{ as } \varepsilon \to 0\right\}$$

leads to the ultra-pseudo-norm on $\widetilde{\mathbb{C}} : |r|_e = e^{-v(r)}$.

It is proved in [16] that a $\widetilde{\mathbb{C}}$-linear map $T : \mathcal{G}_c(\Omega) \to \widetilde{\mathbb{C}}$ is continuous for the above topologies if and only if there exists a finite subset $I_0 \subset I$ and a constant $C > 0$ such that, for all $u \in \mathcal{G}_c(\Omega)$

$$|\langle T, u\rangle|_e \le C \max_{i\in I_0} \mathcal{P}_i(u).$$

In [17], the topological dual $\mathcal{L}\left(\mathcal{G}_c(\Omega), \widetilde{\mathbb{C}}\right)$ is endowed with the topology of uniform convergence on bounded subsets which is defined by the ultra-pseudo-seminorms

$$\mathcal{P}_{\mathcal{B}}(T) = \sup_{u\in\mathcal{B}} |\langle T, u\rangle|_e$$

with $\mathcal{B}$ varying in the family of all bounded subsets of $\mathcal{G}_c(\Omega)$, i.e., for each $i \in I$, $\sup_{u\in\mathcal{B}} \mathcal{P}_i(u) < \infty$.

2.5.2 Localization of $\mathcal{G}$-singularities

The sheaf embedding $\mathcal{G} \to \mathcal{L}\left(\mathcal{G}_c, \widetilde{\mathbb{C}}\right)$ is defined, for each open set Ω of $\mathbb{R}^n$, by the continuous (as recalled in [17]) map

$$\mathcal{G}(\Omega) \ni u \mapsto T_u \in \mathcal{L}\left(\mathcal{G}_c(\Omega), \widetilde{\mathbb{C}}\right)$$

where T_u is defined, for $u = [u_\varepsilon] \in \mathcal{G}(\Omega)$ and each $v = [v_\varepsilon] \in \mathcal{G}_c(\Omega)$, by

$$\langle T_u, v \rangle = \left[\int_K u_\varepsilon(x)\, v_\varepsilon(x)\, dx\right] \in \widetilde{\mathbb{C}}$$

where K is an arbitrary compact set containing supp$v$ in its interior.

From Definition 2.9 in [17], the $\mathcal{G}$-singular support of $T \in \mathcal{L}\left(\mathcal{G}_c(\Omega), \widetilde{\mathbb{C}}\right)$ denoted by $(singsupp_{\mathcal{G}}(T))$ is the complement of the set of all points $x \in \Omega$ such that the restriction of T to some neighborhood V of x belongs to $\mathcal{G}(V)$. Then we still have with our standard notations ($\mathcal{A} = \mathcal{L}\left(\mathcal{G}_c, \widetilde{\mathbb{C}}\right)$, $\mathcal{B} = \mathcal{G}$)

$$singsupp_{\mathcal{G}}(T) = \mathcal{S}^{\mathcal{G}}_{\mathcal{L}(\mathcal{G}_c, \widetilde{\mathbb{C}})}(T).$$

2.6 $\mathcal{F}$-local analysis in $(\mathcal{C}, \mathcal{E}, \mathcal{P})$-algebras $\mathcal{A}(\Omega)$

2.6.1 The algebraic structure of a $(\mathcal{C}, \mathcal{E}, \mathcal{P})$ algebra

We summarize the construction of the so-called $(\mathcal{C}, \mathcal{E}, \mathcal{P})$ algebras [14, 29] which generalize many cases met in the literature. $\mathbb{K}$ is the real or complex field and Λ a set of indices. $\mathcal{C}$ is the factor ring A/I where I is an ideal of A, a given subring of $\mathbb{K}^\Lambda$. $(\mathcal{E}, \mathcal{P})$ is a sheaf of topological $\mathbb{K}$-algebras on a topological space X. A presheaf of $(\mathcal{C}, \mathcal{E}, \mathcal{P})$ algebras on X is a presheaf $\mathcal{A} = \mathcal{H}/\mathcal{J}$ of factor algebras where $\mathcal{J}$ is an ideal of $\mathcal{H}$, a subsheaf of $\mathcal{E}^\Lambda$. The sections of $\mathcal{H}$ (resp. $\mathcal{J}$) of X have to verify some estimates given by means of $\mathcal{P}$ and A (resp. I).

The above construction needs some technical conditions given in [14] on the structure of $\mathcal{C}$ and we suppose that for any open set Ω in X, the algebra $\mathcal{E}(\Omega)$ is endowed with a family $\mathcal{P}(\Omega) = (p_i)_{i \in I(\Omega)}$ of semi-norms. Then, we set

$$\mathcal{H}(\Omega) = \mathcal{H}_{(A,\mathcal{E},\mathcal{P})}(\Omega) = \{(u_\lambda)_\lambda \in [\mathcal{E}(\Omega)]^\Lambda \mid \forall i \in I(\Omega), ((p_i(u_\lambda))_\lambda \in |A|\},$$
$$\mathcal{J}(\Omega) = \mathcal{J}_{(I_A,\mathcal{E},\mathcal{P})}(\Omega) = \left\{(u_\lambda)_\lambda \in [\mathcal{E}(\Omega)]^\Lambda \mid \forall i \in I(\Omega), (p_i(u_\lambda))_\lambda \in |I_A|\right\}.$$

The factor $\mathcal{H}_{(A,\mathcal{E},\mathcal{P})}/\mathcal{J}_{(I_A,\mathcal{E},\mathcal{P})}$ is a presheaf verifying the localization principle (F_1) but generally not the gluing principle (F_2). *The element in* $\mathcal{A}(\Omega)$ *defined by* $(u_\lambda)_{\lambda\in\Lambda} \in \mathcal{H}_{(A,\mathcal{E},\mathcal{P})}(\Omega)$ *is denoted by* $[u_\lambda]$. For $u \in \mathcal{A}(\Omega)$, the notation $(u_\lambda)_{\lambda\in\Lambda} \in u$ means that $(u_\lambda)_{\lambda\in\Lambda}$ is a representative of u.

2.6.2 Association process

We assume further that A is unitary and Λ is left-filtering for the given (partial) order relation $\prec$. Let us denote by:

• $\mathcal{F}$ a given sheaf of topological $\mathbb{K}$-vector spaces (resp. $\mathbb{K}$-algebras) over X containing $\mathcal{E}$ as a subsheaf,

• a a map from $\mathbb{R}_+$ to A_+ such that $a(0) = 1$ (for $r \in \mathbb{R}_+$, we denote $a(r)$ by $(a_\lambda(r))_\lambda$).

For $(u_\lambda)_\lambda \in \mathcal{H}_{(A,\mathcal{E},\mathcal{P})}(\Omega)$, we denote by $\lim\limits_{\Lambda} {}_{\mathcal{F}(\Omega)}\, u_\lambda$ the limit of $(u_\lambda)_\lambda$ for the $\mathcal{F}$-topology when it exists. We recall that $\lim\limits_{\Lambda} {}_{\mathcal{F}(V)}\, u_\lambda = f \in \mathcal{F}(\Omega)$ iff, for each $\mathcal{F}$-neighbourhood W of f, there exists $\lambda_0 \in \Lambda$ such that

$$\lambda \prec \lambda_0 \Longrightarrow u_\lambda \in W.$$

We suppose also that we have, for each open subset $V \subset \Omega$,

$$\mathcal{J}_{(I_A,\mathcal{E},\mathcal{P})}(V) \subset \left\{ (v_\lambda)_\lambda \in \mathcal{H}_{(A,\mathcal{E},\mathcal{P})}(V) : \lim_{\Lambda} {}_{\mathcal{F}(V)}\, v_\lambda = 0 \right\}.$$

Now, consider $u = [u_\lambda] \in \mathcal{A}(\Omega)$, $r \in \mathbb{R}_+$, V an open subset of Ω and $f \in \mathcal{F}(V)$. We say that u is *$a\,(r)$-associated to f in V*:

$$u \underset{\mathcal{F}(V)}{\overset{a(r)}{\sim}} f$$

if $\lim\limits_{\Lambda} {}_{\mathcal{F}(V)} \left(a_\lambda\,(r)\; u_\lambda\,|_V \right) = f$.

In particular, if $r = 0$, u and f are said *associated* in V. To ensure the independence of the definition with respect to the representative of u, we must have, for any $(\eta_\lambda)_\lambda \in \mathcal{J}_{(I_A,\mathcal{E},\mathcal{P})}(\Omega)$, $\lim\limits_{\Lambda} {}_{\mathcal{F}(V)}\, a_\lambda\,(r)\; \eta_\lambda\,|_V = 0$. As $\mathcal{J}_{(I_A,\mathcal{E},\mathcal{P})}(V)$ is an ideal over A, $(a_\lambda\,(r)\; \eta_\lambda\,|_V)_\lambda$ is in $\mathcal{J}_{(I_A,\mathcal{E},\mathcal{P})}(V)$. Thus, our claim follows from the above assumption.

When taking $X = \mathbb{R}^d$, $\mathcal{F} = \mathcal{D}'$, $\Lambda =]0,1]$, $\mathcal{A} = \mathcal{G}$, $V = \Omega$, $r = 0$, the usual association between $u = [u_\varepsilon] \in \mathcal{G}\,(\Omega)$ and $T \in \mathcal{D}'\,(\Omega)$ is defined by

$$u \sim T \Longleftrightarrow u \underset{\mathcal{D}'(\Omega)}{\overset{a(0)}{\sim}} T \Longleftrightarrow \lim_{\varepsilon \to 0} {}_{\mathcal{D}'(\Omega)}\, u_\varepsilon = T.$$

Using the previous notations and according to the previous assumption we have, for any open set Ω in X

$$\mathcal{J}_{(I_A,\mathcal{E},\mathcal{P})}(\Omega) \subset \mathcal{N}_{\mathcal{E}}^{\mathcal{F}}\,(\Omega) = \left\{ (u_\lambda)_\lambda \in \mathcal{H}_{(A,\mathcal{E},\mathcal{P})}(\Omega) \;:\; \lim_{\Lambda} {}_{\mathcal{F}(\Omega)}\, u_\lambda = 0 \right\}.$$

Set

$$\mathcal{F}_{\mathcal{A}}(\Omega) = \left\{ u \in \mathcal{A}(\Omega) \mid \exists\, (u_\lambda)_\lambda \in u,\ \exists f \in \mathcal{F}(\Omega) \;:\; \lim_{\Lambda} {}_{\mathcal{F}(\Omega)}\, u_\lambda = f \right\}.$$

$\mathcal{F}_{\mathcal{A}}(\Omega)$ is well defined because if $(\eta_\lambda)_\lambda$ belongs to $\mathcal{J}_{(I_A,\mathcal{E},\mathcal{P})}(\Omega)$, we have $\lim\limits_{\Lambda} {}_{\mathcal{F}(\Omega)}\, \eta_\lambda = 0$. Moreover, $\mathcal{F}_{\mathcal{A}}$ is a subpresheaf of vector spaces (resp. algebras) of $\mathcal{A}$. Roughly speaking, it is the presheaf whose sections above some open set Ω are the generalized functions in $\mathcal{A}\,(\Omega)$ associated to an element of $\mathcal{F}\,(\Omega)$.

2.6.3 Localization of $\mathcal{F}$-singularities

We refer to definition and results given in section 2 and take here $\mathcal{B} = \mathcal{F}_{\mathcal{A}}$. When u belongs to $\mathcal{A}\,(\Omega)$, we can consider the set $\mathcal{O}_{\mathcal{A}}^{\mathcal{F}}\,(u)$ $(= \mathcal{O}_{\mathcal{A}}^{\mathcal{B}}\,(u))$ of all $x \in \Omega$ having a neighborhood V on which u is associated to $f \in \mathcal{F}\,(V)$, that is:

$$\mathcal{O}_{\mathcal{A}}^{\mathcal{F}}\,(u) = \left\{ x \in \Omega \mid \exists V \in \mathcal{V}_x : u\,|_V \in \mathcal{F}_{\mathcal{A}}(V) \right\},$$

$\mathcal{V}_x$ being the set of all the neighborhoods of x. This leads to the following definition: The $\mathcal{F}$-singular support of $u \in \mathcal{A}(\Omega)$ is denoted $\mathcal{S}_{\mathcal{A}}^{\mathcal{F}}\,(u)$ and defined as

$$\mathcal{S}_{\mathcal{A}}^{\mathcal{F}}\,(u) = \Omega \backslash \mathcal{O}_{\mathcal{A}}^{\mathcal{F}}\,(u).$$

Since the support of $u \in \mathcal{A}(\Omega)$ is defined by

$$\operatorname{supp}(u) = \Omega \backslash \mathcal{O}_{\mathcal{A}}\,(u) \quad \text{with} \quad \mathcal{O}_{\mathcal{A}}\,(u) = \left\{ x \in \Omega \mid \exists V \in \mathcal{V}_x : u\,|_V = 0 \right\},$$

it is clear that $\mathcal{S}_{\mathcal{A}}^{\mathcal{F}}\,(u)$ is a closed subset containing $\operatorname{supp}(u)$.

2.6.4 Some results

We can directly deduce the algebraic properties of $\mathcal{S}_{\mathcal{A}}^{\mathcal{F}}(u)$ (see [14]) from subsection 2.2. For the differential ones we suppose that $\mathcal{F}$ is a sheaf of topological differential vector spaces, with continuous differentiation, admitting $\mathcal{E}$ as a subsheaf of topological differential algebras. Then the presheaf $\mathcal{A}$ is also a presheaf of differential algebras with, for any $\alpha \in \mathbb{N}^n$ and $u \in \mathcal{A}(\Omega)$,

$$\partial^\alpha u = [\partial^\alpha u_\lambda], \text{ where } (u_\lambda)_\lambda \text{ is any representative of } u.$$

The independence of $\partial^\alpha u$ on the choice of representative follows directly from the definition of $\mathcal{J}_{(I_A,\mathcal{E},\mathcal{P})}$.) The behaviour of $\mathcal{S}_{\mathcal{A}}^{\mathcal{F}}(u)$ under differential operations is linked to the following

Proposition 13 *Under the above hypothesis ∂^α is a $\mathcal{F}_\mathcal{A}$-compatible presheaf operator of $\mathcal{A}(\Omega)$.*

Proof. Let V be any open set of Ω. We have

$$\mathcal{F}_\mathcal{A}(V) = \left\{ v \in \mathcal{A}(V) \mid \exists (v_\lambda)_\lambda \in v, \ \exists f \in \mathcal{F}(V) \ : \ \lim_{\Lambda} {}_{\mathcal{F}(V)} \, v_\lambda = f \right\}.$$

Le be $v \in \mathcal{F}_\mathcal{A}(V)$ Then $\partial^\alpha v$ has a representative $\partial^\alpha v_\lambda$ verifying

$$\lim_{\Lambda} {}_{\mathcal{F}(V)} \, \partial^\alpha v_\lambda = \partial^\alpha f \in \mathcal{F}(V)$$

and $\partial^\alpha v$ is in $\mathcal{F}_\mathcal{A}(V)$. ■

This result permits to obtain in the following subsection all the expected results on the propagation of local $\mathcal{F}_\mathcal{A}$-singularities under differential operations.

Example 8 *Taking $\mathcal{E} = \mathrm{C}^\infty$; $\mathcal{F} = \mathcal{D}'$; $\mathcal{A} = \mathcal{G}$ leads to the $\mathcal{D}'$-singular support of an element of the Colombeau algebra. this notion is complementary to the usual concept of local association in the Colombeau sense. We refer the reader to [28, 29] for more details.*

Example 9 *Those following examples are considered for $X = \mathbb{R}^d$, $\mathcal{E} = \mathrm{C}^\infty$ and $\mathcal{A} = \mathcal{G}$.*

(i) Take $u \in \sigma_\Omega(\mathrm{C}^\infty(\Omega))$, where $\sigma_\Omega : \mathrm{C}^\infty(\Omega) \to \mathcal{G}(\Omega)$ is the well known canonical embedding. Then $\mathcal{S}_{\mathcal{G}}^{\mathrm{C}^p}(u) = \emptyset$, for all $p \in \overline{\mathbb{N}}$.

(ii) Take $\varphi \in \mathcal{D}(\mathbb{R})$, with $\int \varphi(x)\, \mathrm{d}x = 1$, and set $\varphi_\varepsilon(x) = \varepsilon^{-1}\varphi(x/\varepsilon)$. As $\varphi_\varepsilon \underset{\mathcal{D}'(\mathbb{R})}{\overset{\varepsilon \to 0}{\longrightarrow}} \delta$, we have: $\mathcal{S}_{\mathcal{G}}^{\mathcal{D}'}([\varphi_\varepsilon]) = \{0\}$. We note also that $\mathcal{S}_{\mathcal{G}}^{\mathrm{C}^p}([\varphi_\varepsilon]) = \{0\}$. (Indeed, for any $K \Subset \mathbb{R}^* = \mathbb{R}\backslash\{0\}$ and ε small enough, φ_ε is null on K and, therefore, $\varphi_\varepsilon \underset{\mathrm{C}^\infty(\mathbb{R}^*)}{\overset{\varepsilon \to 0}{\longrightarrow}} = 0$.)

(iii) Take $u = [u_\varepsilon]$ with $u_\varepsilon(x) = \varepsilon \sin(x/\varepsilon)$. We have, for all $K \Subset \mathbb{R}$, $\lim p_{K,0}(u_\varepsilon) = 0$, for all $K \Subset \mathbb{R}$, whereas $\lim p_{K,1}(u_\varepsilon)$ does not exists, for $l \geq 1$. Therefore

$$\mathcal{S}_{\mathcal{G}}^{\mathrm{C}^0}(u) = \emptyset, \quad \mathcal{S}_{\mathcal{G}}^{\mathrm{C}^1}(u) = \mathbb{R}.$$

Remark that we have, for any $(p,q) \in \overline{\mathbb{N}}^2$, with $p \leq q$, and $u \in \mathcal{G}$, $\mathcal{S}_{\mathcal{G}}^{\mathrm{C}^p}(u) \subset \mathcal{S}_{\mathcal{G}}^{\mathrm{C}^q}(u)$.

Example 10 *:* **$\mathcal{D}'_{3\sigma-1}$-local analysis in $\mathcal{G}^\sigma(\Omega)$**

In [21] Gramchev proves the embedding of some spaces of ultradistributions in $\mathcal{G}(\Omega)$. In [6] Benmeriem and Bouzar prove the imbedding of $E'_{3\sigma-1}(\Omega)$ (the Gevrey ultradistibutions with compact support and $3\sigma - 1$ indice) into $\mathcal{G}^\sigma(\Omega)$ (which is in fact in relation with the index $2\sigma - 1$). The imbeding of $\mathcal{D}'_{3\sigma-1}(\Omega)$ (the Gevrey ultradistibution of $3\sigma - 1$ indice) into $\mathcal{G}^\sigma(\Omega)$ is *also proved. However if* $u \in \mathcal{G}^{\sigma}(\Omega)$ *it is possible to define an association with an ultradistribution (for example of* $\mathcal{D}'_{3\sigma-1}(\Omega)$*) in the following way: for* $T \in \mathcal{D}'_{3\sigma-1}(\Omega)$ *and* $[u_\varepsilon] = u \in \mathcal{G}^{\sigma}(\Omega)$ *we set*

$$u \sim T \Longleftrightarrow \lim_{\varepsilon \underset{\mathcal{D}'_{3s-1}}{\rightarrow} 0} u_\varepsilon = T.$$

It suffices to verify that $\mathcal{N}_{\mathcal{G}^{\sigma}}^{\mathcal{D}'_{3\sigma-1}}(\Omega) = \left\{ (u_\varepsilon)_\varepsilon \in \mathcal{E}_m^{\sigma}(\Omega), \lim_{\varepsilon \underset{\mathcal{D}'_{3\sigma-1}}{\rightarrow} 0} u_\varepsilon = 0 \right\}$ *contains* $\mathcal{N}^{\sigma}(\Omega)$ *to ensure that the previous definition don't depends upon the representative* $(u_\varepsilon)_\varepsilon$ *of* $u \in \mathcal{G}^{\sigma}(\Omega)$*. The subspace of* $\mathcal{G}^{s}(\Omega)$*, the Gevrey generalized ultradistributions associated to ultradistributions* $\mathcal{D}'_{3\sigma-1}(\Omega)$ *is*

$$\mathcal{D}'_{3\sigma-1,\mathcal{G}^{\sigma}}(\Omega) = \left\{ u = [u_\varepsilon] \in \mathcal{G}^{\sigma}(\Omega), \exists T \in \mathcal{D}'_{3\sigma-1}(\Omega) \ \lim_{\varepsilon \underset{\mathcal{D}'_{3\sigma-1}}{\rightarrow} 0} u_\varepsilon = T \right\}$$

$\mathcal{D}'_{3\sigma-1,\mathcal{G}^{\sigma}}(\Omega)$ *is well defined because the limit doesn't depend on the representative of* u.

One can consider $\mathcal{O}_{\mathcal{G}^{\sigma}}^{\mathcal{D}'_{3\sigma-1}}(u)$ *"the set of all* x *in the neighbourhood of which* u *is associated to an ultradistribution" :*

$$\mathcal{O}_{\mathcal{G}^{\sigma}}^{\mathcal{D}'_{3\sigma-1}}(u) = \left\{ x \in \Omega / \exists V \in \mathcal{V}(x) : u\mid_V \in \mathcal{D}'_{3\sigma-1}(\Omega) \right\}$$

where $\mathcal{V}(x)$ *is the set of all the open neighbourhoods of* x*. The* $\mathcal{D}'_{3s-1}$*-asymptotic singular support of* $u \in \mathcal{G}^{s}(\Omega)$ *is obtained by taking* $\mathcal{A} = \mathcal{G}^{\sigma}$ *and* $\mathcal{B} = \mathcal{D}'_{3\sigma-1}$

$$S_{\mathcal{G}^{\sigma}}^{\mathcal{D}'_{3\sigma-1}}(u) = \Omega \backslash \mathcal{O}_{\mathcal{G}^{\sigma}}^{\mathcal{D}'_{3\sigma-1}}(u).$$

2.7 $\mathcal{B}$-compatibility of differential or pseudo-differential operators

Here is a list of particular cases of subpresheaf $\mathcal{B}$ of presheaf $\mathcal{A}$ of interest to us:

$$\begin{array}{lllllllllll} \mathcal{B}: & \mathrm{C}^{\infty} & \mathrm{C}^{L} & \mathcal{G}^{L} & \mathcal{G}^{\infty} & \mathcal{G}^{\mathcal{R}} & \mathcal{G}^{\sigma,\infty} & \mathcal{G}^{r,\mathcal{R}} & \mathcal{G}^{r,\mathcal{R},L} & \mathcal{G} & \mathcal{F}_{\mathcal{A}} \\ \mathcal{A}: & \mathcal{D}' & \mathcal{D}' & \mathcal{G} & \mathcal{G} & \mathcal{G} & \mathcal{G}^{\sigma} & \mathcal{G}^{r} & \mathcal{G}^{r} & \mathcal{L}\left(\mathcal{G}_c(\Omega), \widetilde{\mathbb{C}}\right) & \mathcal{A} \end{array}$$

$\mathcal{B}$ is always a presheaf (or a sheaf) of differential algebras and $\mathcal{A}$ is a differential $\mathcal{B}$-module with a differentiation ∂^{α} ($\alpha \in \mathfrak{A} = \mathbb{N}^n$) extending the $\mathcal{B}$-one. Then in each case and each open set V in Ω (open set of $X = \mathbb{R}^n$) it is easy to prove that $\partial^{\alpha} v$ maps $\mathcal{B}(V)$ into itself. Thus ∂^{α} is a presheaf operator $\mathcal{B}$-compatible in $\mathcal{A}(\Omega)$ according to Definition 6. If we give now a family $(b_\alpha)_{\alpha \in \mathbb{N}^n}$ of elements in $\mathcal{B}(\Omega)$, then, $P(\partial) = \sum_{|\alpha| \leq m} b_\alpha \partial^{\alpha}$ is a $\mathcal{B}$-compatible operator in $\mathcal{A}(\Omega)$ from Corollary 7.

Moreover at least in some cases, when $\mathcal{A}$ is $\mathcal{D}'$, (resp.$\mathcal{G}$, $\mathcal{L}\left(\mathcal{G}_c(\Omega), \widetilde{\mathbb{C}}\right)$), a *pseudo-differential operator* A can be defined in $\mathcal{A}(\Omega)$. Setting $b(x,\xi) = (2\pi)^{-n} \sum_{|\alpha| \leq m} b_\alpha(x)(i\xi)^{\alpha}$ when b_α belongs to $\mathrm{C}^{\infty}(\Omega)$, the differential operator $P(\partial) = \sum_{|\alpha| \leq m} b_\alpha \partial^{\alpha}$ verifying the formula

$$P(\partial)u(x) = \int_{\mathbb{R}^n} e^{ix\xi} b(x,\xi) \widehat{u}(\xi) d\xi = \int\int_{\Omega \times \mathbb{R}^n} e^{i\langle (x-y),\xi \rangle} b(x,\xi) u(y) dy d\xi$$

maps $\mathcal{D}(\Omega)$ into $\mathrm{C}^{\infty}(\Omega)$. Via the theory of oscillatory integrals, the above formula can be extended into

$$Au(x) = \int\int_{\mathbb{R}^n} e^{i\langle (x-y),\xi \rangle} a(x,y,\xi) u(y) dy d\xi$$

which defines a pseudo-differential operator A [23] when $a = a(x, y, \xi)$ in $S^m_{\rho,\delta}(\Omega \times \Omega \times \mathbb{R}^n)$ of Hörmander symbols of order m and type (ρ, δ). A extends continuously to a map $\mathcal{E}'(\Omega) \to \mathcal{D}'(\Omega)$.

We can find (for example in [11]) a definition of generalized pseudo-differential operators with generalized symbols extending the classical one. The set $\widetilde{\mathcal{S}}^m_{\rho,\delta}(\Omega \times \mathbb{R}^n)$ of generalized symbols can be described [17] as the algebra $\mathcal{G}_{\mathcal{S}^m_{\rho,\delta}}(\Omega \times \mathbb{R}^n)$ based on $\mathcal{S}^m_{\rho,\delta}(\Omega \times \mathbb{R}^n)$ and obtained as a $\mathcal{G}_E$-module by choosing $E = \mathcal{S}^m_{\rho,\delta}(\Omega \times \mathbb{R}^n)$. Then, the pseudo differential operator with generalized symbols $b \in \widetilde{\mathcal{S}}^m_{\rho,\delta}(\Omega \times \mathbb{R}^n)$ is the map $\mathcal{G}_c(\Omega) \to \mathcal{G}(\Omega)$ given by

$$Au := \int_{\mathbb{R}^n} e^{ix\xi} b(x,\xi)\widehat{u}(\xi)d\xi := \left[\int_{\mathbb{R}^n} e^{ix\xi} b_\varepsilon(x,\xi)\widehat{u}_\varepsilon(\xi)d\xi\right]$$

One can define more generally pseudo differential operators by means of symbols in $\widetilde{\mathcal{S}}^m_{\rho,\delta}(\Omega \times \Omega \times \mathbb{R}^n)$ and generalized oscillatory integrals (see [15]). Such an operator A is given by

$$Au := \int\int_{\Omega\times\mathbb{R}^n} e^{i\langle(x-y),\xi\rangle} a(x,y,\xi)u(y)dyd\xi := \left[\int\int_{\Omega\times\mathbb{R}^n} e^{i\langle(x-y),\xi\rangle} a_\varepsilon(x,y,\xi)u_\varepsilon(y)dyd\xi\right]$$

which defines a generalized function in $\mathcal{G}(\Omega)$ when u is in $\mathcal{G}_c(\Omega)$.

We can find in ([17], def. 2.5) an extension of the action of A to the dual $\mathcal{L}\left(\mathcal{G}_c(\Omega), \widetilde{\mathbb{C}}\right)$ namely

$$AT(u) = T\left({}^tAu\right), u \in \mathcal{G}_c(\Omega)$$

where ${}^tA$ (the transpose of A) is the pseudo-differential operator defined by

$${}^tAu := \int\int_{\Omega\times\mathbb{R}^n} e^{i\langle(x-y),\xi\rangle} a(x,y,-\xi)u(y)dyd\xi.$$

When $\mathcal{A}$ is $\mathcal{D}'$, (resp.$\mathcal{G}$, $\mathcal{L}\left(\mathcal{G}_c(\Omega), \widetilde{\mathbb{C}}\right)$), it is proved in each case that if A *is a properly supported pseudo-differential operator,* it maps $\mathcal{A}(\Omega)$ into itself. Moreover, when $\mathcal{B}$ is C^∞, (resp.$\mathcal{G}^\infty$, $\mathcal{G}$), for each open set V in Ω, A maps $\mathcal{B}(V)$ into itself. In other words A is a $\mathcal{B}$-compatible operator in $\mathcal{A}(\Omega)$.

Therefore proposition 4 allows to deduce the classical inclusions

$$\mathcal{S}^{\mathcal{B}}_{\mathcal{A}}(P(\partial)u) \subset \mathcal{S}^{B}_{\mathcal{A}}(u)$$
$$\text{or } \mathcal{S}^{\mathcal{B}}_{\mathcal{A}}(Au) \subset \mathcal{S}^{B}_{\mathcal{A}}(u)$$

from the presheaf property of an operator $\mathcal{B}$-compatible.

Through Proposition 9, we can even obtain some non linear results, when $\mathcal{A}$ is a presheaf of algebras as

$$\mathcal{S}^{\mathcal{B}}_{\mathcal{A}}\left(\sum_{|\alpha|\le m} b_\alpha\left(\partial^\alpha u\right)^{p_\alpha}\right) \subset \mathcal{S}^{B}_{\mathcal{A}}(u)$$
$$\text{or } \mathcal{S}^{\mathcal{B}}_{\mathcal{A}}(Au)^p \subset \mathcal{S}^{B}_{\mathcal{A}}(u)$$

where p is any positive integer and $(p_\alpha)_{\alpha\in\mathbb{N}^n}$ any given family of positive integers.

Remark 6 *In Definition 1.3 of [17], a functional* $T \in \mathcal{L}\left(\mathcal{G}_c(\Omega), \widetilde{\mathbb{C}}\right)$ *is said to be basic if it is of the form* $\langle T, u\rangle = [\langle T_\varepsilon, u_\varepsilon\rangle]$ *where* $(T_\varepsilon)_\varepsilon$ *is a net of distributions in* $\mathcal{D}'(\Omega)$ *satisfying the following condition*

$$\forall K \Subset \Omega\ \exists j \in \mathbb{N}\ \exists \eta \in (0,1]\ \ \forall u \in \mathcal{D}_K(\Omega)\ \ \forall\varepsilon \in (0,\eta]\ |T_\varepsilon(u)| \le \sup_{x\in K,\ |\alpha|\le j} |\partial^\alpha u(x)|.$$

When T is a basic functional in $\mathcal{L}\left(\mathcal{G}_c(\Omega),\widetilde{\mathbb{C}}\right)$, Theorem 2.10 proves that the inclusion

$$singsupp_{\mathcal{G}}\,(AT) \subset singsupp_{\mathcal{G}}\,(T)$$

is valid for any pseudo differential operator A with amplitude in $\widetilde{\mathcal{S}}^m_{\rho,\delta}\left(\Omega\times\Omega\times\mathbb{R}^n\right)$ and a fortiori when A is properly supported. In this case our above remark on the presheaf property of an operator $\mathcal{B}$-compatible shows that this result remains valid for any functional in $\mathcal{L}\left(\mathcal{G}_c\left(\Omega\right),\widetilde{\mathbb{C}}\right)$.

3 The frequential microlocal analysis

After a short overview of classical results in distribution theory and propagation of singularities under linear and nonlinear operators, we will give a characterisation of the local regularity in the two more general cases of generalized functions which summarize the other ones. This leads to the definition of corresponding wave front sets which gives a general frequential microanalysis of generalized singularities.

3.1 Microlocal analysis in distribution spaces

3.1.1 Wave front set and microlocal regularity of product

As it is recalled in Introduction it follows from Lemma 8.1.1. in [22] that if Ω is an open set in $\mathbb{R}^n$ and $u \in \mathcal{D}'\left(\Omega\right)$, one can set: $\Sigma_x(u) = \underset{\Phi}{\cap}\Sigma\left(\Phi u\right);\, \Phi \in \mathcal{D}\left(\Omega\right),\, \Phi\left(x\right) \neq 0$ and define the wave front set of u as

$$WF(u) = \left\{(x,\xi) \in \Omega\times\left(\mathbb{R}^n\backslash 0\right); \xi \in \Sigma_x(u)\right\}.$$

Then, if $u \in \mathcal{D}'\left(\Omega\right)$ and $(x,\xi) \in \Omega\times\left(\mathbb{R}^n\backslash 0\right)$, u is said to be in $H^{loc}_{(s)}$ at (x,ξ) if $(x,\xi) \notin WF(u-v)$ for some $v \in H_{(s)}\left(\mathbb{R}^n\right)$. The microlocal regularity of products is proved in

Theorem (8.3.3 in [24]): *Let $u_j \in H_{(s_j)}\left(\mathbb{R}^n\right)$, $j = 1, 2.$ Then*

(i) $u_1u_2 \in H^{loc}_{(s_2)}$ outside $WF(u_1)$ if $s_1 > n/2$ and $s_1 + s_2 > n/2$.

(ii) $u_1u_2 \in H^{loc}_{(s)}$ outside $WF(u_1)$ if $s_1 < n/2$ and $s_1 + s_2 - n/2 > s \geq 0$.

(iii) $u_1u_2 \in H^{loc}_{(s_1+s_2-n/2)}$ outside $WF(u_1) \cup WF(u_2)$ if $s_1 + s_2 > 0$.

3.1.2 Pseudo-differential operators

We recall that the space $\mathcal{S}^m_{\rho,\delta}\left(\mathbb{R}^n\times\mathbb{R}^n\right)$ of symbols of order m and type (ϱ,δ) consist in all $a \in \mathrm{C}^\infty\left(\mathbb{R}^n\times\mathbb{R}^n\right)$ such that

$$\left|\partial_x^\beta\partial_\xi^\alpha a(x,\xi)\right| \leq C_{\alpha\beta}(1+|\xi|^{m-\varrho|\alpha|+\delta|\beta|}$$

with $0 \leq \delta < \rho \leq 1$. We consider here the simplified case $\mathcal{S}^m = \mathcal{S}^m_{1,0}$ which defines a pseudo-differential operator (belonging to $Op\mathcal{S}^m$)

$$a\left(x,D\right)u\left(x\right) = \textstyle\int e^{i\langle x,\xi\rangle}a\left(x,\xi\right)\hat{u}\left(\xi\right)d\xi$$

which maps continuously $\mathcal{S}\left(\mathbb{R}^n\right)$ into $L^\infty\left(\mathbb{R}^n\right)\cap\mathrm{C}\left(\mathbb{R}^n\right)$. It extends for every $s \in \mathbb{R}$ to a continuous map $H_{(s)}\left(\mathbb{R}^n\right) \to H_{(s-m)}\left(\mathbb{R}^n\right)$.

If Ω is open then a continuous linear map $A : \mathcal{D}\left(\Omega\right) \to \mathrm{C}^\infty\left(\Omega\right)$ is said to be a pseudo-differential operator of order m in Ω (an element of $\Psi^m\left(\Omega\right)$) if for any $\varphi,\psi \in \mathcal{D}\left(\Omega\right)$ the operator $\mathcal{S}\left(\mathbb{R}^n\right) \ni u \mapsto \varphi A\left(\psi u\right)$ is in $Op\mathcal{S}^m$. If $u \in \mathcal{E}'\left(\Omega\right)$ and $A \in \Psi^m\left(\Omega\right)$, then $Au \in \mathcal{D}'\left(\Omega\right)$ is well defined and leads to the inclusion

$$\text{sing supp}Au \subset \text{sing supp}u$$

which is the projection of the microlocal property

$$WF\,(Au) \subset WF\,(u)$$

when A is properly supported and u belongs to $\mathcal{D}'\,(\Omega)$.

The propagation of non characteristic regularity for semi-linear equations studied in [35] is given by

Theorem (8.4.13 in [24]): *Let $u \in H^{loc}_{(s+k)}\,(\Omega)$, where Ω is an open set in R^n and $s > n/2$, be a solution of the semi-linear equation*

$$P(x, D)u = f\,(x, J_k u)$$

where $J_k u = (\partial^\alpha u)_{|\alpha \leq k|}$, f and the coefficients of P are C^∞ and k is smaller than the order m of $P(x, D)$. If P is noncharacteristic at $(x, \xi) \in \Omega \times (\mathbb{R}^n \backslash 0)$, it follows that $u \in H^{loc}_{(2s+m-n/2)}$ at (x, ξ).

3.1.3 Application of paradifferential calculus

The paradifferentiel calculus of Bony [4] is based on some regularization of non smooth symbols. We don't intend to develop this theory here but look at it as a powerfull tool to prove good results for nonlinear problems such as the following

Theorem (10.3.6 in [24]): *Let $u \in H^{loc}_{(s+m-1/2)}\,(\Omega)$, $s > \max((n-1)\,/2\ ,\, n/4)$, and assume that u verifies the quasilinear differential equation*

$$\sum_{|\alpha|=m} a_\alpha(x, J_{m-1}u(x))\partial^\alpha u + c(x, J_{m-1}u(x)) =$$

where a_α and c are C^∞. Then it follows that $u \in H^{loc}_{(2s+m-n/2)}$ at every non characteristic point (x, ξ).

3.1.4 Propagation of singularities

Roughly speaking we know that under some conditions for linear or pseudo-differential equations, the singularities of solutions propagate along bicharacteristics This remains valid for nonlinear equations in the sense of

Theorem (11.4.1 in [24]: *Let $u \in H^{loc}_{(s+m)}\,(\Omega)\,, s > n/2 + 1$,be a real valued solution of the differential equation*

$$F(x, J_m u(x)) = 0$$

where $F \in \mathrm{C}^\infty$. If $\sigma \leq 2s - n/2$, then the set of $(x, \eta) \in \Omega \times (\mathbb{R}^n \backslash 0)$ where $u \notin H^{loc}_{(\sigma+m-1)}$ is contained in the characteristic set and it is invariant under the Hamilton flow defined by the principal symbol of the linearized equation.

Beals [2, 3] has studied the case of second order hyperbolic equations (extended by [5] to arbitrary order) for which we can give a special version as

Theorem (11.5.10 in [24]: *Let $u \in H^{loc}_{(s)}\,(\Omega)\,, s > n/2$ be a solution of of the hyperbolic second order semi-linear equation*

$$P(x, \partial) = f(x, u)$$

where $f \in \mathrm{C}^\infty$. If $u \in H^{loc}_{(s)}$ at a characteristic point (x, ξ) and if $s \leq \sigma < 3s - n + 1$,it follows that $u \in H^{loc}_{(s)}$ at the bicharacteristic γ through (x, ξ).

3.2 The frequential microlocal analysis in $\mathcal{G}^r$

We refer the reader to [12, 13] for more details.

3.2.1 Characterization of $\mathcal{G}^{r,\mathcal{R}}$-local regularity

We consider an open subset $\Omega$of $\mathbb{R}^d$ and the Schwartz space $\mathcal{S}(\Omega)$ of rapidly decreasing functions defined on Ω, endowed with the family of seminorms $\mathcal{Q}(\Omega) = (\mu_{q,\alpha})_{(q,\alpha)\in\mathbb{N}\times\mathbb{N}^d}$ defined by

$$\mu_{q,\alpha}(f) = \sup_{x\in\Omega} (1+|x|)^q \left|\partial^\alpha f(x)\right|.$$

The space of "rough"rapidly decreasing functions can be defined as

$$\mathcal{S}_*(\Omega) = \left\{ f \in \mathrm{C}^\infty(\Omega) \mid \forall q \in \mathbb{N},\ \ \mu_{q,0}(f) < +\infty \right\}.$$

In order to make easier the comparison between the distributional case and the generalized case, we begin by recalling the classical theorem and complete it by some equivalent statements given in the following result (Theorem 16 in [12]): for u in $\mathcal{E}'(\mathbb{R}^n)$, the following equivalences hold:

$$\begin{aligned}
(i)\ \ u \in \mathrm{C}^\infty(\mathbb{R}^n) \ \ &\Leftrightarrow (ii) \ \ \ \mathcal{F}(u) \in \mathcal{S}(\mathbb{R}^n) \\
&\Leftrightarrow (iii) \ \ \mathcal{F}(u) \in \mathcal{S}_*(\mathbb{R}^n) \\
&\Leftrightarrow (iv) \ \ \ \mathcal{F}(u) \in \mathcal{O}'_M(\mathbb{R}^n) \\
&\Leftrightarrow (v) \ \ \ \ \mathcal{F}(u) \in \mathcal{O}'_C(\mathbb{R}^n).
\end{aligned}$$

where $\mathcal{F}$ is the classical Fourier transform defined as topological automorphism of $\mathcal{S}\left(\mathbb{R}^d\right)$. The result is based on the following inclusions

$$\begin{gathered}
\mathcal{S}(\mathbb{R}^n) \subset \mathcal{S}_*(\mathbb{R}^n) \subset \mathcal{O}'_M(\mathbb{R}^n) \subset \mathcal{O}'_C(\mathbb{R}^n), \\
\mathcal{F}\left(\mathcal{E}'(\mathbb{R}^n)\right) \subset \mathcal{O}_C(\mathbb{R}^n); \mathcal{O}_C(\mathbb{R}^n) \cap \mathcal{O}'_C(\mathbb{R}^n) = \mathcal{S}(\mathbb{R}^n).
\end{gathered}$$

The Fourier transform has been extended to some spaces of rapidly decreasing generalized functions (like $\mathcal{G}_s(\mathbb{R}^n) = \mathcal{X}_\mathcal{S}(\mathbb{R}^n)/\mathcal{N}_\mathcal{S}(\mathbb{R}^n)$) and more completely described in [12] in the framework of $\mathcal{R}$-regular spaces. We can point out that in any framework, the elements with compact support have always a Fourier transform.

Definition 12 *Let $\mathcal{R}$ be a regular subset of $\mathbb{R}_+^{\mathbb{N}}$ and Ω an open subset or $\mathbb{R}^n$. Set*

$$\begin{aligned}
\mathcal{X}_{\mathcal{S}_*}^{r,\mathcal{R}}(\Omega) &= \left\{ (f_\varepsilon)_\varepsilon \in \mathcal{S}_*(\Omega)^\Lambda \ \middle|\ \exists N \in \mathcal{R},\ \forall q \in \mathbb{N},\ \ \mu_{q,0}(f_\varepsilon) = \mathrm{O}\left(r_\lambda^{-N(q)}\right) \text{ as } \lambda \to 0 \right\}, \\
\mathcal{N}_{\mathcal{S}_*}^{r}(\Omega) &= \left\{ (f_\varepsilon)_\varepsilon \in \mathcal{S}_*(\Omega)^\Lambda \ \middle|\ \forall N \in \mathbb{R}_+^{\mathbb{N}},\ \forall q \in \mathbb{N},\ \ \mu_{q,0}(f_\varepsilon) = \mathrm{O}\left(r_\lambda^{N(q)}\right) \text{ as } \lambda \to 0 \right\}.
\end{aligned}$$

One can show that $\mathcal{X}_{\mathcal{S}_*}^{r,\mathcal{R}}(\Omega)$ is a subalgebra of $\mathcal{S}_*(\Omega)^{(0,1]}$ and that $\mathcal{N}_{\mathcal{S}_*}^{r}(\Omega)$ is an ideal of $\mathcal{X}_{\mathcal{S}_*}^{r,\mathcal{R}}(\Omega)$. (The proof is similar to that of Proposition 1 in [12]).

Definition 13 *The algebra $\mathcal{G}_{s_*}^{r,\mathcal{R}}(\Omega) = \mathcal{X}_{\mathcal{S}_*}^{r,\mathcal{R}}(\Omega)/\mathcal{N}_{\mathcal{S}_*}^{r}(\Omega)$ is called the algebra of $(r,\mathcal{R})$-regular rough rapidly decreasing generalized functions.*

Theorem 14 *Let $x_0 \in \Omega \subset \mathbb{R}^n$ and $u \in \mathcal{G}^r(\Omega)$. Then, u is $\mathcal{G}^{r,L}$ at x_0 (in the sense of definition 2) iff there exist some neighbourhood W of x_0, some $\varphi \in \mathcal{D}(W)$, $\varphi(x_0) \neq 0$, such that $\widehat{\varphi u} \in \mathcal{G}_{s_*}^{r,\mathcal{R}}(\mathbb{R}^n)$*

Proof. (Sketch). Let u be an element in $\mathcal{G}^{r}(\Omega)$ $\mathcal{G}^{r,\mathcal{R}}$ at x_0. There exists a neighbourhood W of x_0 such that $u|_W \in \mathcal{G}^{r,\mathcal{R}}(W)$. We can extend any given $\varphi \in \mathcal{D}(W)$, $\varphi(x_0) \neq 0$, into $\widetilde{\varphi} \in \mathcal{D}(\mathbb{R}^d)$ such that $\widetilde{\varphi}u \in \mathcal{G}_c^{r,\mathcal{R}}(\mathbb{R}^d) = \mathcal{G}_c(\mathbb{R}^n) \cap \mathcal{G}^{r,\mathcal{R}}(\mathbb{R}^n)$. We follow the arguments of Theorem 22 in [12], replacing the ε-estimates by the r_λ-ones. This leads to prove that $\widehat{\widetilde{\varphi}u} \in \mathcal{G}_{s_*}^{r,\mathcal{R}}(\mathbb{R}^n)$. Conversely, this last assertion with the above hypothesis permits to prove that $\widetilde{\varphi}u \in \mathcal{G}_c^{r,\mathcal{R}}(\mathbb{R}^n)$, and then there exits a neighbourhood V of x_0 such that $u|_V \in \mathcal{G}^{r,\mathcal{R}}(V)$. In this last part one needs to define an inverse Fourier transform $\mathcal{F}^{-1}$ in $\mathcal{G}_{s_*}^{r,\mathcal{R}}(\mathbb{R}^n)$ for which one introduces the space $\mathcal{G}_{\mathcal{B}}^{r,\mathcal{R}}(\mathbb{R}^n)$ of $(r,\mathcal{R})$-regular bounded generalized functions such that $\mathcal{F}^{-1}\left(\mathcal{G}_{s_*}^{r,\mathcal{R}}(\mathbb{R}^n)\right) \subset \mathcal{G}_{\mathcal{B}}^{r,\mathcal{R}}(\mathbb{R}^n)$. The result follows from the equality $\mathcal{G}_{\mathcal{B}}^{r,\mathcal{R}}(\mathbb{R}^n) \cap \mathcal{G}_c(\mathbb{R}^n) = \mathcal{G}^{r,\mathcal{R}}(\mathbb{R}^n) \cap \mathcal{G}_c(\mathbb{R}^n)$. ■

3.2.2 The $\mathcal{G}^{r,\mathcal{R}}$-generalized wave front set

Definition 14 *An element $u \in \mathcal{G}^{r}(\Omega)$ is said to be microlocally $(r,\mathcal{R})$-regular at $(x_0,\xi_0) \in \Omega \times (\mathbb{R}^d \backslash 0)$ (we set: $u \in \mathcal{G}^{r,\mathcal{R}}(x_0,\xi_0)$) if there exist some neighbourhood W of x_0, some $\varphi \in \mathcal{D}(W)$, $\varphi(x_0) \neq 0$, some conic neighborhood Γ of ξ_0 such that $\widehat{\varphi u} \in \mathcal{G}_{s_*}^{r,\mathcal{R}}(\Gamma)$.*

Definition 15 *The $\mathcal{G}^{r,\mathcal{R}}$-generalized wave front set of $u \in \mathcal{G}^{r}(\Omega)$, denoted by $WF^{(r,\mathcal{R})}(u)$ is the complement in $\Omega \times (\mathbb{R}^n \backslash 0)$ of the set of all pairs (x_0,ξ_0) such that u is microlocally $(r,\mathcal{R})$-regular at (x_0,ξ_0).*

Theorem 15 *The projection of $WF^{(r,\mathcal{R})}(u)$ in Ω is equal to $sing\,supp^{(r,\mathcal{R})}(u)$.*

The proof follows from the arguments involved in [22] using lemma 8.1.1.

Example 11 *taking $\lambda = \varepsilon \in \left]0,1\right]$, $r_\varepsilon = \varepsilon$ and $\mathcal{R} = \mathcal{B}o$ (the set of bounded sequences), we obtain the $\mathcal{G}^\infty$ microlocal analysis of elements in $\mathcal{G}$ [31, 37].*

taking $\lambda = \varepsilon \in \left]0,1\right]$ and $r_\varepsilon = \varepsilon$, we obtain for any $\mathcal{R}$ the $\mathcal{G}^{\mathcal{R}}$ microlocal analysis of elements in $\mathcal{G}$ [12].

3.2.3 Characterization of $\mathcal{G}^{r,\mathcal{R},L}$-local regularity

When starting from previous cases (like $\mathcal{G}$, $\mathcal{G}^{\mathcal{R}}$ or $\mathcal{G}^{L}$) the problem is to change simultaneously the asymptotic scale into a new one, and the $\mathcal{G}^\infty$-regularity subordinated to L-conditions into $\mathcal{G}^{\mathcal{R}}$-regularity subordinated to L-conditions. To do that we have to mix carefully the techniques used in [12] and [30]. This study is done in [13]. In this subsection, we only give the definitions and results without proofs.

Theorem *Let $x_0 \in \Omega \subset R^n$ and $u \in G^r(\Omega)$. Then, u is $G^{r,\mathcal{R},L}$ at x_0 (in the sense of Definition 2) iff there exist some neighbourhood W of x_0, a compact K such $W \subset K \Subset \Omega$, a sequence of functions χ_k, each in $D_K(\Omega)$ and valued in $[0,1]$ with $\chi_k u = u$ on W, a representative $(u_\lambda)_\lambda$ of u, a regular sequence $N \in R$, a positive constant c, and $\lambda_0 \in \Lambda$ such that for all $\xi \in R$*

$$(*) \qquad \forall k \in \mathbb{N}, \forall \lambda \prec \lambda_0, \quad |\xi|^k \left|\widehat{u_{k,\lambda}}(\xi)\right| \leq c r_\lambda^{-N(k)} (cL_k)^k .$$

3.2.4 The $\mathcal{G}^{r,\mathcal{R},L}$-generalized wave front set

Definition 16 *An element $u \in \mathcal{G}^{r}(\Omega)$ is said to be microlocally $(r,\mathcal{R},L)$-regular at $(x_0,\xi_0) \in \Omega \times (\mathbb{R}^n \backslash 0)$ (we set: $u \in \mathcal{G}^{r,\mathcal{R},L}(x_0,\xi_0)$) if there exist a neighborhood W of x_0, a conic neighborhood Γ of ξ_0, a sequence $(u_k = \chi_k u)_{k\in\mathbb{N}}$ of generalized functions where each χ_k is valued in $[0,1]$ and is in $\mathcal{D}_K(\Omega)$, with $W \subset K \Subset \Omega$, u_k being equal to u in W, a sequence $N \in \mathcal{R}$, a positive constant c, and $\lambda_0 \in \Lambda$ such that (*) holds when $\xi \in \Gamma$.*

Definition 17 *The $\mathcal{G}^{r,\mathcal{R},L}$-generalized wave front set of $u\in \mathcal{G}^{r}(\Omega)$, denoted by $WF^{(r,\mathcal{R},L)}(u)$ is the complement in $\Omega\times(\mathbb{R}^{n}\backslash 0)$ of the set of all pairs (x_0,ξ_0) such that u is microlocally $(r,\mathcal{R},L)$-regular at (x_0,ξ_0).*

$WF^{r,\mathcal{R},L}(u)$ is a closed subset of $\Omega\times(\mathbb{R}^{n}\backslash 0)$, and its projection in Ω is given by the following result:
Theorem The projection of $WF^{(r,\mathcal{R},L)}(u)$ in Ω is equal to $sing\,supp^{(r,\mathcal{R},L)}(u)$.

Example 12 *Taking $\lambda=\varepsilon\in\left]0,1\right]$, $r_{\varepsilon}=\varepsilon$, $\mathcal{R}=\mathcal{B}o$ (the set of bounded sequences), we obtain for any L the $\mathcal{G}^{L}$ microlocal analysis of elements in $\mathcal{G}$ [30].*

Taking $\lambda=\varepsilon\in\left]0,1\right]$, $r_{\varepsilon}=\varepsilon$, $\mathcal{R}=\mathcal{B}o$ and $L_k=k+1$, we get the $\mathcal{G}^{A}$ microlocal analysis of elements in $\mathcal{G}$ [34].

Taking $\lambda=\varepsilon\in\left]0,1\right]$, $r_{\varepsilon}=e^{\varepsilon^{-\frac{1}{2s-1}}}$, $\mathcal{R}=\mathcal{B}o$ and $L_k=(k+1)^{s}$, we obtain the $\mathcal{G}^{s,\infty}$ micolocal analysis of elements in $\mathcal{G}^{s}$ [6].

3.2.5 Propagation of singularities under differential (or pseudo-differential) operators

a) We can summarize the first investigations in the following results proved in [13]
Proposition *Suppose that (a,u) is in $G^{r}(\Omega)\times G^{r}(\Omega)$, we have*
(i) If $a\in G^{r,\mathcal{R}}(\Omega)$ (resp. $a\in G^{r,\mathcal{R},L}(\Omega)$), then $WF^{(r,\mathcal{R})}(au)\subset WF^{(r,\mathcal{R})}(u)$ (resp. $WF^{(r,\mathcal{R},L)}(au)\subset WF^{(r,\mathcal{R},L)}(u)$)
(ii) $WF^{(r,\mathcal{R})}(\partial^{\alpha}u)\subset WF^{(r,\mathcal{R})}(u)$ and $WF^{(r,\mathcal{R},L)}(\partial^{\alpha}u)\subset WF^{(r,\mathcal{R},L)}(u)$.
Proposition *Let $P(\partial)=\sum\limits_{|\alpha|\leq m}a_{\alpha}\partial^{\alpha}$ a differential operator in $G^{r}(\Omega)$.*
If the coefficients a_{α} lie in $G^{r,\mathcal{R}}(\Omega)$ (resp. in $G^{r,\mathcal{R},L}(\Omega)$), then we have

$$WF^{(r,\mathcal{R})}(P(\partial)u)\subset WF^{(r,\mathcal{R})}(u)\ \text{ (resp. } WF^{(r,\mathcal{R},L)}(P(\partial)u)\subset WF^{(r,\mathcal{R},L)}(u).$$

b) In the special case of $\mathcal{G}^{\infty}$ singularities of $\mathcal{G}$, we can quote the results based on pseudodifferential operators and pseudodifferential techniques. In [18] analogues of Hörmander definition of the distributional wave front set given in [23] are obtained by characterizations of generalized wave front set in terms of intersection over some non-ellipticity domains. This intersection is taken over all slow scale pseudo-differential operators $a\in\underline{\widetilde{\mathcal{S}}}^{m}_{sc}(\Omega\times\mathbb{R}^{n})$ (def. 1.1) verifying some other regularity conditions. More precisely, if $Ell_{sc}(a)$ denote the set of all $(x,\xi)\in\Omega\times T^{*}(\Omega)\backslash 0$ where a is slow scale micro-elliptic (def. 1.2), Theorem 2.1 proves that *for all $u\in G(\Omega)$*

$$WF_{g}(u)=WF_{sc}(u):=\bigcap_{\substack{a(x,D)\in\ {}_{pr}\Psi^{0}(\Omega)\\ a(x,D)u\in\mathcal{G}^{\infty}(\Omega)}} Ell_{sc}(a)^{c}$$

where ${}_{pr}\Psi^{0}(\Omega)$ denote the set of all properly supported slow scale operators of order 0.

Another pseudo-differential characterisation of $WF_{g}(u)$ is given by Theorem 2.1.1 which proves that for all $u\in G(\Omega)$

$$WF_{g}(u)=\cap Char(A)$$

where the intersection is taken over all classical properly supported classical pseudo-differential operators A such that Au belongs to $G^{\infty}(\Omega)$.

Following these characterizations some refined results on propagation of singularities can be obtained. For example, Theorem 3.1 proves that *if $A=a(x,D)$* is a properly supported pseudo-differential operator with slow scale symbol and $u\in G(\Omega)$

$$WF_{g}(Au)\subset WF_{g}(u)\subset WF_{g}(Au)\cup Ell_{sc}(a)^{c}.$$

3.3 The frequential microlocal analysis in $\mathcal{L}\left(\mathcal{G}_c(\Omega), \widetilde{\mathbb{C}}\right)$

3.3.1 The generalized wave front set $WF_{\mathcal{G}}(T)$

Inspired by the results and definitions of [18] recalled in the previous subsection, Garetto (Def. 3.3, [17]) defines the $\mathcal{G}$-wave front set of a functional $T \in \mathcal{L}\left(\mathcal{G}_c(\Omega), \widetilde{\mathbb{C}}\right)$ as

$$WF_{\mathcal{G}}(T) := \bigcap_{\substack{a(x,D) \in_{pr} \Psi^0(\Omega) \\ a(x,D)T \in \mathcal{G}(\Omega)}} Ell_{sc}(a)^c .$$

And even the $\mathcal{G}^\infty$-wave front set of T is defined in the same way by replacing $\mathcal{G}(\Omega)$ by $\mathcal{G}^\infty(\Omega)$. Proposition 3.5 shows that the projection on Ω of $WF_{\mathcal{G}}(T)$ is exactly $singsupp_{\mathcal{G}} T$.

When A is a properly supported pseudo-differential operator with symbol $a \in \widetilde{\mathcal{S}}^m_{\rho,\delta}(\Omega \times \mathbb{R}^n)$, the inclusion

$$WF_{\mathcal{G}}(AT) \subset WF_{\mathcal{G}}(T)$$

can be refined by introducting the concept of $\mathcal{G}$-microsupport of a, denoted $\mu supp_{\mathcal{G}}(a)$. It is the complement of all $(x,\xi) \in \Omega \times T^*(\Omega) \backslash 0$ where a is $\mathcal{G}$-smoothing (Def.3.6). Then we have (Corollary 3.9)

$$WF_{\mathcal{G}}(a(x,D)T) \subset WF_{\mathcal{G}}(T) \cap \mu supp_{\mathcal{G}}(a) .$$

This result is reformulated in terms of $\mathcal{G}$-microsupport of the operator A ($\mu supp_{\mathcal{G}}(A)$) in the form

$$WF_{\mathcal{G}}(AT) \subset WF_{\mathcal{G}}(T) \cap \mu supp_{\mathcal{G}}(A)$$

where the $\mathcal{G}$-microsupport of A is defined (Def.3.11) by

$$\mu supp_{\mathcal{G}}(A) := \bigcap_{\substack{a \in \widetilde{\mathcal{S}}^m_{\rho,\delta}(\Omega \times \mathbb{R}^n) \\ a(x,D)=A}} \mu supp_{\mathcal{G}}(a) .$$

3.3.2 Fourier transform characterisation of T and propagation of singularities

When $\varphi \in \mathcal{D}(\Omega)$ and $T \in \mathcal{D}'(\Omega)$, we recall that the regularity of φT can be measured by the rapid decay of its Fourier transform in some conic region $\Gamma \subset \mathbb{R}^n \backslash 0$. Following this idea, Garetto introduces the subset $\mathcal{G}_{\mathcal{S},0}(\Gamma)$ of $\mathcal{G}_\tau(\mathbb{R}^n)$ (algebra of tempered generalized functions) such that

$$\mathcal{G}_{\mathcal{S},0}(\Gamma) = \left\{ u = [u_\varepsilon] \in \mathcal{G}_\tau(\mathbb{R}^n) \;\; \forall l \in \mathbb{R} \; \exists \mathbb{N} \in \mathbb{N} \sup_{x \in \Gamma} (1+|\xi|)^l |u_\varepsilon(x)| = O\left(\varepsilon^{-N}\right) \text{ as } \varepsilon \to 0 \right\}$$

which is similar to $\mathcal{G}^{r,\mathcal{R}}_{s_*}(\Gamma)$ introduced in Definition 13.

This leads to the Fourier transform characterization of T given in

Theorem 3.15 of [18] (or 3.10 of [17]): *Let T be a basic functional in $\mathcal{L}\left(\mathcal{G}_c(\Omega), \widetilde{\mathbb{C}}\right)$. Then $(x,\xi) \notin WF_{\mathcal{G}}(T)$ if and only if there exist a conic neighbourhood of ξ and a cutoff function $\Phi \in D(\Omega)$ with $\Phi(x) = 1$ such that*

$$\mathcal{F}(\Phi T) \subset \mathcal{G}_{\mathcal{S},0}(\Gamma) .$$

Then an extension of Theorem 4.1 in [15] follows:

Theorem 4.1 in [17]: *If* $A = a(x,D)$is a properly supported pseudo-differential operator with symbol $a \in \underline{\widetilde{\mathcal{S}}}^m_{sc}(\Omega \times \mathbb{R}^n)$ and T *a basic functional in* $L\left(\mathcal{G}_c(\Omega), \widetilde{\mathbb{C}}\right)$, then

$$WF_{\mathcal{G}}(AT) \subset WF_{\mathcal{G}}(T) \subset WF_{\mathcal{G}}(AT) \cup Ell_{sc}(a)^c .$$

4 The asymptotic microlocal analysis

Let Ω be an open set in X. Fix $u = [u_\lambda] \in \mathcal{A}(\Omega)$ and $x \in \Omega$. The idea of the $(a,\mathcal{F})$-microlocal analysis is the following: $(u_\lambda)_\lambda$ may not tend to a section of $\mathcal{F}$ above a neighborhood of x, that is, there may not exist $V \in \mathcal{V}_x$ and $f \in \mathcal{F}(V)$ such that $\lim_{\Lambda} {}_{\mathcal{F}(V)}\, u_\lambda = f$. Nevertheless, in this case, there may exist $V \in \mathcal{V}_x$, $r \geq 0$ and $f \in \mathcal{F}(V)$ such that $\lim_{\Lambda} {}_{\mathcal{F}(V)}\, a_\lambda(r) u_\lambda = f$, that is $[a_\lambda(r) u_\lambda |_V]$ is in the subspace (resp. subalgebra) $\mathcal{F}_\mathcal{A}(V)$ of $\mathcal{A}(V)$ introduced in Subsection 2.5. These preliminary remarks lead to the following concept and results which we summarize from the results given in [28, 29, 14].

4.1 The $(a,\mathcal{F})$-singular parametric spectrum

We recall that a is a map from $\mathbb{R}_+$ to A_+ such that $a(0) = 1$ and $\mathcal{F}$ is a presheaf of topological vector spaces (or topological algebras). For any open subset Ω of X, $u = [u_\lambda] \in \mathcal{A}(\Omega)$ and $x \in \Omega$, set

$$\begin{aligned} N_{(a,\mathcal{F}),x}(u) &= \left\{ r \in \mathbb{R}_+ \mid \exists V \in \mathcal{V}_x,\ \exists f \in \mathcal{F}(V) \ :\ \lim_{\Lambda} {}_{\mathcal{F}(V)} \left(a_\lambda(r)\, u_\lambda |_V \right) = f \right\} \\ &= \left\{ r \in \mathbb{R}_+ \mid \exists V \in \mathcal{V}_x \ :\ [a_\lambda(r)\, u_\lambda |_V] \in \mathcal{F}_\mathcal{A}(V) \right\}. \end{aligned}$$

It is easy to check that $N_{(a,\mathcal{F}),x}(u)$ does not depend on the representative of u. If no confusion may arise, we shall simply write

$$N_{(a,\mathcal{F}),x}(u) = N_x(u).$$

Assume that:

(a) For all $\lambda \in \Lambda$

$$\forall (r,s) \in \mathbb{R}_+, \quad a_\lambda(r+s) \leq a_\lambda(r) a_\lambda(s),$$

and, for all $r \in \mathbb{R}_+ \backslash \{0\}$, the net $(a_\lambda(r))_\lambda$ converges to 0 in $\mathbb{K}$

(b) $\mathcal{F}$ is a presheaf of Hausdorff locally convex topological vector spaces.

Then, from Theorem 7 in [14] we have, for $u \in \mathcal{A}(\Omega)$:

(i) If $r \in N_x(u)$, then $[r, +\infty)$ is included in $N_x(u)$. Moreover, for all $s > r$, there exists $V \in \mathcal{V}_x$ such that: $\lim_{\Lambda} {}_{\mathcal{F}(V)} \left(a_\lambda(s)\, u_\lambda |_V \right) = 0$. Consequently, $N_x(u)$ is either empty, or a subinterval of $\mathbb{R}_+$.

(ii) More precisely, suppose that for $x \in \Omega$, there exist $r \in \mathbb{R}_+$, $V \in \mathcal{V}_x$ and $f \in \mathcal{F}(V)$, nonzero on each neighborhood of x included in V, such that $\lim_{\Lambda} {}_{\mathcal{F}(V)} \left(a_\lambda(r)\, u_\lambda |_V \right) = f$. Then $N_x(u) = [r, +\infty)$.

(iii) In the situation of (i) and (ii), we have that $0 \in N_x(u)$ iff $N_x(u) = \mathbb{R}_+$. Moreover, if one of these assertions holds, the limits $\lim_{\Lambda} {}_{\mathcal{F}(V)} \left(a_\lambda(s)\, u_\lambda |_V \right)$ can be non null only for $s = 0$.

Now, we set

$$\begin{aligned} \Sigma_{(a,\mathcal{F}),x}(u) &= \Sigma_x(u) = \mathbb{R}_+ \backslash N_x(u), \\ R_{(a,\mathcal{F}),x}(u) &= R_x(u) = \inf N_x(u). \end{aligned}$$

According to the previous remarks and comments, $\Sigma_{(a,\mathcal{F}),x}(u)$ is an interval of $\mathbb{R}_+$ of the form $\left[0, R_{(a,\mathcal{F}),x}(u)\right)$ or $\left[0, R_{(a,\mathcal{F}),x}(u)\right]$, the empty set, or $\mathbb{R}_+$. This leads to the following

Definition (4 in [14]) *The $(a,\mathcal{F})$-singular spectrum of $u \in A(\Omega)$ is the set*

$$\mathcal{S}_\mathcal{A}^{(a,\mathcal{F})}(u) = \left\{ (x,r) \in \Omega \times \mathbb{R}_+ \mid r \in \Sigma_x(u) \right\}.$$

Example (4 in [14]) *Set* $X=\mathbb{R}^d$, $\mathcal{E}=\mathrm{C}^\infty$, $\mathcal{F}=\mathrm{C}^p$ *(*$p\in\overline{\mathbb{N}}=\mathbb{N}\cup\{+\infty\}$*)*, $f\in\mathrm{C}^\infty(\Omega)$*. Set* $u=\left[\left(\varepsilon^{-1}f\right)_\varepsilon\right]$ *and* $v=\left[\left(\varepsilon^{-1}\left|\ln\varepsilon\right|f\right)_\varepsilon\right]$ *in* $\mathcal{A}(\Omega)=\mathcal{G}(\Omega)$*. Then, for all* $x\in\mathbb{R}$*,*

$$N_{(a,\mathrm{C}^p),x}(u)=[1,+\infty)\ ,\quad N_{(a,\mathrm{C}^p),x}(v)=(1,+\infty)\ ,\quad R_{(a,\mathrm{C}^p),x}(u)=R_{(a,\mathrm{C}^p),x}(v)=1.$$

Remark (5 in [14]) *We have:* $\Sigma_{(a,\mathcal{F}),x}(u)=\varnothing$ *iff* $N_{(a,\mathcal{F}),x}(u)=\mathbb{R}_+$ *and, according to Theorem* 7 in [14], *iff* $0\in N_{(a,\mathcal{F}),x}(u)$*, that is, there exist* $(V,f)\in V_x\times\mathcal{F}(V)$ *such that* $\lim_\Lambda {}_{\mathcal{F}(V)}\left(a_\lambda(0)\,u_\lambda\left|_V\right.\right)=f$*. As* $a_\lambda(0)\equiv1$*, this last assertion is equivalent to* $x\in\mathcal{O}_{\mathcal{A}}^{\mathcal{F}}(u)$*. Thus* $\Sigma_{(a,\mathcal{F}),x}(u)=\varnothing$ *iff* $x\notin\mathcal{S}_{\mathcal{A}}^{\mathcal{F}}(u)$*.*

This remark implies directly the:

Proposition (8 in [14]) *The projection of the* $(a,\mathcal{F})$*-singular spectrum of* u *on* Ω *is the* $\mathcal{F}$*-singular support of* u*.*

4.2 Some properties of the $(a,\mathcal{F})$-singular parametric spectrum

Notation For $u=[u_\lambda]\in\mathcal{A}(\Omega)$, $\lim_\Lambda {}_{\mathcal{F}(V)}\left(a_\lambda(r)\,u_\lambda\left|_V\right.\right)\in\mathcal{F}(V)$ means that there exists $f\in\mathcal{F}(V)$ such that $\lim_\Lambda {}_{\mathcal{F}(V)}\left(a_\lambda(r)\,u_\lambda\left|_V\right.\right)=f$.

4.2.1 Linear and differential properties

It is easy to prove that for any $u,v\in\mathcal{A}(\Omega)$, we have

$$\mathcal{S}_{\mathcal{A}}^{(a,\mathcal{F})}(u+v)\subset\mathcal{S}_{\mathcal{A}}^{(a,\mathcal{F})}(u)\cup\mathcal{S}_{\mathcal{A}}^{(a,\mathcal{F})}(v)\,.$$

As a corollary: for any u, u_0, u_1 in $\mathcal{A}(\Omega)$ with

$$(i)\ \ u=u_0+u_1\qquad(ii)\ \mathcal{S}_{\mathcal{A}}^{(a,\mathcal{F})}(u_0)=\varnothing,$$

we have: $\mathcal{S}_{\mathcal{A}}^{(a,\mathcal{F})}(u)=\mathcal{S}_{\mathcal{A}}^{(a,\mathcal{F})}(u_1)$.

Assume that $\mathcal{F}$ is a sheaf of topological differential vector spaces, with continuous differentiation, admitting $\mathcal{E}$ as a subsheaf of topological differential algebras. Then the sheaf $\mathcal{A}$ is also a sheaf of differential algebras with, for any $\alpha\in\mathbb{N}^d$ and $u\in\mathcal{A}(\Omega)$,

$$\partial^\alpha u=\left[\partial^\alpha u_\lambda\right],\text{ where }\left(u_\lambda\right)_\lambda\text{ is any representative of }u.$$

The independence of $\partial^\alpha u$ on the choice of representative follows directly from the definition of $\mathcal{J}_{(I_A,\mathcal{E},\mathcal{P})}$.) It follows that if u is in $\mathcal{A}(\Omega)$ and g in $\mathcal{E}(\Omega)$, for all ∂^α, $\alpha\in\mathbb{N}^d$, we have

$$\mathcal{S}_{\mathcal{A}}^{(a,\mathcal{F})}(g\partial^\alpha u)\subset\mathcal{S}_{\mathcal{A}}^{(a,\mathcal{F})}(u)\,.$$

This leads to the more general statement: Let $P(\partial)=\sum_{|\alpha|\leq m}C_\alpha\partial^\alpha$ be a differential polynomial with coefficients in $E(\Omega)$. For any $u\in A(\Omega)$, we have

$$\mathcal{S}_{\mathcal{A}}^{(a,\mathcal{F})}(P(\partial)u)\subset\mathcal{S}_{\mathcal{A}}^{(a,\mathcal{F})}(u)\,.$$

4.2.2 Nonlinear properties

When $\mathcal{F}$ is a presheaf of algebras, the $(a,\mathcal{F})$-singular spectrum inherits new properties with respect to nonlinear operations. It is the purpose of following results.

Theorem (15 in [14]) *We suppose that F is a presheaf or algebras. For u and $v \in A(\Omega)$, let D_i $(i = 1, 2, 3)$ be the following disjoint sets:*

$$D_1 = \mathcal{S}^{\mathcal{F}}_{\mathcal{A}}(u)\diagdown(\mathcal{S}^{\mathcal{F}}_{\mathcal{A}}(u) \cap \mathcal{S}^{\mathcal{F}}_{\mathcal{A}}(v)) \ ; \quad D_2 = \mathcal{S}^{\mathcal{F}}_{\mathcal{A}}(v)\diagdown(\mathcal{S}^{\mathcal{F}}_{\mathcal{A}}(u) \cap \mathcal{S}^{\mathcal{F}}_{\mathcal{A}}(v)) \ ; \quad D_3 = \mathcal{S}^{\mathcal{F}}_{\mathcal{A}}(u) \cap \mathcal{S}^{\mathcal{F}}_{\mathcal{A}}(v).$$

Then the (a,F)-singular asymptotic spectrum of uv verifies

$$\mathcal{S}^{(a,\mathcal{F})}_{\mathcal{A}}(uv) \subset \{(x, \Sigma_x(u)), x \in D_1\} \cup \{(x, \Sigma_x(v)), x \in D_2\} \cup \{(x, E_x(u, v)), x \in D_3\}$$

where (for any $x \in D_3$)

$$E_x(u, v) = \begin{cases} [0, \sup \Sigma_x(u) + \sup \Sigma_x(v)] \text{ if } \Sigma_x(u) \neq \mathbb{R}_+ \text{ and } \Sigma_x(v) \neq \mathbb{R}_+ \\ \mathbb{R}_+ \text{ if } \Sigma_x(u) = \mathbb{R}_+ \text{ or } \Sigma_x(v) = \mathbb{R}_+ \end{cases}$$

Corollary (16 in [14]) *When F is a presheaf of topological algebras, for $u \in A(\Omega)$ and $p \in N^*$, we have*

$$\mathcal{S}^{(a,\mathcal{F})}_{\mathcal{A}}(u^p) \subset \left\{(x, H_{p,x}(u)), x \in \mathcal{S}^{\mathcal{F}}_{\mathcal{A}}(u)\right\}$$

where $H_{p,x}(u) = \begin{cases} [0, p \sup \Sigma_x(u)] \text{ if } \Sigma_x(u) \neq \mathbb{R}_+ \\ \mathbb{R}_+ \text{ if } \Sigma_x(u) = \mathbb{R}_+. \end{cases}$

4.3 Some examples and applications to partial differential equations

In this subsection we shall give some examples of $(a,\mathcal{F})$-singular spectra of solutions to nonlinear partial differential equations given in ([14], subsection 4.2). Throughout we shall suppose that $\Lambda =]0, 1]$, $X = \mathbb{R}^d$, $\mathcal{E} = \mathrm{C}^\infty$, $\mathcal{F} = \mathrm{C}^p$ $(1 \leq p \leq \infty)$ or $\mathcal{F} = \mathcal{D}'$, $a_\varepsilon(r) = \varepsilon^r$. The results will hold for any $(\mathcal{C}, \mathcal{E}, \mathcal{P})$-algebra

$$\mathcal{A} = \mathcal{H}_{(A,\mathcal{E},\mathcal{P})}/\mathcal{J}_{(I_A,\mathcal{E},\mathcal{P})}$$

such that $(a_\varepsilon(r))_\varepsilon \in A_+$ for all $r \in \mathbb{R}_+$ and the hypothesis given in 2.6.2 holds.

4.3.1 On the singular spectrum of powers of the delta function

We can compare the (a, C^p)-singular spectrum and the $(a, \mathcal{D}')$-singular spectrum of powers of the delta function. Given a mollifier of the form

$$\varphi_\varepsilon(x) = \frac{1}{\varepsilon^d}\varphi\left(\frac{x}{\varepsilon}\right), \ x \in \mathbb{R}^d \ \text{ where } \varphi \in \mathcal{D}(\mathbb{R}^d), \varphi \geq 0 \text{ and } \textstyle\int \varphi(x)\, dx = 1,$$

its class in $\mathcal{A}(\mathbb{R}^d)$ defines the delta function $\delta(x)$ as an element of $\mathcal{A}(\mathbb{R}^d)$. Its powers are given by $(m \in \mathbb{N})$

$$\delta^m = [\varphi^m_\varepsilon] = \left[\frac{1}{\varepsilon^{md}}\,\varphi^m\left(\frac{\cdot}{\varepsilon}\right)\right].$$

Clearly, the C^0-singular spectrum is given by

$$\mathcal{S}^{(a,\mathrm{C}^0)}_{\mathcal{A}}(\delta^m) = \big(0, [0, md]\big).$$

Differentiating $\varphi^m(x)$ and observing that for each derivative there is a point x at which this function does not vanish we obtain the (a, C^p)-singular spectrum of δ^m :

$$\mathcal{S}^{(a,\mathrm{C}^p)}_{\mathcal{A}}(\delta^m) = \big(0, [0, md + p]\big).$$

Given now a test function $\psi \in \mathcal{D}(\mathbb{R}^d)$, we have

$$\int \varphi_\varepsilon^m(x)\psi(x)\,dx = \int \frac{1}{\varepsilon^{md-d}}\,\varphi^m(x)\psi(\varepsilon x)\,dx,$$

thus the (a, D')-singular spectrum of δ^m is

$$\mathcal{S}_{\mathcal{A}}^{(a,\mathcal{D}')}(\delta^m) = \varnothing \ \text{ for } m = 1, \qquad \mathcal{S}_{\mathcal{A}}^{(a,\mathcal{D}')}(\delta^m) = \big(0, [0, md-d[\big) \ \text{ for } m > 1.$$

4.3.2 The singular spectrum of solutions to semilinear hyperbolic equations

The singular spectrum of solutions of a semilinear transport problem

$$(P_\lambda)\begin{cases} \partial_t u_\varepsilon(x,t) + \lambda(x,t)\partial_x u_\varepsilon(x,t) = F(u_\varepsilon(x,t)), & x \in \mathbb{R},\ t \in \mathbb{R} \\ u_\varepsilon(x,0) = u_{0\varepsilon}(x), & x \in \mathbb{R} \end{cases}$$

where λ and F are smooth functions of their arguments. may decrease or increase with respect to the one of the data, depending on the function F. We observe that by a change of coordinates we may assume without loss of generality that $\lambda \equiv 0$.

a) For $F(u_\varepsilon(x,t)) = -u_\varepsilon^3(x,t)$ (the dissipative case: Example 8 in [14]), the problem (P_0) has the solution

$$u_\varepsilon(x,t) = \frac{u_{0\varepsilon}(x)}{\sqrt{2tu_{0\varepsilon}^2(x)+1}} = \frac{1}{\sqrt{2t + 1/u_{0\varepsilon}^2(x)}}.$$

When the initial data are given by a power of the delta function, $u_{0\varepsilon}(x) = \varphi_\varepsilon^m(x)$, the solution formula shows that $u_\varepsilon(x,t)$ is a bounded function (uniformly in ε) supported on the line $\{x = 0\}$. Thus $u_\varepsilon(x,t)$ converges to zero in $\mathcal{D}'(\mathbb{R}\times]0,\infty[)$, and so

$$\mathcal{S}_{\mathcal{A}}^{(a,\mathcal{D}')}(u_0) = \big(0, [0, m-1[\big), \qquad \mathcal{S}_{\mathcal{A}}^{(a,\mathcal{D}')}(u) = \varnothing.$$

b) For $F(u_\varepsilon(x,t)) = \sqrt{1 + u_\varepsilon^2(x,t)}$, $\ x \in \mathbb{R},\ t > 0$ ([14], Example 9), the problem (P_0) has the solution

$$u_\varepsilon(x,t) = u_{0\varepsilon}(x)\cosh t + \sqrt{1 + u_{0\varepsilon}^2(x)}\,\sinh t.$$

b$_1$) with a delta function as initial value, that is, $u_{0\varepsilon}(x) = \varphi_\varepsilon(x)$ we obtain

$$\begin{aligned}\iint u_\varepsilon(x,t)\psi(x,t)\,dxdt &= \iint \Big(\varphi(x)\cosh t + \sqrt{\varepsilon^2 + \varphi^2(x)}\,\sinh t\Big)\psi(\varepsilon x, t)\,dxdt \\ &\to \iint \Big(\varphi(x)\cosh t + |\varphi(x)|\sinh t\Big)\psi(0,t)\,dxdt\end{aligned}$$

for $\psi \in \mathcal{D}(\mathbb{R}^2)$. Thus in this case

$$\mathcal{S}_{\mathcal{A}}^{(a,\mathcal{D}')}(u_0) = \mathcal{S}_{\mathcal{A}}^{(a,\mathcal{D}')}(u) = \varnothing.$$

b$_2$) with the derivative of a delta function as initial value, $u_{0\varepsilon}(x) = \varphi'_\varepsilon(x)$, a similar calculation shows that

$$\iint u_\varepsilon(x,t)\psi(x,t)\,dxdt = \iint \Big(\varphi(x)\cosh t + \frac{1}{\varepsilon}\sqrt{\varepsilon^4 + (\varphi')^2(x)}\,\sinh t\Big)\psi(\varepsilon x, t)\,dxdt$$

and so

$$\mathcal{S}_{\mathcal{A}}^{(a,\mathcal{D}')}(u_0) = \varnothing, \qquad \mathcal{S}_{\mathcal{A}}^{(a,\mathcal{D}')}(u) = \{(0,t,r) : t > 0, 0 \le r < 1\}.$$

Example 10 in [14] shows that it is quite possible for the singular spectrum to increase with time.
c) when taking $F(u_\varepsilon(x,t)) = \big(u_\varepsilon(x,t)+1)\big)\log\big(u_\varepsilon(x,t)+1\big), \quad x\in\mathbb{R},\ t>0$, the problem (P_0) has the solution

$$u_\varepsilon(x,t) = \big(u_{0\varepsilon}(x)+1\big)^{e^t},$$

provided $u_{0\varepsilon} > -1$ in which case the function on the right hand side of the differential equation is smooth in the relevant region. To demonstrate the effect, we take a power of the delta function as initial value, that is $u_{0\varepsilon}(x) = \varphi_\varepsilon^m(x)$. Then

$$\mathcal{S}_{\mathcal{A}}^{(a,\mathcal{D}')}(u_0) = \{(0,r) : 0\le r < m-1\}, \qquad \mathcal{S}_{\mathcal{A}}^{(a,\mathcal{D}')}(u) = \{(0,t,r) : t>0, 0\le r < me^t-1\}.$$

4.3.3 Blow-up in finite time

In situations where blow-up in finite time occurs, microlocal asymptotic methods allow to extract information beyond the point of blow-up. This can be done by regularizing the initial data and truncating the nonlinear term. This is shown in Example 11 of [14] for a simple situation.

The problem to be treated is formally the initial value problem

$$\begin{aligned}&\partial_t u(x,t) = u^2(x,t), \quad x\in\mathbb{R},\ t>0\\ &u(x,0) = H(x), \quad x\in\mathbb{R}\end{aligned}$$

where H denotes the Heaviside function. Clearly, the local solution $u(x,t) = H(x)/(1-t)$ blows up at time $t=1$ when $x>0$. Choose $\chi_\varepsilon \in \mathrm{C}^\infty(\mathbb{R})$ with

$$0\le \chi_\varepsilon(z)\le 1\ ;\ \chi_\varepsilon(z) = 1 \text{ if } |z|\le \varepsilon^{-s},\ \chi_\varepsilon(z) = 0 \text{ if } |z|\ge 1+\varepsilon^{-s},\ s>0.$$

Further, let $H_\varepsilon(x) = H*\varphi_\varepsilon(x)$ where φ_ε is a mollifier as in 4.3.1. One considers the regularized problem

$$\begin{aligned}&\partial_t u_\varepsilon(x,t) = \chi_\varepsilon\big(u_\varepsilon(x,t)\big)u_\varepsilon^2(x,t), \quad x\in\mathbb{R},\ t>0\\ &u_\varepsilon(x,0) = H_\varepsilon(x), \quad x\in\mathbb{R}.\end{aligned}$$

When $x<0$ and ε is sufficiently small, $u_\varepsilon(x,t) = 0$ for all $t\ge 0$. For $x>0$, $u_\varepsilon(x,t) = 1/(1-t)$ as long as $t\le 1-\varepsilon^s$. The cut-off function is chosen in such a way that $|\chi_\varepsilon(z)z^2| \le (1+\varepsilon^{-s})^2$ for all $z\in\mathbb{R}$. Therefore,

$$\partial_t u_\varepsilon \le (1+\varepsilon^{-s})^2 \text{ always and } \partial_t u_\varepsilon = 0 \text{ when } |u_\varepsilon|\ge 1+\varepsilon^{-s}.$$

Some computations and estimates permit to obtain the following C^0-singular support and $\big(a,\mathrm{C}^0\big)$-singular spectrum (for $a_\varepsilon(r) = \varepsilon^r$) of $u = [u_\varepsilon]$:

$$\mathcal{S}_{\mathcal{A}}^{\mathrm{C}^0}(u) = \mathcal{S}_1(u)\cup\mathcal{S}_2(u) \text{ with } \mathcal{S}_1(u) = \{(0,t) : 0\le t<1\}\ ;\ \mathcal{S}_2(u) = \{(x,t) : x\ge 0, t\ge 1\},$$

$$\mathcal{S}_{\mathcal{A}}^{(a,\mathrm{C}^0)}(u) = (\mathcal{S}_1(u)\times\{0\})\cup(\mathcal{S}_2(u)\times[0,s]).$$

These results give a microlocal precision on the the blow-up: The C^0-singularities (resp. $\big(a,\mathrm{C}^0\big)$-singularities) of u are described by means of two sets: $\mathcal{S}_1(u)$ and $\mathcal{S}_2(u)$ (resp. $\mathcal{S}_1(u)\times\{0\}$ and $\mathcal{S}_2(u)\times[0,s]$). The set $S_1(u)$ (resp. $\mathcal{S}_1(u)\times\{0\}$) is related to the data C^0 (resp. $\big(a,\mathrm{C}^0\big)$)-singularity. The set $\mathcal{S}_2(u)$ (resp. $\mathcal{S}_2(u)\times[0,s]$) is related to the singularity due to the nonlinearity of the equation giving the blow-up at $t=1$. The blow-up locus is the edge $\{x\ge 0, t=1\}$ of $\mathcal{S}_2(u)$ and the strength of the blow-up is measured by the length s of the fiber $[0,s]$ above each point of the blow-up locus. This length is closely related to the diameter of the support of the regularizing function χ_ε and depends essentially on the nature of the blow-up: Changing simultaneously the scales of the regularization and of the cut-off (i.e. replacing ε by some function $h(\varepsilon)\to 0$ in the definition of φ_ε and χ_ε) does not change the fiber and characterizes a sort of moderateness of the strength of the blow-up.

4.3.4 The strength of a singularity and the sum law

We point out the following remark ([14], subsection 4.3): when studying the propagation and interaction of singularities in semilinear hyperbolic systems, Rauch and Reed [36] defined the strength of a singularity of a piecewise smooth function. This notion is recalled in the one-dimensional case. Assume that the function $f : \mathbb{R} \to \mathbb{R}$ is smooth on $]-\infty, x_0]$ and on $[x_0, \infty[$ for some point $x_0 \in \mathbb{R}$. The *strength of the singularity of* f *at* x_0 is the order of the highest derivative which is still continuous across x_0. For example, if f is continuous with a jump in the first derivative at x_0, the order is 0; if f has a jump at x_0, the order is -1. Travers [39] later generalized this notion to include delta functions. Slightly deviating from her definition, but in line with the one of [36], it is possible to define the strength of singularity of the k-th derivative of a delta function at x_0, $\partial_x^k \delta(x - x_0)$, by $-k-2$.

The significance of these definitions is perceived in the description of what Rauch and Reed termed *anomalous singularities* in semilinear hyperbolic systems. This effect is demonstrated in a paradigmatic example, also due to [36], the (3×3)-system

$$(**) \qquad \left\{ \begin{array}{rcll} (\partial_t + \partial_x)u(x,t) &=& 0, & u(x,0) = u_0(x) \\ (\partial_t - \partial_x)v(x,t) &=& 0, & v(x,0) = v_0(x) \\ \partial_t w(x,t) &=& u(x,t)v(x,t), & w(x,0) = 0 \end{array} \right.$$

Assume that u_0 has a singularity of strength $n_1 \geq -1$ at $x_1 = -1$ and v_0 has a singularity of strength $n_2 \geq -1$ at $x_2 = +1$. The characteristic curves emanating from x_1 and x_2 are straight lines intersecting at the point $x = 0$, $t = 1$. Rauch and Reed showed that, in general, the third component w will have a singularity of strength $n_3 = n_1 + n_2 + 2$ along the half-ray $\{(0,t) : t \geq 1\}$. This half-ray does not connect backwards to a singularity in the initial data for w, hence the term *anomalous singularity.* The formula $n_3 = n_1 + n_2 + 2$ is called the *sum law.* Travers extended this result to the case where u_0 and v_0 were given as derivatives of delta functions at x_1 and x_2. We are going to further generalize this result to powers of delta functions, after establishing the relation between the strength of a singularity of a function f at x_0 and the singular spectrum of $f * \varphi_\varepsilon$.

We consider a function $f : \mathbb{R} \to \mathbb{R}$ which is smooth on $(-\infty, x_0]$ and on $[x_0, \infty)$ for some point $x_0 \in \mathbb{R}$; actually only the local behavior near x_0 is relevant. A mollifier $\varphi_\varepsilon(x) = \frac{1}{\varepsilon}\varphi(\frac{x}{\varepsilon})$ is fixed as in 4.3.1 and the corresponding embedding of $\mathcal{D}'(\mathbb{R})$ into the $(\mathcal{C}, \mathcal{E}, \mathcal{P})$-algebra $\mathcal{A}(\mathbb{R})$ is denoted by ι. In particular, $\iota(f) = [f * \varphi_\varepsilon]$.

If f is continuous at x_0, then $\lim_{\varepsilon \to 0} f * \varphi_\varepsilon = f$ in C^0. If f has a jump x_0, this limit does not exist in C^0, but $\lim_{\varepsilon \to 0} \varepsilon^r f * \varphi_\varepsilon = 0$ in C^0 for every $r > 0$. The following result is

Proposition (16 in [14]) *Let* $x_0 \in \mathbb{R}$. *If* $f : \mathbb{R} \to \mathbb{R}$ *is a smooth function on* $(-\infty, x_0]$ *and on* $[x_0, \infty)$ *or* $f(x) = \partial_x^k \delta(x - x_0)$ *for some* $k \in \mathbb{N}$, *then the strength of the singularity of* f *at* x_0 *is* $-n$ *if and only if*

$$\Sigma_{(a,\mathrm{C}^1),x_0}\big(\iota(f)\big) = [0, n].$$

Here $n \in \mathbb{N}$ *and* $a_\varepsilon(r) = \varepsilon^r$.

When returning to the model equation we find that the sum law remains valid when the initial data are powers of delta functions. Suitable $(\mathcal{C}, \mathcal{E}, \mathcal{P})$-algebras $\mathcal{A}(\mathbb{R})$ and $\mathcal{A}(\mathbb{R}^2)$ are exhibited in which the initial value problem can be uniquely solved. When the scale is taken as $a_\varepsilon(r) = \varepsilon^r$, the following result is obtained:

Proposition (17 in [14]) *Let* $u_0(x) = \delta^m(x+1)$, $v_0(x) = \delta^n(x-1)$ *for some* $m, n \in \mathbb{N}^*$. *Let* $w \in \mathcal{A}(\mathbb{R}^2)$ *be the third component of the solution to problem (**). Then* $w(x,t)$ *vanishes at all points* (x,t) *with* $x \neq 0$ *as well as* $(0,t)$ *with* $t < 1$, *and*

$$\Sigma_{(a,\mathrm{C}^1),(0,t)}\big(w\big) \subset [0, m+n]$$

for $t \geq 1$.

4.4 Microlocal characterisation of some regular subalgebras

We recall that the subsheaf $\mathcal{G}^\infty$ of *regular Colombeau functions* of the sheaf $\mathcal{G}$ is defined as follows [32]: Given an open subset Ω of $\mathbb{R}^d$, the algebra $\mathcal{G}^\infty(\Omega)$ comprises those elements u of $\mathcal{G}(\Omega)$ whose representatives $(u_\varepsilon)_\varepsilon$ satisfy the condition

$$\forall K \Subset \Omega,\ \exists m \in \mathbb{N},\ \forall l \in \mathbb{N} : p_{K,l}(u_\varepsilon) = o(\varepsilon^{-m}) \text{ as } \varepsilon \to 0.$$

In relation with regularity theory of solutions to nonlinear partial differential equations, a further subalgebra of $\mathcal{G}(\Omega)$ has been introduced in [33] – the algebra of Colombeau functions of *total slow scale type*. It consists of those elements u of $\mathcal{G}(\Omega)$ whose representatives $(u_\varepsilon)_\varepsilon$ satisfy the condition

$$\forall K \Subset \Omega,\ \forall r > 0,\ \forall l \in \mathbb{N} : p_{K,l}(u_\varepsilon) = o(\varepsilon^{-r}) \text{ as } \varepsilon \to 0.$$

The term *slow scale* refers to the fact that the growth is slower than any negative power of ε as $\varepsilon \to 0$. Both previous properties can be characterized by means of the singular spectrum.

We find in ([14], subsection 4.4) the proof of the corresponding characterisations. Let $u \in \mathcal{G}(\Omega)$, then:

(i) (Proposition 18) u belongs to $\mathcal{G}^\infty(\Omega)$ if and only if $\Sigma_{(a,\mathrm{C}^\infty),x}(u) \neq \mathbb{R}_+$ for all $x \in \Omega$.

(ii) (Proposition 19) u is of total slow scale type if and only if $\Sigma_{(a,\mathrm{C}^\infty),x}(u) \subset \{0\}$ for all $x \in \Omega$.

References

[1] Aragona, J., Biagioni, H.: *Intrinsic definition of the Colombeau algebra of generalized functions.* Anal. Math. **17**, 75–132 (1991)

[2] Beals, M.: Propagation of smoothness for nonlinear second order strictly hyperbolic differential equations A.M.S. Proc. Symp. Pure Math. **43** 21-44 (1985)

[3] Beals, M.: Propagation and interaction of singularities in nonlinear hyperbolic problems. Progress in Nonlinear Differential Equations and their Applications. **3**. Birkhausen Boston, Inc.,Boston, MA-(1989)

[4] Bony, J.-M.: *Calcul symbolique et propagation des singularités pour les équations aux dérivées partielles non linéaires.* Ann. Sc. Ec. Norm. Sup. **14,** 209-246 (1981)

[5] Bony, J.-M.: *Analyse microlocale des équations aux dérivées partielles non linéaires.* Springer Lecture Notes in Math. **1495**, 1-45 (1991)

[6] Bouzar, C., Benmeriem, K.: *Colombeau Generalized Gevrey Ultradistributios and their Microlocal Analysis.* arXiv:0707.0124 vl [math.FA] (2007)

[7] Colombeau, J.F.: *Multiplication of Distributions: a tool in mathematics, numerical engineering and theoretical physics.* Lecture Notes in Mathematics, vol. **1532**. Springer-Verlag, Berlin (1992)

[8] Colombeau, J.F.: *Generalized Functions and infinitesimals.* arXiv:0610264 [math.FA], (2006)

[9] Colombeau, J.F.: *Generalized functions as a tool for nonsmooth nonlinear problems in mathematics and physics.* arXiv:0612077 [math.-ph.], (2006)

[10] Colombeau, J.F.: *Nonlinear generalized functions and nonlinear numerical simulations in fluid and solid continuum mechanics.* arXiv:0702014 [math.-ph.], (2007)

[11] Dapic, N., Pilipovic, S., Scarpalézos, D.: *Microlocal analysis of Colombeau's generalized function: propagation of singularities.* Jour.d'Analyse Math., **75**, 51-66 (1998)

[12] Delcroix, A.: *Regular rapidly decreasing nonlinear generalized functions. Application to microlocal regularity.* J. Math. Anal. Appl. **327**, 564–584 (2007)

[13] Delcroix, A., Marti, J.-A.: $\mathcal{G}^{r,\mathcal{R},L}$*-microlocal analysis of generalized functions.* Preprint

[14] Delcroix, A., Marti, J.-A., Oberguggenberger, M.: *Microlocal Asymptotic Analysis in Algebras of Generalized Functions.* arXiv:0704.1077vl [math.FA] 9 Apr 2007

[15] Garetto, C., Gramchev, T., Oberguggenberger, M.: *Pseudo-Differential operators and regularity theory.* Electron J. Diff. Eqns. (**116**), 1–43 (2005)

[16] Garetto, C.: *Topological structures in Colombeau algebras: topological* $\widetilde{\mathbb{C}}$*-modules and duality theory.* Acta Appl. Math. **80**, no. 1, 81-125 (2005)

[17] Garetto, C.: *Microlocal analysis in the dual of a Colombeau algebra: generalized wave front sets and noncharacteristic regularity.* New York J. Math. **12**, 275-318 (2006)

[18] Garetto, C., Hörmann, G.: *Microlocal analysis of generalized functions: pseudodifferential techniques and propagation of singularities.* Proc. Edinb. Math. Soc. **48**, 603–629 (2005)

[19] Grosser, M., Kunzinger, M., Oberguggenberger, M., Steinbauer, R.: *Geometric Theory of Generalized Functions with Applications to General Relativity.* Kluwer Academic Publ., Dordrecht (2001)

[20] Godement, R.: *Topologie algébrique et théorie des faisceaux.* Hermann, Paris (1958)

[21] Gramchev, T.: *Nonlinear maps in spaces of distributions.* Math. Z. **209**, n°. 1, 101-114 (1992)

[22] Hörmander, L.: *The Analysis of Linear Partial Differential Operators, I: Distribution Theory and Fourier Analysis.* Grundlehren der mathematischen Wissenschaften, vol. **256**, 2nd edition. Springer-Verlag, Berlin (1990)

[23] Hörmander, L.: *Fourier Integral Operators I.* Acta Math., **127**, 79-183 (1971)

[24] Hörmander, L.: *Lectures on Nonlinear Hyperbolic Differential Equations.* Springer-Verlag, Berlin Heidelberg (1997)

[25] Hörmann, G., De Hoop, M.V.: *Microlocal analysis and global solutions for some hyperbolic equations with discontinuous coefficients.* Acta Appl. Math. **67**, 173–224 (2001)

[26] Hörmann, G., Kunzinger, M.: *Microlocal properties of basic operations in Colombeau algebras.* J. Math. Anal. Appl. **261**, 254–270 (2001)

[27] Hörmann, G., Oberguggenberger, M., Pilipović, S.: *Microlocal hypoellipticity of linear differential operators with generalized functions as coefficients.* Trans. Amer. Math. Soc. **358**, 3363–3383 (2006)

[28] Marti, J.-A.: $(\mathcal{C},\mathcal{E},\mathcal{P})$*-Sheaf structure and applications.* In: Grosser, M., Hörmann, G., Kunzinger, M., Oberguggenberger, M. (Eds.) *Nonlinear Theory of Generalized Functions.* Chapman & Hall/CRC Research Notes in Mathematics, vol. **401**, 175–186. Boca Raton (1999)

[29] Marti, J.-A.: *Nonlinear algebraic analysis of delta shock wave to Burgers' equation*. Pacific J. Math. **210** (1), 165–187 (2003)

[30] Marti, J.-A.: *$\mathcal{G}^L$-microlocal analysis of generalized functions*. Integral Transf. Spec. Funct. **2–3**, 119–125 (2006)

[31] Nedeljkov, M., Pilipović, S., Scarpalézos, D.: The linear theory of Colombeau generalized functions. Pitman Research Notes in Mathematics, vol. **385**. Longman Scientific & Technical, Harlow (1998)

[32] Oberguggenberger, M.: *Multiplication of Distributions and Applications to Partial Differential Equations*. Pitman Research Notes in Mathematics, vol. **259**. Longman Scientific & Technical, Harlow (1992)

[33] Oberguggenberger, M.: *Generalized solutions to nonlinear wave equations*. Matemática Contemporânea **27**, 169–187 (2004)

[34] Pilipovic, S., Scarpalézos, D., Valmorin,V. *Real Analytic Generalized Functions*. Preprint

[35] Rauch, J.: *Singularities of solutions to semilinear wave equation*. J. Math. Pures et Appl. **58**, 299-308,

[36] Rauch, J., Reed, M.: Jump discontinuities of semilinear, strictly hyperbolic equations in two variables: Creation and propagation. Comm. Math. Phys. **81**, 203-227 (1981)

[37] Scarpalézos, D.: Colombeau's generalized functions: topological structures; microlocal properties. A simplified point of view. Bull. Cl. Sci. Math. Nat. Sci. Math. **25**, 89–114 (2000)

[38] Schwartz, L.: Théorie des Distributions. Hermann, Paris (1966)

[39] Travers, K.: Semilinear hyperbolic systems in one space dimension with strongly singular initial data. Electron. J. Diff. Eqns. **14**, 1–11 (1997)