

A formula for the spectral projection of the time operator

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Abstract: In this paper, we study the one-level Friedrichs model with using the quantum time super-operator that predicts the excited state decay inside the continuum. Its survival probability in long time limit is an algebraically decreasing function and an exponentially decreasing multiplied by the oscillating functions.

1 Introduction

In this paper we shall study the concept of survival probability of an unstable quantum system introduced in [1] and we shall test it in the Friedrichs model [2]. The survival probability should be a monotonically decreasing time function and this property could not exist in the framework of the usual Weisskopf-Wigner approach [3, 4, 5, 6]. It could only be properly treated if it is defined through an observable time operator T whose eigenprojections provide the probability distribution of the time of decay. The equation defining time operator is the following:

$$U_{-t}TU_t = T + tI \quad (1.1)$$

where U_t is the unitary group of states evolution. It is known that such an operator cannot exist when the evolution is governed by the Schrödinger equation, since the Hamiltonian has a bounded spectrum from below, and this contradicts the equation:

$$[H, T] = iI \quad (1.2)$$

in the Hilbert space of pure states \mathcal{H} . However, a time operator can exist under some conditions, for mixed states. They can be embedded [1, 7, 8] in The “Liouville space”, denoted \mathcal{L} , that is the space of Hilbert-Schmidt operators ρ on \mathcal{H} such that $Tr(\rho^*\rho) < \infty$, equipped with the scalar product: $\langle \rho, \rho' \rangle =$

$Tr(\rho^* \rho')$. The time evolution of these operators is given by the Liouville von-Neumann group of operators:

$$U_t \rho = e^{-itH} \rho e^{itH} \quad (1.3)$$

The infinitesimal self-adjoint generator of this group is the Liouville von-Neumann operator L given by:

$$L\rho = H\rho - \rho H \quad (1.4)$$

That is, $U_t = e^{-itL}$. The states of a quantum system are defined by normalized elements $\rho \in \mathcal{L}$ with respect to the scalar product, the expectation of T in the state ρ is given by:

$$\langle T \rangle_\rho = \langle \rho, T\rho \rangle \quad (1.5)$$

and the ‘‘uncertainty’’ of the observable T as its fluctuation in the state ρ :

$$(\Delta T)_\rho = \sqrt{\langle T^2 \rangle_\rho - (\langle T \rangle_\rho)^2} \quad (1.6)$$

Let \mathcal{P}_τ denote the family of spectral projection operators of T defined by:

$$T = \int_{\mathbb{R}} \tau d\mathcal{P}_\tau \quad (1.7)$$

It is shown that [1] the unstable states are those states verifying $\rho = \mathcal{P}_0 \rho$. Let ΔE be the usual energy uncertainty in the state M given by:

$$\Delta E = \sqrt{\text{Tr}(M.H^2) - (\text{Tr}(M.H))^2} \quad (1.8)$$

and $\Delta T = (\Delta T)_{M^{1/2}}$ be the uncertainty of T in the state M defined as in (1.6). It has been shown that:

$$\Delta E \Delta T \geq \frac{1}{2\sqrt{2}} \quad (1.9)$$

This uncertainty relation leads to the interpretation of T as the time occurrence of specified random events. The time of occurrence of such events fluctuates and we speak of the probability of its occurrence in a time interval $I =]t_1, t_2]$. The observable T' associated to such event in the initial state ρ_0 has to be related to the time parameter t by:

$$\langle T' \rangle_{\rho_t} = \langle T' \rangle_{\rho_0} - t \quad (1.10)$$

where $\rho_t = e^{-itL} \rho_0$. Comparing this condition with the above Weyl relation we see that we have to define T' as: $T' = -T$. Let \mathcal{Q}_τ be the family of spectral projections of T' , then, in the state ρ , the probability of occurrence of the event in a time interval I is given, as in the usual von Neumann formulation, by:

$$\mathcal{P}(I, \rho) = \|\mathcal{Q}_{t_2} \rho\|^2 - \|\mathcal{Q}_{t_1} \rho\|^2 = \|(\mathcal{Q}_{t_2} - \mathcal{Q}_{t_1}) \rho\|^2 := \|\mathcal{Q}(I) \rho\|^2 \quad (1.11)$$

The unstable ‘‘undecayed’’ states prepared at $t_0 = 0$ are the states ρ such that $\mathcal{P}(I, \rho) = 0$ for any negative time interval I , that is:

$$\|\mathcal{Q}_\tau \rho\|^2 = 0, \quad \forall \tau \leq 0 \quad (1.12)$$

In other words, these are the states verifying $\mathcal{Q}_0\rho = 0$. It is straightforwardly checked that the spectral projections \mathcal{Q}_τ are related to the spectral projections \mathcal{P}_τ by the following relation:

$$\mathcal{Q}_\tau = 1 - \mathcal{P}_{-\tau} \quad (1.13)$$

Thus, the unstable states are those states verifying: $\rho = \mathcal{P}_{00}\rho$ and they coincide with our subspace \mathcal{F}_0 ¹. For these states, the probability that a system prepared in the undecayed state ρ is found to decay sometime during the interval $I =]0, t]$ is $\|\mathcal{Q}_t\rho\|^2 = 1 - \|\mathcal{P}_{-t}\rho\|^2$ a monotonically nondecreasing quantity which converges to 1 as $t \rightarrow \infty$ for $\|\mathcal{P}_{-t}\rho\|^2$ tends monotonically to zero. As noticed by Misra and Sudarshan [3], such quantity could not exist in the usual quantum mechanical treatment of the decay processes and could not be related to the “survival probability” for it is not a monotonically decreasing quantity in the Hilbert space formulation. In the Liouville space, given any initial state ρ , its survival probability in the unstable space is given by:

$$p_\rho(t) = \|\mathcal{P}_0 e^{-iLt} \rho\|^2 \quad (1.14)$$

Hence, in the Liouville space, given any initial state ρ , its survival probability in the unstable space is given by:

$$\begin{aligned} p_\rho(t) &= \|\mathcal{P}_0 e^{-itL} \rho\|^2 \\ &= \|U_{-t} \mathcal{P}_0 U_t \rho\|^2 \\ &= \|\mathcal{P}_{-t} \rho\|^2 \end{aligned} \quad (1.15)$$

Then, the survival probability is monotonically decreasing to 0 as $t \rightarrow \infty$. As \mathcal{P}_t is a spectral family of projections $p_\rho(t) \rightarrow 1$ when $t \rightarrow -\infty$. This survival probability and the probability of finding the system to decay sometime during the interval $I =]0, t]$, $q_\rho(t) = \|\mathcal{Q}_\rho(t)\|^2$ are related by:

$$q_\rho(t) = 1 - p_\rho(t) \quad (1.16)$$

2 Spectral projections of time operator

The expression of time operator is given in a spectral representation of H . As shown in [1], H should have an unbounded absolutely continuous spectrum. In the simplest case, we shall suppose that H is represented as the multiplication operator on $\mathcal{H} = L^2(\mathbb{R}^+)$:

$$H\psi(\lambda) = \lambda\psi(\lambda) \quad (2.17)$$

the Hilbert-Schmidt operators on $L^2(\mathbb{R}^+)$ correspond to the square-integrable functions $\rho(\lambda, \lambda') \in L^2(\mathbb{R}^+ \times \mathbb{R}^+)$ and the Liouville-Von Neumann operator L is given by :

$$L\rho(\lambda, \lambda') = (\lambda - \lambda')\rho(\lambda, \lambda') \quad (2.18)$$

¹We define the subspace \mathcal{F}_{t_0} to the set of decaying states prepared at time t_0

Then we obtain a spectral representation of L via the change of variables:

$$\nu = \lambda - \lambda' \quad (2.19)$$

and

$$E = \min(\lambda, \lambda') \quad (2.20)$$

This gives a spectral representation of L :

$$L\rho(\nu, E) = \nu\rho(\nu, E), \quad (2.21)$$

where $L\rho(\nu, E) \in L^2(\mathbb{R} \times \mathbb{R}^+)$. In this representation $T\rho(\nu, E) = i\frac{d}{d\nu}\rho(\nu, E)$ so that the spectral representation of T is obtained by the inverse Fourier transform:

$$\hat{\rho}(\tau, E) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{i\tau\nu} \rho(\nu, E) d\nu = (\mathcal{F}^* \rho)(\tau, E) \quad (2.22)$$

and

$$T\hat{\rho}(\tau, E) = \tau\hat{\rho}(\tau, E). \quad (2.23)$$

The spectral projection operators \mathcal{P}_s of T are given in the (τ, E) -representation by:

$$\mathcal{P}_s \hat{\rho}(\tau, E) = \chi_{]-\infty, s]}(\tau) \hat{\rho}(\tau, E) \quad (2.24)$$

where $\chi_{]-\infty, s]}$ is the characteristic function of $]-\infty, s]$. So that we obtain in the (ν, E) -representation the following expression of these spectral projection operators:

$$\begin{aligned} \mathcal{P}_s \hat{\rho}(\nu, E) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^s e^{-i\nu\tau} \hat{\rho}(\tau, E) d\tau \\ &= e^{-i\nu s} \int_{-\infty}^0 e^{-i\nu\tau} \hat{\rho}(\tau + s, E) d\tau. \end{aligned} \quad (2.25)$$

Let us denote the Fourier transform $\mathcal{F}f(\nu) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\nu\tau} f(\tau) d\tau$ and remind the Paley-Wiener theorem which says that a function $f(\nu)$ belongs to the Hardy class H^+ (i.e. the limit as $y \rightarrow 0^+$ of an analytic function $\Phi(\nu + iy)$ such that $\int_{-\infty}^{\infty} |\Phi(\nu + iy)|^2 dy < \infty$) if and only if it is of the form $f(\nu) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^0 e^{-i\nu\tau} \hat{f}(\tau) d\tau$ where $\hat{f} \in L^2(\mathbb{R}^+)$ [9]. Using the Hilbert transform:

$$Hf(x) = \frac{1}{\pi} \text{P} \int_{-\infty}^{\infty} \frac{f(t)}{t-x} dt \quad (2.26)$$

for $f \in L^2(\mathbb{R})$ we can write the decomposition:

$$\begin{aligned} f(x) &= \frac{1}{2}[f(x) - iHf(x)] + \frac{1}{2}[f(x) + iHf(x)] \\ &= f_+(x) + f_-(x) \end{aligned} \quad (2.27)$$

According to the theorem, $f_+(x)$ (resp. $f_-(x)$) belongs to the Hardy class H^+ (resp. H^-). This decomposition is unique as a result of Paley-Wiener theorem. Thus taking the Fourier transformation of f we obtain :

$$\mathcal{F}(f)(\nu) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^0 e^{-i\nu\tau} \hat{f}(\tau) d\tau + \frac{1}{\sqrt{2\pi}} \int_0^{\infty} e^{-i\nu\tau} \hat{f}(\tau) d\tau.$$

It follows that:

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^0 e^{-i\nu\tau} \hat{f}(\tau) d\tau = \frac{1}{2}(\mathcal{F}(f) - iH\mathcal{F}(f)). \quad (2.28)$$

Now, using the well known property of the translated Fourier transformation $\sigma_s \hat{f}(\tau) = \hat{f}(\tau + s)$ we have :

$$\mathcal{F}(\sigma_s \hat{f})(\nu) = e^{i\nu s} \mathcal{F}.\hat{f}(\nu) = e^{i\nu s} f(\nu), \quad (2.29)$$

this and (2.26) yields:

$$\mathcal{P}_s \rho(\nu, E) = \frac{1}{2} e^{-i\nu s} [e^{i\nu s} \rho(\nu, E) - iH(e^{i\nu s} \rho(\nu, E))]. \quad (2.30)$$

Thus:

$$\mathcal{P}_s \rho(\nu, E) = \frac{1}{2} [\rho(\nu, E) - ie^{-i\nu s} H(e^{i\nu s} \rho(\nu, E))]. \quad (2.31)$$

It is clear from (1.15) that $\mathcal{P}_s \rho(\nu, E)$ is in the Hardy class H^+ .

3 Computation of spectral projections of T in a Friedrichs model

The one-level Friedrichs model is a simple model Hamiltonian in which a discrete eigenvalue the free Hamiltonian H_0 . It has been often used as a simple model of decay of unstable states illustrating the Weisskopf-Wigner theory of decaying quantum systems. The Hamilton operator H is an operator on the Hilbert space of the wave functions of the form $|\psi\rangle = \{f_0, g(\omega)\}$, $f_0 \in \mathbb{C}$, $g \in L^2(\mathbb{R}^+)$,

$$H = H_0 + \lambda V, \quad (3.32)$$

where λ is a positive coupling constant, and

$$H_0 |\psi\rangle = \{\omega_1 f_0, \omega g(\omega)\}, (\omega_1 > 0). \quad (3.33)$$

We shall denote the eigenfunction of H_0 by $\chi = \{1, 0\}$. The operator V is given by:

$$V\{f, g(\omega)\} = \{\langle v(\omega), g(\omega) \rangle, f_0.v(\omega)\}. \quad (3.34)$$

Thus H can be represented as a matrix :

$$H = \begin{pmatrix} \omega_1 & \lambda v^*(\omega) \\ \lambda v(\omega) & \omega \end{pmatrix}, \quad (3.35)$$

where $v(\omega) \in L^2(\mathbb{R}^+)$ and it is called a factor form. It has been shown than for λ small enough, H has no eigenvalues and that the spectrum of H is continuous extending over \mathbb{R}^+ . It is also shown that in the outgoing spectral representation of H , the vector χ is represented by:

$$f_1(\omega) = \frac{\lambda v(\omega)}{\eta^+(\omega + i\epsilon)}, \quad (3.36)$$

where

$$\eta^+(\omega + i\epsilon) = \omega - \omega_1 + \lambda^2 \lim_{\epsilon \rightarrow 0} \int_0^\infty \frac{|v(\omega)|^2}{\omega' - \omega - i\epsilon} d\omega' \quad (3.37)$$

and $H\chi$ is represented $\omega f_1(\omega)$. The quantity $\langle \chi, e^{-iHt}\chi \rangle$ is usually called the decay law and $|\langle \chi, e^{-iHt}\chi \rangle|^2 = \int_0^\infty |f_1(\omega)|^2 e^{-i\omega t} d\omega$ is called the survival probability at time t . It is however clear that this is not a true probability, since it is not a monotonically decreasing quantity, although it tends to zero as a result of the Riemann-Lebesgue lemma. Let us now identify the state χ with element $\rho = |\chi\rangle\langle\chi|$ of the Liouville space, that is, to the kernel operator:

$$\rho_{11}(\omega, \omega') = f_1(\omega) \overline{f_1(\omega')}. \quad (3.38)$$

We shall compute first the unstable component $\mathcal{P}_0\rho_{11}$ and show that $\mathcal{P}_0\rho_{11} \neq \rho_{11}$. Then we shall compute the survival probability in the state ρ .

$$\lim_{s \rightarrow \infty} \|\mathcal{P}_{-s}\rho\|^2 \rightarrow 0. \quad (3.39)$$

4 Computation of $\mathcal{P}_s\rho_{11}$

As explained above the Liouville operator is given by:

$$L\rho(\omega, \omega') = (\omega - \omega')\rho(\omega, \omega') \quad (4.40)$$

and that the spectral representation of L is given by the change of variables:

$$\nu = \omega - \omega' \quad (4.41)$$

and

$$E = \min(\omega, \omega'). \quad (4.42)$$

Thus we obtain for $\rho_{11}(\nu, E)$:

$$\rho_{11}(\nu, E) = \begin{cases} \lambda^2 \frac{v(E)}{\eta^-(E)} \frac{v^*(E+\nu)}{\eta^+(E+\nu)} & \nu > 0 \\ \lambda^2 \frac{v^*(E)}{\eta^+(E)} \frac{v(E-\nu)}{\eta^-(E-\nu)} & \nu < 0. \end{cases} \quad (4.43)$$

where η^- is the complex conjugate of η^+ .

$$\eta^+(\omega) \simeq \omega - z_1, \quad z_1 = \tilde{\omega}_1 - i\frac{\gamma}{2} \quad (4.44)$$

where z_1 is called the resonance with energy $\tilde{\omega}_1$ and a lifetime γ [10]. It is believed that this form results from weak coupling approximations. It can be shown $\rho_{11}(\nu, E)$ in the following form:

$$\rho_{11}(\nu, E) = \frac{\gamma}{2} f(\nu), \quad (4.45)$$

where

$$f(\nu) = \begin{cases} \frac{1}{\nu_0^*(\nu + \nu_0)} & \nu > 0 \\ \frac{1}{\nu_0(\nu_0^* - \nu)} & \nu < 0. \end{cases} \quad (4.46)$$

where $\nu_0 = a + ib = (E - \tilde{\omega}_1) + i\frac{\gamma}{2}$. For obtaining $\mathcal{P}_s(f)(\nu)$, we shall use the formula (2.31) and we obtain

$$\begin{aligned} \mathcal{P}_s f(\nu) = & i e^{-is\nu} \left[\frac{-1}{2\pi\nu_0(\nu_0^* - \nu)} \left(\int_{-\infty}^0 \frac{e^{-sy}}{y + i\nu_0^*} dy - \int_{-\infty}^0 \frac{e^{-sy}}{y + i\nu} dy \right) \right. \\ & \left. + \frac{1}{2\pi\nu_0^*(\nu + \nu_0)} \left(\int_{-\infty}^0 \frac{e^{-sy}}{y - i\nu_0} dy - \int_{-\infty}^0 \frac{e^{-sy}}{y + i\nu} dy \right) \right] \\ & + \begin{cases} e^{-is\nu} \left[\frac{e^{is\nu_0^*}}{\nu_0(\nu_0^* - \nu)} - \frac{e^{-is\nu_0}}{\nu_0^*(\nu_0 + \nu)} \right], & E < \tilde{\omega}_1 \\ 0, & E > \tilde{\omega}_1. \end{cases} \end{aligned} \quad (4.47)$$

In this equation the non integrals terms yield a poles and lead to the resonance shown in equation (4.53), and the integral terms yield an algebraical term analog to the background in the Hamiltonian theories [6, 11, 12]. We can also compute the same result for the case $\nu < 0$.

4.1 Case $s = 0$

In this case (4.47) can be obtained as:

$$\begin{aligned} \mathcal{P}_0 f(\nu) = & \frac{i}{\nu_0(\nu_0^* - \nu)} \log^+ \left(\frac{\nu}{\nu_0^*} \right) - \frac{i}{\nu_0^*(\nu + \nu_0)} \log^+ \left(-\frac{\nu}{\nu_0} \right) \\ & + \begin{cases} \left[\frac{1}{\nu_0(\nu_0^* - \nu)} - \frac{1}{\nu_0^*(\nu_0 + \nu)} \right], & E < \tilde{\omega}_1 \\ 0, & E > \tilde{\omega}_1. \end{cases} \end{aligned} \quad (4.48)$$

where $\log^+ z$ is the complex analytic function with cut-line along the negative axis:

$$\log^+ z = \log |z| + i \arg(z), \quad \arg(z) \in] -\frac{\pi}{2}, \frac{3\pi}{2}[. \quad (4.49)$$

Also, we used $\lim_{R \rightarrow \infty} \log^+ \left(\frac{i\nu - R}{i\nu_0^* - R} \right) \rightarrow 0$ and $\lim_{R \rightarrow \infty} \log^+ \left(\frac{i\nu - R}{-i\nu_0 - R} \right) \rightarrow 0$.

We see that $\mathcal{P}_0 f(\nu)$ is an upper Hardy class function. This verified the general theorem about the properties of the unstable states associated to time operator, as being in the upper Hardy class.

4.2 Asymptotical behavior of the survival probability

First, using the following approximation, for $s \rightarrow -\infty$

$$\begin{aligned}
\int_{-\infty}^0 \frac{e^{-sz}}{y+z} dy &= e^{sz} \int_{-\infty}^z \frac{e^{-su}}{u} du \\
&= e^{sz} \left\{ \left[\frac{e^{-su}}{-su} \right]_{-\infty}^z - \int_{-\infty}^z \frac{e^{-su}}{su^2} du \right\} \\
&= \frac{1}{(-zs)} \left[1 + \frac{1}{(-zs)} + \frac{2!}{(-zs)^2} + \cdots + \frac{n!}{(-zs)^n} + r_n(-zs) \right]
\end{aligned} \tag{4.50}$$

where the last result was obtained by integral part by part repetitions, z can be a complex number, and

$$r_n(z) = (n+1)! z e^{-z} \int_{-\infty}^z \frac{e^t}{t^{n+2}} dt. \tag{4.51}$$

and we have [13]

$$|r_n(z)| \leq \frac{(n+1)!}{|z|^{n+1}}. \tag{4.52}$$

Thus, by using the above approximation in the equations (4.47) and (4.45) for $s \rightarrow -\infty$ we obtain an estimate of the survival probability:

$$\int_0^\infty \int_{-\infty}^{+\infty} |\mathcal{P}_s \rho_{11}(\nu, E)|^2 d\nu dE \leq \frac{\gamma^2}{4} \left[\frac{h(\gamma, \tilde{\omega}_1)}{\gamma^4 s^4} + e^{\gamma s} h_1(s, \gamma, \tilde{\omega}_1) \right]. \tag{4.53}$$

where:

$$\begin{aligned}
h(\gamma, \tilde{\omega}_1) &= \left(\frac{256}{\pi \gamma^2} \right) \left[\frac{7\pi}{64} + \frac{7}{32} \arctan \frac{2\tilde{\omega}_1}{\gamma} - \frac{1}{12} \sin^3(2 \arctan \frac{2\tilde{\omega}_1}{\gamma}) + \frac{1}{4} \sin(2 \arctan \frac{2\tilde{\omega}_1}{\gamma}) \right. \\
&\quad \left. - \frac{1}{16} \sin(4 \arctan \frac{2\tilde{\omega}_1}{\gamma}) + \frac{1}{256} \sin(8 \arctan \frac{2\tilde{\omega}_1}{\gamma}) \right]
\end{aligned} \tag{4.54}$$

and

$$h_1(s, \gamma, \tilde{\omega}_1) = 2 \left[\frac{\pi}{\gamma} \arctan \frac{2\tilde{\omega}_1}{\gamma} + \frac{\gamma \sin(2\tilde{\omega}_1 s) - 2\tilde{\omega}_1 \cos(2\tilde{\omega}_1 s)}{s(\tilde{\omega}_1^2 + \frac{\gamma^2}{4})} \right] \tag{4.55}$$

Here we have an algebraically decreasing function and an exponentially decreasing multiplied by the oscillating functions.

5 Conclusion

We have shown that the pure initial state $\rho(t) = |\psi_t\rangle\langle\psi_t|$, decomposes into decaying state and a background, $\rho(t) \rightarrow \mathcal{P}_0 \rho(t) + (1 - \mathcal{P}_0) \rho(t)$. In the other

hand, our result shows that the survival probability is decreasing for long time exponentially and algebraically, i.e. we do not have a Zeno effect [14, 15] for our survival probability.

Recently, we have studied 2-level Friedrichs model with weak coupling interaction constants for a decay phenomena in the Hilbert space for kaonic system [16, 17]. In future, we shall consider 2-level or n -level Friedrichs by using time super-operator in the Liouville space to study in order an irreversible decay description.

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