



## HYPERBOLICITY OF GENERAL DEFORMATIONS: PROOFS

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ABSTRACT. We modify the deformation method from [9] in order to construct further examples of Kobayashi hyperbolic surfaces in  $\mathbb{P}^3$  of any even degree  $d \geq 8$ .

Given a hypersurface  $X_d = f_d^*(0)$  in  $\mathbb{P}^n$  of degree  $d$ , we say that a (very) general small deformation of  $X_d$  is hyperbolic if for any (very) general degree  $d$  hypersurface  $X_\infty = g_d^*(0)$  and for all sufficiently small  $\varepsilon \in \mathbb{C} \setminus \{0\}$  (depending on  $X_\infty$ ) the hypersurface  $X_{d,\varepsilon} = (f_d + \varepsilon g_d)^*(0)$  is Kobayashi hyperbolic. With this definition let us formulate the following version of the Kobayashi Conjecture.

**Weak Kobayashi Conjecture.** *For every hypersurface  $X_d$  in  $\mathbb{P}^n$  of degree  $d \geq 2n - 1$ , a (very) general small deformation of  $X_d$  is Kobayashi hyperbolic.*

The original Kobayashi Conjecture claims, in particular, that a (very) general surface  $X_d$  of degree  $d \geq 5$  in  $\mathbb{P}^3$  is Kobayashi hyperbolic. This is known to hold indeed for a very general surface of degree at least 21 (see McQuillan [7] and Demailly-El Goul [2]).

By Brody's Theorem, a compact complex space  $X$  is hyperbolic if and only if any holomorphic map  $\mathbb{C} \rightarrow X$  is constant. Hence the proof of hyperbolicity reduces to a certain degeneration principle for entire curves in  $X$ . The Green-Griffiths' proof of Bloch's Conjecture [6] provides a kind of such degeneration principle. According to this principle, every entire curve  $\varphi : \mathbb{C} \rightarrow X$  in a very general surface  $X \subseteq \mathbb{P}^3$  of degree  $d \geq 21$  satisfies an algebraic differential equation [2, 7]. See also [8, 12] for recent advances in higher dimensions.

The deformation method showed to be quite effective to construct examples of low degree hyperbolic surfaces in  $\mathbb{P}^3$ . A nice construction due to J. Duval [3] of a hyperbolic sextic  $X_\varepsilon \subseteq \mathbb{P}^3$  uses this method iteratively in 5 steps, so that  $\varepsilon = (\varepsilon_1, \dots, \varepsilon_5)$  has 5 subsequently small enough components. Hence  $X_\varepsilon$  belongs to a 5-dimensional linear system; however the deformation of  $X_0$  to  $X_\varepsilon$  neither is linear nor very generic.

In [9] we exhibited examples of some special surfaces  $X_d$  in  $\mathbb{P}^3$  of any given degree  $d \geq 8$  such that a *general* small deformation of  $X_d$  is Kobayashi hyperbolic. In these examples  $X_d = X'_{d'} \cup X''_{d''}$ , where  $d = d' + d''$ , is a union of two cones in  $\mathbb{P}^3$  with distinct vertices over plane hyperbolic curves in general position.

Let us indicate briefly the deformation method used in [9] (see also the references in [9, 10]). Given two hypersurfaces  $X_{d,0}$  and  $X_{d,\infty}$  in  $\mathbb{P}^n$  of the same degree  $d$ , we consider the pencil of hypersurfaces  $\{X_{d,\varepsilon}\}_{\varepsilon \in \mathbb{C}}$  generated by  $X_{d,0}$  and  $X_{d,\infty}$ . Assuming that for a sequence  $\varepsilon_n \rightarrow 0$ , the hypersurfaces  $X_{d,\varepsilon_n}$  are not hyperbolic, there exists a sequence of Brody entire curves  $\varphi_n : \mathbb{C} \rightarrow X_{d,\varepsilon_n}$  which converges to a (non-constant) Brody curve  $\varphi : \mathbb{C} \rightarrow X_{d,0}$ . Suppose in addition that the hypersurface  $X_{d,0}$  admits a rational map to a hyperbolic variety

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$\pi : X_{d,0} \dashrightarrow Y_0$  (to a curve  $Y_0$  of genus  $\geq 2$  in case where  $\dim X_{d,0} = 2$ ). Then necessarily  $\pi \circ \varphi = \text{cst}$ , provided that the composition  $\pi \circ \varphi$  is well defined. Anyhow, the limiting Brody curve  $\varphi : \mathbb{C} \rightarrow X_{d,0}$  degenerates.

For a union  $X_{d,0} = X'_{d',0} \cup X''_{d'',0}$  of two cones in general position in  $\mathbb{P}^3$  as in [9], there is a further degeneration principle. It prohibits to the image  $\varphi(\mathbb{C})$  to meet the double curve  $D = X'_{d',0} \cap X''_{d'',0}$  outside the points of  $D \cap X_{d,\infty}$ . Using the assumptions that  $d', d'' \geq 4$  and  $X_{d,\infty}$  is general this forces  $\varphi$  to be constant, contrary to our construction.

This applies in particular to the union of two quartic cones  $X'_{4,0} \cup X''_{4,0}$  in  $\mathbb{P}^3$  in general position. Modifying the construction in [9], in the present note we establish, in particular, hyperbolicity of a general deformation of a double quartic cone in  $\mathbb{P}^3$ , see Example 2.3 below.

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## 1. SOME TECHNICAL LEMMAS

Here we expose some preliminary facts that will be used in the next section. We let  $\Delta$  denote the unit disc in  $\mathbb{C}$ ,  $B^n$  the unit ball in  $\mathbb{C}^n$  and  $\text{Hol}(B^n)$  the space of all holomorphic functions on  $B^n$ . For two complex spaces  $X$  and  $Y$ ,  $\text{Hol}(X, Y)$  stands for the space of all holomorphic maps  $X \rightarrow Y$  with the usual topology.

**Lemma 1.1.** *Let  $f_0, f_\infty \in \text{Hol}(B^n)$  be such that  $f_0(0) = f_\infty(0) = 0$  and the divisors  $X_0 = f_0^*(0)$  and  $X_\infty = f_\infty^*(0)$  have no common component passing through 0. Let  $\Gamma = X_0 \cap X_\infty$  and  $X_\varepsilon = f_\varepsilon^{-1}(0)$ , where  $f_\varepsilon = f_0 + \varepsilon f_\infty$ . We assume that  $\nabla f_0|_\Gamma = 0$ . Let further  $\varphi_n \in \text{Hol}(\Delta, X_{\varepsilon_n})$ , where  $\varepsilon_n \rightarrow 0$ , be a sequence of holomorphic discs which converges to  $\varphi \in \text{Hol}(\Delta, X_0)$  with  $\varphi(0) = 0$ . Then necessarily  $d\varphi(0) \in T_0 X_\infty$ .*

*Proof.* The assertion is clearly true in the case where  $\varphi(\Delta) \subseteq \Gamma$ . So we will assume further that  $\varphi(\Delta) \not\subseteq \Gamma$ .

*Claim 1.* *Under the assumptions as above  $\varphi_n(t_n) \in \Gamma$  for some sequence  $t_n \rightarrow 0$ .*

*Proof of Claim 1.* Let us consider the holomorphic map

$$F : B^n \rightarrow \mathbb{C}^2, \quad z \mapsto (f_0(z), f_\infty(z)).$$

It is easily seen that  $F$  possesses the following properties:

- $F(0) = 0$ ;
- $F^{-1}(0) = \Gamma$ ;
- $F(X_{\varepsilon_n}) \subseteq l_n$ , where  $l_n := \{x + \varepsilon_n y = 0\} \subseteq \mathbb{C}^2$ ;
- $F(X_0) \subseteq l_0 := \{x = 0\}$ ;
- $F \circ \varphi_n(\Delta) \subseteq l_n$ ;
- $F \circ \varphi(\Delta) \subseteq l_0$ ,  $F \circ \varphi(0) = 0$ ,  $F \circ \varphi \not\equiv 0$ .

We let  $F \circ \varphi_n = (x_n(t), y_n(t))$  and  $F \circ \varphi = (0, y(t))$ . Thus  $x_n \rightarrow 0$  and  $y_n \rightarrow y$  as  $n \rightarrow \infty$ . Since  $y(0) = 0$  and  $y \not\equiv 0$ , we have  $y_n \not\equiv 0$ . By Rouché's Theorem there exists a sequence  $t_n \rightarrow 0$  such that  $y_n(t_n) = 0$ , so also  $x_n(t_n) = -\varepsilon_n y_n(t_n) = 0$ . Hence  $\varphi_n(t_n) \in \Gamma = X_0 \cap X_\infty$ , as claimed.  $\square$

It will be convenient for the rest of the proof to replace the given sequence  $(\varphi_n)$  by a new one  $(\psi_n)$ . We let  $\psi_n(t) = \varphi_n(a_n t + t_n)$  with  $(t_n)$  as in Claim 1 and  $a_n := 1 - |t_n| \rightarrow 1$ . Then  $\psi_n \in \text{Hol}(\Delta, X_{\varepsilon_n})$  and  $\psi_n \rightarrow \varphi$  as  $n \rightarrow \infty$ . Moreover  $p_n := \psi_n(0) = \varphi_n(t_n) \in \Gamma \forall n \geq 1$

and  $v_n := d\psi_n(0) \longrightarrow v := d\varphi(0)$  when  $n \longrightarrow \infty$ . Now the assertion follows immediately from the next claim.

*Claim 2.*  $v_n \in T_{p_n}X_\infty \forall n \geq 1$ .

*Proof of Claim 2.* We have:

$$\psi_n(t) = p_n + tv_n + \text{HOT}(t) \quad \text{and} \quad f_{\varepsilon_n}(x) = \langle \nabla f_{\varepsilon_n}(p_n), x - p_n \rangle + \text{HOT}(x - p_n),$$

where HOT means “the higher order terms”. Hence

$$(1) \quad f_{\varepsilon_n} \circ \psi_n(t) = \langle \nabla f_{\varepsilon_n}(p_n), v_n \rangle \cdot t + \text{HOT}(t).$$

Using (1) and the identity  $f_{\varepsilon_n} \circ \psi_n \equiv 0$  we obtain

$$0 = \langle \nabla f_{\varepsilon_n}(p_n), v_n \rangle = \langle \nabla f_0(p_n), v_n \rangle + \varepsilon_n \langle \nabla f_\infty(p_n), v_n \rangle = \varepsilon_n \langle \nabla f_\infty(p_n), v_n \rangle.$$

Indeed, by our assumption  $\nabla f_0|_\Gamma = 0$ , in particular  $\nabla f_0(p_n) = 0 \forall n \geq 1$ . This proves the claim.  $\square$

Consider, for instance, a pencil of degree  $d$  hypersurfaces

$$X_\varepsilon = (f_0 + \varepsilon f_\infty)^*(0) \quad \text{in} \quad \mathbb{P}^{n+1}$$

generated by

$$X_0 = X'_0 \cup X''_0 = f_0^*(0) \quad \text{and} \quad X_\infty = f_\infty^*(0).$$

Assume that  $D := X'_0 \cap X''_0 \subseteq X_\infty$ . Then for any sequence of entire curves  $\varphi_n : \mathbb{C} \rightarrow X_{\varepsilon_n}$  which converges to  $\varphi : \mathbb{C} \rightarrow X'_0$  we have by Lemma 1.1:

$$d\varphi(t) \in T_P X'_0 \cap T_P X_\infty \quad \forall P = \varphi(t) \in D.$$

Next we study an enumeration problem, which deals with the intersection of a general hypersurface and generators of a given cone in  $\mathbb{P}^{n+1}$ .

**Proposition 1.2.** *We let  $\widehat{Y} \subseteq \mathbb{P}^{n+1}$  be a cone over a variety  $Y \subseteq \mathbb{P}^n$ . We consider also a general hypersurface  $X \subseteq \mathbb{P}^{n+1}$  of degree  $e \geq 2 \dim Y$ . Then  $X$  meets every generator  $l = (PQ)$  of  $\widehat{Y}$ , where  $P$  is the vertex of the cone and  $Q$  runs over  $Y$ , in at least  $k = e - 2 \dim Y$  points transversally.*

*Proof.* We use below the following notation. For a pair  $(n, e) \in \mathbb{N}^2$  we let  $\mathbb{F}(n+1, e)$  denote the vector space of all homogeneous forms in  $n+2$  variables of degree  $e$  and  $\mathbb{P}(n+1, e)$  its projectivization. We let  $CY$  denote the affine cone over  $Y$  and  $CY^* = CY \setminus \{0\}$  the same cone with the vertex deleted. Let us fix coordinates in  $\mathbb{P}^{n+1}$  in such a way that  $P = (0 : \dots : 0 : 1)$  and  $Y \subseteq \{z_{n+1} = 0\}$ . If  $Q = (z_0 : \dots : z_n : 0) = (z' : 0) \in Y$  then

$$(PQ) = \{(z' : z_{n+1}) \mid z_{n+1} \in \mathbb{C}\} \cup \{P\}.$$

For a hypersurface  $X$  in  $\mathbb{P}^{n+1}$  of degree  $e$  its defining equation  $f = 0$  can be written in the form

$$(2) \quad f(z', z_{n+1}) = \sum_{i=0}^e a_i(z') z_{n+1}^{e-i} = 0,$$

where  $a_i$  is a homogeneous form in  $z'$  of degree  $i$ . Assuming that  $P \notin X$  i.e.,  $a_0 \neq 0$ , we can normalize the equation so that  $a_0 = 1$ . Fixing  $z' \in \mathbb{A}^{n+1}$  we specialize  $f$  to a monic polynomial  $f_{z'} \in \mathbb{C}[z_{n+1}]$  of degree  $e$ . In these terms the proposition asserts that for

$k = e - 2 \dim Y$  and for a general  $f \in \mathbb{F}(n+1, e)$ , the specialization  $f_{z'}$  has at least  $k$  simple roots whatever is the choice of  $z' \in CY^* \subseteq \mathbb{A}^{n+1}$ .

The affine chart

$$U = \mathbb{P}(n+1, e) \setminus \{a_0 = 0\}$$

can be identified with the affine space of all sequences of homogeneous forms  $a = (a_1, \dots, a_e)$  with  $\deg a_i = i$ . The specialization  $(f, z') \mapsto f_{z'}$  defines a morphism

$$\tilde{\rho} : U \times CY \rightarrow \text{Poly}_e,$$

where  $\text{Poly}_e$  stands for the affine variety of all monic polynomials of degree  $e$ . In turn  $\text{Poly}_e$  can be identified with  $\text{Symm}_e(\mathbb{A}^1) \cong \mathbb{A}^e$ .

Let us consider further the Vieta map

$$\nu : \mathbb{A}^e \rightarrow \text{Poly}_e, \quad (\lambda_1, \dots, \lambda_e) \mapsto p(z) = \prod_{i=1}^e (z - \lambda_i).$$

This is a ramified covering of degree  $e!$ . For a multi-index  $\bar{n} = (n_1, \dots, n_s)$  with  $\sum_{i=1}^s n_i = e$  we let

$$\Sigma'_{\bar{n}} = \nu(D_{\bar{n}}) \subseteq \text{Poly}_e,$$

where  $D_{\bar{n}}$  is the linear subspace of  $\mathbb{A}^e$  given by equations

$$\lambda_1 = \dots = \lambda_{n_1}, \quad \lambda_{n_1+1} = \dots = \lambda_{n_1+n_2}, \quad \dots, \quad \lambda_{n_1+\dots+n_{s-1}+1} = \dots = \lambda_e.$$

Clearly both  $D_{\bar{n}}$  and  $\Sigma'_{\bar{n}}$  have pure dimension  $s$ . Letting

$$\Sigma'_k = \bigcup_{n_k \geq 2} \Sigma'_{\bar{n}} \subseteq \text{Poly}_e$$

denote the variety of all monic polynomials of degree  $e$  with at most  $k-1$  simple roots, we have

$$\dim \Sigma'_k = \max_{n_k \geq 2} \{\dim \Sigma'_{\bar{n}}\} = k-1 + \left\lfloor \frac{e-k+1}{2} \right\rfloor.$$

If  $e-k+1$  is even then the latter maximum is achieved for

$$n_1 = \dots = n_{k-1} = 1, \quad n_k = \dots = n_s = 2,$$

and otherwise for

$$n_1 = \dots = n_{k-2} = 1, \quad n_{k-1} = \dots = n_s = 2.$$

Anyhow

$$\text{codim}(\Sigma'_k, \text{Poly}_e) = 1 + \left\lfloor \frac{e-k}{2} \right\rfloor.$$

*Claim 1. The restriction  $d\tilde{\rho}|_{TU}$  is surjective at every point  $(a, z') \in U \times CY^*$ . In particular  $d\tilde{\rho}$  has maximal rank  $e$  at every such point.*

*Proof of Claim 1.* For a point  $(a, z') = (a_1, \dots, a_e, z_0, \dots, z_n) \in U \times CY^*$  we let

$$a^0 = (a_1^0, \dots, a_e^0) \in \mathbb{A}^e, \quad \text{where } a_i^0 = a_i(z'), \quad i = 1, \dots, e.$$

Since  $z' \neq 0$ , for an arbitrary tangent vector  $b^0 = (b_1^0, \dots, b_e^0) \in \mathbb{A}^e$  there exists a  $e$ -tuple of homogeneous forms  $b = (b_1, \dots, b_e)$  with  $\deg b_i = i$  such that  $b(z') = b^0$ . Therefore

$$(a + tb)(z') = a^0 + tb^0 \quad \text{and so} \quad d\tilde{\rho}(a^0, z')(b, 0) = b^0.$$

This proves Claim 1. □

By virtue of Claim 1,

$$\text{codim}(\tilde{\rho}^{-1}(\Sigma'_k), U \times CY^*) = \text{codim}(\Sigma'_k, \text{Poly}_e) = 1 + \left\lfloor \frac{e-k}{2} \right\rfloor.$$

Since

$$f_{\lambda z'}(z_{n+1}) = \lambda \cdot f_{z'}(z_{n+1}) = \lambda^{-e} f_{z'}(\lambda z_{n+1}) \quad \forall \lambda \in \mathbb{C}^*,$$

the subvariety  $\tilde{\rho}^{-1}(\Sigma'_k)$  of  $U \times CY^*$  is stable under the natural  $\mathbb{C}^*$ -action on the second factor. Hence

$$\text{codim}(\tilde{\rho}^{-1}(\Sigma'_k)/\mathbb{C}^*, U \times Y) = \text{codim}(\tilde{\rho}^{-1}(\Sigma'_k), U \times CY^*) = 1 + \left\lfloor \frac{e-k}{2} \right\rfloor.$$

Thus the general fibers of the projection

$$\text{pr}_2 : U \times Y \rightarrow U$$

do not meet  $\tilde{\rho}^{-1}(\Sigma'_k)/\mathbb{C}^* \subseteq U \times Y$  provided that

$$\dim Y \leq \left\lfloor \frac{e-k}{2} \right\rfloor.$$

The latter inequality is equivalent to  $k \leq e - 2 \dim Y$ , which fits our assumption. Now the proposition follows.  $\square$

**Remark 1.3.** Let us indicate an alternative approach. Given a projective variety  $Y \subseteq \mathbb{P}^n$  and a cone  $X \subseteq \mathbb{P}^{n+1}$  over  $Y$  with vertex  $P$ , for every  $k \geq 1$  we consider the subset  $\mathbb{F}(Y, e, k) \subseteq \mathbb{F}(n+1, e)$  of all forms  $f \in \mathbb{F}(n+1, e)$  such that the intersection divisor  $f^*(0) \cdot (PQ)$  has at most  $k-1$  reduced points on at least one generator  $l = (PQ)$  ( $Q \in Y$ ) of  $X$ . We let  $\mathbb{P}(Y, e, k)$  denote the projectivization of  $\mathbb{F}(Y, e, k)$ . Proposition 1.2 asserts that the complement  $\mathbb{P}(n+1, e) \setminus \mathbb{P}(Y, e, k)$  is a nonempty Zariski open subset of  $\mathbb{P}(n+1, e)$  provided that  $e \geq 2 \dim Y + k$ . We divide this into two claims; the first one is proved in a general setting, while for the second one we provide a simple argument in dimension 3 only.

*Claim 1.*  $\mathbb{P}(Y, e, k)$  is a Zariski closed subset of  $\mathbb{P}(n+1, e)$ .

*Proof of Claim 1.* Blowing up  $\mathbb{P}^{n+1}$  with center at  $P$  yields a fiber bundle  $\xi : \widehat{\mathbb{P}}^{n+1} \rightarrow \mathbb{P}^n$  with fiber  $\mathbb{P}^1$ . We let  $\text{Sym}_e(\xi)$  denote the  $e$ th symmetric power<sup>1</sup> of  $\xi$  over  $\mathbb{P}^n$ . Its fiber over a point  $Q \in \mathbb{P}^n$  consists of all effective divisors on  $\xi^{-1}(Q) \cong \mathbb{P}^1$  of degree  $e$ . Given a partition

$$e = \sum_{i=1}^k n_i \quad \text{with} \quad 1 \leq n_1 \leq n_2 \leq \dots \leq n_s$$

we let  $\Sigma_{\bar{n}}$ , where  $\bar{n} = (n_1, \dots, n_s)$ , denote the closed subbundle of  $\text{Sym}_e(\xi)$  whose fiber over  $Q$  consists of all effective divisors on  $\xi^{-1}(Q)$  of the form

$$\sum_{i=1}^s n_i [p_i], \quad \text{where} \quad p_i \in \xi^{-1}(Q).$$

We also let

$$\Sigma_k = \bigcup_{\bar{n}: n_k \geq 2} \Sigma_{\bar{n}}.$$

<sup>1</sup>That is the  $e$ th Cartesian power factorized by the natural action of the symmetric group of degree  $e$ .

The restriction map

$$\rho : f \longmapsto f^*(0) \cdot (PQ), \quad Q \in Y,$$

associates to  $f$  a section  $\rho(f)$  of  $\text{Symm}_e(\xi)$  over  $Y$ . It is easily seen that  $f \in \mathbb{F}(n+1, e)$  belongs to  $\mathbb{F}(Y, e, k)$  if and only if  $\rho(f)$  meets  $\Sigma_k$ .

We claim that the set, say,  $\Gamma_{e,k}$  of all sections of  $\text{Symm}_e(\xi)|_Y$  meeting  $\Sigma_k$  is a Zariski closed subset of  $\Gamma(Y, \mathcal{O}(\text{Symm}_e(\xi)|_Y))$ . More generally, given projective varieties  $X$  and  $Y$  and a subvariety  $S \subset Y$ , the set  $\mathcal{M}_S$  of all morphisms  $f : X \rightarrow Y$  such that the image  $f(X)$  meets  $S$  is a Zariski closed subset of  $\text{Mor}(X, Y)$ . Indeed, let us consider the incidence relation

$$I = \{(f, x, y) \in \text{Mor}(X, Y) \times X \times Y \mid f(x) = y\}.$$

Then  $\mathcal{M}_S = \pi_1(\pi_3^{-1}(S) \cap I)$  is Zariski closed, as claimed.

Consequently,  $\mathbb{P}(Y, e, k)$  is Zariski closed in  $\mathbb{P}(n+1, e)$ , as stated.  $\square$

*Claim 2.*  $\mathbb{P}(n+1, e) \setminus \mathbb{P}(Y, e, k) \neq \emptyset$  if  $n = 3$ .

Indeed, it is easy to see that the union  $X'$  of  $e$  planes in  $\mathbb{P}^3$  in general position belongs to this complement.  $\square$

Presumably the same holds in higher dimensions for unions of  $e$  hyperplanes in general position. However the latter is much less evident, so we've chosen above a different approach.

## 2. EXAMPLES

**Theorem 2.1.** *Let  $Y_0$  be a Kobayashi hyperbolic hypersurface in  $\mathbb{P}^n$  ( $n \geq 2$ ), where  $\mathbb{P}^n$  is realized as the hyperplane  $H = \{z_{n+1} = 0\}$  in  $\mathbb{P}^{n+1}$ . Then a general small deformation  $X_\varepsilon \subseteq \mathbb{P}^{n+1}$  of the double cone  $X_0 = 2\widehat{Y}_0$  over  $Y_0$  is Kobayashi hyperbolic.*

*Proof.* Suppose the contrary. Then letting  $X_\infty$  be a general hypersurface of degree  $2d = 2 \deg Y_0$  and  $(X_t)_{t \in \mathbb{P}^1}$  the pencil generated by  $X_0$  and  $X_\infty$ , we can find a sequence  $\varepsilon_n \rightarrow 0$  and a sequence of Brody curves  $\varphi_n : \mathbb{C} \rightarrow X_{\varepsilon_n}$  such that  $\varphi_n \rightarrow \varphi$ , where  $\varphi : \mathbb{C} \rightarrow \widehat{Y}_0$  is non-constant. We let  $\pi : \widehat{Y}_0 \dashrightarrow Y_0$  be the cone projection. Since  $Y_0$  is assumed to be hyperbolic we have  $\pi \circ \varphi = \text{cst}$ . In other words  $\varphi(\mathbb{C}) \subseteq l$ , where  $l \cong \mathbb{P}^1$  is a generator of the cone  $\widehat{Y}_0$ .

Letting  $Y_0 = f_0^*(0)$ , where  $f$  is a homogeneous form of degree  $d$  in  $z_0, \dots, z_n$ , we note that  $\nabla f_0^2|_{\widehat{Y}_0} = 0$ . If  $l$  and  $X_\infty$  meet transversally in a point  $\varphi(t) \in l \cap X_\infty$  then  $d\varphi(t) = 0$  by virtue of Lemma 1.1.

Since  $Y_0 \subseteq \mathbb{P}^n$  is hyperbolic and  $n \geq 2$  we have  $d \geq n + 2$ . In particular

$$\deg X_\infty = 2d \geq 2n + 4 \geq 2 \dim Y_0 + 5.$$

By Proposition 1.2,  $l$  and  $X_\infty$  meet transversally in at least 5 points. Hence the nonconstant meromorphic function  $\varphi : \mathbb{C} \rightarrow l \cong \mathbb{P}^1$  possesses at least 5 multiple values. Since the defect of a multiple value is  $\geq 1/2$ , this contradicts the Defect Relation.  $\square$

**Remark 2.2.** Given a hyperbolic hypersurface  $Y \subseteq \mathbb{P}^n$  of degree  $d$ , Theorem 2.1 provides a hyperbolic hypersurface  $X \subseteq \mathbb{P}^{n+1}$  of degree  $2d$ . Iterating the construction yields hyperbolic hypersurfaces in  $\mathbb{P}^n \forall n \geq 3$ . However, their degrees  $d(n)$  grow exponentially with  $n$ , whereas the best asymptotic achieved so far is  $d(n) = 4(n-1)^2$  (see e.g., [11]).

**Example 2.3.** Let  $C \subseteq \mathbb{P}^2$  be a hyperbolic curve of degree  $d \geq 4$ , and let  $\widehat{C} \subseteq \mathbb{P}^3$  be a cone over  $C$ . Then a general small deformation of the double cone  $X_0 = 2\widehat{C}$  is a Kobayashi hyperbolic surface in  $\mathbb{P}^3$  of even degree  $2d \geq 8$ .

The following degeneration principle can be proved along the same lines as Theorem 2.1.

**Proposition 2.4.** *Let  $(X_t)_{t \in \mathbb{P}^1}$  be a pencil of hypersurfaces in  $\mathbb{P}^{n+1}$  generated by two hypersurfaces  $X_0$  and  $X_\infty$  of the same degree  $d$ , where  $X_0 = kQ$  with  $k \geq 2$  for some hypersurface  $Q \subseteq \mathbb{P}^{n+1}$ , and  $X_\infty = \bigcup_{i=1}^d H_{a_i}$  ( $a_1, \dots, a_d \in \mathbb{P}^1$ ) is the union of  $d \geq 5$  distinct hyperplanes from a pencil of hyperplanes  $(H_a)_{a \in \mathbb{P}^1}$ . If a sequence of entire curves  $\varphi_n : \mathbb{C} \rightarrow X_{\varepsilon_n}$ , where  $\varepsilon_n \rightarrow 0$ , converges to an entire curve  $\varphi : \mathbb{C} \rightarrow X_0$ , then  $\varphi(\mathbb{C}) \subseteq X_0 \cap H_a$  for some  $a \in \mathbb{P}^1$ .*

**Examples 2.5.** Given a pencil of planes  $(H_a)_{a \in \mathbb{P}^1}$  in  $\mathbb{P}^3$ , using Proposition 2.4 one can deform

- $X_0 = 5Q$ , where  $Q \subseteq \mathbb{P}^3$  is a plane,
- a triple quadric  $X_0 = 3Q \subseteq \mathbb{P}^3$ , or
- a double cubic, quartic, etc.  $X_0 = 2Q \subseteq \mathbb{P}^3$

to an irreducible surface  $X_\varepsilon \in \langle X_0, X_\infty \rangle$  of the same degree  $d$ , where as before  $X_\infty = \bigcup_{i=1}^d H_{a_i}$ , so that every limiting entire curve  $\varphi : \mathbb{C} \rightarrow X_0$  is contained in a section  $X_0 \cap H_a$  for some  $a \in \mathbb{P}^1$ .

The famous Bogomolov-Green-Griffiths-Lang Conjecture on strong algebraic degeneracy (see e.g., [1, 6]) suggests that every surface  $S$  of general type possesses only finite number of rational and elliptic curves and, moreover, the image of any nonconstant entire curve  $\varphi : \mathbb{C} \rightarrow S$  is contained in one of them. In particular, this should hold for any smooth surface  $S \subseteq \mathbb{P}^3$  of degree  $\geq 5$ , which fits the Kobayashi Conjecture. Indeed, by Clemens-Xu-Voisin's Theorem, a general smooth surface  $S \subseteq \mathbb{P}^3$  of degree  $\geq 5$  does not contain rational or elliptic curves, hence it should be hyperbolic provided that the above conjecture holds indeed.

Anyhow, the deformation method leads to the following result, which is an immediate consequence of Proposition 2.4.

**Corollary 2.6.** *Let  $S \subseteq \mathbb{P}^3$  be a surface and  $Z \subset S$  be a curve such that the image of any nonconstant entire curve  $\varphi : \mathbb{C} \rightarrow S$  is contained in  $Z$ <sup>2</sup>. Let  $X_\infty$  be the union of  $d = 2 \deg S$  planes from a general pencil of planes in  $\mathbb{P}^3$ . Then any small enough linear deformation  $X_\varepsilon$  of  $X_0 = 2S$  in direction of  $X_\infty$  is hyperbolic.*

Along the same lines, Proposition 2.4 can be applied in the following setting.

**Example 2.7.** Let us take for  $X_0$  a double cone in  $\mathbb{P}^3$  over a plane hyperbolic curve of degree  $d \geq 4$ , and for  $X_\infty$  the union of  $2d$  distinct planes from a general pencil of planes  $(H_a)_{a \in \mathbb{P}^1}$ . Then small deformations  $X_\varepsilon$  of  $X_0$  in direction of  $X_\infty$  provide examples of hyperbolic surfaces of any even degree  $2d \geq 8$ . In suitable coordinates in  $\mathbb{P}^3$  such a surface can be given by equation

$$(3) \quad Q(X_0, X_1, X_2)^2 - P(X_2, X_3) = 0,$$

where  $P, Q$  are generic homogeneous forms of degree  $d = 2k$  and  $k$ , respectively. The latter are actually the Duval-Fujimoto examples [4, 5].

<sup>2</sup>The latter holds, for instance, if  $S$  is hyperbolic modulo  $Z$ .

Let us finally turn to the Kobayashi problem on hyperbolicity of complements of general hypersurfaces. By virtue of Kiernan-Kobayashi-M. Green's version of Borel's Lemma, the complement  $\mathbb{P}^n \setminus L$  of the union  $L = \bigcup_{i=1}^{2n+1} L_i$  of  $2n+1$  hyperplanes in  $\mathbb{P}^n$  in general position is Kobayashi hyperbolic. In particular, this applies to the union  $l$  of 5 lines in  $\mathbb{P}^2$  in general position. Moreover [13]  $l$  can be deformed to a smooth quintic curve with hyperbolic complement via a small deformation. This deformation proceeds in 5 steps and neither is linear nor very generic. So the following question arises.

**2.8. Question.** Let  $L$  ( $M$ ) stands for the union of  $2n+1$  ( $2n-1$ , respectively) hyperplanes in  $\mathbb{P}^n$  in general position. Is the complement of a general small linear deformation of  $L$  Kobayashi hyperbolic? Is a general small linear deformation of  $M$  Kobayashi hyperbolic? In particular, does the union of 5 lines in  $\mathbb{P}^2$  (of 5 planes in  $\mathbb{P}^3$ ) in general position admit a general small linear deformation to an irreducible quintic curve with hyperbolic complement (to a hyperbolic quintic surface, respectively)?

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