Existence results and a priori bounds for higher order elliptic equations and systems

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Abstract. We apply degree theory to prove the existence of positive solutions of semilinear elliptic systems. As an application we obtain a number of new results for higher order equations which appear frequently in applications. In particular, we extend to these equations and systems the notions of sublinearity and superlinearity, classical in the setting of second order equations.

Résumé. On utilise la théorie du degré topologique pour montrer l'existence de solutions de certaines classes de systèmes elliptiques semi-linéaires. En tant qu'application on obtient des résultats d'existence pour équations d'ordre supérieur, qui apparaissent fréquemment dans les applications. En particulier, on étend à ces équations et systèmes les notions de sous-linéarité et sur-linéarité, classiques dans le cadre des équations scalaires d'ordre deux.

Keywords : elliptic systems, degree theory, existence.

1 Introduction

In recent years there has been a great deal of work on elliptic equations like

$$(-\Delta)^m u = g(x,u), \tag{1}$$

or

$$\Delta^2 u + \beta \Delta u = g(x,u), \tag{2}$$

in a domain $\Omega \subseteq \mathbb{R}^N$, $N \geq 1$, for $m \in \mathbb{N}$, $\beta \in \mathbb{R}$, where g is some continuous function. This paper is a contribution to the study of existence and properties of positive solutions to problems for which (1) and (2) are model cases.

Two types of boundary conditions for (1) are most often considered, when $\Omega \neq \mathbb{R}^N$. These are Dirichlet boundary conditions $u = Du = \ldots = D^{m-1}u=0$ on $\partial\Omega$, and Navier boundary conditions

$$u = \Delta u = \ldots = \Delta^{m-1} u = 0 \qquad \text{on } \partial\Omega. \tag{3}$$

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Each of these conditions presents its own type of difficulties. We concentrate on the Navier problem here, consequently most of the references we give concern that problem. There are various motivations for studying such higher order boundary value problems. On one hand, (2) and its generalizations have recently been proposed as models for some phenomena in complex spatio-temporal pattern formation (see the reviews [37], [38], and more specifically [8], [39]). On the other hand, equations like (1) and (2) appear when studying the so-called Paneitz-Branson operator and its generalizations, which have many geometric properties (in particular, conformal invariance), and are important in mathematical physics – see [10], [12], [15], [21], and the references there. Higher order problems appear also in other areas of physics, for instance, the hinged plate problem – [34], [44].

Note that (1) and (2) are very particular cases of an elliptic system

$$-L_i u_i = f_i(x, u_1, \ldots, u_n), \quad i = 1, \ldots, n, \tag{4}$$

with, for example for (1), $L_i = \Delta$; $f_i(x, u) = u_{i+1}, i < m$; $f_m(x, u) = f(x, u_1)$. Apart from their applications mentioned above, systems of type (4) appear in many other situations, for instance in probability theory (switched diffusion processes, [13]) and stochastic control (switching costs problem, [28]). In 1982 P.L. Lions asked whether and to what extent known results for scalar equations can be extended to systems of this type - see open problem 4.2 (c) in [31]. This work is a part of a series devoted to providing answers to that question in some cases. In the next sections we give results on existence of solutions to system (4), whereas in the introduction we put the emphasis on applications to higher order equations, because of their importance. We stress that our results are new even for (1) and (2).

Let us describe our setting. Suppose Ω is a smooth bounded domain in $\mathbb{R}^N$, $N \geq 1$, and let L be a uniformly elliptic second order operator with Hölder continuous coefficients,

$$L = \sum_{i,j=1}^{N} a_{ij}(x)\partial_{ij} + \sum_{i=1}^{N} b_i(x)\partial_i + c(x),$$

with $\lambda|\xi|^2 \leq \sum a_{ij}\xi_i\xi_j$, $\xi \in \mathbb{R}^N$, for some $0 < \lambda \leq \nu$, and $|a_{ij}|, |b_i|, |c| \leq \nu$ in $\overline{\Omega}$. Assume L has a positive first eigenvalue $\lambda_1 = \lambda_1(L, \Omega)$ (that is, $-L\varphi_1 = \lambda_1\varphi_1$ for some $\varphi_1 > 0$ in Ω, $\varphi_1 = 0$ on $\partial\Omega$, see [6]). Of course $\lambda_1 > 0$ if $c \leq 0$.

Suppose f is a nonnegative Hölder function on $\overline{\Omega} \times \mathbb{R}_+$, and $\alpha_1 \in \mathbb{R}$,

$\alpha_i \geq 0, i = 2, \ldots, m-1$ (if $m \geq 3$) are constants. We consider the problem

$$(-L)^m u = \sum_{i=1}^{m-1} \alpha_i (-L)^{m-i} u + f(x,u) \quad \text{in } \Omega \tag{5}$$

$$(-L)^k u = 0,\ k \in \{0, 1, \ldots, m-1\} \qquad \text{on } \partial\Omega. \tag{6}$$

In the second order case $-Lu = f(u)$ (i.e. $m = 1$) there is a well developed existence theory when the behaviour of $f(u)/u$ at zero and at infinity is different with respect to λ_1. In particular, the development of degree theory (we refer to the classical works of Leray-Schauder [29], Krasnoselskii [27], Amann [1], Nussbaum [35]) permitted to show that if a second-order equation is *sublinear*, that is, $\liminf_{u\to 0} u^{-1} f(u) > \lambda_1 > \limsup_{u\to\infty} u^{-1} f(u)$, then it always has a positive solution, while if it is *superlinear* in the sense that $\limsup_{u\to 0} u^{-1} f(u) < \lambda_1 < \liminf_{u\to\infty} u^{-1} f(u)$, then it has a positive solution, provided it admits a priori bounds (see below).

Our first principal result is that the higher order equation (5)-(6) has the same property, and the dividing number is

$$\lambda^* = \max\left\{0, \lambda_1^m - \sum_{i=1}^{m-1} \alpha_i \lambda_1^{m-i}\right\}.$$

To give an example, let us compute λ^* for the standard Paneitz equation with constant coefficients $\Delta^2 u + \alpha\Delta u + au = f(x,u)$, $\alpha \in \mathbb{R}$, $a \geq 0$ (if $a < 0$ we replace f by $f - au$; to avoid confusion, note that here $\Delta = \sum \partial_{ii}$, and not $-\sum \partial_{ii}$ as in some other works on the subject). This equation can be recast in the form (5) if $\delta = \alpha^2 - 4a \geq 0$, for $L = \Delta + c_0$, with $2c_0 = \alpha \pm \sqrt{\delta}$, $\alpha_1 = \mp\sqrt{\delta}$. Hence by $\lambda_1(L) = \lambda_1(\Delta) - c_0$ we get $\lambda^* = (\lambda_1(\Delta)^2 - \alpha\lambda_1(\Delta) + a)_+$.

Theorem 1 *Suppose there exist $a, b \in \mathbb{R}$ such that*

$$\infty \geq \liminf_{u\to 0} \frac{f(x,u)}{u} \geq a > \lambda^* > b \geq \limsup_{u\to\infty} \frac{f(x,u)}{u}, \qquad \text{for } x \in \overline{\Omega}. \tag{7}$$

Then problem (5)-(6) has a positive solution in $C^{2m}(\overline{\Omega})$, with $(-L)^k u > 0$ in Ω, for all $k \in \{0, \ldots, m-1\}$.

Theorem 2 *Suppose there exist $a, b \in \mathbb{R}$ such that*

$$\limsup_{u\to 0} \frac{f(x,u)}{u} \leq a < \lambda^* < b \leq \liminf_{u\to\infty} \frac{f(x,u)}{u} \leq \infty, \qquad \text{for } x \in \overline{\Omega}. \tag{8}$$

Suppose in addition that problem (5) admits a priori bounds, in the following sense : for each $t_0 \geq 0$ there exists a constant C depending only on

$t_0, \lambda, \nu, m, N, \alpha_i, \Omega, f$, such that if $t \in [0, t_0]$, $u \in C^{2m}(\overline{\Omega})$ is a solution of (5) with $f(x,u)$ replaced by $f(x, u+t)$, and $(-L)^k u > 0$ in Ω, $(-L)^k u = 0$ on $\partial\Omega$, $k = 0, \ldots, m-1$, then $\|u\|_{L^\infty(\Omega)} \le C$.

Then problem (5)-(6) has a positive solution in $C^{2m}(\overline{\Omega})$, with $(-L)^k u > 0$ in Ω, for all $k \in \{0, \ldots, m-1\}$.

Remark 1. Extending the established terminology for the second order case (when $\lambda^* = \lambda_1$), we say that (7) (resp. (8)) means f is *sublinear* (resp. *superlinear*) in u. To our knowledge, these notions are being defined here for the first time for higher order elliptic partial differential equations.

Remark 2. Theorems 1 and 2 are consequences of results on system (4), which rely on degree theory and linear programming (for a general survey on use of degree theory in differential equations see [32]). It follows from these results, see Sections 2 and 4, that we can consider equations in which, instead of taking the powers of a given operator, we iterate different elliptic operators. We can also replace the constants α_i in (5) by functions. Further, we can get existence results for systems of higher order equations. All these statements have been postponed to the next sections, for the sake of conciseness and clarity of the introduction.

Remark 3. Theorems 1 and 2 hold if we suppose only that $A(x) \in C(\overline{\Omega})$, $b_i, c \in L^\infty(\Omega)$, and $f \in C(\overline{\Omega} \times \mathbb{R})$. In this case the solutions we obtain belong to $W^{2,p}(\Omega) \cap C^{2m-2}(\overline{\Omega}), p < \infty$.

Theorem 2 settles the existence question in the superlinear case, provided a priori bounds can be proved. Consequently, in the second part of the paper we study the availability of such bounds – an important question in itself. Here we shall concentrate on the situation where the nonlinearity f has *power growth* in u. Note that, when $\lambda_* > 0$, the model case $f(x,u) = u^p$ satisfies hypothesis (7) when $p \in (0,1)$, and hypothesis (8) when $p \in (1, \infty)$.

A large amount of work has been devoted to equations (1) and (2) with superlinear power-like nonlinearity, see for example [2], [3], [4], [5], [21], [23], [24], [26], as well as the references there. As is well-known, in (1) the growth of f with respect to u plays a crucial role. Specifically, for f behaving as u^p the number $p^* = (N+2m)/(N-2m)$ ($p^* = \infty$ if $N \le 2m$), plays the role of a critical exponent, similarly to the case $m = 1$. A large part of the previous works concentrate on the critical case, $p = p^*$. The supercritical case has recently been considered in [22].

In view of the historical development of the second-order case, it may seem surprising that there has been relatively little work on subcritical f, i.e. $p < p^*$. The reason for this is that previous works on (1) make use of variational methods and, generally, in the subcritical case these methods extend quite simply from $m = 1$ to any $m \in \mathbb{N}$. For instance, if f in (1) is subcritical and satisfies the conditions of the Mountain Pass theorem, then this theorem easily implies that problem (1)-(3) has a positive solution.

On the other hand, in the second-order case it has long been known that degree theory permits to prove existence of solutions in situations where variational methods cannot be employed. For instance, this is the case for most systems of equations. For higher order problems topological methods have been used in [14] , [36] and [41], where equation (1) in a *convex* domain was studied, obtaining a direct extension of the results for $m = 1$ in [19]. In these works the variational nature (divergence form) of the Laplacian was used in an essential way. We have the following result, which applies to general operators and domains.

Theorem 3 *Suppose f in (5) has subcritical power growth, that is, for some function $b \in C(\overline{\Omega})$ such that $b > 0$ in $\overline{\Omega}$, and for some $p \in (1, p^*)$*

$$\lim_{u\to\infty} \frac{f(x,u)}{u^p} = b(x), \qquad \text{for} \;\; x \in \overline{\Omega}.$$

Then (5) admits an a priori bound, as defined in Theorem 2.

A result of this type was obtained in [9], for the particular equation $(-L)^m u = f(x,u)$, under the stronger hypothesis $p \in (1, N/(N-2m)]$ - see Theorem 6.3 in that paper.

It is very well known that a priori bounds cannot be proven for the equation $-\Delta u = u^p$, provided $p \geq (N+2)/(N-2)$, more generally for $(-\Delta)^m u = u^p$, $p \geq (N+2m)/(N-2m)$. So the range in Theorem 3 is optimal.

Remark 4. Combining Theorems 2 and 3 yields an existence result for superlinear higher order equations with subcritical power growth. Note this result could not be obtained by variational methods, both because of the form of L and the form of f. In addition, by combining the methods and the results of this paper with those in [20], it is only a matter of technique to extend Theorem 3 to systems of higher order equations with power growth nonlinearities.

Remark 5. In some important in practice cases our results can be used to get existence in the whole space – see Theorem 5 in the next section.

In the proof of Theorem 3 we use the blow-up method of Gidas and Spruck, developed in [25] for the second-order scalar case, and recently extended to some systems of second-order equations in [20], [46] (see also the references in these works). We will show that this method can be used for another large class of systems, which covers the case we are interested in

Theorem 3. We note that the blow-up method contains a contradiction argument, which in turn relies on Liouville (nonexistence) theorems in $\mathbb{R}^N$ or in a half-space of $\mathbb{R}^N$, see Section 5.

The paper is organized as follows. In the next section we give more general existence results for systems, which contain Theorems 1 and 2 as particular cases. The proofs of these results are given in Section 3, while Section 4 contains some extensions and comments. In Section 5 we develop the blow-up method for a class of systems which include (5)-(6), and give the proof of Theorem 3. Finally, in Section 6 we prove some existence results in $\mathbb{R}^N$.

2 Elliptic systems – more principal results

We consider the system

$$(\mathcal{P}_{\mathbf{t}}) \left\{ \begin{array}{rcllll} -L_i u_i & = & f_i(x, u_1+t, \ldots, u_n+t) & \text{in} & \Omega, & i=1,\ldots,n \\ u_i & \geq & 0 & \text{in} & \Omega, & i=1,\ldots,n \\ u_i & = & 0 & \text{on} & \partial\Omega, & i=1,\ldots,n, \end{array} \right.$$

where $t \geq 0$, $n \in \mathbb{N}$, and $L_k = \sum_{i,j=1}^N a_{ij}^{(k)}(x)\partial_{ij} + \sum_{i=1}^N b_i^{(k)}(x)\partial_i + c^{(k)}(x)$ satisfy the hypotheses we made on L in Section 1.

Let us introduce some notations and conventions. We denote $\lambda_k = \lambda_1(L_k, \Omega) > 0$ and $\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_n) \in \mathcal{M}_n(\mathbb{R})$. We shall use the matrix notation $U = (u_1, \ldots, u_n)^T \in \mathbb{R}^n$, $F = (f_1, \ldots, f_n)^T$, $L = \mathrm{diag}(L_1, \ldots, L_n)$. We set $\vec{1} = (1, \ldots, 1) \in \mathbb{R}^n$. On $\mathbb{R}^n$ we use the norm $\|U\| = \max_{1\leq i\leq n} |u_i|$. Throughout the paper equalities and inequalities between vectors or matrices will be understood to hold component-wise. We define the following relation between matrices : if A and B are two $n \times n$ matrices,

$$A \prec B \iff \forall U \in \mathbb{R}^n \quad : \quad \left\{ \begin{array}{rcl} BU & \leq & AU \\ U & \geq & 0 \end{array} \right. \quad \text{implies} \quad U = 0. \tag{9}$$

Geometrically, if $B - A$ is invertible, (9) means that $A \prec B$ if the (closed) positive cone generated by the columns of $B - A$ does not meet the negative hyper-quadrant $\{U \leq 0\}$, except at the origin.

We suppose that f_i are Hölder (or just continuous, see Remark 3 in Section 1) functions, and for some $\xi_i \geq 0$

$$f_i(x, U) \geq -\xi_i u_i, \qquad \text{for all } U \in \mathbb{R}^n_+ := \{U \in \mathbb{R}^N \ : \ u_i \geq 0\}.$$

Replacing $c^{(i)}$ by $c^{(i)} - \xi_i$ (resp. λ_i by $\lambda_i + \xi_i$) we can assume $\xi_i = 0$.

We have the following result on existence of solutions of $(\mathcal{P}_0)$.

Theorem 4 *Suppose $L_1 \equiv \ldots \equiv L_n$ and either (sublinear case)*

(H_0) *there exist $r > 0$ and a matrix $B \in \mathcal{M}_n(\mathbb{R})$ such that for $x \in \overline{\Omega}$*

$$B \succ \Lambda \qquad \text{and} \qquad F(x,U) \geq BU \qquad \text{if} \quad \|U\| \leq r,\; U \in \mathbb{R}^n_+,$$

(H_∞) *there exist $k > 0$ and a matrix $A \in \mathcal{M}_n(\mathbb{R})$ such that for $x \in \overline{\Omega}$*

$$A \prec \Lambda \qquad \text{and} \qquad F(x,U) \leq AU + k\vec{1} \qquad \text{for all} \quad U \in \mathbb{R}^n_+,$$

or (superlinear case)

(H^0) *there exist $r > 0$ and a matrix $A \in \mathcal{M}_n(\mathbb{R})$ such that for $x \in \overline{\Omega}$*

$$A \prec \Lambda \qquad \text{and} \qquad F(x,U) \leq AU \qquad \text{if} \quad \|U\| \leq r,\; U \in \mathbb{R}^n_+,$$

(H^∞) *there exist $R > 0$ and a matrix $B \in \mathcal{M}_n(\mathbb{R})$ such that for $x \in \overline{\Omega}$*

$$B \succ \Lambda \qquad \text{and} \qquad F(x,U) \geq BU \qquad \text{if} \quad \min\{u_1,\ldots,u_n\} \geq R,$$

(APB) *for any $t_0 \geq 0$ there exists a constant M, depending only on t_0, Ω, n, N, λ, ν, and on the functions f_i, such that $\max\limits_{1\leq i\leq n} \sup\limits_{x\in\Omega} u_i(x) \leq M$ for any $t \in [0,t_0]$ and any solution $(u_1,\ldots,u_n)$ of $(\mathcal{P}_t)$.*

Then $(\mathcal{P}_0)$ has a nonnegative solution, such that $u_k > 0$ in Ω, for at least one $k \in \{1,\ldots,n\}$.

Remark 1. We note that we need the operators to coincide in the above theorem – this is the price to pay to have the nice and explicit hypotheses given by the relation "$\succ$". A more general (but less explicit) result for different L_i is given in Section 4, Theorem 8 (see also the remark following this theorem ; in Section 4 we also comment on issues of coercivity of matrix and higher order operators). Note that if L_i differ only in their zero-order coefficients, we can always make them equal by changing f_i.

Remark 2. A weaker variant of Theorem 4 for the case $L_i = \Delta$ was proved in [40], where the divergence form of the Laplacian was used in an essential way (note [40] was not written viewing applications to higher order equations). Here we employ a very different approach, relying on Farkas' lemma – quite an untypical tool in the field of elliptic PDE's – and on results on existence and properties of first eigenvalues of vector operators, obtained in [11].

Remark 3. Note that if F is differentiable at $u = 0$ then (H_0) (resp. (H^0)) reduces to $F'(x,0) \succ \Lambda$ (resp. $F'(x,0) \prec \Lambda$), for $x \in \overline{\Omega}$.

One gets Theorems 1 and 2 by applying Theorem 4 to the system

$$\begin{cases} -L_i u_i &= u_{i+1}, \quad i = 1, \dots, m-1 \\ -L_m u_m &= f(x, u_1) + \alpha_{m-1} u_2 + \dots + \alpha_1 u_m, \\ u_i &\geq 0 \quad \text{in } \Omega, \quad i = 1, \dots, m \\ u_i &= 0 \quad \text{on } \partial\Omega, \quad i = 1, \dots, m, \end{cases} \tag{10}$$

and by using the following proposition (obtained by a simple computation and the definition of the relation "$\succ$")

Proposition 2.1 *For any $n \in \mathbb{N}$, $\mu_i \geq 0, 1 \leq i \leq n-1, \mu_n \in \mathbb{R}$, the $n \times n$ matrix*

$$M = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 0 & 1 \\ \mu_1 & \mu_2 & \mu_3 & \dots & \mu_{n-1} & \mu_n \end{pmatrix}$$

is such that $M \prec \Lambda$ (resp. $M \succ \Lambda$) provided

$$\sum_{j=1}^{n} \frac{\mu_j}{\lambda_j \dots \lambda_n} < 1 \qquad \left(\text{resp. } \mu_1 > 0 \text{ and } \sum_{j=1}^{n} \frac{\mu_j}{\lambda_j \dots \lambda_n} > 1 \right).$$

Note that a simple application of the maximum principle and the strong maximum principle for scalar elliptic operators shows that if $(u_1, \dots, u_n)$ is a solution of (10) then either $u_1 = \dots = u_n \equiv 0$ or $u_i > 0$ in Ω for all i.

In the end we state an existence result in the whole space for a class of systems which include a number of important models, for instance the Paneitz equation. See [8] or [39] for various applications.

Theorem 5 *Suppose $a_1, \dots, a_n$ are positive numbers, and $f(u)$ is a nondecreasing locally Lipschitz function such that*

$$\limsup_{u \to 0} \frac{f(u)}{u} < \prod_{i=1}^{n} a_i, \qquad \lim_{u \to \infty} \frac{f(u)}{u^p} = c > 0, \tag{11}$$

for some $c > 0, p \in (1, p^*)$. Then the system

$$\begin{cases} -\Delta u_i + a_i u_i &= u_{i+1}, \quad i = 1, \dots, n-1, \\ -\Delta u_n + a_n u_n &= f(u_1), \end{cases} \tag{12}$$

has a solution in $\mathbb{R}^N$, such that $u_i > 0$ in $\mathbb{R}^N$ and $u_i \to 0$ as $|x| \to \infty$.

Note that many higher order equations like the ones studied in the introduction can be factorized in the form (12). Various extensions of Theorem 5 can be deduced from its proof. See also Proposition 6.1 in Section 6.

3 Proof of Theorem 4

In this section we prove Theorem 4. We recall that Theorems 1 and 2 are particular cases of this theorem.

3.1 Preliminaries

The following result, due to Krasnoselskii and Benjamin (see Proposition 2.1 and Remark 2.1 in [19]) has nowadays become a classical tool in proving existence results.

Theorem 6 *Let K be a closed cone in a Banach space X, and let $B_R = \{x \in K \,:\, \|x\| < R\}$. Let $T \,:\, \overline{B_R} \to K$ be a compact mapping. Suppose $\sigma, \rho \in (0,R), \sigma \neq \rho$ are such that*

(i) *$Tx \neq tx$ for all $x \in \partial B_\sigma$ and all $t \geq 1$;*

and there exists a mapping $H \,:\, \overline{B_\rho} \times [0,\infty) \to K$ such that

(ii) *$H(x,0) = Tx$ for all $x \in \partial B_\rho$;*

(iii) *$H(x,t) \neq x$ for all $x \in \partial B_\rho$, and all $t \geq 0$;*

(iv) *$\exists\, t_0 \in \mathbb{R}_+ \,:\, H(x,t) \neq x$ for all $x \in \overline{B_\rho}$, and all $t \geq t_0$.*

Then there exists a fixed point x of T (i.e. $Tx = x$), such that $\|x\|_X$ is between σ and ρ.

We denote with X the space $\left(C^0(\overline{\Omega})\right)^n$ and introduce the linear mapping $S \,:\, X \to X$, such that for any $\Psi = (\psi_1, \ldots, \psi_n)^T, W = (w_1, \ldots, w_n)^T \in X$,

$$S(\Psi) = W \qquad \Longleftrightarrow \qquad \left\{ \begin{array}{rcllr} -L_i w_i & = & \psi_i & \text{in} & \Omega, \quad i = 1, \ldots, n \\ w_i & = & 0 & \text{on} & \partial\Omega, \quad i = 1, \ldots, n. \end{array} \right.$$

The mapping S is well-defined, since $\lambda_1(L_i, \Omega) > 0$. Properties of scalar operators with a positive first eigenvalue were studied in [7]. Below (Theorem 7) we recall some of these properties, obtained in [11] in the more general setting of a cooperative system.

We set $T(U) = S(F(U))$ and note that T maps compactly X into itself, by standard regularity and imbedding theorems. With this notation, solving $(\mathcal{P}_0)$ clearly amounts to finding a fixed point of T in the cone

$$K = \{U \in X \,:\, u_i \geq 0, \quad i = 1, \ldots, n\}.$$

Of course T maps K into itself, by the maximum principle, which is verified by L_i in Ω, since $\lambda_1(L_i, \Omega) > 0$. Consequently, finding a nontrivial fixed point of T in K, that is, verifying the four hypotheses of Theorem 6, will be our task in this section.

We shall need some results, and consequences of results from [11] (see in particular Sections 8, 13 and 14 in that paper).

Let $c_{ij}(x)$ be bounded functions and set $\mathcal{C}(x) = (c_{ij}(x))_{i,j=1}^n$. Suppose $g_i(x) \in L^N(\Omega)$. Consider a linear system in the form

$$LU + \mathcal{C}U = G, \tag{13}$$

where $L = \mathrm{diag}(L_1, \ldots, L_n)$, $\mathcal{C}(x) = (c_{ij}(x))_{i,j=1}^n$, $U = (u_1, \ldots, u_n)^T$, and $G = (g_1, \ldots, g_n)^T$.

Since we are going to use Alexandrov-Bakelman-Pucci estimates and Maximum Principles we shall need to consider cooperative systems. System (13) is called *cooperative* (or *quasi-monotone*) if $c_{ij} \geq 0$ for all $i \neq j$.

We recall that a system of this type is called *fully coupled* (and the matrix $\mathcal{C}$ is called *irreducible*) provided for any non-empty sets $I, J \subset \{1, \ldots, n\}$ such that $I \cap J = \emptyset$ and $I \cup J = \{1, \ldots, n\}$, there exist $i_0 \in I$ and $j_0 \in J$ for which

$$\mathrm{meas}\{x \in \Omega \mid c_{i_0 j_0}(x) > 0\} > 0. \tag{14}$$

When (14) holds we write $c_{i_0 j_0} \not\equiv 0$ in Ω. Simply speaking, a system is fully coupled provided it cannot be split into two subsystems, one of which does not depend on the other.

As explained in [11], any matrix $\mathcal{C}(x)$ can have its lines and columns renumbered in such a way that it is in block triangular form, with each block on the main diagonal being fully coupled. More precisely, $\mathcal{C} = (\mathcal{C}_{kl})_{k,l=1}^q$, for some $1 \leq q \leq n$, $\mathcal{C}_{kl}$ are $t_k \times t_l$ matrices for some $t_k \leq n$ with $\sum_{k=1}^q t_k = n, \mathcal{C}_{kk}$ is an *irreducible* matrix for all $k = 1, \ldots, q$, and $\mathcal{C}_{kl} \equiv 0$ in Ω, for all $k, l \in \{1, \ldots, q\}$ with $k < l$. Note that $q = 1$ means $\mathcal{C}$ itself is irreducible, while $q = n$ means $\mathcal{C}$ is in triangular form. We set $s_0 = 0$, $s_k = \displaystyle\sum_{j=1}^q t_j$, and $S_k = \{s_{k-1} + 1, \ldots, s_k\}$.

For instance, any 1×1 matrix is irreducible. Then, up to renumbering, when $n = 2$ we divide the set of matrices $\mathcal{C}$ into two parts : matrices of the form $\begin{pmatrix} * & a \\ b & * \end{pmatrix}$ and matrices of the form $\begin{pmatrix} * & 0 \\ * & * \end{pmatrix}$, where $a, b \not\equiv 0$ and $*$ stands for an arbitrary function. The first of these matrices is irreducible, the second is not, and has two 1×1 irreducible blocks. Respectively, for

$n = 3$, there are four types of matrices (of course, up to renumbering again) :

$$\begin{pmatrix} * & a & * \\ * & * & b \\ c & * & * \end{pmatrix}, \quad \begin{pmatrix} * & a & 0 \\ b & * & c \\ 0 & d & * \end{pmatrix}, \quad \begin{pmatrix} * & a & 0 \\ b & * & 0 \\ * & * & * \end{pmatrix}, \quad \begin{pmatrix} * & 0 & 0 \\ * & * & 0 \\ * & * & * \end{pmatrix}, \qquad a, b, c, d \not\equiv 0.$$

The first two of these matrices are irreducible, the third has one 2×2 and one 1×1 irreducible blocks, and the fourth has three 1×1 irreducible blocks.

It was proved in Theorem 13.1 in [11] that the matrix operator $L + \mathcal{C}$ admits a *principal eigenvalue* with all the usual properties of the principal eigenvalue of a scalar operator, provided $\mathcal{C}$ is *cooperative and irreducible.* We recall that this eigenvalue is defined by:

$$\begin{aligned} \lambda_1 &= \lambda_1(L + \mathcal{C}) \\ &= \sup\{\lambda \in \mathbb{R} : \exists \Psi \in W^{2,N}_{loc}(\Omega, R^n),\ \Psi > 0 \ \&\ (L + \mathcal{C} + \lambda I)\Psi \le 0 \text{ in } \Omega\}. \end{aligned}$$

Hence, using the explained above block triangular representation of the cooperative matrix $\mathcal{C}$, we can associate to $\mathcal{C}$ a set of eigenvalues $\lambda_1^{(1)}, \dots, \lambda_1^{(q)}$, where $\lambda_1^{(k)}$ is the principal eigenvalue of $L^{(k)} + \mathcal{C}_{kk}$. Here we have denoted $L^{(k)} = \mathrm{diag}(L_{s_{k-1}+1}, \dots, L_{s_k})$ (see above for the notations).

A combination of Theorems 8.1, 12.1, 13.1, 13.2, 14.1 and Lemma 14.1 in [11] yields the following result (we shall provide a brief proof, for convenience). Note that, when $q = n = 1$, it reduces to the well-known results on scalar equations from [7].

Theorem 7 **(i)** *The following are equivalent :*

(a) $\lambda_1^{(k)} > 0$ *for all* $k = 1, \dots, q$ *;*

(b) there exists a vector $\Psi(x) \in C^2(\Omega)$ *(or* $\Psi(x) \in W^{2,p}(\Omega) \cap C(\overline{\Omega})$*) such that* $C_0\vec{1} \ge \Psi \ge \vec{1}$ *and* $L\Psi + \mathcal{C}\Psi \le 0$ *in* Ω*, where* C_0 *depends only on* Ω *and the coefficients of* L *and* $\mathcal{C}$ *;*

(c) (Alexandrov-Bakelman-Pucci inequality) for any $G \in L^N(\Omega, \mathbb{R}^n)$ *and any subsolution* U *of (13) (i.e.* $LU + \mathcal{C}U \ge G$*) there holds*

$$\sup_{\Omega} \max\{u_1, \dots, u_n\} \le C \left(\sup_{\partial\Omega} \max\{u_1, \dots, u_n, 0\} + \|\max_{1\le i\le n} g_i\|_{L^N(\Omega)} \right),$$

where C *depends only on* Ω *and on the coefficients of* L *and* $\mathcal{C}$*.*

(d) the operator $L + \mathcal{C}$ *satisfies the maximum principle in* Ω*, that is, if* $LU + \mathcal{C}U \le 0$ *in* Ω *and* $U \ge 0$ *on* $\partial\Omega$*, then* $U \ge 0$ *in* Ω*.*

(ii) *if* $\lambda_1^{(k)} > 0$ *for all* $k = 1, \ldots, q$*, then for any* $G \in C^\alpha(\Omega)$ *(or* $G \in L^p(\Omega), p \geq N$*) there exists a unique classical (resp. in* $W^{2,p}(\Omega) \cap C(\overline{\Omega})$*) solution of (13), such that* $u = 0$ *on* $\partial\Omega$ *; in addition* $\|U\|_{W^{2,p}(\Omega)} \leq C\|G\|_{L^p(\Omega)}$.

(iii) *Suppose* $\psi \in C(\overline{\Omega}, \mathbb{R}^d)$ *is such that* $\psi \geq 0$ *and* $L\psi + \mathcal{C}\psi \leq 0$ *in* Ω*. If* $\psi_j \not\equiv 0$ *in* $\overline{\Omega}$ *for some* $j \in S_k$ *and some* $k \in \{1 \ldots, q\}$*, then* $\lambda_1^{(k)} \geq 0$.

Sketch of the proof of Theorem 7.(i) Theorem 14.1 and Lemma 14.1 in [11] give $(a) \Leftrightarrow (b) \Leftrightarrow (d)$. Theorem 8.1 in [11] gives $(b) \Rightarrow (c)$, and $(c) \Rightarrow (d)$ is obvious.
(ii) If $q = 1$ this is Theorem 13.2 in [11] (due to Sweers [42]). If $q > 1$ we apply this theorem q times : using the block-diagonal structure of $\mathcal{C}$, first we solve $(L^{(1)} + \mathcal{C}_{11})u^{(1)} = g^{(1)}$, then $(L^{(2)} + \mathcal{C}_2)u^{(2)} = g^{(2)} - \mathcal{C}_{21}u^{(1)}$, etc. The last inequality in (ii) follows from standard regularity results and (i)-(c).
(iii) This follows from the cooperativeness of $\mathcal{C}$ and the definition of the first eigenvalue, together with Theorem 14.1 in [11]. □

3.2 Proof of Theorem 4 in the sublinear case

In this section we show that problem $(\mathcal{P}_0)$ has a nontrivial solution in K, provided (H_0) and (H_∞) hold.

For any $U \in K$ and any $t \in [0, \infty)$ we define

$$H(U, t) = T(U) + t\widetilde{\Phi}_1,$$

where $\Phi_1 = (\varphi_{1,1}, \ldots, \varphi_{1,n})^T$, $\widetilde{\Phi}_1 = (\frac{1}{\lambda_1}\varphi_{1,1}, \ldots, \frac{1}{\lambda_n}\varphi_{1,n})^T$ and $\varphi_{1,i}$ denotes the positive eigenfunction of L_i in Ω (corresponding to λ_i). Note that, for later use, we keep working with different elliptic operators wherever it is possible.

We are going to show that the hypotheses of Theorem 6 are satisfied by the mappings T and H, under (H_0) and (H_∞). Note that $H(U, t) = S(F(U) + t\Phi_1)$, hence

$$H(U,t) = U \quad \Leftrightarrow \quad \left\{ \begin{array}{rcllll} -LU & = & F(x,U) + t\Phi_1 & & \text{in} & \Omega \\ u_i & \geq & 0, & i = 1, \ldots, n, & \text{in} & \Omega \\ u_i & = & 0, & i = 1, \ldots, n, & \text{on} & \partial\Omega \end{array} \right. \tag{15}$$

First, hypothesis (ii) in Theorem 6 is clearly verified by H. Let us now show that hypotheses (iii) and (iv) in Theorem 6 hold with $\rho = r$, where r is the number which appears in (H_0).

Suppose that $H(U, t) = U$ for some $U \in K$, $\|U\| \leq r$, and some $t \in [0, \infty)$. By (H_0) and (15) we have

$$LU + BU \leq -t\Phi_1 \leq 0 \qquad \text{in } \Omega. \tag{16}$$

We use the following simple lemma.

Lemma 3.1 *Let D be a real $n\times n$ matrix. Then*

(a) *the set $\{Dx\,:\,x\in\mathbb{R}^n, x\geq 0\}$ is closed ;*

(b) *if D is such that*

$$\forall\, U\in\mathbb{R}^n\ :\qquad \left\{\begin{array}{rcl} DU & \leq & 0\\ U & \geq & 0\end{array}\right. \qquad \textit{implies}\quad U=0, \tag{17}$$

then there exists $\varepsilon>0$ such that $D-\varepsilon I$ has the same property.

Proof. Statement (a) is a very standard fact from linear optimization, while (b) follows from a simple contradiction argument. Indeed if for each ε there exists a vector $U_\varepsilon\geq 0, U_\varepsilon\neq 0$ such that $(D-\varepsilon I)U_\varepsilon\leq 0$ then a subsequence of $V_\varepsilon=\|U_\varepsilon\|^{-1}U_\varepsilon$ converges to a vector V such that $V\geq 0, \|V\|=1$ and $DV\leq 0$, a contradiction. □

So, since $B\succ\Lambda$, there exists $\varepsilon>0$, such that $B\succ\Lambda+\varepsilon I$. Hence we can rewrite inequality (16) as

$$\widetilde{L}U+\widetilde{B}U\leq -t\Phi_1\leq 0 \qquad \text{in }\ \Omega, \tag{18}$$

with $\widetilde{L}=L+\Lambda+\varepsilon I$, $\widetilde{B}=B-(\Lambda+\varepsilon I)$, so $\lambda_1(\widetilde{L}_i,\Omega)=-\varepsilon<0$ and $\widetilde{B}\succ 0$.

We want to infer that $U\equiv 0$. Then from (18) $t=0$ as well, so (iii) and (iv) of Theorem 6 hold. Suppose for contradiction that there exists an index j and a point $x_0\in\Omega$ such that $u_j(x_0)>0$.

We are going to make use of the following variant of a basic result from linear programming, known as Farkas' Lemma. Since it is not usually encountered in this form in the literature, for the reader's convenience we provide a proof at the end of this section.

Proposition 3.1 *Suppose $k,l\in\mathbb{N}$, D is a $k\times l$ real matrix, and let $d\in\mathbb{R}^k$. Then* exactly one *of the following systems of linear inequalities has a solution $\alpha\in\mathbb{R}^k,\beta\in\mathbb{R}^l$ (the dot will denote scalar product) :*

$$\left\{\begin{array}{rcl} D^T\alpha & \geq & 0\\ \alpha & \geq & 0\\ d.\alpha & > & 0,\end{array}\right. \qquad \textit{and} \qquad \left\{\begin{array}{rcl} D\beta & \leq & -d\\ \beta & \geq & 0.\end{array}\right. \tag{19}$$

We apply this proposition with $D=\widetilde{B}=(\widetilde{b}_{ij})_{i,j=1}^n$ and $d=e_j$, the unitary vector with j-th coordinate equal to 1 and all other coordinates equal to 0. The hypothesis $\widetilde{B}\succ 0$ implies that the second system in (19) does not have a solution. Hence we can find nonnegative numbers $\alpha_1, \ldots, \alpha_n$, with $\alpha_j > 0$, such that $\displaystyle\sum_{i=1}^{n} \alpha_i \widetilde{b}_{ik} \geq 0$, for all $k = 1, \ldots, n$.

We multiply the i-th equation in (18) by α_i, for each i, and sum up the resulting equations. We obtain that the function $u = \sum_{i=1}^{n} \alpha_i u_i$ is such that

$$\begin{cases} \widetilde{L}_1 u & \leq & 0 & \text{in} & \Omega \\ u & \geq & 0 & \text{in} & \Omega \\ u(x_0) & > & 0 & & \end{cases}$$

(at this moment we use the hypothesis that the elliptic operators coincide). By the strong maximum principle $u > 0$ in Ω. By the definition of the principal eigenvalue (or Theorem 7 (iii) for $n = 1$), the existence of such a function u implies $\lambda_1(\widetilde{L}_1, \Omega) \geq 0$, which is a contradiction.

It remains to verify condition (i) of Theorem 6, under (H_0) and (H_∞). Suppose for contradiction that (i) does not hold, that is, for any $\sigma > \rho$ we can find a vector U and a number $t \geq 1$ such that $\|U\|_{L^\infty(\Omega)} = \sigma$ and

$$\begin{cases} -LU & = & t^{-1}\, F(x, U) & & \text{in} & \Omega \\ u_i & \geq & 0, & i = 1, \ldots, n, & \text{in} & \Omega \\ u_i & = & 0, & i = 1, \ldots, n, & \text{on} & \partial\Omega \end{cases} \tag{20}$$

By (H_∞) and $t \geq 1$, this implies

$$LU + AU \geq -k\vec{1} \qquad \text{in } \Omega, \tag{21}$$

where A is a matrix such that $A \prec \Lambda$ and k is a constant. We fix $\varepsilon > 0$ such that $A \prec \Lambda - \varepsilon I$. Note that $AU + k\vec{1} \geq 0$ for each $U \geq 0$ implies $A \geq 0$.

Hence Theorem 7 can be applied to the operator $L + A$. Specifically, we are going to show that this operator satisfies the maximum principle in Ω, i.e. that condition (i)-(d) of this theorem is verified. Then, by the equivalence in Theorem 7 (i), statement (i)-(c) will also hold. Hence, by applying (i)-(c) to (21) we get $\|U\|_{L^\infty(\Omega)} \leq C_0$ (here C_0 depends on k, n, N, L, A, Ω), which is a contradiction, since we can take $\sigma > C_0$.

So let is show that, given a function V for which

$$\begin{cases} LV + AV & \leq & 0 & \text{in} & \Omega \\ V & \geq & 0 & \text{on} & \partial\Omega, \end{cases} \tag{22}$$

we necessarily have $V \geq 0$ in Ω.

We recast (22) as

$$\widetilde{L}V + \widetilde{A}V \leq 0 \tag{23}$$

with $\widetilde{L} = L + \Lambda - \varepsilon I$ (so $\lambda_1(\widetilde{L}_i, \Omega) = \varepsilon > 0$) and $\widetilde{A} = A - \Lambda + \varepsilon I$ (so $\widetilde{A} \prec 0$).

Now, since $\lambda_1(\widetilde{L}_1, \Omega) > 0$ there exists a function ψ such that $C_1 \geq \psi \geq 1$ in $\overline{\Omega}$ and $\widetilde{L}_1\psi \leq 0$ in Ω – this is for instance Theorem 7 (i)-(b) for $n = 1$, or Proposition 6.1 in [7]. By the construction of ψ and standard regularity results $\|\psi\|_{C^{1,\alpha}(\Omega)} \leq C_2$ (the constants C_1, C_2 depend only on the coefficients of $\widetilde{L}_1$ and Ω). We set $V = \psi W$(here we use the fact that the operators L_i coincide, more precisely, that ψ is the same for all of them).

A simple computation transforms (23) into

$$\left\{ \begin{array}{rcccc} \overline{L}_1 w_1 + \widetilde{a}_{11} w_1 + \widetilde{a}_{12} w_2 + \ldots + \widetilde{a}_{1n} w_n & \leq & 0 & \text{in} & \Omega \\ \cdots & \cdots & \cdots & & \\ \overline{L}_1 w_n + \widetilde{a}_{n1} w_1 + \widetilde{a}_{n2} w_2 + \ldots + \widetilde{a}_{nn} w_n & \leq & 0 & \text{in} & \Omega \\ W & \geq & 0 & \text{on} & \partial\Omega\,, \end{array} \right. \tag{24}$$

where

$$\overline{L}_1 = \sum_{i,j=1}^{N} a_{ij}(x)\partial_{ij} + \sum_{i=1}^{N} \left(b_i(x) + 2\sum_{j=1}^{N} a_{ij}(x) \frac{\partial_j \psi(x)}{\psi(x)} \right) \partial_i + \frac{\widetilde{L}_1 \psi}{\psi}.$$

In particular, the zero-order coefficient of $\overline{L}_1$ is nonpositive.

Recall that our goal is to show that $W \geq 0$ in Ω. Suppose this is not true and set

$$J = \{ j \in \{1, \ldots, n\} \ : \ w_j < 0 \ \text{ somewhere in } \ \Omega \}, \qquad I = \{1, \ldots, n\} \setminus J$$

(I can be empty, but $J \neq \emptyset$).

We remove from (24) all inequations in Ω with indices in I. Then if we remove from the remaining inequations all terms where appears a function w_i with $i \in I$, the inequalities remain true, since $\widetilde{A}$ is cooperative (all off-diagonal terms of $\widetilde{A}$ are nonnegative), recall $A \geq 0$. In this way we see that we can suppose $I = \emptyset$ in (24), by taking a smaller n, if necessary. Here we have to note the simple fact that if a cooperative matrix $D \in \mathcal{M}_n(\mathbb{R})$ is such that $D \prec 0$, then any minor $D_k \in \mathcal{M}_{n-k}(\mathbb{R})$ of D obtained by removing from D lines and columns with the same indices is such that $D_k \prec 0$. Indeed, if not, take a vector $z \in \mathbb{R}^{n-k}$, $z \geq 0$, $z \neq 0$, such that $D_k z \geq 0$; then adding k zero coordinates to z leads to a contradiction with $D \prec 0$.

Let $x_j \in \Omega$ be a point where w_j attains its negative minimum, for all $j = 1, \ldots, n$. We set $W_0 = (w_1(x_1), \ldots, w_n(x_n)) \in \mathbb{R}^n$.

Since x_j is point of negative minimum, and $\overline{L}_1$ is elliptic with a nonpositive zero-order term, we clearly have

$$\overline{L}_1 w_j(x_j) \geq 0,$$

for all $j = 1, \ldots, n$. Hence

$$\sum_{l=1}^{n} \widetilde{a}_{kl} w_l(x_k) \leq 0, \qquad \text{for all } \; k = 1, \ldots, n.$$

By the minimal choice of x_j ($w_l(x_k) \geq w_l(x_l)$ for all k, l) and $\widetilde{a}_{kl} \geq 0$ for $k \neq l$, this implies

$$\sum_{l=1}^{n} \widetilde{a}_{kl} w_l(x_l) \leq 0, \qquad \text{for all } \; k = 1, \ldots, n.$$

In other words, we have

$$\begin{cases} -\widetilde{A}(-W_0) & \leq & 0 \\ -W_0 & > & 0. \end{cases}$$

So $-\widetilde{A} \succ 0$ implies $W_0 = 0$, a contradiction.

This finishes the proof of Theorem 4 under (H_0) and (H_∞).

3.3 Proof of Theorem 4 in the superlinear case

In this section we prove that problem $(\mathcal{P}_0)$ has a nontrivial solution in K, assuming (H^0), (H^∞), and (APB).

We are going to use Theorem 6 again. First we show that (H^0) permits to verify hypothesis (i) in Theorem 6. Suppose U is a solution of $TU = tU$ with $t \geq 1$, that is, (20) holds. By (H^0) we have, for all U with $\|U\|_{L^\infty(\Omega)} \leq r$,

$$\begin{cases} (L+A)U & \geq & 0 & \text{in} & \Omega \\ U & \geq & 0 & \text{in} & \Omega \\ U & = & 0 & \text{on} & \partial\Omega \end{cases} \tag{25}$$

where A is a constant matrix such that $A \prec \Lambda$ and $AU \geq 0$ for all U, $\|U\| \leq r$, which implies $A \geq 0$.

We claim that if U satisfies (25) then $U \equiv 0$ (so hypothesis (i) in Theorem 6 is verified with $\sigma = r$). Like in the considerations which lead us to (24) we introduce $\widetilde{L} = L + \Lambda - \varepsilon I$ (so $\lambda_1(\widetilde{L}_i, \Omega) = \varepsilon > 0$), $\widetilde{A} = A - \Lambda + \varepsilon I$ (so $\widetilde{A} \prec 0$), the function ψ with $C \geq \psi \geq 1$ and $\widetilde{L}_1 \psi \leq 0$, the operator $\overline{L}_1$ (with a nonpositive zero-order coefficient), and set $U = \psi W$. So

$$\begin{cases} \overline{L}_1 w_1 + \widetilde{a}_{11} w_1 + \widetilde{a}_{12} w_2 + \ldots + \widetilde{a}_{1n} w_n & \geq & 0 & \text{in} & \Omega \\ \cdots & \cdots & \cdots & & \\ \overline{L}_1 w_n + \widetilde{a}_{n1} w_1 + \widetilde{a}_{n2} w_2 + \ldots + \widetilde{a}_{nn} w_n & \geq & 0 & \text{in} & \Omega \\ W & \geq & 0 & \text{in} & \partial\Omega \\ W & = & 0 & \text{on} & \partial\Omega\,, \end{cases} \tag{26}$$

We take $x_j \in \Omega$ to be points of maximum for w_j, $j = 1, \dots, n$, and set $W_0 = (w_1(x_1), \dots, w_n(x_n)) \in \mathbb{R}^n$. Then, clearly,

$$\overline{L}_1 w_j(x_j) \leq 0,$$

so by $A \geq 0$ and $w_j(x_i) \leq w_j(x_j)$ we get from (26)

$$\begin{cases} \widetilde{A}W_0 & \geq & 0 \\ W_0 & \geq & 0, \end{cases}$$

which implies $W_0 = 0$, by $\widetilde{A} \prec 0$.

We now turn to the remaining three conditions required for Theorem 6 to hold. Here we define, for $t \geq 0$,

$$H(U,t) = T(U + t\,\vec{1}\,).$$

Note that now

$$H(U,t) = U \quad \Longleftrightarrow \quad \begin{cases} -LU & = & F(x, U + t\,\vec{1}\,) & & \text{in} & \Omega \\ u_i & \geq & 0, & i = 1, \dots, n, & \text{in} & \Omega \\ u_i & = & 0, & i = 1, \dots, n, & \text{on} & \partial\Omega \end{cases} \tag{27}$$

Hypothesis (ii) of Theorem 6 is again trivially satisfied by H. Let us show that the equation $H(U,t) = U$ does not have solutions in K for $t \geq R$, where R is the number that appears in (H^∞). This will then imply hypotheses (iii) of Theorem 6 for $t \geq R$, and (iv) with $t_0 = R$.

Indeed, if $t \geq R$, then (H^∞) and (27) yield

$$-LU \geq B(U + t\,\vec{1}\,) \qquad \text{in} \ \ \Omega. \tag{28}$$

We can repeat a reasoning we used in the sublinear case, setting $\widetilde{L} = L + \Lambda + \varepsilon I$, $\widetilde{B} = B - (\Lambda + \varepsilon I)$, so $\lambda_1(\widetilde{L}_i, \Omega) = -\varepsilon < 0$ and $\widetilde{B} \succ 0$. Supposing that $u_j(x_0) > 0$ for some j and $x_0 \in \Omega$, by Proposition 3.1 we can find nonnegative numbers $\alpha_1, \dots, \alpha_n$, with $\alpha_j > 0$, such that $\displaystyle\sum_{i=1}^n \alpha_i \widetilde{b}_{ik} \geq 0$, for all $k = 1, \dots, n$.

Multiplying the i-th equation in (28) by α_i, for each i, and summing up the resulting equations (note that the terms coming from $t\widetilde{B}\vec{1}$ become positive, by the choice of α, so we can remove them from the resulting inequality), we obtain that the function $u = \sum_{i=1}^n \alpha_i u_i$ is such that $\widetilde{L}_1 u \leq 0$, $u > 0$ in Ω, which leads to a contradiction with $\lambda_1(\widetilde{L}_1, \Omega) < 0$.

Finally, the validity of hypothesis (iii) of Theorem 6 for $t < R$ is a consequence of the a priori estimate for $(\mathcal{P}_t)$, hypothesis (APB) with $t_0 = R$, which we assume in Theorem 2 – specifically, we take ρ in Theorem 6 (iii) to be larger than this a priori bound.

Theorem 4 is proved. □

Proof of Proposition 3.1. We recall that Farkas' lemma in its classical form, to be found in most textbooks on linear programming, states that for any $k, l \in \mathbb{N}$, any real $k \times l$ matrix A, and any $b \in \mathbb{R}^k$, exactly one of the following systems has a solution $x \in \mathbb{R}^l, y \in \mathbb{R}^k$:

$$\left\{ \begin{array}{rcl} Ax & = & b \\ x & \geq & 0 \end{array} \right. \qquad \text{and} \qquad \left\{ \begin{array}{rcl} A^T y & \geq & 0 \\ b.y & < & 0\,, \end{array} \right. \tag{29}$$

or, equivalently,

$$\exists\, x \in \mathbb{R}^l \;:\; \left\{ \begin{array}{rcl} Ax & = & b \\ x & \geq & 0 \end{array} \right. \quad \Longleftrightarrow \quad \big(\forall\, y \in \mathbb{R}^k \;:\; A^T y \geq 0 \;\Rightarrow\; b.y \geq 0\big)\,. \tag{30}$$

Suppose that the first problem in (19) has a solution $\alpha_0 \geq 0$. This obviously implies that for any $z \in \mathbb{R}, z \geq 0$, the vector α_0 is a solution of the problem

$$\left\{ \begin{array}{rcl} D^T y & \geq & 0 \\ (-d - z).y & < & 0 \end{array} \right.$$

By (29) this implies that the problem

$$\left\{ \begin{array}{rcl} Dx & = & -d - z \\ x & \geq & 0 \end{array} \right.$$

has no solution for all $z \geq 0$, which is equivalent to saying that the second problem in (19) has no solutions.

Next, suppose that the first problem in (19) has no solutions. This means that for any $x \in \mathbb{R}^k$ the inequalities $D^T x \geq 0, x \geq 0$ imply $d.x \leq 0$. In other words, setting

$$D' = \begin{pmatrix} D^T \\ I \end{pmatrix}$$

(D' is a $(l + k) \times k$ matrix), the inequality $D'x \geq 0$ implies $d.x \leq 0$. By (30) this implies that the problem

$$\left\{ \begin{array}{rcl} (D')^T y' & = & -d \\ y' & \geq & 0 \end{array} \right.$$

has a solution $y\prime \in \mathbb{R}^{k+l}$. Hence the vector $\beta \in \mathbb{R}^l$ containing the first l coordinates of y' is a solution of the second problem in (19). □

4 More general results and comments

In this section we give results on existence of solutions for $(\mathcal{P}_0)$ with different elliptic operators. As explained in Section 3.1, to any uniformly elliptic operators $L_1, \ldots, L_n$ and any cooperative matrix $\mathcal{C}(x)$ we can associate a set of first eigenvalues in Ω of the irreducible blocks of the matrix operator $L+\mathcal{C}$, denoted by $\lambda_1^{(1)}, \ldots, \lambda_1^{(q)}$. We set $\overline{\lambda}_1(L+\mathcal{C},\Omega)$ and $\underline{\lambda}_1(L+\mathcal{C},\Omega)$ to be respectively the largest and the smallest of these numbers.

We suppose that the operators L_i have no zero-order terms ($c^{(i)} \equiv 0$ for all i). This will not introduce any restriction whatsoever, since in the next theorem we shall allow the matrices A, B, which appear in its hypotheses, to depend on x – that is, $c^{(i)}(x)$ are incorporated in the coefficients on the main diagonal of A, B.

Theorem 8 *Suppose that either (sublinear case)*

$(\overline{H_0})$ *there exist $r > 0$ and a cooperative matrix $B(x)$ such that we have $F(x,U) \geq B(x)\,U$ if $\|U\| \leq r, x \in \overline{\Omega}$, and*

$$\overline{\lambda}_1(L+B(x),\Omega) < 0.$$

$(\overline{H_\infty})$ *there exist $k > 0$ and a cooperative matrix $A(x)$ such that we have $F(x,U) \leq A(x)\,U + k\vec{1}$ for all $U \in \mathbb{R}^n_+, x \in \overline{\Omega}$, and*

$$\underline{\lambda}_1(L+A(x),\Omega) > 0\,;$$

or (APB) holds and (superlinear case)

$(\overline{H^0})$ *there exist $r > 0$ and a cooperative matrix $A(x)$ such that we have $F(x,U) \leq A(x)\,U$ if $\|U\| \leq r,\ x \in \overline{\Omega}$, and*

$$\underline{\lambda}_1(L+A(x),\Omega) > 0.$$

$(\overline{H^\infty})$ *there exist $R > 0$ and a cooperative matrix $B(x)$ such that we have $F(x,U) \geq B(x)\,U$ if $\min\{U_1,\ldots,U_n\} \geq R, x \in \overline{\Omega}$, and*

$$\overline{\lambda}_1(L+B(x),\Omega) < 0.$$

Then $(\mathcal{P}_0)$ has a nonnegative solution, such that $u_k > 0$ in Ω, for at least one $k \in \{1,\ldots,n\}$.

Remark 1. To verify the hypotheses of Theorem 8 one can use the upper and lower bounds on $\underline{\lambda}_1(L+C,\Omega), \overline{\lambda}_1(L+C,\Omega)$ in terms of Ω and the coefficients of L, C, given in [11] (actually, as explained there, all bounds from [7] can be extended to matrix eigenvalues). See also Proposition 4.1 below.

Proof of Theorem 8. The proof is an application of the definition of the eigenvalues and Theorem 7, which permit to us to verify the hypotheses of Theorem 6. We shall sketch this proof, mostly in its parts where it is different from the proof of Theorem 4 above. Just like in the Theorem 4 we need to deal with problems (16), (21), (22), (25), (28).

In the sublinear case we again set $H(U,t) = T(U)+t\widetilde{\Phi}_1$. Then hypotheses (iii) and (iv) from Theorem 6 are verified thanks to Theorem 7 (iii), by which the existence of a nontrivial solution of (16) implies $\overline{\lambda}_1(L+B(x),\Omega) \geq 0$. Hypothesis (i) from Theorem 6 follows from the equivalence between parts (i)-(a) and (i)-(c) of Theorem 7, applied to (21).

In the superlinear case hypothesis (i) from Theorem 6 is verified, since if a function U satisfies (25) then by the maximum principle (Theorem 7 (i)-(d), here we use $\underline{\lambda}_1(L+A(x),\Omega) > 0$, which is (i)-(a) in Theorem 7) we have $U \leq 0$, but $U \geq 0$ by (25), so $U \equiv 0$. To verify the other hypotheses of Theorem 6, take again $H(U,t) = T(U+t\,\vec{1}\,)$ and note that if U satisfies (28) then

$$(L+B(x))(U+t\vec{1}) \leq 0,$$

and the existence of such a function $U+t\vec{1} \geq 0$ implies that either $U+t\vec{1} \equiv 0$ or $\overline{\lambda}_1 \geq 0$, by Theorem 7 (iii). □

Actually, studying the proof of Theorem 4, we see that in its course we have proved the following result, of clear independent interest. It gives conditions for a matrix operator with a constant matrix to have positive or negative first eigenvalues.

Proposition 4.1 *Assume $\mathcal{C} \in \mathcal{M}_n(\mathbb{R})$ is cooperative.*

(a) *Suppose there exists a function $\psi \in W^{2,p}(\Omega)$, $p > N$, such that $\psi \geq 1$ and $L_i\psi \leq 0$ in Ω, for all $i = 1,\dots,n$ (note this hypothesis is satisfied if all L_i have nonpositive zero-order coefficients, then we can take $\psi \equiv 1$). Then*

$$\mathcal{C} \prec \Lambda \quad \Longrightarrow \quad \underline{\lambda}_1(L+\mathcal{C}-\Lambda,\Omega) > 0.$$

(b) *Suppose $L_1 \equiv \ldots \equiv L_n$. Then*

$$\mathcal{C} \succ \Lambda \quad \Longrightarrow \quad \overline{\lambda}_1(L+\mathcal{C}-\Lambda,\Omega) < 0.$$

(that is, $\overline{\lambda}_1(L+\mathcal{C},\Omega) < \lambda_1(L_1,\Omega)$).

Remark. We have obtained Theorem 4 in the sublinear case as a combination of Theorem 8 and Proposition 4.1, except that in Theorem 4 we need not suppose that the matrix B is cooperative. Here this hypothesis guarantees the existence of first eigenvalues.

Proof of Proposition 4.1. Set $\mathcal{L} = L + \mathcal{C} - \Lambda$. Recall that by Theorem 7 (i)

$$\underline{\lambda}_1(\mathcal{L},\Omega) > 0 \quad \Longleftrightarrow \quad \left(\forall U : \quad \left\{\begin{array}{rcll} \mathcal{L}U & \leq & 0 & \text{in} \quad \Omega \\ U & \geq & 0 & \text{on} \quad \partial\Omega \end{array}\right. \Rightarrow U \geq 0 \text{ in } \Omega \right),$$

and that by Theorem 7 (iii)

$$\overline{\lambda}_1(\mathcal{L},\Omega) < 0 \quad \Longleftrightarrow \quad \left(\forall U : \quad \left\{\begin{array}{rcll} \mathcal{L}U & \leq & 0 & \text{in} \quad \Omega \\ U & \geq & 0 & \text{in} \quad \Omega \end{array}\right. \Rightarrow U \equiv 0 \text{ in } \Omega \right).$$

The validity of the right-hand sides of these equivalences, under $\mathcal{C} \prec \Lambda$ (resp. $\mathcal{C} \succ \Lambda$), was established in the course of the proof in Section 3.2. □

In view of Theorem 7, part (a) of Proposition 4.1 contains a condition for the operator $L + \mathcal{C}$ to be coercive. As a consequence of this result, we get conditions for the coerciveness of various higher order operators. For instance, $\Delta^2 + \alpha\Delta + a$ with $\alpha^2 \geq 4a$ is equivalent to $\Delta\vec{1} + \mathcal{C}$, with

$$\mathcal{C} = \frac{1}{2}\begin{pmatrix} \alpha & \sqrt{\alpha^2 - 4a} \\ \sqrt{\alpha^2 - 4a} & \alpha \end{pmatrix}.$$

5 Proof of Theorem 3

This section is devoted to verifying (APB) for a general class of superlinear systems, of which (10) is a very particular case. We show how to apply the Gidas-Spruck blow-up method to these systems. This result is of independent importance and applies to other problems as well. The widely used Gidas-Spruck method first appeared in [25] for scalar equations, we refer to [17] and [20] for a review of its use for systems of two equations. For some higher order problems it was used in [9] ; we note that a different class of systems was recently studied in [46], for a special type of domains.

We first prove the following combinatorial lemma.

Lemma 5.1 *Given two sequences of positive numbers* $\{z_k\}_{k=1}^n$ *and* $\{\beta_k\}_{k=1}^n$, *there exists at least one* $k \in \{1,\ldots,n\}$, *such that*

$$z_k^{\beta_j} \geq z_j^{\beta_k} \qquad \textit{for all } \; j \in \{1,\ldots,n\}.$$

Proof. Suppose the lemma is false, that is, for each k we can find an index $j_k \neq k$ such that $z_k^{\beta_{j_k}} < z_{j_k}^{\beta_k}$. We apply this with $k = 1$, then with $k = j_1 \neq 1$ which yields $j_2 \neq j_1$ such that $z_{j_1}^{\beta_{j_2}} < z_{j_2}^{\beta_{j_1}}$, then with $k = j_2$, etc. We get

$$z_1^{\beta_{j_1}} < z_{j_1}^{\beta_1} < z_{j_2}^{\beta_{j_1}\frac{\beta_1}{\beta_{j_2}}}.$$

If $j_2 = 1$ this is a contradiction. If not, we find $j_3 \neq j_2$ such that

$$z_1^{\beta_{j_1}} < z_{j_1}^{\beta_1} < z_{j_2}^{\beta_{j_1}\frac{\beta_1}{\beta_{j_2}}} < z_{j_3}^{\beta_{j_1}\frac{\beta_1}{\beta_{j_3}}}.$$

Again if $j_3 = 1$ or $j_3 = j_1$ we get a contradiction. If not, we continue the process, which will clearly lead to a contradiction after a finite number of steps. □

Recall we have to prove an a priori bound for solutions of (we shall write n instead of m, to conform with the notations in the previous sections)

$$(\mathcal{P}_t)\left\{\begin{array}{rcl} -L_i u_i &=& u_{i+1} + t, \quad i = 1, \ldots, n-1, \\ -L_n u_n &=& f(x, u_1 + t) + a_2 u_2 + \ldots + a_n u_n + t\sum_{i=2}^n a_i, \\ u_i &\geq& 0 \quad \text{in } \Omega, \quad i = 1, \ldots, n \\ u_i &=& 0 \quad \text{on } \partial\Omega, \quad i = 1, \ldots, n, \end{array}\right. \tag{31}$$

where $t \in [0, t_0]$. Note that if $(u_1, \ldots, u_n)$ is a solution of (31) and $u_j \equiv 0$ for some j then $u_k \equiv 0$ for all $k \geq j$ (by using successively the j-th till the $(n-1)$-th equation in (31)). On the other hand, if $u_j \equiv 0$ then by applying successively the Alexandrov-Bakelman-Pucci inequality (Theorem (7) (i)-(c) for $n = 1$) to the $(j-1)$-th equation $-L_{j-1}u_{j-1} = t$, then to the $(j-2)$-th, till the first, we see that u_k are uniformly bounded for all $k < j$. The desired a priori bound is then true, so we can suppose in what follows that $u_j \not\equiv 0$ for all $j = 1, \ldots, n$.

Suppose there is not such bound, that is, for all $l \in \mathbb{N}$ there exist $t_l \in [0, t_0]$ and $u^{(l)}$ which solves (31) with $t = t_l$, such that

$$\max_{1\leq i\leq n} \|u_i^{(l)}\|_{L^\infty(\Omega)} \to \infty \qquad \text{as} \quad l \to \infty. \tag{32}$$

We shall now develop the blow-up method – in a more general context than (31), in view of some further applications.

Let us have the system (setting $u_{n+1} = u_1$)

$$\left\{\begin{array}{rcll} -L_i u_i &=& f_i(x, u_1, \ldots, u_n), & i = 1, \ldots, n \\ u_i &\geq& 0 \quad \text{in } \Omega, & i = 1, \ldots, n \\ u_i &\not\equiv& 0 \quad \text{in } \Omega, & i = 1, \ldots, n \\ u_i &=& 0 \quad \text{on } \partial\Omega, & i = 1, \ldots, n. \end{array}\right. \tag{33}$$

We suppose the functions f_i are such that for some matrices $Q \in \mathcal{M}_n(\mathbb{R})$, $Q \geq 0$ and $A(x) \in C(\overline{\Omega})$, $A(x) \geq 0$ in Ω, there exist continuous functions $h_i(x, u_1, \ldots, u_n)$ for which

$$f_i(x,s) = \sum_{j=1}^{n} a_{ij}(x) s_j^{q_{ij}} + h_i(x,s), \qquad \lim_{s^2\to\infty} \frac{h_i(x,s)}{\sum_{j=1}^{n} a_{ij}(x) s_j^{q_{ij}}} = 0, \qquad (34)$$

uniformly in $x \in \overline{\Omega}$, for all i. We suppose that $p_i := q_{i,i+1} > 0$ (of course again the index $n+1$ replaces 1) and $b_i(x) := a_{i,i+1}(x) \geq b_0 > 0$, for all i.

Further, we assume that the second order coefficients of L_k coincide, that is,

$$a_{ij}^{(k)}(x) \quad \text{is independent of } k, \text{ for all } i,j = 1,\ldots,N. \qquad (35)$$

We set

$$\gamma_1 = 1 + \sum_{i=1}^{n-1} p_1 \ldots p_i, \qquad \gamma_2 = 1 + \sum_{i=2}^{n} p_2 \ldots p_i,$$

$$\gamma_k = 1 + p_k + p_k p_{k+1} + \ldots + p_k \ldots p_n \left(1 + \sum_{i=1}^{k-2} p_1 \ldots p_i\right), \quad 3 \leq k \leq n.$$

Suppose the exponents p_i satisfy

$$\delta_0 := p_1 p_2 \ldots p_n - 1 > 0. \qquad (36)$$

Note this is the weakest possible superlinearity condition for the limiting system (39) below. This condition is standard for $n = 2$, when one can employ variational methods to treat (39), but is not studied for larger n.

Set

$$\beta_i = \frac{2\gamma_i}{\delta_0}. \qquad (37)$$

We shall suppose in addition that

$$q_{ij}\beta_j - \beta_i < 2, \quad \text{for all } j \neq i+1,\ i,j = 1,\ldots,n \qquad (38)$$

(obviously this hypothesis is satisfied if $q_{ij} = 0$). Then we have the following result.

Lemma 5.2 *Suppose $(u_1^{(l)}, \ldots, u_n^{(l)})$ is a solution of (33) satisfying (32), and (34), (35), (36), (38) hold. Then the system (setting $v_{n+1} = v_1$)*

$$\left\{\begin{array}{rcll} -\Delta v_i & = & v_{i+1}^{p_i} & i = 1,\ldots,n \\ v_i & > & 0 \quad \text{in } G, & i = 1,\ldots,n \\ v_i & = & 0 \quad \text{on } \partial G, & \textit{if} \ \ \partial G \neq \emptyset, \end{array}\right. \qquad (39)$$

has a solution with $G = \mathbb{R}^N$ or $G = \mathbb{R}^N_+ = \{x \in \mathbb{R}^N \,:\, x_N > 0\}$.

Proof. For any fixed l we apply Lemma 5.1 to the sequences $z_{kl} := \|u_k^{(l)}\|_{L^\infty(\Omega)}$ and β_k given by (37). Note that all z_{kl} are positive, since $u_k^{(l)} \not\equiv 0$. So by Lemma 5.1 for any l there exists $k = k(l) \in \{1, \ldots, n\}$, such that $z_{kl}^{\beta_j} \geq z_{jl}^{\beta_k}$, for all $j \in \{1, \ldots, n\}$. By taking a subsequence of $l \to \infty$, we can suppose that k is independent of l, say $k = 1$. We set

$$\nu_l = \|u_1^{(l)}\|_{L^\infty(\Omega)}^{-\frac{1}{\beta_1}}, \qquad \text{and} \qquad v_i^{(l)}(x) = \nu_l^{\beta_i} u_i^{(l)}(\nu_l x + x_l),$$

where x_l is a point of maximum of $u_1^{(l)}$ in Ω. Recall that $\nu_l \to 0$ as $l \to \infty$, by (32). Then, setting $\Omega_l = \dfrac{1}{\nu_l}(\Omega - x_l)$, by the above considerations we have

$$v_1^{(l)}(0) = 1 \quad \text{and} \quad v_i^{(l)} \leq 1 \ \text{ in } \Omega_l, \text{ for all } i \in \{1, \ldots, n\},\ l \geq 1. \tag{40}$$

It is trivial to check that the sequence $(v_1^{(l)}, \ldots, v_n^{(l)})$ satisfies the system

$$\left\{ \begin{array}{rcl} -\widetilde{L}_i^{(l)} v_i^{(l)} & = & \displaystyle\sum_{j \neq i+1} a_{ij}(\cdot) \nu_l^{\beta_i + 2 - q_{ij}\beta_j} v_i^{(l)} + b_i(\cdot) \nu_l^{\beta_i + 2 - p_i \beta_{i+1}} v_{i+1}^{(l)} + \widetilde{h_i^{(l)}} \\ v_i^{(l)} & > & 0 \quad \text{in } \Omega_l \\ v_i^{(l)} & = & 0 \quad \text{on } \partial\Omega_l, \end{array} \right. \tag{41}$$

in the domain Ω_l, where the dot stands for $\nu_l x + x_l$, we have denoted $\widetilde{h_i^{(l)}} = h_i\left(\cdot, \nu_l^{-\beta_1} v_1^{(l)}, \ldots, \nu_l^{-\beta_n} v_n^{(l)}\right)$, and

$$\widetilde{L}_k^{(l)} = \sum_{i,j=1}^N a_{ij}^{(k)}(\cdot)\partial_{ij} + \nu_l \left(\sum_{i=1}^N b_i^{(k)}(\cdot)\partial_i + \nu_l c^{(k)}(\cdot) \right).$$

By compactness we can assume that $\{x_l\}$ tends to some point $x_0 \in \overline{\Omega}$.

It is then a very standard fact that the domain Ω_l tends either to the whole space or to a half-space, when $l \to \infty$. Note that in (37) we have chosen β_i to be the solution of the linear system

$$\beta_i - p_i \beta_{i+1} = -2, \quad i = 1, \ldots, n$$

(we have set $\beta_{n+1} = \beta_1$; the determinant of this system is $1 - p_1 \ldots p_n$, which is strictly negative, by (36)). In addition (38) guarantees that for these β_i the powers of ν_l which appear in the sums in the right-hand sides of (41) are strictly positive.

Therefore, thanks to the uniform boundedness of $(v_1^{(l)}, \ldots, v_n^{(l)})$ in $L^\infty(\Omega_l)$, elliptic regularity theory permits to us to pass to the limit in (41) (we recall once more that $\nu_l \to 0$). We thus obtain a vector $(v_1, \ldots, v_n)$, which satisfies the limiting system

$$\left\{\begin{array}{rcll} -\mathrm{tr}(AD^2 v_i) &=& v_{i+1}^{p_i} & i=1,\ldots,n \\ v_i &\geq& 0 \quad \text{in } G, & i=1,\ldots,n \\ v_i &=& 0 \quad \text{on } \partial G, & \text{if } \partial G \neq \emptyset, \end{array}\right. \tag{42}$$

where G is either $\mathbb{R}^N$ or $\mathbb{R}^N_+$, $A = (a_{ij}(x_0))_{i,j=1}^N$ is a constant positive definite matrix, and $D^2 v_i$ stands for the matrix of the second derivatives of v_i. Note also that $v_1(0) = 1$, so all v_i are strictly positive, by the strong maximum principle, applied to all equations in (42), starting from the last, and going to the first. Finally, by rotating and stretching the coordinates, we obtain a solution of (39). □

It is obvious that if (32) holds then system (31) satisfies the hypotheses of Lemma 5.2 with $p_i = 1$, $i = 1, \ldots, n-1$, $p_n = p > 1$, $q_{ij} = 0$ for all $j \geq i+2$ and all $j \leq i < n$, while $q_{nj} = 1$ for all $2 \leq j \leq n$. This lemma implies that the problem

$$\left\{\begin{array}{rcll} (-\Delta)^n v &=& v^p, & \text{in} \quad G \\ (-\Delta)^i v &>& 0 & \text{in} \quad G, \quad i=0,\ldots,n-1 \\ v &=& 0 & \text{on} \quad \partial G \end{array}\right. \tag{43}$$

has a bounded solution with either $G = \mathbb{R}^N$ or $G = \mathbb{R}^N_+$. The first of these is impossible by Theorem 1.4 of [45], which states that $(-\Delta)^n v = v^p$ has no positive classical solutions in $\mathbb{R}^N$, for $p \in (1, p^*)$, $p^* = (N+2n)/(N-2n)$. Note that when $N \leq 2n$ and $p > 1$ even the inequality $(-\Delta)^n v \geq v^p$ does not have nontrivial solutions such that $(-\Delta)^k u > 0$, $k \in \{0, \ldots, n-1\}$, see for instance [33].

The following result provides a contradiction in the case $G = \mathbb{R}^N_+$ and proves the a priori bound, Theorem 3.

Theorem 9 *Problem (43) with $G = \{x \in \mathbb{R}^N \ : \ x_N > 0\}$ does not have bounded solutions, provided*

$$p < \frac{N+2n-1}{N-2n-1} \qquad (\textit{or } \ p < \infty \ \ \textit{if } \ N \leq 2n+1).$$

The proof of this theorem relies on an idea by Dancer [16], which consists in the following : if there is a solution of (43) with $G = \mathbb{R}^N_+$, and if one is able to show that any such solution is increasing in the x_N-direction, then, after eventually some supplementary work, one should be able to pass at the limit as $x_N \to \infty$ and thus get a solution of the same equation in $\mathbb{R}^{N-1}$, which in turn permits to use nonexistence results for the whole space (note the exponent in Theorem 9 is p^* with N replaced by $N-1$).

General monotonicity results for second-order scalar equations were obtained in [6]. Corresponding results for systems of two scalar equations were recently proved in [20]. The reasoning for the system we are interested in here uses an approach similar to the one in [20] and relies on a moving planes argument and the Harnack-Krylov-Safonov estimates for nonlinear elliptic systems obtained in [11].

We have the following monotonicity result.

Theorem 10 *Suppose $U=(u_1,\dots,u_n)$ is a bounded solution to the problem*

$$\left\{\begin{array}{rcl} -\Delta u_i &=& u_{i+1}, \quad i=1,\dots,n-1,\\ -\Delta u_n &=& u_1^p,\\ u_i &>& 0 \quad in\, G, \quad\ i=1,\dots,n \end{array}\right. \tag{44}$$

with $G=\{x\in\mathbb{R}^N\ :\ x_N>0\}$ and $p\geq 1$. Suppose $U=0$ on $\{x_N=0\}$. Then $\dfrac{\partial u_i}{\partial x_N}>0$ in G, for all i.

This theorem was proved in [20] for $n=2$, more precisely, for the system of two equations $-\Delta u_1=u_2^q$, $-\Delta u_2=u_1^p$ (see Theorem 1.2 in that paper). Although the condition $q>1$ is stated there, absolutely the same proof works for $q=1$, and in an even simpler way. We shall omit the extension to arbitrary n, since it involves only trivial technicalities.

We note that an essential role in the proof of Theorem 10 is played by the following Harnack inequality, which is a consequence of the results in [11]. The reader can find a simple proof in this particular case in [20], for $n=2$ (extension to arbitrary n is rather straightforward).

Theorem 11 *Let $(u_1,\dots,u_n)$ be a positive solution of*

$$\left\{\begin{array}{rcl} -L_i u_i &=& u_{i+1}, \quad i=1,\dots,n-1\\ -L_n u_n &=& u_1^p,\\ u_i &>& 0 \quad in\, G, \quad\ i=1,\dots,n \end{array}\right. \tag{45}$$

in some domain G. Suppose K is a compact set properly included in G and

$$\max\left\{\inf_{x\in K} u_1,\dots,\inf_{x\in K} u_n\right\}\leq 1, \qquad \max\left\{\sup_{x\in G} u_1,\dots,\sup_{x\in G} u_n\right\}\leq M.$$

Then

$$\sup_{x\in K}\max\{u_1,\dots,u_n\}\leq C\min\left\{\left(\inf_{x\in K} u_1\right)^{\frac{1}{p}},\inf_{x\in K} u_2,\dots,\inf_{x\in K} u_n\right\}.$$

where C depends only on N,M,G,Ω.

Finally, the nonexistence result in a half-space, Theorem 9, is obtained by combining the nonexistence result in $\mathbb{R}^{N-1}$ from [45] (Theorem 1.4 in that paper) with the following theorem.

Theorem 12 *If there exists a bounded solution U of (44) with $G = \mathbb{R}^N_+$, such that $U = 0$ on $\{x_N = 0\}$, then there exists a solution of (44) with $G = \mathbb{R}^{N-1}$.*

Proof. This theorem can be proved by a somewhat standard and tedious argument involving multiplication by test functions and integration by parts, see for instance [20] for such a reasoning. We shall give here a simpler proof, which also applies to elliptic operators in non-divergence form. Suppose U is a solution of (44) with $G = \mathbb{R}^N_+$, $0 \le U \le M\vec{1}$ in $\mathbb{R}^N_+$, $U = 0$ on $\{x_N = 0\}$ (so all components of U are monotonous in the x_N direction, by Theorem 10).

For each $x = (x_1, \ldots, x_N)$ in the strip $\Sigma_1 = \{x \,:\, 0 < x_N < 1\}$ and each $l \in \mathbb{N}$ we set

$$U^{(l)}(x) = U(x_1, \ldots, x_{N-1}, x_N + l).$$

Now $U^{(l)}$ satisfies the same system as U so, using once more the elliptic regularity and convergence results, we see that the bounded vector $U^{(l)}$ converges uniformly on compact subsets of Σ_1 to a vector function $\widetilde{U}$ which satisfies (44) with $G = \Sigma_1$. However, the monotonicity of $U^{(l)}$ in x_N trivially implies that $\widetilde{U}$ is independent of the x_N-variable. This means that (44) is actually satisfied with $G = \mathbb{R}^{N-1}$. □

6 Existence results in $\mathbb{R}^N$

In this section we prove Theorem 5. One of our main observations is contained in the following proposition.

Proposition 6.1 *For $G \subseteq \mathbb{R}^N$ we consider the system*

$$\begin{cases} -L_i u_i = f_i(x, u_1, \ldots, u_n) & in \quad G, \quad i = 1, \ldots, n \\ \quad u_i > 0 & in \quad G, \quad i = 1, \ldots, n, \end{cases} \tag{46}$$

with elliptic operators L_i in general non-divergence form, as in Section 2, and $f_i \in C(\overline{\Omega} \times \mathbb{R}^n_+)$. We suppose that for some positive constants $a, c_1, \ldots, c_n$ and some function g

$$c^{(i)}(x) \le -c_i < 0, \; , i = 1, \ldots, n, \tag{47}$$

$$f_i(x, u_1, \ldots, u_n) \le u_{i+1}, \; i = 1, \ldots, n-1, \tag{48}$$

$$f_n(x,u_1,\ldots,u_n)\le g(u_1) \quad \textit{and} \quad \limsup_{t\to 0}\frac{g(t)}{t}\le a<\prod_{i=1}^n c_i\,. \tag{49}$$

Assume also that (46) with $G=B_R:=\{x\in\mathbb{R}^N : |x|<R\}$ has a solution u_R, such that $\|u_R\|_{L^\infty(B_R)}$ is uniformly bounded in R, for $R\ge R_0>0$, and $u_{i,R}$ has a point of maximum $x_{i,R}\in B_R$, such that $x_{1,R}$ belongs to a fixed ball B_d, for all $R\ge R_0$. Then (46) with $G=\mathbb{R}^N$ has a solution such that $u_1\not\equiv 0$.

Proof. First, by elliptic theory, the uniform boundedness of u_R in L^∞ implies that this sequence is bounded in $W^{2,p}_{\rm loc}(\mathbb{R}^N)$, $p<\infty$, and converges, up to a subsequence, locally uniformly to a solution of (46) in $\mathbb{R}^N$. We only have to show that the limit function u_1 is not identically zero. This will certainly be the case if $u_{1,R}(x_{1,R})\ge\varepsilon>0$ for some subsequence of $R\to\infty$ (we can assume that $x_{1,R}$ tends to a point $x_0\in B_d$). So we suppose for contradiction that $u_{1,R}(x_{1,R})\to 0$ as $R\to\infty$, that is, u_1 converges uniformly to zero (we do not write the subscript R for u_i in what follows).

We evaluate the first equation in (46) at $x_{1,R}$. Since $x_{1,R}$ is a point of maximum of u_1, by (47) we have $-L_1u_1(x_{1,R})\ge c_1u_1(x_{1,R})$. Hence, by (48),

$$c_1u_1(x_{1,R})\le u_2(x_{1,R})\le u_2(x_{2,R})$$

(the last inequality follows from the definition of $x_{2,R}$). Then we evaluate the second equation at $x_{2,R}$, and get in the same way

$$c_1c_2u_1(x_{1,R})\le c_2u_2(x_{2,R})\le u_3(x_{2,R})\le u_3(x_{3,R}),\quad \text{etc.}$$

We repeat the same procedure $n-1$ times and at the end evaluate the n-th equation at $x_{n,R}$, to get

$$\begin{aligned}\left(\prod_{i=1}^n c_i\right)u_1(x_{n,R}) &\le \left(\prod_{i=1}^n c_i\right)u_1(x_{1,R})\\ &\le c_nu_n(x_{n,R})\le f_n(x_{n,R},u_1(x_{n,R}),\ldots,u_n(x_{n,R})).\end{aligned}$$

This is a contradiction with $u_1\rightrightarrows 0$ and (49). □

Proof of Theorem 5. System (12) clearly satisfies (47)–(49).

Let us show that (12), written in the matrix form

$$-\Delta U=F(U),$$

where

$$f_i(U)=-a_iu_i+u_{i+1},\quad i<n,\qquad f_n(U)=-a_nu_n+f(u_1),$$

satisfies conditions $(H^0), (H^\infty), (APB)$ in Theorem 4 if $\Omega = B_R$, for any fixed R. Set $A_0 := \limsup_{u\to 0} u^{-1}f(u)$ and $\lambda_R = \lambda_1(-\Delta, B_R)$.

Then it is easy to see that checking (H^0) amounts to verifying the condition $M \prec \lambda_R I$, where

$$M = \begin{pmatrix} -a_1 & 1 & 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & -a_{n-1} & 1 \\ m & 0 & 0 & \dots & 0 & -a_n \end{pmatrix},$$

with $m = A_0$. A trivial computation shows this is equivalent to

$$A_0 < (a_1 + \lambda_R)\dots(a_n + \lambda_R),$$

which is a consequence of hypothesis (11), for all $R > 0$.

Similarly, by (26) we have $\lim_{u\to\infty} u^{-1}f(u) = \infty$, so (H^∞) is satisfied if $M \succ \lambda_R I$ for some (large) number m, which is also easy to check.

We have already verified (APB), for any fixed R. Indeed, system (12) is of type (31), which we already studied. Therefore, by Theorem 4, (12) has a solution $u_R > 0$ (we have already explained the fact that if one component of a nonnegative solution of (12) is not identically zero, then all components are strictly positive).

Next, by a result of Troy [43], all components of any positive solution of (12) in a ball are radially symmetric, and attain their unique maximum at the origin. So, to verify the hypotheses of Proposition 6.1, it only remains to show that $u_{i,R}(0)$ is bounded as $R \to \infty$, for all i. This can be done by exactly the same contradiction argument as the one we used in order to prove an priori bound for (31). More precisely, we suppose that $u_{i,R_l}(0) \to \infty$ for some $R_l \to \infty$ (as $l \to \infty$), we introduce the normalized functions

$$v_i^{(l)}(x) = \nu_l^{\beta_i} u_{i,R_l}(\nu_l x),$$

and pass to the limit as $l \to \infty$, thus getting a solution of $(-\Delta)^n u = u^p$ in $\mathbb{R}^N$ or in a half-space (note that in the passage to the limit $R_l \to \infty$ only improves the speed of convergence), which is a contradiction.

Finally, the L^∞-a priori bound together with the variational structure of (12) easily imply an a priori bound in $H_0^1(B_R)$ for solutions of (12), from which the condition $u_i \to 0$ as $|x| \to \infty$ is an easy consequence. □

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