Normalization in Supernatural deduction and in Deduction modulo

Paul Brauner[1], Gilles Dowek[2], Benjamin Wack[3]

[1] École polytechnique and INRIA,
LIX, École polytechnique, 91128 Palaiseau Cedex, France
gilles.dowek@polytechnique.edu http://www.lix.polytechnique.fr/~dowek/
[2] Loria, Campus Scientifique - BP 239 - 54506 Vandoeuvre-lès-Nancy Cedex, France
paul.brauner@loria.fr
http://www.loria.fr/~brauner/
[3] Loria, Campus Scientifique - BP 239 - 54506 Vandoeuvre-lès-Nancy Cedex, France

Abstract. Deduction modulo and Supernatural deduction are two extentions of predicate logic with computation rules. Whereas the application of computation rules in deduction modulo is transparent, these rules are used to build non-logical deduction rules in Supernatural deduction. In both cases, adding computation rules may jeopardize proof normalization, but various conditions have been given in both cases, so that normalization is preserved. We prove in this paper that normalization in Supernatural deduction and in Deduction modulo are equivalent, *i.e.* the set of computation rules for which one system strongly normalizes is the same as the set of computation rules for which the other is.

1 Introduction

Integrating a theory into a deduction system has several advantages over working with axioms. It allows proof-search algorithms to avoid some redundancies, it permits in some cases to get witnesses from constructive proofs and hence to program with proofs, etc.

One way to do so is to formulate the theory of interest as a rewrite system and identify congruent formulae. This is the idea of deduction modulo. For instance, the axiom $\forall X \forall Y ((X \subseteq Y) \Leftrightarrow \forall z\ (z \in X \Rightarrow z \in Y))$ may be formulated as the rewrite rule $(X \subseteq Y) \rightarrow \forall z\ (z \in X \Rightarrow z \in Y)$ and

$$(\forall I)\ \frac{\Gamma \vdash x \in A \Rightarrow x \in B}{\Gamma \vdash A \subseteq B}$$

is then a valid deduction step. Thus the natural deduction rules have to be reformulated in order to take the congruence into account.

An more deterministic alternative to deduction modulo is *Supernatural Deduction* where the rewrite rules are translated into new deduction rules in the logic. For instance, the rewrite rule $(X \subseteq Y) \rightarrow \forall z\ (z \in X \Rightarrow z \in Y)$ yields the following introduction and elimination rules

$$\forall\text{-INTRO}\ \frac{\Gamma \vdash (z \in X \Rightarrow z \in Y)}{\Gamma \vdash X \subseteq Y}\ z \notin \mathcal{FV}(\Gamma) \qquad \forall\text{-ELIM}\ \frac{\Gamma \vdash X \subseteq Y}{\Gamma \vdash t \in X \Rightarrow t \in Y}$$

where all connectors and quantifiers have disappeared. It may be argued that these rules are much closer to usual mathematical practice than either predicate logic with axioms or Deduction Modulo.

Since soundness and completeness with respect to predicate logic with axioms has been proved for both systems, they prove the same theorems but we are interested in a stronger equivalence property.

To explain this strong equivalence property, we must first notice that in both cases, proof normalization and cut elimination are jeopardized. However, various conditions have been given so that normalization is preserved. For deduction modulo, such criteria have been developed in [1,2,3]. For SND, criteria have been developed in [4].

This raises the following problem: are there some rewrite system for which one deduction system has the proof normalization property but not the other or does proof normalization depend only on the rewrite system itself ? We prove in this paper that this is indeed the case: given an orthogonal rewrite system $\mathcal{R}$ rewriting atomic formulae to arbitrary formulae, proofs normalize in Deduction Modulo($\mathcal{R}$) if and only if they normalize in Supernatural Deduction($\mathcal{R}$). As a consequence, the semantic criteria developed for Deduction Modulo [3] may be transferred to supernatural deduction.

Our result is limited to minimal logic (*i.e.* our connectives are restricted to implication and universal quantification). The result can be extended to conjonction but the extension to the disjunction, the existential quantifier and contradiction is more challenging. Indeed, Supernatural Deduction rules are already trickier to define when such connectors and quantifiers are involved. A possible solution may be to move from natural deduction to sequent calculus [5].

To prove this equivalence between Supernatural Deduction and Deduction Modulo, we shall prove that they are both equivalent to a third and older system due to Prawitz: Natural Deduction with folding and unfolding rules. Actually, the equivalence of Deduction modulo and Prawitz' system has already been proved in [6]. Thus, all that remains to be done is to prove the equivalence between Prawitz' system and Supernatural Deduction. Fortunately, this can be achieved by a simple syntactic translation of proofs.

This paper contributes to show that the choice of a formalism or another seems to be a rather superficial choice, as properties such as strong normalization are quite stable when one switches from one system to another.

2 From fold/unfold to supernatural deduction

2.1 Natural deduction

Our starting point is natural deduction for minimal logic with five rules: (Ax), $(\Rightarrow I)$, $(\Rightarrow E)$, $(\forall I)$, $(\forall E)$. We use a lambda calculus notation for proofs:

$$\mathcal{T} ::= \alpha \mid \lambda\alpha : A.\mathcal{T} \mid \mathcal{T}\,\mathcal{T} \mid \lambda x.\mathcal{T} \mid \mathcal{T}\,t$$

The deduction rules enriched with proof-terms are the following: but these rules are typed with different typing rules

$$(Ax)\ \frac{}{\Gamma, \alpha : A \vdash \alpha : A}$$

$$(\Rightarrow I)\ \frac{\Gamma, \alpha : A \vdash \pi : B}{\Gamma \vdash (\lambda\alpha : A.\pi) : A \Rightarrow B} \qquad (\Rightarrow E)\ \frac{\Gamma \vdash \pi_1 : A \Rightarrow B \qquad \Gamma \vdash \pi_2 : A}{\Gamma \vdash (\pi_1\ \pi_2) : B}$$

$$(\forall I)\ \frac{\Gamma \vdash \pi : A}{\Gamma \vdash (\lambda x.\pi) : \forall x A}\ x \notin \mathcal{FV}(\Gamma) \qquad (\forall E)\ \frac{\Gamma \vdash \pi : \forall x A}{\Gamma \vdash (\pi\ t) : A\{x := t\}}$$

The reduction rules on these proof-terms are the two kinds of β-reduction

$$((\lambda\alpha : A\ \pi_1)\ \pi_2) \rhd \pi_1\{\alpha := \pi_2\}$$

$$((\lambda x\ \pi)\ t) \rhd \pi\{x := t\}$$

2.2 Natural deduction with folding and unfolding rules

Consider an orthogonal rewrite system $\mathcal{R}$ rewriting atomic formulae to formulae.

For a rewrite rule $P \to \varphi$, we may add Prawitz' *folding* and *unfolding* rules [7] replacing φ by P and P by φ. To do so, we enrich the proof-terms language.

$$\mathcal{T} ::= \alpha \mid \lambda\alpha : A.\mathcal{T} \mid \mathcal{T}\,\mathcal{T} \mid \lambda x.\mathcal{T} \mid \mathcal{T}\,t \mid \uparrow_R \mathcal{T} \mid \downarrow_R \mathcal{T}$$

And we introduce two deduction rules.

$$\textsc{fold}\ \frac{\Gamma \vdash \pi : \varphi}{\Gamma \vdash \uparrow_R \pi : P} \qquad \textit{and} \qquad \textsc{unfold}\ \frac{\Gamma \vdash \pi : P}{\Gamma \vdash \downarrow_R \pi : \varphi}$$

We also inroduce the following reduction rule.

$$\downarrow_R \uparrow_R \pi \rhd \pi$$

2.3 Supernatural deduction

Supernatural deduction is to goes one step further and incorporates in these rules the introductions and eliminations for all the connectives of φ. More formally, we define the super-rules as follows.

Super-rules computation

Definition 1 (Computation of the introduction super-rules). *Consider a rewrite rule $R : P \to \varphi$. Consider a sequence of variables $l = x_1, x_2, \ldots$ that do not occur in the rule. We associate to R an introduction rule of the form*

$$\frac{premiseI(\Gamma, \varphi, l)}{\Gamma \vdash P}\ cond(\Gamma,\varphi,l)$$

Where the sequent $premiseI(\Gamma, \varphi, l)$ and the condition $cond(\Gamma, \varphi, l)$ are defined by induction on the stucture of φ as follows

- *if φ is atomic, then $premiseI(\Gamma, \varphi, l) = (\Gamma \vdash \varphi)$ and $cond(\Gamma, \varphi, l) = \varnothing$,*
- *if $\varphi = \varphi_1 \Rightarrow \varphi_2$ then $premiseI(\Gamma, \varphi, l) = premiseI((\Gamma, \varphi_1), \varphi_2, l)$ and $cond(\Gamma, \varphi, l) = cond((\Gamma, \varphi_1), \varphi_2, l)$*
- *if $\varphi = \forall x\ \varphi_1$ then $premiseI(\Gamma, \varphi, y.l) = premiseI(\Gamma, \varphi_1\{x := y\}, l)$ and $cond(\Gamma, \varphi, y.l) = cond(\Gamma, \varphi_1\{x := y\}, l) \cup \{y \notin \mathcal{FV}(\Gamma)\}$*

Definition 2 (Computation of the elimination super-rules). *Consider a rewrite rule $R : P \to \varphi$. Consider a sequence of names $l = t_1, t_2, \ldots$. We associate to R an elimination rule of the form*

$$\frac{\Gamma \vdash P \qquad premisesE(\Gamma, \varphi, l)}{conclusion(\Gamma, \varphi, l)}$$

Where the multiset of sequents $premisesE(\Gamma, \varphi, l)$ and the sequent $conclusion(\Gamma, \varphi)$ are defined by induction on the stucture of φ as follows

- *if φ is atomic then $premisesE(\Gamma, \varphi, l) = \varnothing$ and $conclusion(\Gamma, \varphi) = (\Gamma \vdash \varphi)$*
- *if $\varphi = \varphi_1 \Rightarrow \varphi_2$ then $premisesE(\Gamma, \varphi, l) = \{\Gamma \vdash \varphi_1\} \cup premisesE(\Gamma, \varphi_2, l)$ and $conclusion(\Gamma, \varphi, l) = conclusion(\Gamma, \varphi_2, l)$*
- *if $\varphi = \forall x\ \varphi_1$ then let $premisesE(\Gamma, \varphi, t.l) = premisesE(\Gamma, \varphi_1\{x := t\}, l)$ and $conclusion(\Gamma, \varphi, t.l) = conclusion(\Gamma, \varphi_1\{x := t\}, l)$*

Example 1 (Super-rules for inclusion definition). Given the rewrite rule $\subseteq: X \subseteq Y \to \forall z.(z \in X \Rightarrow z \in Y)$, the associated super deduction rules are:

$$(\subseteq I)\ \frac{\Gamma, z \in X \vdash z \in Y}{\Gamma \vdash X \subseteq Y}\ z \notin \mathcal{FV}(\Gamma) \qquad (\subseteq E)\ \frac{\Gamma \vdash X \subseteq Y \qquad \Gamma \vdash t \in X}{\Gamma \vdash t \in Y}$$

A proof-term language for Supernatural deduction The proof-term language is that of proof-terms for predicate logic, enhanced with a pattern language and the corresponding abstraction:

$$\mathcal{T} ::= \alpha \mid \lambda\alpha : A.\mathcal{T} \mid \mathcal{T}\,\mathcal{T} \mid \lambda x.\mathcal{T} \mid \mathcal{T}\,t \mid \lambda R(\overline{m}).\mathcal{T} \mid \mathcal{T}\,R(\overline{t})$$

The variables $x, y, \ldots$ are variables of the theory while $\alpha, \beta, \ldots$ are proof variables. The two last constructs allow to interprete the super-rules. In the pattern $R(\overline{m})$ the constructor R is applied to a sequence of variables that may be either term or proof variables and in the term $R(\overline{t})$ it is applied to a sequence of terms that may be either terms of the theory or proof-terms.

We can now define the typing rules that correspond to the super-rules above.

Definition 3. *The* arity *of a formula* φ *is a sequence of* $\forall$ *and* $\Rightarrow$ *symbols defined by induction on* φ *as follows*

- *if* φ *is atomic* $arity(\varphi) = [\,]$,
- *if* $\varphi = \varphi_1 \Rightarrow \varphi_2$ *then* $arity(\varphi) = (\Rightarrow .(arity(\varphi_2)))$,
- *if* $\varphi = \forall x\ \varphi_1$ *then* $arity(\varphi) = (\forall.(arity(\varphi_1)))$.

Let φ *be a formula, a sequence for* φ *is a sequence of distinct variables such that the* n*-th variable of the sequence is a proof variable if the* n*-th element of the arity of* φ *is* $\Rightarrow$ *and a term variable otherwise.*

Definition 4 (Computation of the abstraction rules). *Consider a rewrite rule* $R : P \to \varphi$. *Consider a sequence* l *for* φ *of variables that do not occur in the rule. We associate to* R *an abstraction rule of the form*

$$\frac{premiseI(\Gamma, \varphi, l)}{\Gamma \vdash (\lambda R(l).\pi) : P}\ cond(\Gamma,\varphi,l)$$

Where the sequent $premiseI(\Gamma, \varphi, l)$ *and the condition* $cond(\Gamma, \varphi, l)$ *are defined by induction on the stucture of* φ *as follows*

- *if* φ *is atomic, then* $premiseI(\Gamma, \varphi, l) = (\Gamma \vdash \pi : \varphi)$ *and* $cond(\Gamma, \varphi, l) = \varnothing$,
- *if* $\varphi = \varphi_1 \Rightarrow \varphi_2$ *then* $premiseI(\Gamma, \varphi, \alpha.l) = premiseI((\Gamma, \alpha : \varphi_1), \varphi_2, l)$ *and* $cond(\Gamma, \varphi, \alpha.l) = cond((\Gamma, \varphi_1), \varphi_2, l)$
- *if* $\varphi = \forall x\ \varphi_1$ *then* $premiseI(\Gamma, \varphi, y.l) = premiseI(\Gamma, \varphi_1\{x := y\}, l)$ *and* $cond(\Gamma, \varphi, y.l) = cond(\Gamma, \varphi_1\{x := y\}, l) \cup \{y \notin \mathcal{FV}(\Gamma)\}$

Definition 5 (Computation of the application rules). *Consider a rewrite rule* $R : P \to \varphi$. *Consider a sequence* l *for* φ *of names. We associate to* R *the application rule of the form*

$$\frac{\Gamma \vdash \pi : P \qquad premisesE(\Gamma, \varphi, l)}{conclusion(\Gamma, (\pi\ R(l)), \varphi, l)}$$

Where the multiset of sequents $premisesE(\Gamma, \varphi, l)$ *and the sequent* $conclusion(\Gamma, \pi', \varphi, l)$ *are defined by induction on the stucture of* φ *as follows*

- *if* φ *is atomic then* $premisesE(\Gamma, \varphi, l) = \varnothing$ *and* $conclusion(\Gamma, \pi', \varphi, l) = (\Gamma \vdash \pi' : \varphi)$
- *if* $\varphi = \varphi_1 \Rightarrow \varphi_2$ *then* $premisesE(\Gamma, \varphi, \tau.l) = \{\Gamma \vdash \tau : \varphi_1\} \cup premisesE(\Gamma, \varphi_2, l)$ *and* $conclusion(\Gamma, \pi', \varphi, \tau.l) = conclusion(\Gamma, \pi', \varphi_2, l)$
- *if* $\varphi = \forall x\ \varphi_1$ *we let* $premisesE(\Gamma, \varphi, t.l) = premisesE(\Gamma, \varphi_1\{x := t\}, l)$ *and* $conclusion(\Gamma, \pi', \varphi, t.l) = conclusion(\Gamma, \pi', \varphi_1\{x := t\}, l)$

Example 2 (Proof-terms for the inclusion). Our definition of $\subseteq$ uses a witness and charges an assumption into the context. Thus, the associated proof-terms are those given by the following typing rules:

$$(\subseteq I)\ \frac{\Gamma, \alpha{:}(x \in X) \vdash A : (x \in Y)}{\Gamma \vdash \lambda_{\subseteq}(x, \alpha).A : (X \subseteq Y)} \qquad (\subseteq E)\ \frac{\Gamma \vdash A : (X \subseteq Y) \qquad \Gamma \vdash B : (t \in X)}{\Gamma \vdash A_{\subseteq}(t, B) : (t \in Y)}$$

Definition 6 (Generalized cut elimination). *The elimination of a generalized cut is represented by a reduction which transmits the witnesses and the lemmas to the proof.*

$$\lambda R(\overline{m}).\pi \ R(\overline{t}) \rhd_\rho \pi\overline{\{m := t\}}$$

When seeing Supernatural deduction proof-terms as very simple ρ-terms of the rewriting calculus, the generalized cut elimination is then a ρ-reduction step. Hence the notation.

3 Normalization

We prove that strong normalization with respect to a rewrite system $\mathcal{R}$ is equivalent in SND and in fold/unfold. To do so, we introduce translations beetween from each system to the other, such that if $\pi \rhd_{S_1} \pi'$, then $[\![\pi]\!] \rhd^+_{S_2} [\![\pi']\!]$. The existence of such a translation is a sufficient condition for the normalization in S_2 to imply that in S_1.

3.1 From fold/unfold to SND

Definition 7. *To each rule $R : P \to \varphi$, we associate two proof-terms in Supernatural deduction of type $\varphi \Rightarrow P$ and $P \Rightarrow \varphi$*

$$\chi^R_\downarrow = \lambda f \ \lambda R(m_1, \ldots, m_n).(f \ m_1 \ldots m_n)$$

and

$$\chi^R_\uparrow = \lambda p \ \lambda m_1 \ldots \lambda m_n.(p \ R(m_1, \ldots, m_n))$$

where $m_1, \ldots, m_n$ is a sequence for φ.

Proposition 1. *The term $\chi^R_\uparrow \ (\chi^R_\downarrow \ x)$ reduces in two steps to $\lambda m_1 \ldots \lambda m_n.(x \ m_1 \ldots m_n)$.*

Definition 8 (Translation). *To each proof π of a formula φ in fold/unfold, we associate a proof of φ in Supernatural deduction by induction on the structure of π as follows.*

- *if α is a variable then $[\alpha] = \alpha$*
- $[\lambda\alpha : A.\pi'] = \lambda\alpha : A.[\pi']$
- $[\lambda x.\pi'] = \lambda x.[\pi']$
- $[(\pi_1 \ \pi_2)] = [\pi_1] \ [\pi_2]$
- $[(\pi' \ t)] = [\pi'] \ t$
- $[\uparrow_R \pi] = \chi^R_\uparrow \ [\pi]$
- $[\downarrow_R \pi] = \chi^R_\downarrow \ [\pi]$

Definition 9 (another translation). *Let π a proof of a formula φ and $m_1, \ldots, m_n$ a sequence for φ. We let $[\![\pi]\!] = [\pi] \ m_1 \ldots m_n$.*

Proposition 2. *If π is well-typed in Fold, then $[\![\pi]\!]$ is well-typed in Supernatural deduction.*

Proof. By induction on the structure of π we prove that $[\pi]$ is well-typed and has the same type than π. Let us detail the case of the fold. The type of π is A and the type of $\uparrow_R \pi$ is P. $[\pi]$ has type A. $\chi^R_\uparrow$ has type $A \Rightarrow P$. Then $\chi^R_\downarrow\ [\pi]$ has type P. Thus $[\![\pi]\!]$ is well-typed.

Proposition 3. *Let π and π' be two proofs in fold such that $\pi \rhd_{\beta+fold} \pi'$, then $[\![\pi]\!] \rhd^+_\rho [\![\pi']\!]$ where $m_1 \dots m_n$ is a sequence for the type of π.*

Proof. The only non-trivial case is that of the fold-unfold cut. $[\![\uparrow_R\downarrow_R\ \pi]\!] = \chi^R_\uparrow(\chi^R_\downarrow\ [\pi])\ m_1 \dots m_n \rhd_\rho \lambda m_1 \dots \lambda m_n([\pi] m_1 \dots m_n)\ m_1 \dots m_n \rhd_\rho [\![\pi]\!]$.

3.2 From SND to DM

Definition 10 (Translation). *To each proof π of a formula φ in SND, we associate a proof of φ in fold/unfold by induction on the structure of π as follows.*

- *if α is a variable then $[\![\alpha]\!] = \alpha$*
- *if t is a term then $[\![t]\!] = t$*
- $[\![\lambda\alpha : A.\pi']\!] = \lambda\alpha : A.[\![\pi']\!]$
- $[\![\lambda x.\pi']\!] = \lambda x.[\![\pi']\!]$
- $[\![(\pi_1\ \pi_2)]\!] = [\![\pi_1]\!]\ [\![\pi_2]\!]$
- $[\![(\pi'\ t)]\!] = [\![\pi']\!]\ t$
- $[\![\lambda R(x_1, \dots, x_n).\pi]\!] = \uparrow_R (\lambda x1 \dots x_n\ [\![\pi]\!])$
- $[\![(\pi\ R(x_1, \dots, x_n))]\!] = (\downarrow_R [\![\pi]\!]\ x_1\ \dots\ x_n)$

We leave λ-terms as usual and call τ the translation which translates:

- *an abstraction $\lambda R(\overline{m}).\pi$ as the λ-proof-term corresponding to the same introductions in Deduction modulo.*
- *an application $\pi\, R(\bar{t})$ as the λ-proof-term corresponding to the same eliminations in Deduction modulo.*

Proposition 4. *If π is well-typed in SND, then $[\![\pi]\!]$ is well-typed in fold/unfold.*

Proof. By induction on the structure of π.

Proposition 5. *If $\pi \rhd_\rho \pi'$ then $[\![\pi]\!] \rhd^+_\beta [\![\pi']\!]$.*

Proof. The only non-trivial case is that of the SND cut. $[\![(\lambda R(x_1, \dots, x_n)\ \pi)\ R(t_1, \dots, t_n)]\!] = (\downarrow_R\uparrow_R \lambda x_1 \dots x_n[\![\pi]\!])\ [\![t_1]\!]\ \dots\ [\![t_n]\!]$ which reduces to $[\![\pi]\!]$ in $n+1$ reduction steps.

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