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Isometric embeddings of subdivided complete graphs in the hypercube

Laurent Beaudou Sylvain Gravier Kahina Meslem

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Abstract

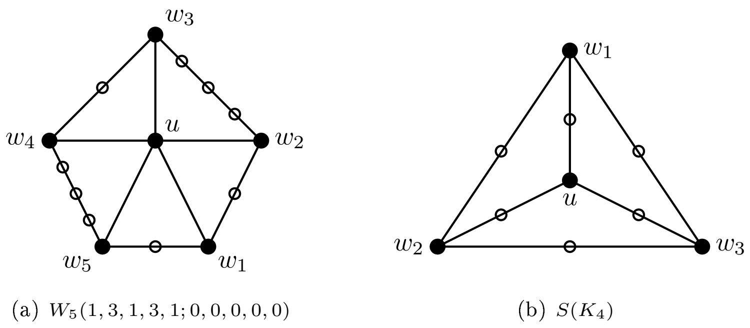
Isometric subgraphs of hypercubes are known as partial cubes. These graphs have first been investigated by Graham and Pollack [4] and Djokovič [3]. Several papers followed with various characterizations of partial cubes. In this paper, we prove that a subdivision of a complete graph of order n ($n \geq 4$) is a partial cube if and only if this one is isomorphic to $S(K_n)$ or there exist $n-1$ non-subdivided edges of K_n adjacent to a common vertex in the subdivision and the other edges of K_n are subdivided an odd number of times.

Introduction

For a graph G, the distance $d_G(u,v)$ between vertices u and v is defined as the number of edges on a shortest uv-path. A subgraph H of G is called isometric if and only if $d_G(u,v) = d_H(u,v)$ for all $u, v \in V(H)$. Isometric subgraphs of hypercubes are called *partial cubes*. Partial cubes have first been investigated by Graham and Pollak [4] and Djokovič [3]. Later, several characterizations were shown using a relation defined on the edges set or constructive operations. Partial cubes have found different applications, for instance, in [5], interesting applications in chemical graph theory were established.

Clearly, partial cubes are bipartite. The simple way to obtain a bipartite graph is to subdivide every edge of G by a single vertex. Such a graph is a subdivision of G and denote $S(G)$. However, the main question is how to determine which subdivision is a partial cube. In this paper, we are dealing with subdivisions of complete graphs. Our goal is to determine among all the subdivisions of a complete graph, which ones are partial cubes. Until now, low-density graphs had been studied (trees, cycles, wheels). We have decided to see what we could say on the other side of the problem, with high-density graphs, and their most known representatives : complete graphs.

In literature, the subdivision of a given graph has been treated as partial cubes and important results were provided. The subdivided wheels result was interesting since it consists in answering in negative a question of Chepoi and Tardif [2] whether partial cubes are precisely bipartite graphs with convex intervals :



(a) $W_5(1,3,1,3,1;0,0,0,0,0)$ (b) $S(K_4)$

Figure 1:

In [6], the authors characterize the partial cubes that are subdivided wheels (see 2).
In this paper we prove a conjecture due to Aïder, Gravier and Meslem [1] which characterizes all the subdivisions of a clique that are a partial cube. Either it is $S(K_n)$, or one of the vertices has no incident subdivided edge and all other edges are subdivided an odd number of times.

1 Preliminary definitions and main result

We only consider finite, simple, loopless, connected and undirected graphs G=(V,E) where V is the vertex set and E is the edge set. A *subgraph* of G is a graph having all its vertices and edges in G. The *neighborhood* of a vertex u, denoted by $N(u)$, consists in all the vertices v which are adjacent to u. Given a subset S of V, the *induced subgraph* $\langle S\rangle$ of G is the maximal subgraph of G with vertex set S. A *complete graph* of order n, denoted K_n, is a graph having n vertices such that each two distinct vertices are adjacent.
A *walk* is a sequence of vertices $v_1, v_2, ..., v_n$ and edges v_iv_{i+1}, $1 \leq i \leq n-1$. A *path* on n vertices, denoted P_n, is a walk on n different vertices. A closed walk, in which all vertices (except the first and the last) are different, is a *cycle*. The cycle on n vertices is denoted C_n. For a graph G, the *distance* $d_G(u,v)$ between vertices u and v is defined as the number of edges on a shortest uv-path (or uv-*geodesic*). A subgraph H of G is called *isometric* if $d_G(u,v) = d_H(u,v)$ for all distinct vertices u and v in $V(H)$.
The vertex set of the n-*cube* (or the *hypercube*) Q_n consists of all n-tuples $b_1, b_2, ..., b_n$ with $b_i \in \{0,1\}$. Two vertices are adjacent in Q_n if the corresponding tuples differ precisely in one place. Q_n is a bipartite graph. An isometric subgraph of Q_n is called *partial cube.* A graph G is an *isometric embedding* in the hypercube if it is isomorphic to a partial cube. A *subdivision* of a graph G, noted sub(G), is a graph obtained from G by adding vertices to the edges of G. A vertex v in G which is adjacent to all its neighbors of G in sub(G) is said *universal* in sub(G) . That means that all the edges of G incident to v, are not subdivided. $S(G)$ is the subdivision of G where each edge of G contains exactly one added vertex.
W_k be the k-wheel, that is, the graph obtained as a join of the one vertex graph K_1 and all the vertices of the cycle C_k. We denote the central vertex of W_k by u and the remaining vertices by $w_1, ..., w_k$. $W_k(m_1, ..., m_k; n_1, ..., n_k)$ is the graph obtained by subdividing edges of W_k, where m_i is the number of vertices added on the edge w_iw_{i+1}, and n_i the number of vertices added on the inner edge uw_i. See Fig. 1(a).
Our proposal is to demonstrate the following theorem conjectured in [1]

Theorem 1. *Let G be a subdivision of a complete graph K_n $(n \geq 4)$. G is a partial cube if and only if either G is isomorphic to $S(K_n)$ or G contains a universal vertex u and the number of added vertices to each edge no incident to u in K_n is odd.*

2 Proof of the main result

In this section, we provide the validity of the Theorem 1. Thus, we use the following terminology to prove this theorem. G is a subdivision of K_n also denoted as sub(K_n). A vertex u in G is said *principal* in G if u belongs to K_n (it has not been added to subdivide an edge). We have to note that in our proof, we only use principal vertices. We will be interested about paths that join principal vertices in G. Thus, a path of order n, $P_n(x_1, x_2, \ldots, x_n)$ is a path that joins principal vertices x_1, x_2,$\ldots$,x_n in G. An edge that joins two principal vertices in G, x and y is said *plain* (it has not been subdivided). We denote by $G\backslash u$ the subdivision of K_{n-1} induced by $V(G)\backslash u$. For each x, y and z principal vertices in G, we say that x *sees* y if the path joining these vertices in G is geodesic.
Notice that in our figures, a line (resp. a dotted line) represents a geodesic (resp. no geodesic) path between two principal vertices in G. A bold line represents a plain edge. A dashed line represents a subdivided edge with undetermined status.

2.1 Proof of the sufficient condition

Theorem 2. [6] Let $k \geq 3$. Then a subdivided k-wheel W is a partial cube if and only if W is isomorphic to $W_k(m_1, ..., m_k; 0, ..., 0)$, where m_i is odd for $i = 1, ..., k$, or $W = W_3(1,1,1;1,1,1)$.

Proposition 3. [8] For any $n \geq 1$, $S(K_n)$ is a partial cube.

Lemma 4. *[1] Let G be a subdivision of K_n $(n \geq 4)$ where each edge in K_n is an isometric path in G. G is a partial cube if and only if G contains a universal vertex and the other edges of K_n have exactly one added vertex or G is isomorphic to $S(K_n)$.*

According to Proposition 3, a graph G isomorphic to a $S(K_n)$ is a partial cube. Thus, it remains to show that a subdivided complete graph $G = \text{sub}(K_n)$ having a universal vertex u and odd added vertices to each edge of K_n not incident to u is a partial cube, for each $n \geq 4$. Let $n \geq 4$, and let G be such a graph. Thanks to Lemma 4, we can embed the subdivision of K_n with u as universal vertex and exactly one added vertex to each edge not incident to u in a hypercube. Then, successively, for each edge of G which is subdivided more than once, we proceed as follows. We consider that the current graph can be embedded in Q_m. Let x and y be the principal vertices of the subdivided edge, and let us suppose that this edge is subdivided $2k+1$ times $(k > 1)$. We remove the subdivision vertex from the graph and we assume that the components of x, u and y in Q_m are : $x = (a_1, a_2, \ldots, a_i, a_j, \ldots, a_m)$, $u = (a_1, a_2, \ldots, \overline{a}_i, a_j, \ldots, a_m)$, $y = (a_1, a_2, \ldots, \overline{a_i}, \overline{a_j}, \ldots, a_m)$. We embed the same graph where the edge xy is subdivided $2k+1$ times in Q_{m+k}. In fact, the first m components of each vertex in the embedding which belongs to Q_m are the same in Q_{m+k} and the remaining ones are null. For each $i = 1, ..., 2k+1$, we can attribute to the vertex v_i the following components in Q_{m+k}:

$$\begin{cases} v_i = (a_1, a_2, \ldots, a_i, a_j, \ldots, a_m, \overbrace{1, \ldots, 1}^{i \text{ times}}, 0, \ldots, 0) & 1 \leq i \leq k \\ v_i = (a_1, a_2, \ldots, a_j, \overline{a_j}, \ldots, a_m, 1, 1, \ldots, 1, 1) & i = k+1 \\ v_i = (a_1, a_2, \ldots, \overline{a_i}, \overline{a_j}, \ldots, a_m, \underbrace{0, \ldots, 0}_{i-k-1 \text{ times}}, 1, \ldots, 1) & k+2 \leq i \leq 2k+1 \end{cases}$$

The distances between vertices from the precedent embedding are preserved. Besides it is straightforward to see that a shortest path from any vertex of the precedent embedding to a vertex v_i goes through x or y so that the resulting graph is also a partial cube. By doing this transformation for every edge subdivided more than once, we obtain that G is a partial cube.

2.2 Proof of the necessary condition

We will proceed by induction. We first study the subdivisions of K_4 and K_5.

2.2.1 First steps

We have the following results concerning the subdivision of a complete graph:
The theorem for K_4 is contained in Theorem 2 (a 3-wheel is isomorphic to K_4).

Proposition 5. *[1] Let G be a subdivision of K_5. G is a partial cube if and only if G is isomorphic to $S(K_5)$ or G contains a universal vertex u and the number of the added vertices to each edge no incident to u in K_5 is odd.*

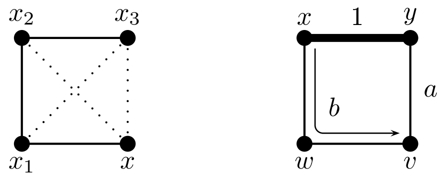
2.2.2 Useful minor results

Proposition 6. *Let x, y be principal vertices of G, then a xy-geodesic is either isomorphic to P_2 or P_3.*

Proof. Clearly, there exists $p \geq 2$ such that a xy-geodesic is isomorphic to P_p.
For a contradiction, assume that $p \geq 4$; we now consider the first four vertices of $geo(x, y)$: x, x_1, x_2 and x_3. They induce in G an isometric subdivision of K_4 (see Fig.2(a)). Then, by Theorem 2, either $\{x, x_1, x_2, x_3\}$ is isometric to $S(K_4)$ (impossible because $P(x, x_3)$ would be isometric), either there is a

universal vertex in it. But $P(x, x_2)$ cannot be a plain edge because $d_G(x, x_2) = d_G(x, x_1) + d_G(x_1, x_2) \geq 2$. Besides, $P(x_1, x_3)$ cannot be a plain edge for the same reason, so that there is no universal vertex in this subgraph which is a contradiction.

□



(a) (b)

Figure 2:

From now on, for any x and y principal vertices of G, we will note $x \to y$ if $P(x, y)$ is geodesic and $x \xrightarrow{v} y$ if the path along x, v and y is a xy-geodesic.

Remark 2.1. *If G is a partial cube, then G is also bipartite, and all its cycles are even.*

Lemma 7. *If $P(x, y)$ is plain and $y \to v$ then a xv-geodesic is contained in $\langle x, y, v \rangle$.*

Proof. For a contradiction let us suppose that $x \xrightarrow{w} v$ for some w distinct from x, y and v. We denote by a the distance between y and v, and b the distance between x and v (see Fig.2(b)). Then we must have $b < a + 1$ (else, $x \xrightarrow{y} v$). Moreover, if $b \leq a - 1$, the sequence y, x, w, v would be a yv-geodesic isomorphic to P_4, which is impossible by Proposition 6. Finally, we have $b = a$ that leads to an odd cycle of length $2a + 1$ which is also impossible.

□

2.2.3 Proof of the induction

We can now suppose that there exists $n \geq 6$ such that the partial cube G is a subdivision of K_n. Moreover, by Lemma 4 we can assume that G is not isomophic to $S(K_n)$, and that the theorem is proven for any $m < n$.

Proposition 8. *If there exists u of G a principal vertex, such as $G \backslash u$ is isometric, then there exists a universal vertex in G.*

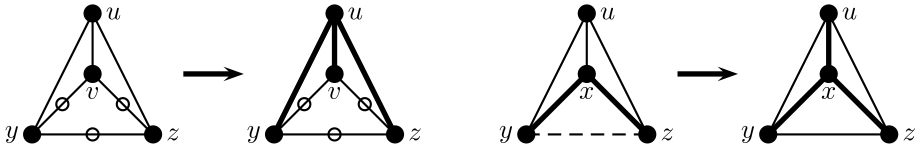
Proof. As $G \backslash u$ is isometric in G which is a partial cube, $G \backslash u$ is also a partial cube. With the induction hypothesis, there exists $x \in G \backslash u$ a universal vertex in $G \backslash u$ or $G \backslash u$ is isomorphic to $S(K_{n-1})$.
We first consider the case when $P(u, v)$ is isometric for any $v \in G \backslash u$.

- If $G \backslash u$ is isomorphic to $S(K_{n-1})$, let us prove that u is universal in G. As G is not isomophic to $S(K_n)$ there exists $v \in G \backslash u$ such that $P(u, v)$ is not subdivided exactly once.

  Let y, z be vertices in $G \backslash \{u, v\}$ ($n \geq 6$, so that we can assume that u, v, y, z are distinct), then $\langle u, v, y, z \rangle$ is clearly isometric in G. Therefore, it is a partial cube and a subdivision of K_4 not isomorphic to $S(K_4)$ because of $P(u, v)$. Then, by Theorem 2, u is the only possible universal vertex in this subgraph and $P(u, v)$, $P(u, y)$ and $P(u, z)$ are plain edges. Therefore, u is a universal vertex in G (see Fig.3(a)).

- If there exists $x \in G \backslash u$ universal in $G \backslash u$; as $n \geq 6$, there exist $y, z \in G \backslash u$ distinct.

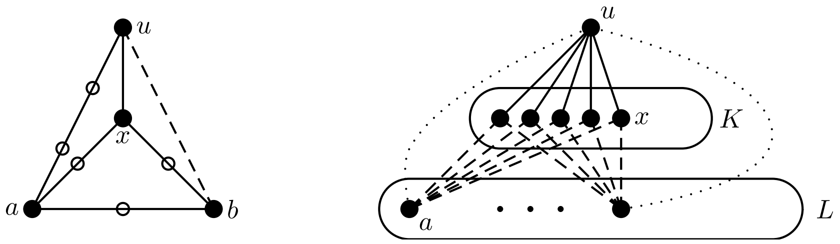
  As x is universal in $G \backslash u$, we can assume that a shortest path from y to z is contained in $\langle x, y, z \rangle$. Therefore, $\langle u, x, y, z \rangle$ is isometric in G and, as a subdivision of K_4, it contains, by the Theorem 2, a universal vertex which must be x (it cannot be isomorphic to $S(K_4)$ because of the plain edge $P(x, y)$). Therefore, $P(u, x)$ is a plain edge and x is a universal vertex in G (see Fig.3(b)).



(a) (b)

Figure 3:

Let us now consider the case when there exists $a \in G$ principal vertex, such that $P(u,a)$ is not isometric. Let us demonstrate that $G\backslash u$ is not isomorphic to $S(K_{n-1})$ by contradiction. Let us suppose it is, and let x be a vertex of $G\backslash u$ that minimize $d_G(u,x)$. There exists $b \in G\backslash\{u,a,x\}$ $(n \geq 6)$. We can assume that $u \xrightarrow{x} a$, and either $u \rightarrow b$ or $u \xrightarrow{x} b$. These four vertices induce an isometric subdivision of K_4 in G. But $P(u,a)$ is at least subdivided twice (or it would be isometric). Therefore, it neither is isomorphic to $S(K_4)$ nor has a universal vertex which is impossible (see Fig. 4(a))



(a) (b)

Figure 4:

$G\backslash u$ contains then a universal vertex x.

We can split vertices of $G\backslash u$ in two non-empty sets K containing the principal vertices y such that $P(u,y)$ is isometric and L containing principal vertices y such that $P(u,y)$ is not isometric (for instance we know that $a \in L$ and $x \in K$, see Fig. 4(b)).

Let us prove that $x \in K$. For a contradiction, let us assume that $x \in L$. Then we can choose y a nearest vertex of u (therefore, $u \rightarrow y$ and $y \in K$). This implies that $u \xrightarrow{y} x$ as $P(x,y)$ is plain. Now let us pick another vertex z in K if it is possible or in L if $K = \{y\}$. Clearly (x is universal in $G\backslash u$), we have either $y \rightarrow z$ or $y \xrightarrow{x} z$. z can be in K or in L :

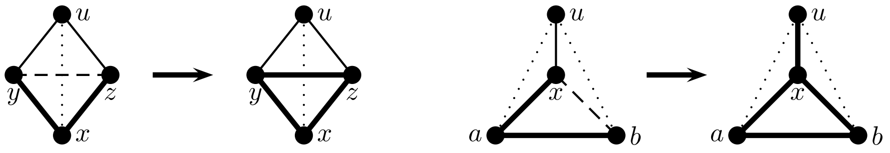
- If $z \in K$, then $\langle u,x,y,z\rangle$ is an isometric subdivision of K_4 which implies that $P(y,z)$ is plain. We have then a triangle (x,y,z) and we know we cannot have any odd cycle. Therefore, this is impossible (see Fig.5(a)).

- If $z \in L$, then $K = \{y\}$ so that $u \xrightarrow{y} z$. Therefore, $\langle x,u,y,z\rangle$ is an isometric subdivision of K_4 which implies that y is universal in it. It would mean that (x,y,z) is a triangle (see Fig.5(b)). This contradicts the fact that G is a partial cube.

We can now assume that $x \in K$. Let us prove that $P(u,x)$ is plain. For that, we consider a geodesic between u and a. It goes through K by a vertex y.

- If $y \neq x$, then $\langle u,x,a,y\rangle$ is an isometric subdivison of K_4 ; x is the only universal vertex that can be chosen so that $P(u,x)$ is plain.

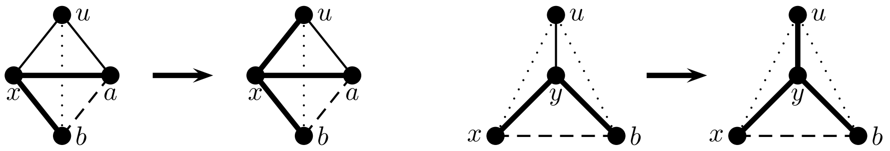
- If $x = y$



(a) (b)

Figure 5:

- let us pick b another vertex of K if it exists. Then we also have an isometric subdivision of K_4 with $\langle a, x, b, u\rangle$. Once more, x has to be the universal vertex in it and $P(u, x)$ is plain (see Fig.6(a)).
- if $|K| = 1$, let us pick b in L. Then $u \xrightarrow{x} b$ and by the way we have an isometric subdivision of K_4 with $\langle a, x, b, u\rangle$. x has to be the universal vertex and $P(u, x)$ is plain(see Fig.6(b)).



(a) (b)

Figure 6:

We now have proven that there exists a universal vertex in G. ☐

We still have to consider the case when there is no u in G such that $G\backslash u$ is isometric.

Proposition 9. *If G is a partial cube, then there exists u in G such that $G\backslash u$ is isometric in G.*

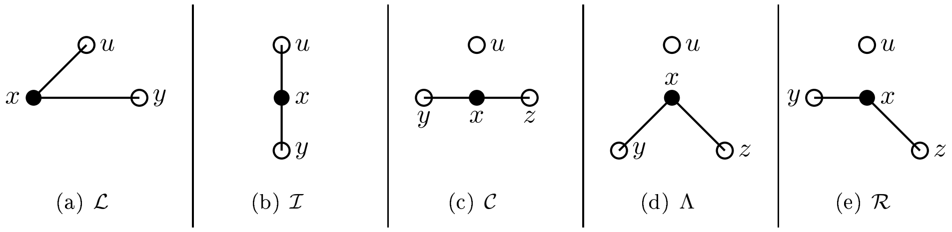
Proof. Let us suppose that there is not any u in G that can be removed. It means that, for any vertex u in G, there exist vertices x, y in G such that $x \xrightarrow{u} y$ is the only xy-geodesic.

Definition 2.1. *For the rest of the proof, we will classify vertices x of K as follows (see also Fig.7):*

- *x has type $\mathcal{L}$ if there exists y in K such that $u \xrightarrow{x} y$.*
- *x has type $\mathcal{I}$ if there exists y in L such that $u \xrightarrow{x} y$.*
- *x has type $\mathcal{C}$ if there exist y, z in K such that $y \xrightarrow{x} z$.*
- *x has type Λ if there exist y, z in L such that $y \xrightarrow{x} z$.*
- *x has type $\mathcal{R}$ if there exist y in K and z in L such that $y \xrightarrow{x} z$.*

Remark 2.2. *Clearly, every vertex of K has one of these types. Moreover, there is no vertex with only type $\mathcal{L}$ because $P(u, y)$ is also geodesic, and $G\backslash x$ would be isometric ; we have supposed it was not.*

Lemma 10. *There is no vertex with type $\mathcal{C}$ in K.*



(a) $\mathcal{L}$ (b) $\mathcal{I}$ (c) $\mathcal{C}$ (d) Λ (e) $\mathcal{R}$

Figure 7:

Proof. Let us suppose that there exists a vertex x in K such that $y \xrightarrow{x} z$ where y and z are in K. The subdivided K_4 -$\langle u, x, y, z\rangle$- is isometric. Hence, neither z nor y is universal vertex in this sub(K_4) since $d_G(y, z) \geq 2$. The vertex u is not universal too, otherwise $y \xrightarrow{x} z$ is not geodesic. Consequently x is universal in sub(K_4).

Now, let us show for each vertex t in K distinct from x,y and z, if z sees t then $P(z, t)$ is plain. In fact, if z sees t, let us denote $a = d_G(z, t)$; then $d_G(u, t) = a$ (See Fig.8(a)). On the one hand, $a - 2 < d_G(u, t) < a + 2$ otherwise a ut-geodesic or a zt-geodesic would be isomorphic to P_4 (forbidden by Proposition 6). On the other hand, $d_G(u, t) \notin \{a - 1, a + 1\}$ otherwise the cycle (u, z, t, x) would be odd, and by Remark 2.1, G would not be a partial cube.

Consider now an xt-geodesic. It is not $x \xrightarrow{z} t$ otherwise z would have type $\mathcal{C}$ and thus, $P(u, z)$ would be plain giving birth to a triangle, (z, u, x). Consequently, $d_G(x, t) \leq a$. Furthermore, this xt-geodesic has a length at least $a - 1$ otherwise we would not have $z \xrightarrow{x} t$ (a shorter way would exist through x. $d_G(x, t) \neq a$ otherwise the vertices (x, t, z) would make an odd cycle in G, which is a contradiction. Consequently, $d_G(x, t) = a - 1$. Moreover, we can assume that $x \rightarrow t$, otherwise there would be a zt-geodesic going through four principal vertices of G, which is forbidden by Proposition 6. $\langle x, z, t, u\rangle$ is an isometric subdivision of K_4 which cannot be isomorphic to $S(K_4)$ ($P(x, z)$ is plain). There must be a universal vertex which can only be x (or else, it would lead to triangles). Thus $P(x, t)$ is plain and $a = 2$.

Now, let us consider the type of vertex y in K.

- The vertex y is not of type $\mathcal{C}$ otherwise we would have $P(u, y)$ plain that would lead to a triangle which is forbidden.

- If the vertex y has type $\mathcal{I}$, then there exists a vertex v in L such that $u \xrightarrow{y} v$. A xv-geodesic is contained in $\langle x, y, v\rangle$ thanks to Lemma 7. Then, the subdivided K_4 - $\langle u, y, x, v\rangle$ - is isometric. The vertices u and v are not universal since $P(u, v)$ is not isometric. The vertex x can not be universal, otherwise we would not have $u \xrightarrow{y} v$. If y is a universal vertex in sub(K_4) then G contains a triangle (y, x, u). Contradiction. Consequently, y is not a vertex of type $\mathcal{I}$.

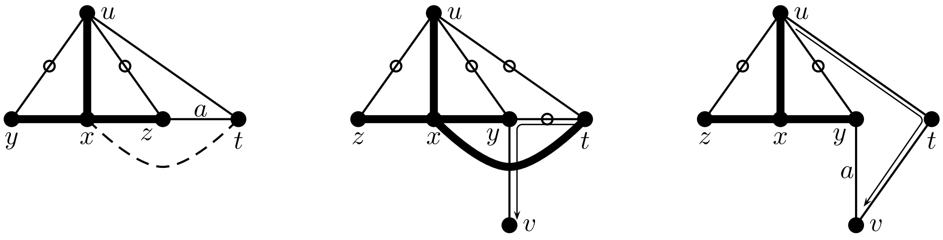
- Let us show that the vertex y is not of type Λ. If there exists two vertices v and w in L, such that $v \xrightarrow{y} w$, then a xv-geodesic (resp. xw-geodesic) is contained in $\langle x, v, y\rangle$ (resp. $\langle x, w, y\rangle$), thanks to Lemma 7. Consequently, the subdivided K_4 - $\langle x, y, v, w\rangle$ - is isometric. The vertex v (resp. w) is not universal otherwise (x, y, v) (resp. (x, y, w)) is a triangle in G. This is a contradiction. The vertex x is not universal otherwise G contains a triangle (x, y, v). If y is a universal vertex in sub(K_4), then the distance between u and v is less or equal than 3. If it is 3, we would have a uv geodesic going through u, x, y, v which would be isomorphic to P_4, this is forbidden. If it is 2, an odd cycle will arise, which also forbidden. If it is 1, v would not be in L which contradicts our hypothesis. Finally, we can assume that y has not the type Λ.

- The vertex y is not of type $\mathcal{R}$. If y has type $\mathcal{R}$, then either $t \xrightarrow{y} v$ or $x \xrightarrow{y} v$ where t is a vertex in K different than x, y and z and v is a vertex in L. In the first case, a tx-geodesic going through t, x, y, v is isomorphic P_4 (see Fig.8(b)). This is a contradiction. In the second case, we cannot have $u \xrightarrow{y} v$, otherwise y would have type $\mathcal{I}$ treated above. Neither can we have $u \xrightarrow{x} v$ (else, the sequence u, x, y, v would be a geodesic isomorphic to P_4). Then there exists t in K such that

$u \xrightarrow{t} v$ (see Fig. 8(c)). Let a be the distance between v and y. We denote l the distance between u and v. Then $l \leq a+1$, else we would have $u \xrightarrow{y} v$ which contradicts precedent conclusions. Moreover, $a-1 \leq l$, else we would have a yv-geodesic going through y, u, t, v isomorphic to P_4 which is forbidden. Finally, to avoid odd cycle, we must have $l = a$. But $d(x, v) = a+1$, thus a xv-geodesic going through x, u, t, v is isomorphic to P_4. This is a contradiction. Therefore, y cannot have type $\mathcal{R}$.

We conclude that the vertex y has none of the types $\mathcal{C}$,$\mathcal{I}$, Λ ,$\mathcal{R}$, Contradiction to the previous lemma. Consequently, the vertex x is not of type $\mathcal{C}$.

□



(a) (b) (c)

Figure 8:

Proposition 11. *Each vertex a of L sees exactly one vertex of K*

Proof. Existence : We just have to consider a ua-geodesic, it goes through K by visiting a vertex x which is seen by a.
Unicity : For a contradiction, let us suppose that a sees two distinct vertices x and y of K. We can suppose that $u \xrightarrow{x} a$. Furthermore, as it exists we can consider that (a, u, x, y) realize : $u \xrightarrow{x} a$, $a \rightarrow y$ and $d_G(u, x) + d_G(x, a) + d_G(a, y)$ is minimum for G and u.

Claim 12. *$\langle u, a, x, y \rangle$ is an isometric subdivision of K_4 in G.*

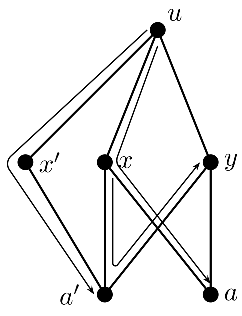
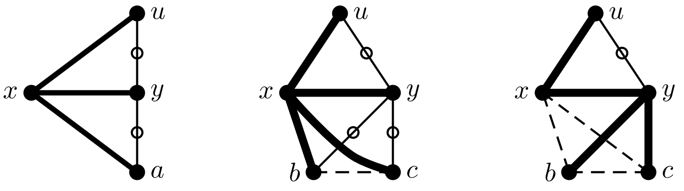


Figure 9:

Proof. For a contradiction, let us suppose that it is not isometric. It would mean that there exists a' principal vertex of G distinct from u and a such that $x \xrightarrow{a'} y$. Thanks to the Lemma 10, we can assume that $a' \in L$ (otherwise it would have type $\mathcal{C}$). Moreover, we are sure that $d_G(u,x)+d_G(x,a')+d_G(a',y) < d_G(u,x) + d_G(x,a) + d_G(a,y)$. As (a, u, x, y) is a minimum for this quantity, it means that a ua'-geodesic does not go through x or y. There exists a vertex x' in K distinct from the others such that $u \xrightarrow{x'} a'$(see Fig.9). We then have : $u \xrightarrow{x'} a'$, $a' \rightarrow y$ and $d_G(u,x') + d_G(x',a') + d_G(a',y) < d_G(u,x) + d_G(x,a) + d_G(a,y)$. This is a contradiction that proves our Claim 12.

$\square$

The induction hypothesis for $n = 4$ implies that x or y is universal in this sub(K_4) (it cannot be isomorphic to $S(K_4)$ since $P(u,a)$ is not geodesic and thus it is at least subdivided twice). Let us assume x is this universal vertex, then $P(a,y)$ and $P(u,y)$ are both subdivided exactly once as they are geodesics (they cannot be plain because it would lead to a triangle, see Fig.10(a)). We will now demonstrate that this case can never happen, by using the following Lemma :



(a) (b) (c)

Figure 10:

Lemma 13. *There are no vertices x and y in K such that $P(x,u)$ and $P(x,y)$ are plain.*

Proof. For a contradiction, let us suppose it was possible. Thus, we consider a type of y, which cannot be $\mathcal{C}$ by Lemma 10.
ăy *has type* $\mathcal{I}$ *:* then there exists $b \in L$ such that $u \xrightarrow{y} b$. The distance between u and b is then $d(y,b) + 2$. Then the sequence u,x,y,b would also be a ub-geodesic. But the Proposition 6 forbids geodesics isomorphic to P_4. This contradicts the type $\mathcal{I}$ for y.
y *has type* Λ *:* then there exist b and c in L such that $b \xrightarrow{y} c$. By Lemma 7, we know that a xb-geodesic and a xc-geodesic are contained in $\langle x,y,b,c\rangle$. Then the subdivision of K_4 induced is isometric and distinct from $S(K_4)$ so that it has a universal vertex which can either be x or y (else it would lead to a triangle). If x is universal, then $P(b,y)$ and $P(c,y)$ are both subdivided twice because they are geodesics (they cannot be plain to avoid triangles, see Fig.10(b)). This contradicts $b \xrightarrow{y} c$; there is a shorter walk through x. If y is universal (see Fig.10(c)), let us suppose $d_G(u,c) = 3$, then u,x,y,c is a uc-geodesic isomorphic to P_4 which is impossible by Proposition 6 ; therefore, $d_G(u,c) = 2$ (it cannot equal 1 because $c \in L$). The geodesic of length 2 and the walk of length 3 between u and c induce an odd cycle of length 5. We can then assume there is no y with type Λ.
y *has type* $\mathcal{R}$ *:* then there exists $b \in L$ and $t \in K$ such that $t \xrightarrow{y} b$.

- If $t = x$, we denote by l the distance between y and b. Then $d_G(x,b) = l+1$. A ub-geodesic cannot goes through x (it would be isomorphic to a P_4 forbidden by Proposition 6) or y (it would have type $\mathcal{I}$). Therefore, there exists a vertex z in K such that $u \xrightarrow{z} b$, we denote by p the distance between u and b. Then $p > l$, else, the sequence x,u,z,b would be a geodesic isomorphic to P_4. Besides, $p \leq l+1$, else, we would have $u \xrightarrow{y} b$ and y would have type $\mathcal{I}$. Therefore $p = l+1$ which is also impossible because it induces an odd cycle (b,y,u,z) of length $2l+3$ (see Fig.11(a)).

- If $t \neq x$, let l be the distance between y and t. We then study $d_G(x,t)$, denoted by p (see Fig.11(b)). On the one hand, $p \leq l$, else we would have a path of length $l+1$ between x and t going through y and thus, it would have type $\mathcal{C}$ forbidden by Lemma 10. Furthermore, p cannot equal l because it would lead to an odd cycle (x,t,y) of length $2l+1$. On the other hand we know that $p > l-2$ because $P(t,y)$ is a ty-geodesic. By consequence, $p = l-1$; which implies that $l \geq 2$ (it means, $P(y,t)$ is not plain). From this we can also conclude that $x \rightarrow t$ because if we had $x \xrightarrow{v} t$, the path (y,x,v,t) of length l would then be a geodesic isomorphic to P_4 which is impossible (by Proposition 6). Finally, $\langle y,x,b,t\rangle$ is an isometric sub(K_4) in G. It cannot be

$S(K_4)$ because $P(x,y)$ is plain ; and, as $P(y,t)$ is not plain, x must be the universal vertex in this subgraph. Then $d_G(t,b) \leq d_G(t,x) + d_G(x,b) = 2 \leq l < d_G(t,y) + d_G(y,b) = d_G(t,b)$. This is a contradiction.

This means y cannot not have any of the mandatory types. It finishes the proof of Lemma 13

□

We then have proven the Propositionă11, each vertex a of L sees exactly one vertex x of K; besides, we have $u \xrightarrow{x} a$ which is the only ua-geodesic (if not, any other ua-geodesic would be isomorphic to P_4).

□

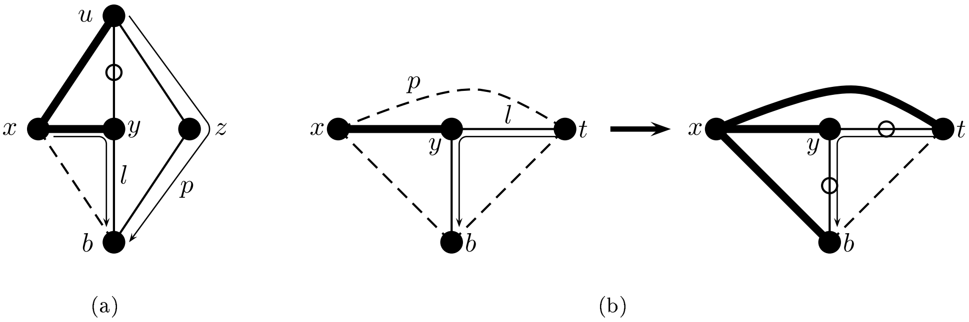


Figure 11:

We will now prove that $|K| = 1$.
For this, we will proceed by contradiction. Thus, let us suppose that there exist x and y distinct vertices of K. Both must have a type $\mathcal{R}$,$\mathcal{I}$ or Λ (by Remark 2.2 and Lemma 10). Each one of these types, implies that x and y sees at least one vertex in L. Let a be a vertex such that x sees a. Then we consider a shortest path from a to y. It cannot be direct because of Proposition 11. It cannot go through x, else it would induce an isometric subdivision of K_4 and x would be universal : $P(x,u)$ and $P(x,y)$ would then be plain which is forbidden by Lemma 13. So there exists b in L such that $a \xrightarrow{b} y$. As b sees y we can assume that $u \xrightarrow{y} b$. We may then consider that (x,a,y,b) are the vertices of that kind ($x, y \in K$, $a, b \in L$, $u \xrightarrow{x} a$, $u \xrightarrow{y} b$ and $a \xrightarrow{b} y$) that minimize the quantity $d_G(x,a) + d_G(a,b) + d_G(b,y)$.

Claim 14. *$\langle u,x,y,a,b\rangle$, is isometric.*

Proof. If the subdivided K_5 -$\langle u,x,y,a,b\rangle$- is not isometric, then any shortest xb-path does not belong to this sub(K_5). According to Proposition 11, the vertex b does not see any other vertex in K than y. Then, there exists a vertex a' in L such that $b \xrightarrow{a'} x$. Since x is the unique vertex of K that x sees (Proposition 11), then $u \xrightarrow{x} a'$. Therefore, $u \xrightarrow{x} a'$,$u \xrightarrow{y} b$ and $b \xrightarrow{a'} x$. Furthermore, $d_G(x,a') + d_G(a',b) + d_G(b,y) < d_G(x,a) + d_G(a,b) + d_G(b,y)$. Contradiction to the hypothesis. Our Claim is proven. □

Then, this subdivision of K_5 is a partial cube and by the induction hypothesis, it is either isomorphic to $S(K_5)$ or has a universal vertex. It cannot be isomorphic to $S(K_5)$ because $P(u,a)$ would then be a ua-geodesic and it is not. By consequence it must have a universal vertex which can neither be u (it does not see a), nor a or b (they do not see u), nor x or y (then we would have $P(x,u)$ and $P(x,y)$ plain which is forbidden by Lemma 13). Thus, this subgraph is not a partial cube which contradicts our hypotheses.
As a conclusion, we can assume $|K| = 1$ and then, u sees only one vertex in G which means that no geodesic goes through it. It implies that $G\backslash u$ is isometric. This contradicts our hypothesis. Finally, we have proven Proposition 9.

□

We then have a universal vertex u in G and to avoid odd cycles, there has to be an odd number of added vertices in edges that are not incident to u.
Consequently, the Theorem 1 is proven.

2.3 A corollary

Corollary 15. *Let sub(G) be a subdivision of a graph such that each edge contains odd added vertices. K is a graph obtained from sub(G) by joining a vertex u adjacent to each principal vertex of sub(G). Then, K is a partial cube.*

Proof. The proof is included in the sufficient condition. We first embed isometrically the star with u as a central vertex. Then, we can add isometrically every odd paths between two vertices of G following the construction made in Paragraph 2.1.

□

Conclusion

A brief summary of the proof could be the following. We first prove that if a vertex can be removed isometrically, we then have a universal vertex thanks to the induction hypothesis. Then we still have to prove that we can always remove a vertex. We consider that if every vertex is needed, they all have a type among $\mathcal{C}$, Λ , $\mathcal{R}$, $\mathcal{I}$. We prove that none can have type $\mathcal{C}$. After that, we exhibit an isometric subdivision of K_4 to show that any vertex of L cannot see two vertices of K. And we conclude by finding an isometric subdivision of K_5 that cannot have any universal vertex and is distinct from $S(K_5)$.
In the end, we have a structural characterization of every subdivisons of complete graphs that are partial cubes.

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