

# Interaction transform for bi-set functions over a finite set

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## Abstract

Set functions appear as a useful tool in many areas of decision making and operations research, and several linear invertible transformations have been introduced for set functions, such as the Möbius transform and the interaction transform. The present paper establish similar transforms and their relationships for bi-set functions, i.e. functions of two disjoint subsets. Bi-set functions have been recently introduced in decision making (bi-capacities) and game theory (bi-cooperative games), and appear to open new areas in these fields.

*Key words:* set function, bi-set function, Möbius transform, interaction transform

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## 1 Introduction

In the field of decision theory and operations research, set functions vanishing on the empty set are an important mathematical tool. In cooperative game theory, they are called *games in characteristic form* (see e.g. Owen [18]), while in operations research they correspond to *pseudo-Boolean functions* [16]. If in addition we require monotonicity with respect to inclusion, we get *capacities* as defined by Choquet [4], or *fuzzy measures* (Sugeno [23]), which happened

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to be very useful in decision under risk and uncertainty [20], and multicriteria decision making [10]. Well-known particular cases of capacities are belief functions (Shafer [21]), possibility measures (Dubois and Prade [8]), etc.

In the case where the underlying set is finite, there exist close connections with combinatorics. The first one, known since Rota [19], is the *Möbius transform*, which has been widely used in the field of belief functions (under the name *probabilistic mass assignment*), capacities [3], and game theory since the Möbius transform of a game  $v$  is the coordinates of  $v$  in the basis of unanimity games [22]. The second one, which has been developed in [7] by Denneberg and Grabisch, is the *interaction transform*. It can be viewed as a generalization of the Shapley value [22], and brings very useful tools to multicriteria decision making [11].

Recently, set functions with two arguments have begun to play an important rôle in decision theory, leading to the concepts of *bi-cooperative games* [1], ternary voting games [9], and bi-capacities [13,14]. Let us describe first the motivation behind bi-capacities, as given in [14] in the framework of multicriteria decision making. We consider a set  $X$  of alternatives in a multicriteria decision problem, where each alternative is described by a set of  $n$  real-valued scores  $(a_1, \dots, a_n)$ . Suppose one wants to compute a global score of this alternative by the Choquet integral w.r.t. a capacity  $\mu$ , namely  $\mathcal{C}_\mu(a_1, \dots, a_n)$ . Then it is well known that the correspondence between the capacity and the Choquet integral is  $\mu(A) = \mathcal{C}_\mu(1_A, 0_{A^c})$ ,  $\forall A \subseteq N$ , where  $(1_A, 0_{A^c})$  is an alternative having 1 as score on all criteria in  $A$ , and 0 otherwise. Such an alternative is called *binary alternative*, and the above result says that the capacity represents the overall score of all binary alternatives.

However, in many practical situations, it is suitable to score alternatives on a bipolar scale, i.e., with a central value 0 having the meaning of a borderline between positive scores, considered as good, and negative scores, considered as bad. It has been observed that most often human decision makers have a different behaviour when faced with alternatives having positive or negative scores, which means that a decision model based solely on the classical Choquet integral, hence on binary alternatives, is no more sufficient. One should, in the general case, consider all *ternary alternatives*, i.e., alternatives of the form  $(1_A, -1_B, 0_{(A \cup B)^c})$ . Clearly, we need two arguments to denote the overall score of ternary alternatives, namely  $v(A, B)$ , with  $A, B \subseteq N$  being disjoint. This defines bi-capacities, by analogy with capacities.

Motivations in game theory are similar. In classical voting games,  $v(A)$  represents the result of a vote concerning some bill, if all voters in  $A$  vote in favor of the bill, the remaining voters being against. In ternary voting games, each voter has three alternatives: voting in favor, against, or abstain. Then  $v(A, B)$  depicts the result of the vote when voters in  $A$  vote in favor, voters in  $B$  vote

against, and the other ones abstain.

Hence, such “bi-set” functions enable a richer modelling of situations in decision making. The question is then to recover the usual tools associated with set functions, namely the Möbius and interaction transforms. The aim of this paper is precisely to fill this gap. We provide a construction of these two transforms, in the same spirit as the one done by Denneberg and Grabisch in [7], so that the present paper can be seen as a natural continuation of the former. For this reason, we will remain at a general level and deal with *bi-set functions*, instead of more specific cases, as bi-capacities, bi-cooperative games, etc. We will see that analogous results are obtained, despite the fact the underlying structure is very different.

The organization of the paper is as follows. Section 2 provides necessary background on set functions and bi-set functions, Section 3 recalls the construction of the interaction transform for set functions as done in [7], Section 4 introduces operators on  $\mathcal{Q} \times \mathcal{Q}$ , where  $\mathcal{Q}$  is the set of pairs of disjoint subsets of  $N$ , while Sections 5 and 6 introduce particular cases of such operators, called level operators and cardinality operators. Section 7 gives the expression of the inverse interaction transform, which enables to have a commutative diagram between bi-set functions and their Möbius and interaction transform.

To simplify notations, cardinality of sets  $S, T, \dots$  will be denoted by the corresponding lower case letter  $s, t, \dots$ .

## 2 Set functions and bi-set functions

We introduce necessary concepts for the sequel. We consider a finite set  $N := \{1, \dots, n\}$  which can be thought as the set of criteria, states of nature, voters, etc., depending on the application. We set  $\mathcal{P} := \mathcal{P}(N)$ . We know that  $(\mathcal{P}, \subseteq)$  is the Boolean lattice  $2^n$ , and any  $A \in \mathcal{P}$  can be written as a *binary tuple*  $x = (x_1, \dots, x_n) \in \{0, 1\}^n$ , where  $x_i = 1$  iff  $i \in A$ .

A *set function*  $v$  on  $N$  is a real-valued mapping on  $\mathcal{P}$ . Several particular cases are of interest. A set function vanishing on the empty set is called a *game*, while a game satisfying monotonicity, i.e.  $v(A) \leq v(B)$  whenever  $A \subseteq B$ , is called a *capacity* [4], or *non-additive measure* [6], or *fuzzy measure* [23]. Note that when subsets are considered as binary tuples, set functions are called *pseudo-Boolean functions* [16].

For any  $C \subseteq N$ , the *unanimity game*  $u_C$  is defined as:

$$u_C(A) := \begin{cases} 1, & \text{if } A \supseteq C \\ 0, & \text{otherwise} \end{cases}, \quad A \subseteq N.$$

Remark that  $u_\emptyset$  is not a game since  $u_\emptyset(\emptyset) = 1$ .

The *Möbius transform*  $m^v : \mathcal{P} \rightarrow N$  [19] of a set function  $v$  is the unique solution of the equation

$$v(A) = \sum_{B \subseteq A} m^v(B), \quad \forall A \subseteq N, \quad (1)$$

and is given by

$$m^v(A) := \sum_{C \subseteq A} (-1)^{a-c} v(C), \quad \forall A \subseteq N. \quad (2)$$

Eq. (1) can be rewritten as, using unanimity games:

$$v(A) = \sum_{C \in \mathcal{P}} m^v(C) u_C(A), \quad A \subseteq N. \quad (3)$$

Hence, the set of unanimity games forms a  $2^n$ -dimensional basis of set functions, and the Möbius transform represents the coordinates of  $v$  in that basis.

For any  $S$  belonging to  $\mathcal{P} \setminus \{\emptyset\}$ , the *derivative* of  $v$  with respect to  $S$  at point  $K \in \mathcal{P}(N \setminus S)$  is given by [15]:

$$\Delta_S v(K) := \sum_{S' \subseteq S} (-1)^{s-s'} v(K \cup S').$$

We set  $\Delta_\emptyset v(K) := v(K)$ , for any  $K \subseteq N$ .

The *interaction index* has been proposed by Grabisch [12] and expresses the interaction among a coalition (group)  $S \subseteq N$  of elements:

$$I^v(S) := \sum_{K \subseteq N \setminus S} \frac{(n-k-s)!k!}{(n-s+1)!} \Delta_S v(K). \quad (4)$$

This definition extends in fact the *Shapley value* [22]  $\phi^v$  and the interaction index  $I_{ij}$  for a pair of elements  $i, j$  in  $N$ , introduced by Murofushi and Soneda [17]. In particular, the Shapley value is defined by

$$\phi_i^v := \sum_{K \subseteq N \setminus i} \frac{(n-k-1)!k!}{n!} \Delta_i v(K), \quad i \in N.$$

We have  $I^v(\{i\}) = \phi_i^v$ . As it will be explained in the next section,  $I^v$  can be seen as a transform of  $v$ , like the Möbius transform.

Let us denote by  $\mathcal{Q}(N)$  or simply  $\mathcal{Q}$  if there is no fear of ambiguity the set of all pairs of disjoint subsets:

$$\mathcal{Q} := \{(A, B) \in \mathcal{P} \times \mathcal{P} \mid A \cap B = \emptyset\}.$$

We endow  $\mathcal{Q}$  with the following partial order:

$$(A, B) \sqsubseteq (C, D) \Leftrightarrow A \subseteq C \text{ and } B \supseteq D.$$

It is easy to see that  $(\mathcal{Q}, \sqsubseteq)$  is the lattice  $3^n$ , noting that any element  $(A, B)$  of  $\mathcal{Q}$  can be written as a *ternary tuple*  $x = (x_1, \dots, x_n) \in \{-1, 0, 1\}^n$ , where  $x_i = 1$  iff  $i \in A$  and  $x_i = -1$  iff  $i \in B$  [14]<sup>1</sup>. Supremum and infimum are respectively

$$\begin{aligned} (A, B) \sqcup (C, D) &= (A \cup C, B \cap D), \\ (A, B) \sqcap (C, D) &= (A \cap C, B \cup D), \quad (A, B), (C, D) \in \mathcal{Q}. \end{aligned}$$

Top and bottom of  $\mathcal{Q}$  are respectively denoted by  $\top := (N, \emptyset)$  and  $\perp := (\emptyset, N)$ . We give as an illustration  $(\mathcal{Q}, \sqsubseteq)$  for  $n = 3$  in Fig. 1.

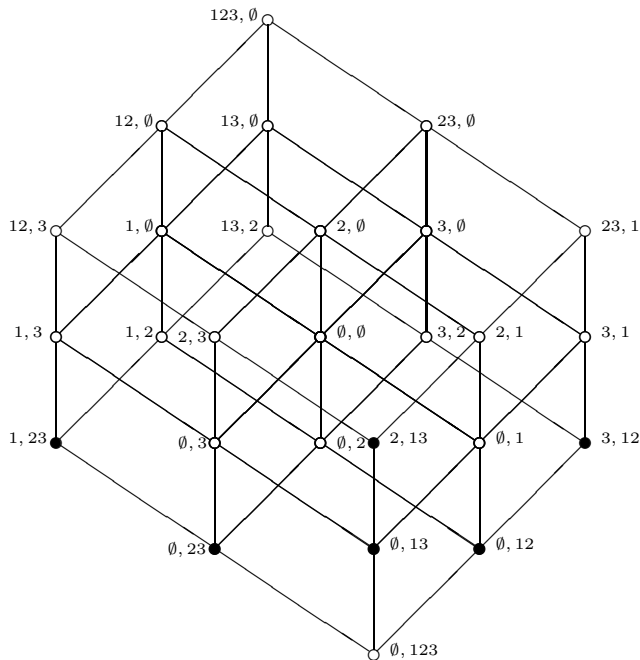


Fig. 1. The lattice  $\mathcal{Q}$  for  $n = 3$

A *bi-set function*  $v$  on  $N$  is a real-valued mapping on  $\mathcal{Q}$ . As explained in the introduction, particular cases of interest are bi-cooperative games, where

<sup>1</sup> Equivalently, one could have chosen a different coding, as 0, 1 and 2 instead of  $-1, 0$  and 1. Our choice, which is unimportant in this paper, is just suited to the original motivation in decision making explained in the introduction.

it is required that  $v(\emptyset, \emptyset) = 0$ , and bi-capacities which require in addition monotonicity, i.e.  $(A, B) \sqsubseteq (C, D)$  implies  $v(A, B) \leq v(C, D)$ .

The lattice  $(\mathcal{Q}, \sqsubseteq)$  being distributive, by Birkhoff's theorem [2], any element of the lattice can be written as a unique irredundant supremum over a set of *join-irreducible elements* (elements having only one predecessor). In the case of  $(\mathcal{Q}, \sqsubseteq)$  the set of all join-irreducible elements (which are represented by black circles in Fig. 1) is

$$\mathcal{J}(\mathcal{Q}) = \{(\emptyset, i^c), (i, i^c), i \in N\},$$

and the unique irredundant decomposition writes [14]:

$$(A, B) = \bigsqcup_{i \in A} (i, i^c) \sqcup \bigsqcup_{j \in N \setminus (A \cup B)} (\emptyset, j^c).$$

This permits to define *layers* in  $\mathcal{Q}$  as follows: for  $k$  in  $N_0 := \{0, \dots, n\}$ , layer  $k$  contains all elements  $(A, B)$  whose decomposition has exactly  $k$  join-irreducible elements, which is equivalent to say that  $|B| = n - k$ . We denote by  $\|\cdot\|$  the function which maps to every element of  $\mathcal{Q}$  the layer to which it belongs.

It is convenient for the sequel to define the following linear order  $\leq$  on  $\mathcal{Q}$ . Recalling that any element of  $\mathcal{Q}$  can be written as a ternary tuple  $x \in \{-1, 0, 1\}^n$  or equivalently in  $\{0, 1, 2\}^n$ , we can assign to each element  $(A, B)$  of  $\mathcal{Q}$  an integer  $n_{(A,B)}$  whose coding in basis  $\{0, 1, 2\}$  is precisely the ternary tuple  $x$  associated to  $(A, B)$ . For example, taking  $n = 4$  and the element  $(\{1\}, \{3\})$ , the corresponding tuple is  $(2, 1, 0, 1)$ , which gives the number  $2 \times 3^0 + 1 \times 3^1 + 0 \times 3^2 + 1 \times 3^3 = 32$ . Obviously, the correspondence between integers and elements of  $\mathcal{Q}$  is unique. Hence, we say that  $(A, B) \leq (C, D)$  iff  $n_{(A,B)} \leq n_{(C,D)}$ . This leads to the following order:

$$\dots (2, 3) (12, 3) \left[ (\emptyset, 12) (\emptyset, 2) (1, 2) \left[ (\emptyset, 1) \left[ (\emptyset, \emptyset) (1, \emptyset) \right] (2, 1) (2, \emptyset) (12, \emptyset) \right] (3, 12) (3, 2) \dots \right. \quad (5)$$

The brackets are there to enhance the fact that this order is in some sense “recursive”, since it can be built by an initial pattern (which is  $(\emptyset, \emptyset)$ ) and a systematic way of augmenting the current pattern, which is to add a new element of  $N$  either to the left part or to the right part of any element of the current pattern.

The Möbius transform  $m^v$  of  $v$  is the unique solution of the equation

$$v(A, B) = \sum_{(C,D) \sqsubseteq (A,B)} m^v(C, D), \quad (A, B) \in \mathcal{Q}, \quad (6)$$

and is given by [14]:

$$m^v(A, B) := \sum_{\substack{(C, D) \sqsubseteq (A, B) \\ D \cap A = \emptyset}} (-1)^{a-c+d-b} v(C, D), \quad (A, B) \in \mathcal{Q}. \quad (7)$$

We extend the notion of derivative of a set function to bi-set functions. As bi-set functions are defined on  $\mathcal{Q}$ , so should be the variables used in the derivation. For any  $i \in N$ , the derivatives with respect to any join-irreducible elements  $(i, i^c)$  and  $(\emptyset, i^c)$  of  $v$  at point  $(K, L)$  are given by [14]:

$$\begin{aligned} \forall (K, L) \in \mathcal{Q}(N \setminus i), \quad \Delta_{(i, i^c)} v(K, L) &:= v(K \cup i, L) - v(K, L), \quad \text{and} \\ \forall (K, L) \in \mathcal{Q} \text{ with } i \in L, \quad \Delta_{(\emptyset, i^c)} v(K, L) &:= v(K, L \setminus i) - v(K, L). \end{aligned}$$

These derivatives are non negative whenever  $v$  is monotonic. Higher order derivatives can be defined recursively for any  $(S, T) \in \mathcal{Q} \setminus \{(\emptyset, N)\}$  by:

$$\Delta_{(S, T)} v(K, L) := \Delta_{(i, i^c)} (\Delta_{(S \setminus i, T \cup i)} v(K, L))$$

if  $(i, i^c)$  belongs to the irredundant decomposition of  $(S, T)$ , or

$$\Delta_{(S, T)} v(K, L) := \Delta_{(\emptyset, i^c)} (\Delta_{(S, T \cup i)} v(K, L)),$$

if  $(\emptyset, i^c)$  belongs to the irredundant decomposition of  $(S, T)$ . We set  $\Delta_{(\emptyset, N)} v(K, L) = v(K, L)$ , for any  $(K, L) \in \mathcal{Q}$ .

In [14], the following definition of the interaction index for bi-set functions has been given, as a natural generalization of the definition for set functions:

$$I^v(S, T) := \sum_{K \subseteq T} \frac{(t-k)!k!}{(t+1)!} \Delta_{(S, T)} v(K, (K \cup S)^c), \quad \forall (S, T) \in \mathcal{Q}. \quad (8)$$

### 3 Interaction transform for set functions

We recall in this section main results given in [7], where a new invertible transform of set functions is introduced, called the *interaction transform*. The authors lay down a general framework of transformations of set functions by introducing an algebraic structure on set functions and operators (set functions of two variables), which enable the writing of the formulae given in the previous section under a simplified algebraic form.

In the first place, we recall main definitions. We call *operator* on  $\mathcal{P}$  a real-valued function on  $\mathcal{P} \times \mathcal{P}$ , and introduce a multiplication  $\star$  between operators,

and between operators and set functions as follows. Let  $v$  be a set function and  $\Phi, \Psi$  some operators; for  $A_1, A_2$  belonging to  $\mathcal{P}$ , we have:

$$\begin{aligned}(\Phi \star \Psi)(A_1, A_2) &:= \sum_{C \in \mathcal{P}} \Phi(A_1, C) \Psi(C, A_2), \\(\Phi \star v)(A_1) &:= \sum_{C \in \mathcal{P}} \Phi(A_1, C) v(C), \\(v \star \Phi)(A_2) &:= \sum_{C \in \mathcal{P}} v(C) \Phi(C, A_2).\end{aligned}$$

Let us now consider a subset  $\mathcal{G}_{\mathcal{P}}$  of these operators<sup>2</sup>, defined by the operators  $\Phi$  which have the property

$$\Phi(A_1, A_2) = \begin{cases} 1, & \text{if } A_1 = A_2 \\ 0, & \text{if } A_1 \not\subseteq A_2, \end{cases}$$

for any  $A_1, A_2 \in \mathcal{P}$ . The family  $\mathcal{G}_{\mathcal{P}}$  endowed with the operation  $\star$  is a group. Into, we found the so-called operators Zeta ( $Z_{\mathcal{P}}$ ) and Möbius ( $Z_{\mathcal{P}}^{-1}$ ) defined by:

$$\begin{aligned}Z_{\mathcal{P}}(A_1, A_2) &:= \begin{cases} 1, & \text{if } A_1 \subseteq A_2 \\ 0, & \text{otherwise} \end{cases}, \quad A_1, A_2 \in \mathcal{P}, \\Z_{\mathcal{P}}^{-1}(A_1, A_2) &= \begin{cases} (-1)^{a_2 - a_1}, & \text{if } A_1 \subseteq A_2 \\ 0, & \text{otherwise} \end{cases}, \quad A_1, A_2 \in \mathcal{P}.\end{aligned}$$

Then, Equations (3) and (2) can be written as:

$$v = m^v \star Z_{\mathcal{P}} \quad \text{and} \quad m^v = v \star Z_{\mathcal{P}}^{-1},$$

A central role is played by the operator  $\Gamma_{\mathcal{P}} \in \mathcal{G}_{\mathcal{P}}$ ,

$$\Gamma_{\mathcal{P}}(A_1, A_2) := \begin{cases} \frac{1}{a_2 - a_1 + 1}, & \text{if } A_1 \subseteq A_2 \\ 0, & \text{else} \end{cases}, \quad (9)$$

which is called the *inverse Bernoulli operator*. This name will be justified in Section 7. Actually, we have the relation

$$I^v = \Gamma_{\mathcal{P}} \star m^v.$$

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<sup>2</sup> The sets and functions denoted with the suffix  $_{\mathcal{P}}$  are sets and functions defined on  $\mathcal{P}$  referring to [7].

We call *level operator* an operator  $\Phi$  satisfying

$$\Phi(A_1, A_2) = \begin{cases} \Phi(\emptyset, A_2 \setminus A_1), & \text{if } A_1 \subseteq A_2 \\ 0, & \text{otherwise} \end{cases},$$

and the set of all level operators is denoted by  $\mathcal{G}'_{\mathcal{P}}$ . Endowed with  $\star$ ,  $\mathcal{G}'_{\mathcal{P}}$  is a subgroup of  $\mathcal{G}_{\mathcal{P}}$ . Let us introduce

$$\mathbf{g}_{\mathcal{P}} := \{\varphi : \mathcal{P} \rightarrow \mathbb{R} \mid \varphi(\emptyset) = 1\},$$

and associate with any level operator  $\Phi$  the function  $\varphi_{\Phi}$  of  $\mathbf{g}_{\mathcal{P}}$  defined by  $\varphi_{\Phi}(\cdot) := \Phi(\emptyset, \cdot)$ . Indeed, it is easy to see that  $\varphi_{\Phi}$  determines  $\Phi$  uniquely: let  $\varphi$  be in  $\mathbf{g}_{\mathcal{P}}$ ; if we define

$$\Phi_{\varphi}(A_1, A_2) := \begin{cases} \varphi(A_2 \setminus A_1), & \text{for } A_1 \subseteq A_2, \\ 0, & \text{else} \end{cases}, \quad A_1, A_2 \in \mathcal{P},$$

we have  $\Phi_{\varphi} = \Phi$  iff  $\varphi := \varphi_{\Phi}$ . Now, if we define the operation  $\star$  between two elements  $\varphi, \psi$  of  $\mathbf{g}_{\mathcal{P}}$  by

$$\varphi \star \psi(A) := \Phi_{\varphi} \star \Phi_{\psi}(\emptyset, A), \quad A \in \mathcal{P},$$

then  $(\mathcal{G}'_{\mathcal{P}}, \star)$  and  $(\mathbf{g}_{\mathcal{P}}, \star)$  are isomorphic.  $\varphi \star \psi$  is the *convolution* of  $\varphi, \psi \in \mathbf{g}_{\mathcal{P}}$ ,

$$\varphi \star \psi(A) = \sum_{C \subseteq A} \varphi(C) \psi(A \setminus C), \quad A \in \mathcal{P}.$$

Since the inverse Bernoulli operator is a level operator, its corresponding function  $\gamma_{\mathcal{P}} := \varphi_{\Gamma_{\mathcal{P}}}$  is:

$$\gamma_{\mathcal{P}}(A) = \frac{1}{a+1}, \quad A \in \mathcal{P}.$$

A *cardinality function* on  $\mathcal{P}$  is an element of  $\mathbf{g}_{\mathcal{P}}$  that only depends on the cardinality of the variable. The above inverse Bernoulli function is an example of cardinality function. We denote by  $\mathbf{c}_{\mathcal{P}}$  the set of all cardinality functions, and endowed with  $\star$ ,  $\mathbf{c}_{\mathcal{P}}$  is subgroup of  $\mathbf{g}_{\mathcal{P}}$ . To each cardinality function  $\varphi$  we associate its *cardinal representation*  $f_{\varphi}$  in the set

$$\mathbf{r} := \{f : N_0 \rightarrow \mathbb{R} \mid f(0) = 1\}$$

in a bijective way; for any  $A \in \mathcal{P}$

$$f_{\varphi}(|A|) = \varphi(A).$$

Conversely, for  $f \in \mathbf{r}$ , we put

$$\varphi_{f, \mathcal{P}}(A) = f(a), \quad A \in \mathcal{P}.$$

Once more,  $\mathfrak{r}$  is an Abelian group with  $f_\delta := 1_{\{0\}}$  as neutral element, and where the convolution operation is, for any  $f, g \in \mathfrak{r}$ ,  $A \in \mathcal{P}$

$$f \star g(|A|) := \varphi_{f, \mathcal{P}} \star \varphi_{g, \mathcal{P}}(A),$$

that is to say, for any  $m \in N_0$ :

$$f \star g(m) = \sum_{k=0}^m \binom{m}{k} f(k) g(m-k). \quad (10)$$

Hence,  $(\mathfrak{r}, \star)$  is isomorphic to  $(\mathfrak{c}_{\mathcal{P}}, \star)$ . We will denote  $f^{\star^{-1}}$  the inverse of an element  $f$  of  $\mathfrak{r}$ .

#### 4 Operators on $\mathcal{Q} \times \mathcal{Q}$

We will proceed in the same way for bi-set functions, the basis working set being  $\mathcal{Q}$ . We consider real-valued functions on  $\mathcal{Q}$  in one and two variables, the latter ones being called operators, and we introduce a multiplication  $\star$  between operators, and between a bi-set function and an operator. Let  $v$  be a bi-set function and  $\Phi, \Psi$  some operators; for  $(A_1, B_1), (A_2, B_2)$  belonging to  $\mathcal{Q}$ , we define:

$$\begin{aligned} (\Phi \star \Psi)((A_1, B_1), (A_2, B_2)) &:= \sum_{(C, D) \in \mathcal{Q}} \Phi((A_1, B_1), (C, D)) \Psi((C, D), (A_2, B_2)), \\ (\Phi \star v)(A_1, B_1) &:= \sum_{(C, D) \in \mathcal{Q}} \Phi((A_1, B_1), (C, D)) v(C, D), \\ (v \star \Psi)(A_2, B_2) &:= \sum_{(C, D) \in \mathcal{Q}} v(C, D) \Psi((C, D), (A_2, B_2)). \end{aligned}$$

Endowed with  $\star$ , the set of these operators contains the neutral element  $\Delta$  defined by

$$\Delta((A_1, B_1), (A_2, B_2)) := \begin{cases} 1, & \text{if } (A_1, B_1) = (A_2, B_2) \\ 0, & \text{else} \end{cases}, \quad (A_1, B_1), (A_2, B_2) \in \mathcal{Q},$$

and satisfies associativity. When it exists, we will denote  $\Phi^{-1}$  the inverse of an operator  $\Phi$ , that is to say the operator verifying  $\Phi \star \Phi^{-1} = \Phi^{-1} \star \Phi = \Delta$ .

The following proposition deals with a subset of the set of operators which is important for our study.

**Proposition 1** *The family  $\mathcal{G}$  of operators defined by:*

$$\Phi \in \mathcal{G} \iff \Phi((A_1, B_1), (A_2, B_2)) = \begin{cases} 1, & \text{if } (A_1, B_1) = (A_2, B_2) \\ 0, & \text{if } (A_1, B_1) \not\sqsubseteq (A_2, B_2) \end{cases},$$

$$(A_1, B_1), (A_2, B_2) \in \mathcal{Q},$$

*endowed with the operation  $\star$  is a group. The inverse  $\Phi^{-1}$  of  $\Phi$  in  $\mathcal{G}$  computes recursively through*

$$\Phi^{-1}((A_1, B_1), (A_1, B_1)) = 1,$$

$$\Phi^{-1}((A_1, B_1), (A_2, B_2)) = - \sum_{\substack{(C,D) \in \\ [(A_1, B_1), (A_2, B_2)[}} \Phi^{-1}((A_1, B_1), (C, D)) \Phi((C, D), (A_2, B_2)).$$

**PROOF.**  $(\mathcal{G}, \star)$  is a group if:

- $\mathcal{G}$  is stable under  $\star$ .
- $\star$  is an associative operation.
- $\Delta$  is the neutral element.
- If  $\Phi \in \mathcal{G}$ , there is an inverse  $\Phi^{-1}$  in  $\mathcal{G}$ :  $\Phi \star \Phi^{-1} = \Phi^{-1} \star \Phi = \Delta$ .

If we fix a linear order on  $\mathcal{Q}$  (see for instance the linear order  $\leq$  p.6), we can identify  $\mathcal{Q}$  with  $\{1, 2, \dots, 3^n\}$  and the operation  $\star$  becomes ordinary multiplication of square matrices or of a vector with a matrix. This shows that the operation  $\star$  is distributive with respect to the usual sum of functions.  $\star$  is also associative. Furthermore, operator  $\Delta$  defined above is clearly the unique left and right neutral element since it corresponds to the identity matrix.

Concerning the fourth property, it is sufficient to show that there is an element  $\Phi^{-1}$  belonging to  $\mathcal{G}$  verifying  $\Phi^{-1} \star \Phi = \Delta$ ; under this condition,  $\Phi \star \Phi^{-1} = \Phi \star \Delta \star \Phi^{-1} = \Phi \star \Phi^{-1} \star \Phi \star \Phi^{-1} = \Delta \star \Delta = \Delta$ . Let us construct  $\Phi^{-1}$ . If  $\Phi^{-1}$  exists (and belongs to  $\mathcal{G}$ ):

$$\Phi^{-1} \star \Phi((A_1, B_1), (A_2, B_2)) = \sum_{\substack{(C,D) \in \\ [(A_1, B_1), (A_2, B_2)[}} \Phi^{-1}((A_1, B_1), (C, D)) \Phi((C, D), (A_2, B_2)).$$

If  $(A_1, B_1) \sqsubseteq (A_2, B_2)$  with  $(A_1, B_1) \neq (A_2, B_2)$ , then:

$$\begin{aligned} & \Phi^{-1}((A_1, B_1), (A_2, B_2)) \Phi((A_2, B_2), (A_2, B_2)) + \\ & \sum_{\substack{(C,D) \in \\ [(A_1, B_1), (A_2, B_2)[}} \Phi^{-1}((A_1, B_1), (C, D)) \Phi((C, D), (A_2, B_2)) = \\ & \Delta((A_1, B_1), (A_2, B_2)) = 0. \end{aligned}$$

That is to say:

$$\Phi^{-1}((A_1, B_1), (A_2, B_2)) = - \sum_{\substack{(C,D) \in \\ [(A_1, B_1), (A_2, B_2)[}} \Phi^{-1}((A_1, B_1), (C, D)) \Phi((C, D), (A_2, B_2)).$$

Conversely, the operator given above, taking value 1 for every  $((A, B), (A, B))$  of  $\mathcal{Q} \times \mathcal{Q}$  and 0 for every  $(A_1, B_1), (A_2, B_2)$  such that  $(A_1, B_1) \not\sqsubseteq (A_2, B_2)$ , satisfies  $\Phi^{-1} \star \Phi = \Delta$ .  $\square$

We can find inside the set  $\mathcal{G}$  the *Zeta operator*  $Z$ , defined by:

$$Z((A_1, B_1), (A_2, B_2)) := \begin{cases} 1, & \text{if } (A_1, B_1) \sqsubseteq (A_2, B_2) \\ 0, & \text{otherwise} \end{cases}, \quad (A_1, B_1), (A_2, B_2) \in \mathcal{Q},$$

which allows us to rewrite Equation (6) as:

$$v = m^v \star Z.$$

Similarly, the *Möbius operator*, defined as the inverse of  $Z$ , permits to rewrite Equation (7) as:

$$m^v = v \star Z^{-1},$$

and Proposition 1 gives, for any  $(A_1, B_1), (A_2, B_2) \in \mathcal{Q}$

$$Z^{-1}((A_1, B_1), (A_2, B_2)) = \begin{cases} (-1)^{a_2 - a_1 + b_1 - b_2}, & \text{if } \begin{cases} (A_1, B_1) \sqsubseteq (A_2, B_2) \\ B_1 \cap A_2 = \emptyset \end{cases} \\ 0, & \text{otherwise} \end{cases},$$

as expected (see (7)).

In the previous section, the interaction index of a set function was expressed through the  $\mathcal{G}_{\mathcal{P}}$  operator  $\Gamma_{\mathcal{P}}$  (see (9)), which facilitated the inversion of (4). We shall undertake to do the same thing for bi-set functions. From formula (8) and according to an expression of the derivatives based on Möbius transform [14], we have for every  $(S, T) \in \mathcal{Q}$ :

$$I^v(S, T) = \sum_{\substack{(S', T') \in \\ [(S, T), (S \cup T, \emptyset)]]} \frac{1}{t - t' + 1} m^v(S', T'). \quad (11)$$

As a result, if we set down:

$$\Gamma((A_1, B_1), (A_2, B_2)) := \begin{cases} \frac{1}{b_1 - b_2 + 1}, & \text{if } (A_1, B_1) \sqsubseteq (A_2, B_2) \sqsubseteq (A_1 \cup B_1, \emptyset), \\ 0, & \text{otherwise} \end{cases}, \quad (12)$$

we can write from (11) the relation:

$$I^v = \Gamma \star m^v. \quad (13)$$

Let us notice that  $\Gamma$  is an operator in  $\mathcal{G}$ . As  $\Gamma_{\mathcal{P}}$ , we call it the *inverse Bernoulli operator*.

$\Gamma$  has a similar expression to that of  $\Gamma_{\mathcal{P}}$  (see (9)):

$$\Gamma_{\mathcal{P}}(A_1, A_2) := \begin{cases} \frac{1}{a_2 - a_1}, & \text{if } A_1 \subseteq A_2, \\ 0, & \text{otherwise} \end{cases}, \quad A_1, A_2 \in \mathcal{P},$$

with however a rather unexpected inequality  $(A_2, B_2) \sqsubseteq (A_1 \cup B_1, \emptyset)$  which will complicate the continuation of the work. Nevertheless, at this point, we can set the following fundamental result, already known in the case of set functions (see Fig. 2)

**Theorem 2** *For any bi-set function  $v$ , the triangular diagram where appear the functions  $v, m^v, I^v$  and the operators of transition  $Z, \Gamma$  is commutative.*

**PROOF.** Commutativity between  $v$  and  $m^v$  is clear according to Equation (6). The one between  $m^v$  and  $I^v$  is known due to (13). By transitivity, the result follows.  $\square$

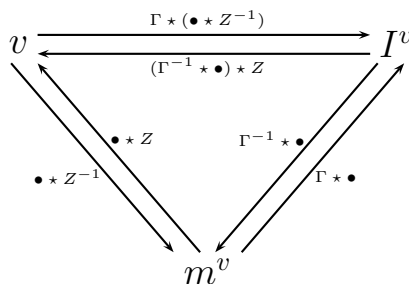


Fig. 2. Three ways of representing bi-set functions

## 5 Level operators

Our aim being now the inversion of  $\Gamma$ , a few results about lattice theory need to be brought in. First, the double inequality in (12) suggests us to introduce

a new binary relation on  $\mathcal{Q}$ , denoted by  $\preceq$ :

$$(A_1, B_1) \preceq (A_2, B_2) \text{ if and only if}$$

$$\left\{ \begin{array}{l} (A_1, B_1) \sqsubseteq (A_2, B_2) \\ A_2 \subseteq A_1 \cup B_1 \end{array} \right. \text{ or equivalently } \left\{ \begin{array}{l} A_1 \subseteq A_2 \subseteq A_1 \cup B_1 \\ B_1 \supseteq B_2. \end{array} \right.$$

It is easy to see that  $\preceq$  is an ordered relation included in  $\sqsubseteq$ .

Moreover, as we use the notation  $[(A_1, B_1), (A_2, B_2)]$  to denote the closed interval of  $(\mathcal{Q}, \sqsubseteq)$  delimited by  $(A_1, B_1)$  and  $(A_2, B_2)$ , we will use the notation  $\llbracket (A_1, B_1), (A_2, B_2) \rrbracket$  for the same of  $(\mathcal{Q}, \preceq)$  — by replacing, if needed,  $\llbracket$  by  $\llbracket$  or  $\rrbracket$  by  $\rrbracket$  if we deprive the interval of the associated bound. We also note

$$\mathcal{Q}_{(A,B)} := \llbracket \perp, (A, B) \rrbracket.$$

We have the following proposition which is useful for the sequel:

**Proposition 3** *For any  $(A, B)$  of  $\mathcal{Q}$ , the ordered subset  $(\mathcal{Q}_{(A,B)}, \preceq)$  of  $(\mathcal{Q}, \preceq)$  is a Boolean lattice isomorphic to  $(\mathcal{P}(B^c), \subseteq)$  by the mapping:*

$$\begin{aligned} q_{(A,B)} : \mathcal{Q}_{(A,B)} &\rightarrow \mathcal{P}(B^c) \\ (C, D) &\mapsto D^c. \end{aligned} \tag{14}$$

*In particular,  $(\mathcal{Q}_\top, \preceq)$  is a Boolean lattice isomorphic to  $(\mathcal{P}, \subseteq)$ .*

**PROOF.** We know that  $(\mathcal{P}(B^c), \subseteq)$  is a Boolean lattice. To show that  $(\mathcal{Q}_{(A,B)}, \preceq)$  is a lattice isomorphic to  $(\mathcal{P}(B^c), \subseteq)$ , it suffices to show that  $q_{(A,B)}$  as defined above is an order-isomorphism, i.e., that  $q_{(A,B)}$  is a bijection from  $(\mathcal{Q}_{(A,B)}, \preceq)$  to  $(\mathcal{P}(B^c), \subseteq)$ , and that for any pair of elements  $(C, D), (C', D')$  of  $(\mathcal{Q}_{(A,B)}, \preceq)$  we have  $(C, D) \preceq (C', D')$  iff  $q_{(A,B)}(C, D) \subseteq q_{(A,B)}(C', D')$  [5].

First, observe the following equivalences:

$$(C, D) \in \mathcal{Q}_{(A,B)} \text{ iff } \left\{ \begin{array}{l} C \subseteq A \subseteq C \cup D \\ B \subseteq D \\ C \cap D = \emptyset \end{array} \right. \text{ iff } \left\{ \begin{array}{l} A \subseteq C \cup D \\ B \subseteq D \\ A^c \subseteq C^c \\ C \cap D = \emptyset \end{array} \right. \text{ iff } \left\{ \begin{array}{l} A = C \cup (A \cap D) \\ B \subseteq D \\ C \cap D = \emptyset \end{array} \right. .$$

Let us show that  $q_{(A,B)}$  is a bijection. Obviously  $q_{(A,B)}$  is onto  $(\mathcal{P}(B^c), \subseteq)$ . Moreover,  $D^c \subseteq B^c$  has a unique antecedent by  $q_{(A,B)}$ , which is  $(C, D)$ , with  $C := A \setminus (A \cap D)$ , by the above equivalence.

Secondly, consider  $(C, D) \sqsubseteq (C', D')$ . Then  $D \supseteq D'$  which means that  $q_{(A,B)}(C, D) = D^c \subseteq D'^c = q_{(A,B)}(C', D')$ . Conversely, if  $D^c \subseteq D'^c$ , the inverse images are  $(A \setminus (A \cap D), D)$  and  $(A \setminus (A \cap D'), D')$ . Since  $A \setminus (A \cap D) \subseteq A \setminus (A \cap D') \subseteq A \cup D$ ,  $q_{(A,B)}^{-1}(D) \sqsubseteq q_{(A,B)}^{-1}(D')$ , and  $q_{(A,B)}$  is order-isomorphic.  $\square$

Endowed with this new order relation, we can define the following operation in  $\mathcal{Q}$ :

**Definition 4** *The strict difference operation in  $\mathcal{Q}$  is defined for every  $((A_1, B_1), (A_2, B_2)) \in \mathcal{Q} \times \mathcal{Q}$  such that  $(A_1, B_1) \sqsubseteq (A_2, B_2)$  by:*

$$(A_2, B_2) \setminus\!\!(A_1, B_1) := (A_2 \setminus A_1, (B_1 \setminus B_2)^c).$$

Note that  $[(A_2, B_2) \setminus\!\!(A_1, B_1)] \sqcup (A_1, B_1) = (A_2, B_2)$ .

One can give a graphic interpretation of the  $\sqsubseteq$  order and the  $\setminus\!\!$  operation: we call *vertices* of  $\mathcal{Q}$  any element  $(A, B)$  such that  $A \cup B = N$ , since they coincide with the vertices of  $[-1, 1]^n$ . In the same way, we define the vertices of any sub-lattice of  $\mathcal{Q}$ . So, for any  $(A, B) \in \mathcal{Q}$ ,  $\mathcal{Q}_{(A,B)}$  is the set of vertices of the sub-lattice  $[\perp, (A, B)]$ . Moreover, two elements  $(C_1, D_1), (C_2, D_2)$  of  $\mathcal{Q}$  are said *complementary* w.r.t. an element  $(A, B)$  of  $\mathcal{Q}$  if  $(C_1, D_1), (C_2, D_2) \in \mathcal{Q}_{(A,B)}$  and:

$$\begin{aligned} (A, B) \setminus\!\!(C_1, D_1) &= (C_2, D_2), \text{ which is equivalent to} \\ (A, B) \setminus\!\!(C_2, D_2) &= (C_1, D_1). \end{aligned}$$

In particular, the pairs of elements which are complementary w.r.t.  $\top$  are the pairs  $\{(A, A^c), (A^c, A)\}$ , for every  $A \subseteq N$ . As a consequence, for  $(A, B) \in \mathcal{Q}$ , complementarity w.r.t. an element of  $\mathcal{Q}_{(A,B)}$  entails the same property than complementarity w.r.t. an element of  $\mathcal{P}$ : if  $(C, D)$  belongs to  $\mathcal{Q}_{(A,B)}$ ,  $(C, D)$  and its complement w.r.t.  $(A, B)$  are opposite vertices in the sub-lattice  $\mathcal{Q}_{(A,B)}$ .

Like the set difference in  $\mathcal{P}$  (cf. [7]), the strict difference operation in  $\mathcal{Q}$  will allow us to transform some  $\mathcal{G}$  operators into operators of a single variable. In addition, the  $\star$  operation will be transformed into a convolution operation.

Now, let us derive results for  $\mathcal{Q} \times \mathcal{Q}$  operators.

**Definition 5** *A level operator  $\Phi$  is an operator in  $\mathcal{G}$  satisfying for any  $(A_1, B_1), (A_2, B_2)$  belonging to  $\mathcal{Q}$ :*

$$\Phi((A_1, B_1), (A_2, B_2)) = \begin{cases} \Phi(\perp, (A_2, B_2) \setminus\!\!(A_1, B_1)), & \text{if } (A_1, B_1) \sqsubseteq (A_2, B_2) \\ 0, & \text{otherwise} \end{cases}.$$

We will denote  $\mathcal{G}'$  the set of level operators.

We can notice that  $\Gamma$  is a level operator, contrary to  $Z$  and  $Z^{-1}$ , even if in the case of set functions, we can find in the  $\mathcal{G}'_{\mathcal{P}}$  set the  $\Gamma_{\mathcal{P}}$  operator but also the Zeta and Möbius  $\mathcal{P} \times \mathcal{P}$  operators.

Let us introduce  $\mathbf{g} := \{\varphi : \mathcal{Q} \rightarrow \mathbb{R} \mid \varphi(\perp) = 1\}$ , and the mapping  $\Lambda : \mathcal{G}' \longrightarrow \mathbf{g}$ , which associates to any level operator  $\Phi$  the function  $\Lambda(\Phi)$ , also denoted  $\varphi_{\Phi}$  for convenience:

$$\varphi_{\Phi}(A, B) := \Phi(\perp, (A, B)), \quad (A, B) \in \mathcal{Q}.$$

Then  $\varphi$  determines  $\Phi$  uniquely: let  $\varphi \in \mathbf{g}$ , if we define

$$\Phi_{\varphi}((A_1, B_1), (A_2, B_2)) = \begin{cases} \varphi((A_2, B_2) \setminus\setminus (A_1, B_1)), & \text{for } (A_1, B_1) \sqsubseteq (A_2, B_2) \\ 0, & \text{otherwise} \end{cases},$$

we have  $\Phi_{\varphi} = \Phi$  iff  $\varphi := \varphi_{\Phi}$ . Hence,  $\Lambda$  is a bijection.

We define the operation  $\star$  on  $\mathbf{g}$  by

$$\varphi \star \psi(A, B) := \Phi_{\varphi} \star \Phi_{\psi}(\perp, (A, B)), \quad (A, B) \in \mathcal{Q}.$$

We have

$$\begin{aligned} \varphi \star \psi(A, B) &= \sum_{(C, D) \sqsubseteq (A, B)} \Phi_{\varphi}(\perp, (C, D)) \Phi_{\psi}((C, D), (A, B)) \\ &= \sum_{(C, D) \sqsubseteq (A, B)} \Phi_{\varphi}(\perp, (C, D)) \Phi_{\psi}((C, D), (A, B)) \\ &= \sum_{(C, D) \sqsubseteq (A, B)} \varphi(C, D) \psi((A, B) \setminus\setminus (C, D)). \end{aligned}$$

$\varphi \star \psi$  is the *convolution* of  $\varphi, \psi \in \mathbf{g}$ .

**Proposition 6**  $(\mathcal{G}', \star)$  is an Abelian group isomorphic to  $(\mathbf{g}, \star)$  through the group isomorphism  $\Lambda$ .

**PROOF.**

- Let us show that  $\mathcal{G}'$  is subgroup of  $\mathcal{G}$ .  $\Delta \in \mathcal{G}'$  is obvious.  $\mathcal{G}'$  is stable for the operation  $\star$ : let  $(A_1, B_1), (A_2, B_2)$  be  $\mathcal{Q}$  such that  $(A_1, B_1) \sqsubseteq (A_2, B_2)$  and  $\Phi, \Psi \in \mathcal{G}'$ :

$$\begin{aligned} &\Phi \star \Psi((A_1, B_1), (A_2, B_2)) \\ &= \sum_{\substack{(C, D) \in \\ [(A_1, B_1), (A_2, B_2)]}} \Phi((A_1, B_1), (C, D)) \Psi((C, D), (A_2, B_2)) \\ &= \sum_{\substack{(C, D) \in \\ [(A_1, B_1), (A_2, B_2)]}} \Phi(\perp, ((C, D) \setminus\setminus (A_1, B_1))) \Psi(\perp, [(A_2, B_2) \setminus\setminus (C, D)]). \end{aligned}$$

Let  $(C', D') := (C, D) \setminus\!\!(A_1, B_1)$ , we have

$$\begin{aligned}
& \Phi \star \Psi((A_1, B_1), (A_2, B_2)) \\
&= \sum_{\substack{(C', D') \in \\ [\perp, (A_2, B_2) \setminus\!\!(A_1, B_1)]}} \Phi(\perp, (C', D')) \Psi(\perp, (A_2, B_2) \setminus\!\![(C', D') \sqcup (A_1, B_1)]) \\
&= \sum_{\substack{(C', D') \in \\ [\perp, (A_2, B_2) \setminus\!\!(A_1, B_1)]}} \Phi(\perp, (C', D')) \Psi(\perp, [(A_2, B_2) \setminus\!\!(A_1, B_1)] \setminus\!\!(C', D')) \\
&= \sum_{\substack{(C', D') \in \\ [\perp, (A_2, B_2) \setminus\!\!(A_1, B_1)]}} \Phi(\perp, (C', D')) \Psi((C', D'), (A_2, B_2) \setminus\!\!(A_1, B_1)) \\
&= \Phi \star \Psi(\perp, (A_2, B_2) \setminus\!\!(A_1, B_1)).
\end{aligned}$$

We now show that the inverse of an element of  $\mathcal{G}'$  lies in  $\mathcal{G}'$ . By Prop. 1, we know that the inverse of any  $\Phi \in \mathcal{G}$  is such that  $\Phi((A, B), (A, B)) = 1$ , and defined recursively as

$$\begin{aligned}
\Phi^{-1}((A_1, B_1), (A_2, B_2)) &= \\
&- \sum_{(C, D) \in [(A_1, B_1), (A_2, B_2)[} \Phi^{-1}((A_1, B_1), (C, D)) \Phi((C, D), (A_2, B_2)),
\end{aligned}$$

for  $(A_1, B_1) \sqsubset (A_2, B_2)$ , and 0 otherwise. We suppose now that  $\Phi \in \mathcal{G}'$ . Then  $\Phi^{-1}((A, B), (A, B)) = 1 = \Phi^{-1}(\perp, \perp)$ , hence  $\Phi^{-1}$  has the property of level operators for the pair  $((A, B), (A, B))$ . Using the above formula and the definition of level operators, we obtain:

$$\begin{aligned}
\Phi^{-1}((A_1, B_1), (A_2, B_2)) &= \\
&- \sum_{(C, D) \in [(A_1, B_1), (A_2, B_2)[} \Phi^{-1}((A_1, B_1), (C, D)) \Phi(\perp, (A_2, B_2) \setminus\!\!(C, D)),
\end{aligned}$$

for  $(A_1, B_1) \triangleleft (A_2, B_2)$ , and 0 otherwise. It remains to show that  $\Phi^{-1}((A_1, B_1), (A_2, B_2)) = \Phi^{-1}(\perp, (A_2, B_2) \setminus\!\!(A_1, B_1))$ , and we do it by recurrence on  $(A_2, B_2)$ . We know it is already true for  $(A_2, B_2) = (A_1, B_1)$ , and let us suppose it is true for any  $(C, D)$  such that  $(A_1, B_1) \trianglelefteq (C, D) \triangleleft (A_2, B_2)$ . Then the above formula writes:

$$\begin{aligned}
& \Phi^{-1}((A_1, B_1), (A_2, B_2)) \\
&= - \sum_{(C, D) \in [(A_1, B_1), (A_2, B_2)[} \Phi^{-1}(\perp, (C, D) \setminus\!\!(A_1, B_1)) \Phi(\perp, (A_2, B_2) \setminus\!\!(C, D)) \\
&= \sum_{(C', D') \in [\perp, (A_2, B_2) \setminus\!\!(A_1, B_1)[} \Phi^{-1}(\perp, (C', D')) \Phi((C', D'), (A_2, B_2) \setminus\!\!(A_1, B_1)) \\
&= \Phi^{-1}(\perp, (A_2, B_2) \setminus\!\!(A_1, B_1)),
\end{aligned}$$

where we have put as above  $(C', D') := (C, D) \setminus\!\!(A_1, B_1)$ .

- We show that  $\Lambda$  is a group isomorphism. We already know that  $\Lambda$  is bijective, it remains to show that  $\Lambda(\Phi \star \Psi) = \Lambda(\Phi) \star \Lambda(\Psi)$ , with  $\Phi, \Psi \in \mathcal{G}'$ . Using

previous notations we have:

$$\begin{aligned}
\varphi_\Phi \star \varphi_\Psi(A, B) &= \sum_{(C,D) \trianglelefteq (A,B)} \varphi_\Phi(C, D) \varphi_\Psi((A, B) \setminus\setminus (C, D)) \\
&= \sum_{(C,D) \trianglelefteq (A,B)} \Phi(\perp, (C, D)) \Psi(\perp, (A, B) \setminus\setminus (C, D)) \\
&= \sum_{(C,D) \trianglelefteq (A,B)} \Phi(\perp, (C, D)) \Psi((C, D), (A, B)) \\
&= \Phi \star \Psi(\perp, (A, B)) \\
&= \varphi_{\Phi \star \Psi}(A, B).
\end{aligned}$$

The neutral element  $\Delta$  of  $\mathcal{G}$  becomes the neutral element  $\delta$  of  $\mathbf{g}$ ,

$$\delta(A, B) := \varphi_\Delta(A, B) = \begin{cases} 1 & \text{if } (A, B) = \perp, \\ 0 & \text{otherwise.} \end{cases}$$

- The convolution being a commutative operation,  $(\mathbf{g}, \star)$  is an Abelian group and so is  $(\mathcal{G}', \star)$ : let  $\varphi, \psi$  be in  $\mathbf{g}$  and  $(A, B)$  in  $\mathcal{Q}$ :

$$\begin{aligned}
\psi \star \varphi(A, B) &= \sum_{(C,D) \trianglelefteq (A,B)} \psi(C, D) \varphi((A, B) \setminus\setminus (C, D)) \\
&= \sum_{(C',D') \trianglelefteq (A,B)} \psi((A, B) \setminus\setminus (C', D')) \varphi(C', D'),
\end{aligned}$$

where  $(C', D') := (A, B) \setminus\setminus (C, D)$ , thus

$$\psi \star \varphi(A, B) = \varphi \star \psi(A, B).$$

□

The inverse of  $\varphi \in \mathbf{g}$  will be denoted  $\varphi^{\star-1}$  as it is common for convolutions.

As a consequence, we can express the inverse Bernoulli function  $\gamma := \varphi_{\Gamma}$ . Thanks to what we have seen before, we can directly write

$$\gamma(A, B) = \frac{1}{n - b + 1}, \quad (A, B) \in \mathcal{Q}. \quad (15)$$

$\gamma^{\star-1}$  is called the *Bernoulli function*.

## 6 Cardinality operators

A real-valued function on  $\mathcal{Q}$  is called a *cardinality function* if it only depends on the layer of the variable, and is equal to 1 at  $\perp$ . We denote by  $\mathbf{c}$  the set of these functions. We recall that  $\mathbf{r} := \{f : N_0 \rightarrow \mathbb{R} \mid f(0) = 1\}$ , and introduce the mapping  $\lambda : \mathbf{c} \rightarrow \mathbf{r}$ , which associates to each cardinality function  $\varphi$  its *cardinal representation*  $\lambda(\varphi)$ , also denoted by  $f_{\varphi}$  for convenience, defined by:

$$f_{\varphi}(\|(A, B)\|) = \varphi(A, B), \quad (A, B) \in \mathcal{Q}.$$

Conversely, for any  $f \in \mathbf{r}$  we define  $\varphi_f((A, B)) := f(\|(A, B)\|)$ ,  $(A, B) \in \mathcal{Q}$ . Thus,  $\lambda$  is bijective.

Furthermore, we call *cardinality operator* of  $\mathcal{Q} \times \mathcal{Q}$  (resp.  $\mathcal{P} \times \mathcal{P}$ ) any level operator  $\Phi$  whose associated function  $\varphi_{\Phi}$  of  $\mathbf{g}$  (resp.  $\mathbf{g}_{\mathcal{P}}$ ) belongs to  $\mathbf{c}$  (resp.  $\mathbf{c}_{\mathcal{P}}$ ). We denote by  $\mathcal{G}''$  (resp.  $\mathcal{G}_{\mathcal{P}}''$ ) the set of cardinality operators. As shown by the following Lemma,  $(\mathbf{c}, \star)$  is a subgroup of  $(\mathbf{g}, \star)$ .

**Lemma 7 (fundamental)**  *$(\mathbf{c}, \star)$  is an Abelian group isomorphic to  $\mathbf{r}$  through the group isomorphism  $\lambda$ , and the triangular diagram representing  $\mathbf{c}_{\mathcal{P}}$ ,  $\mathbf{c}$  and  $\mathbf{r}$  is commutative.*

### PROOF.

- We already know that  $\mathbf{c}_{\mathcal{P}}$  is isomorphic to  $\mathbf{r}$  (cf. section 3).
- Let us show that  $\lambda : \mathbf{c} \rightarrow \mathbf{r}$  is a group isomorphism.  $\lambda$  being bijective, it remains to show that  $\lambda(\varphi \star \psi) = \lambda(\varphi) \star \lambda(\psi)$ , for any  $\varphi, \psi$  in  $\mathbf{c}$ . Using previous notations, for  $m$  in  $N_0$ , by (10), we have

$$f_{\varphi} \star f_{\psi}(m) = \sum_{k=0}^m \binom{m}{k} f_{\varphi}(k) f_{\psi}(m - k).$$

On the other hand, for  $(A, B) \in \mathcal{Q}$  such that  $\|(A, B)\| = m$  (i.e.,  $b = n - m$ ):

$$\begin{aligned}
f_{\varphi \star \psi}(m) &= \varphi \star \psi(A, B) \\
&= \sum_{(C, D) \trianglelefteq (A, B)} \varphi(C, D) \psi(\underbrace{(A, B) \setminus (C, D)}_{(A \setminus C, (D \setminus B)^c)}) \\
&= \sum_{(C, D) \trianglelefteq (A, B)} f_{\varphi}(n - d) f_{\psi}(d - b).
\end{aligned}$$

Now, by the isomorphism  $q_{(A, B)}$  of Proposition 3, we know that in the lattice  $(\mathcal{Q}_{(A, B)}, \trianglelefteq)$ ,  $(C, D) \trianglelefteq (A, B)$  corresponds to  $D^c \subseteq B^c$ , which entails:

$$\begin{aligned}
f_{\varphi \star \psi}(m) &= \sum_{D^c \subseteq B^c} f_{\varphi}(n - d) f_{\psi}(d - b) \\
&= \sum_{\substack{d \leq n \\ n - d \leq n - b}} \binom{n - b}{d - b} f_{\varphi}(n - d) f_{\psi}(d - b) \\
&= \sum_{d=b}^n \binom{n - b}{d - b} f_{\varphi}(n - d) f_{\psi}(d - b) \\
&= \sum_{k=0}^{n-b} \binom{n - b}{k} f_{\varphi}(n - b - k) f_{\psi}(k) \\
&= \sum_{k=0}^m \binom{m}{k} f_{\varphi}(k) f_{\psi}(m - k) \text{ because } (\mathfrak{r}, \star) \text{ is Abelian,} \\
&= f_{\varphi} \star f_{\psi}(m).
\end{aligned}$$

$\mathfrak{c}$  is indeed a group, and thus a subgroup of  $\mathfrak{g}$ .

□

Therefore, by (15) the inverse Bernoulli function for bi-set functions has cardinal representation  $f_{\gamma}(m) = \frac{1}{m+1}$ ,  $m \in N_0$ . In fact, it appears that  $f_{\gamma} = f_{\gamma \mathcal{P}}$ . This link with the previous result is fundamental in our work.

As a conclusion to these three sections, we can give the following recapitulative result, illustrated by Figure 3.

**Proposition 8** *The diagram successively representing  $\mathcal{G}_{\mathcal{P}}''$  (cardinality operators for set functions),  $\mathfrak{c}_{\mathcal{P}}$  (cardinality functions for set functions),  $\mathfrak{r}$  (functions defined on  $N_0$ , equal to 1 in 0),  $\mathfrak{c}$  (cardinality functions for bi-set functions) and  $\mathcal{G}''$  (cardinality operators for bi-set functions) sets, is commutative.*

**PROOF.** We have shown commutativity of the triangle  $\mathfrak{c}_{\mathcal{P}}, \mathfrak{c}, \mathfrak{r}$  in Lemma 7 (in particular,  $\mathfrak{c}_{\mathcal{P}}$  and  $\mathfrak{c}$  isomorphic through  $\lambda^{-1} \circ \lambda_{\mathcal{P}}$ ). Commutativity between



Yet, since it is the inversion in  $\mathcal{G}$  that interests us, we use Proposition 8 which provides us the required inversion automorphism of  $\mathcal{G}''$ . On the other hand, since  $f_\gamma = f_{\gamma^p}$ , it is easy to find step by step, the inverse of  $\Gamma$ :

$$\Gamma \mapsto \gamma \mapsto \underline{f_\gamma = f_{\gamma^p}} \mapsto \underline{f_\gamma^{\star-1} = f_{\gamma^p}^{\star-1}} = (\mathfrak{b}_m)_{m \in \mathbb{N}_0} \mapsto \gamma^{\star-1} \mapsto \Gamma^{-1}$$

This immediately implies:

**Proposition 9** *The Bernoulli operator  $\Gamma^{-1}$  is given by:*

$$\Gamma((A_1, B_1), (A_2, B_2)) = \begin{cases} \mathfrak{b}_{b_1-b_2} & \text{if } (A_1, B_1) \preceq (A_2, B_2) \\ 0 & \text{otherwise} \end{cases},$$

$$(A_1, B_1), (A_2, B_2) \in \mathcal{Q},$$

where  $(\mathfrak{b}_m)_{m \in \mathbb{N}}$  is the sequence of Bernoulli numbers.

As a consequence, thanks to the inversion of (13), we can write:

$$\begin{aligned} m^v(A, B) &= \sum_{(C,D) \succeq (A,B)} \mathfrak{b}_{b-d} I^v(C, D) \\ &= \sum_{m=0}^b \mathfrak{b}_m \sum_{\substack{(C,D) \succeq (A,B) \\ b-d=m}} I^v(C, D), \quad (A, B) \in \mathcal{Q}. \end{aligned}$$

Finally, we obtain:

**Theorem 10** *For any bi-set function  $v$ , we have:*

$$v(A, B) = \sum_{(C,D) \in \mathcal{Q}} b_{n-|D|}^{n-|B \cup D \cup (A^c \cap C)|} I^v(C, D), \quad (A, B) \in \mathcal{Q},$$

where

$$b_m^p := \sum_{k=0}^m \binom{m}{k} \mathfrak{b}_{p-k}$$

for  $0 \leq m \leq p$ , and  $(\mathfrak{b}_m)_{m \in \mathbb{N}}$  is the sequence of Bernoulli numbers.

**PROOF.** For all  $(A, B) \in \mathcal{Q}$ , according to (6) and Proposition 9 we have

$$\begin{aligned} v(A, B) &= \sum_{(C,D) \sqsubseteq (A,B)} m^v(C, D) \\ &= \sum_{(C,D) \sqsubseteq (A,B)} \sum_{(E,F) \succeq (C,D)} \mathfrak{b}_{d-f} I^v(E, F) \\ &= \sum_{(E,F) \in \mathcal{Q}} \left( \sum_{\substack{(C,D) \preceq (E,F) \\ (C,D) \sqsubseteq (A,B)}} \mathfrak{b}_{d-f} \right) I^v(E, F). \end{aligned}$$

Let us show that  $\begin{cases} (C, D) \trianglelefteq (E, F) \\ (C, D) \sqsubseteq (A, B) \end{cases}$  iff  $(C, D) \trianglelefteq (A \cap E, B \cup F \cup (E \cap A^c))$ :

First consider the “if” part. If we assume that  $\begin{cases} C \subseteq A \cap E \subseteq C \cup D \\ B \cup F \cup (E \cap A^c) \subseteq D \end{cases}$ , we

have easily  $C \subseteq E$ ,  $F \subseteq D$ ,  $C \subseteq A$  and  $B \subseteq D$ . Moreover,  $A \cap E \subseteq C \cup D$  and  $A^c \cap E \subseteq D \subseteq C \cup D$  thus  $E \subseteq C \cup D$ . For the “only if” part, if

$\begin{cases} C \subseteq E \subseteq C \cup D \\ F \subseteq D \\ C \subseteq A \\ B \subseteq D \end{cases}$ , then  $C \subseteq A \cap E$  and  $A \cap E \subseteq C \cup D$  are obvious. Next,

$E \subseteq C \cup D$  and  $A^c \subseteq C^c$  thus  $E \cap A^c \subseteq (C \cup D) \cap C^c = D$  since  $C \cap D = \emptyset$ . Finally,  $B \cup F \cup (E \cap A^c) \subseteq D$  is verified.

Therefore, we can write thanks to Proposition 3

$$\begin{aligned} \sum_{\substack{(C,D) \trianglelefteq (E,F) \\ (C,D) \sqsubseteq (A,B)}} \mathfrak{b}_{d-f} &= \sum_{(C,D) \trianglelefteq (A \cap E, B \cup F \cup (E \cap A^c))} \mathfrak{b}_{d-f} \\ &= \sum_{D^c \subseteq (B \cup F \cup (E \cap A^c))^c} \mathfrak{b}_{d-f} \\ &= \sum_{k=0}^{n-|B \cup F \cup (E \cap A^c)|} \binom{n-|B \cup F \cup (E \cap A^c)|}{k} \mathfrak{b}_{n-f-k} \\ &= b_{n-|B \cup F \cup (E \cap A^c)|}^{n-f}. \end{aligned}$$

The result follows.  $\square$

Let us notice that numbers  $b_m^p$  have been introduced in [7] to express a set function  $v$  from its interaction index  $I^v$ :

$$v(A) = \sum_{C \in \mathcal{P}} b_{|C \cap A|}^{|C|} I^v(C), \quad A \in \mathcal{P}.$$

It is easy to compute them from the sequence of Bernoulli ( $b_0^p = \mathfrak{b}_p$  for any  $p \in \mathbb{N}$ ), and thanks to the recursion of the “Pascal’s triangle”:

$$b_{m+1}^{p+1} = b_m^{p+1} + b_m^p, \quad 0 \leq m \leq p.$$

Furthermore, the coefficients  $b_m^p$  show the following symmetry:

$$b_m^p = (-1)^p b_{p-m}^p, \quad 0 \leq m \leq p.$$

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