Reconstructing Small Perturbations of Scatterers from Electric or Acoustic Far-Field Measurements

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Abstract

In this paper we consider the problem of determining the boundary perturbations of an object from far-field electric or acoustic measurements. Assuming that the unknown scatterer boundary is a small perturbation of a circle, we develop a linearized relation between the far-field data and the shape of the object. This relation is used to find the Fourier coefficients of the perturbation of the shape.

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1 Introduction

The field of inverse shape problems has been an active research area for several decades. Several related scalar problems belong to this field: electric and acoustic scattering form two large classes. In direct problems one wants to calculate the field outside a given object. In two common situations, one knows either the values of the field on the object (the Dirichlet problem), or the values of the normal derivative of the field on the boundary (the Neumann problem). Inverse shape problems involve reconstructing the object shape from measurements of the electric or acoustic field. Differently from Direct problems which are usually well posed, inverse problems are ill posed: the solution has an unstable dependence on the input data.

The formulation of the electric scattering problem is based on the quasi-static approximation and the related Laplace equation for the electric scalar potential. When a perfect conductor is exposed to extremely low-frequency electric fields, the problem is equivalent to the Dirichlet boundary value problem for the Laplace operator.

The sound-soft acoustic scattering problem is characterized by the condition that the total field vanishes on the boundary of the scatterer. Thus, acoustic scattering is equivalent to the Dirichlet boundary value problem for the Helmholtz operator, with the scattered field equal to the negative of the known incident field.

These two problems are frequently solved by methods of potential theory. The single- and double-layer potentials relate a charge density on the object boundary to the limiting values of the field and its normal derivative. The resulting integral equations are then solved in an appropriate function space, a common choice being the Lebesgue space L^2.

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In this paper, assuming that the unknown object boundary is a small perturbation of a unit circle, we develop for both electric and acoustic problems a linearized relation between the the far-field data and the shape of the scatterer. Under this purpose, we investigate the Dirichlet boundary value problem outside the object entering the Dirichlet data as parameters and the shape of the object as variables.

The linearized relation between the far-field data and the object shape is used to find the Fourier coefficients of the boundary perturbation of the object. Suppose that the angular oscillations in the perturbation are less than $1/n$. In order to detect that perturbation, it turns out that one needs to use the first n eigenvectors of the Dirichlet-to-Neumann operator corresponding to the unperturbed shape as the Dirichlet boundary data. We may think that this result is quite general. When the unknown object is a C^2-perturbation of a disk, we obtain asymptotic formulae for the Dirichlet-to-Neumann operator in terms of the small perturbations of the object shape, and it is worth mentioning the expansions of Dirichlet-to-Neumann operators for rough non-periodic surfaces [8, 4] and for periodic interfaces [9].

Our approach relies on asymptotic expansions of the far-field data with respect to the perturbations in the boundary, in much the same spirit as the recent work [2] and the text [1]. We consider only the two-dimensional case, the extension to three dimensions being obvious. In connection with our work, we should also mention the paper by Kaup and Santosa [6] on detecting corrosion from steady-state voltage boundary perturbations and the work by Tolmasky and Wiegmann [10] on the reconstruction of small perturbations of an interface for the inverse conductivity problem.

We deal with electric problems in section 2 and 3, and acoustic problems in 4 and 5.

2 Formulation of the Electric Problem

We consider the reconstructing problem of the perfect conductor D_ϵ which is the small perturbation of the unit disk D described by a Lipschitz function f and a small scale factor ϵ, that is

$$\partial D_\epsilon (= \partial D + \epsilon f e_\theta) := \Big\{ (1 + \epsilon f(\theta)) e_\theta \ , \quad \theta \in [0, 2\pi] \Big\}, \tag{2.1}$$

where $e_\theta = (\cos\theta, \sin\theta)$.

2.1 Electric Scattering problem

If we apply an initial potential v^i to $\mathbb{R}^2$ which is homogeneous except for the perfect conductor D_ϵ, then the derived electric potential v is given by $v = v^i + v^s$, where v^s is the solution to

$$\begin{cases} \Delta v^s = 0, & \text{in } \mathbb{R}^2 \setminus \overline{D}_\epsilon, \\ v^s = -v^i + C(\text{constant}), & \text{on } \ \partial D_\epsilon, \\ v^s(x) \to 0, & \text{as } |x| \to \infty. \end{cases} \tag{2.2}$$

We denote v_0^s as the perturbation of electric potential due to the conductor D, i.e.,

$$\begin{cases} \Delta v_0^s = 0, & \text{in } \mathbb{R}^2 \setminus \overline{D}, \\ v_0^s = -v^i + C(\text{constant}), & \text{on } \ \partial D, \\ v_0^s(x) \to 0, & \text{as } |x| \to \infty, \end{cases} \tag{2.3}$$

By obtaining a linearized relation between ϵf and the ϵ-order term of $(v^s - v_0^s)(r, \theta)$ as $r \to \infty$, we try to recover D_ϵ.

It follows from the Taylor series expansion of v^i near D that

$$v^i(1+\epsilon f(\theta),\theta) = v^i(1,\theta) + \epsilon f \partial_r v^i(1,\theta) + O(\epsilon^2). \tag{2.4}$$

Here we used the polar coordinates $x(r,\theta) = (r\cos\theta, r\sin\theta)$. We investigate v^s by considering two exterior boundary value problems, one with Dirichlet value $(-v^i(1,\theta))$ and the other with $(-\epsilon f \partial_r v^i(1,\theta))$ on ∂D_ϵ. To do that, we formulate the fixed boundary value problem.

2.2 Fixed Dirichlet boundary value problem

When the boundary value is prescribed as the 2π-periodic function Ψ, the voltage potential outside the conductor D_ϵ is given by the harmonic function u which satisfies the following:

$$\begin{cases} \Delta u = 0, & \text{in } \mathbb{R}^2 \setminus \overline{D}_\epsilon, \\ u(1+\epsilon f(\theta),\theta) = \Psi(\theta), & \text{for } \theta \in [0,2\pi], \\ u(x) = \text{constant} + O(1/|x|), & \text{as } |x| \to +\infty. \end{cases} \tag{2.5}$$

We let u_0 be the voltage potential outside the unit disk D with the fixed Dirichlet data Ψ on the boundary, i.e.,

$$\begin{cases} \Delta u_0 = 0, & \text{in } \mathbb{R}^2 \setminus \overline{D}, \\ u_0(1,\theta) = \Psi(\theta), & \text{for } \theta \in [0,2\pi], \\ u_0(x) = \text{constant} + O(1/|x|), & \text{as } |x| \to +\infty. \end{cases} \tag{2.6}$$

We obtain the linearized relation between the boundary interface of the conductor D_ϵ and $(u-u_0)$ at infinity, especially when Ψ is given by a C^4-function or a Lipschitz function.

3 Electric far-field formula and Inversion algorithm

3.1 Linearized relation for the Dirichlet problem

We start by explaining the main idea to obtain the linearized relation.

To derive the asymptotic expansion of the solution u to (2.5) with the given boundary data Ψ, we apply the field expansion method (F.E) (see [9]). Firstly, we expand u in powers of ϵ, i.e.,

$$u(r,\theta) = \sum_{n=0}^{+\infty} u_n(r,\theta)\epsilon^n. \tag{3.1}$$

Now expanding in terms of r and evaluating (3.1) at $r = 1+\epsilon f$, we obtain that

$$u(1+\epsilon f(\theta),\theta) = u_0(1,\theta) + \epsilon\Big(u_1(1,\theta) + \partial_r u_0(1,\theta) f(\theta)\Big) + O(\epsilon^2).$$

Since $u(1+\epsilon f(\theta),\theta)$ and $u_0(1,\theta)$ have the same value, u_1 can be considered as the decaying harmonic function which satisfies

$$u_1(1,\theta) = -\mathcal{N}_0(\Psi)(\theta) f(\theta), \tag{3.2}$$

where

$$\mathcal{N}_0(\Psi)(\theta) = \partial_r u_0(1,\theta).$$

In other words, $\mathcal{N}_0$ is the Dirichlet-to-Neumann operator of D, and it can be expressed as

$$\mathcal{N}_0(\Psi)(\theta) = -\sum_{n=1}^{+\infty}\Big[n\hat{a}_n(\Psi)\cos n\theta + n\hat{b}_n(\Psi)\sin n\theta\Big], \tag{3.3}$$

where $\hat{a}_n(\Psi)$ and $\hat{b}_n(\Psi)$ are the fourier coefficients, that is

$$\hat{a}_n(\Psi) = \frac{1}{\pi}\int_0^{2\pi}\Psi(\theta)\cos(n\theta)d\theta,\ \hat{b}_n(\Psi) = \frac{1}{\pi}\int_0^{2\pi}\Psi(\theta)\sin(n\theta)d\theta.$$

From (3.2) and the expansion of the harmonic function outside a disk, we have

$$u_1(r,\theta) = -\sum_{n=0}^{+\infty}\frac{1}{r^n}\Big[\hat{a}_n(\mathcal{N}_0(\Psi)f)\cos n\theta + \hat{b}_n(\mathcal{N}_0(\Psi)f)\sin n\theta\Big], \quad r \geq 1.$$

Therefore,

$$(u-u_0)(r,\theta) \sim -\frac{\epsilon}{r}\Big[\hat{a}_1(\mathcal{N}_0(\Psi)f)\cos\theta + \hat{b}_1(\mathcal{N}_0(\Psi)f)\sin\theta\Big] + C, \quad r \gg 1,$$

where C is a constant. More precisely, we have the following theorem and give the proof in Subsection 3.3.

Theorem 3.1 *For a 2π-periodic function Ψ, we let u and u_0 be the solution to* (2.5) *and* (2.6)*, respectively.*

1. *Let $\Psi \in C^4([0,2\pi])$. For $r \gg 1$, we have*

$$(u-u_0)(r,\theta) = -\frac{\epsilon}{r}\Big[\hat{a}_1(\mathcal{N}_0(\Psi)f)\cos\theta + \hat{b}_1(\mathcal{N}_0(\Psi)f)\sin\theta\Big] + C + O(\epsilon^{\frac{3}{2}}/r + \epsilon/r^2), \tag{3.4}$$

where C is a constant, and $O(\epsilon^{\frac{3}{2}}/r + \epsilon/r^2)$ depends on the Lipschitz constant of f and $\|\Psi\|_{C^4}$.

2. *For a Lipschitz function Ψ, we have that*

$$(u-u_0)(r,\theta) = C + O(\epsilon^{\frac{1}{2}}/r + 1/r^2), \quad \text{for } r \gg 1, \tag{3.5}$$

where C is a constant, and $O(\epsilon^{\frac{1}{2}}/r + 1/r^2)$ depends on the Lipschitz constant of f and Ψ.

Remark 3.2 *For the case of C^2-perturbation of the interface, i.e., $f \in C^2([0,2\pi])$, the error term of* (3.4) *and* (3.5) *can be replaced by $O(\epsilon^2/r + \epsilon/r^2)$ and $O(\epsilon/r + 1/r^2)$.*

In connection with the results for rough non-periodic surfaces [8, 4] and for periodic interfaces [9], we expand the Dirichlet-to-Neumann operator $\mathcal{N}_{\epsilon f}$ of D_ϵ which is defined by

$$\mathcal{N}_{\epsilon f}(\Psi)(\theta) := \frac{\partial u}{\partial \nu_y}(y), \quad y = (1+\epsilon f(\theta))e_\theta,$$

where ν_y is the outward unit normal vector to D_ϵ.

Note that ν_y is given by

$$\nu_y = \frac{N_\theta}{|N_\theta|}, \tag{3.6}$$

where

$$N_\theta = (1+\epsilon f(\theta))e_\theta - \epsilon\dot{f}\tau_\theta,\ \tau_\theta = (-\sin\theta, \cos\theta).$$

Here $\dot{f}$ is the derivative of f with respect to θ. From the fact that

$$\frac{1}{|N_\theta|} = 1 - \epsilon f + O(\epsilon^2), \tag{3.7}$$

it follows that

$$\begin{aligned}
\mathcal{N}_{\epsilon f}(\Psi)(\theta) &= (1-\epsilon f)\langle \nabla u, N_\theta\rangle + O(\epsilon^2) \\
&= (1-\epsilon f)\Big[(1+\epsilon f)\frac{\partial u}{\partial r}\Big|_{r=1+\epsilon f} - \frac{\epsilon \dot{f}}{1+\epsilon f}\frac{\partial u}{\partial \theta}\Big|_{r=1+\epsilon f}\Big] + O(\epsilon^2) \\
&= \frac{\partial u}{\partial r}\Big|_{r=1+\epsilon f(\theta)} - \epsilon \dot{f}\frac{\partial u}{\partial \theta}\Big|_{r=1+\epsilon f(\theta)} + O(\epsilon^2).
\end{aligned}$$

Applying (3.1), we obtain

$$\mathcal{N}_{\epsilon f}(\Psi)(\theta) \sim \partial_r u_0(1,\theta) + \epsilon\Big(\partial_r u_1(1,\theta) + \partial_r^2 u_0(1,\theta) f(\theta) - \partial_\theta u_0(1,\theta)\dot{f}(\theta)\Big).$$

Defining an operator $\mathcal{D}_0$ by

$$\mathcal{D}_0(\Psi)(\theta) := -\sum_{n=1}^{+\infty}\Big[(n+1)\hat{a}_n(\Psi)\cos n\theta + (n+1)\hat{b}_n(\Psi)\sin n\theta\Big], \tag{3.8}$$

we have

$$\partial_r^2 u_0(1,\theta) = \mathcal{D}_0 \mathcal{N}_0(\Psi)(\theta). \tag{3.9}$$

Lemma 3.3 *For $f \in C^2([0,2\pi])$ and $\Psi \in \mathcal{C}^4([0,2\pi])$, we have*

$$\mathcal{N}_{\epsilon f}(\Psi) = \mathcal{N}_0(\Psi) + \epsilon \mathcal{N}_f^1(\Psi) + O(\epsilon^{\frac{3}{2}}),$$

where

$$\mathcal{N}_f^1(\Psi) = \mathcal{D}_0\mathcal{N}_0(\Psi) f - \mathcal{N}_0(\mathcal{N}_0(\Psi) f) - \dot{f}\dot{\Psi}. \tag{3.10}$$

We give the proof in Subsection 3.3.

3.2 Algorithm for the Inverse Shape Problem

For an entire harmonic function v^i, we let v^s and v_0^s be the solution to (2.2) and (2.3), respectively. The Dirichlet values of the solutions are given by

$$\begin{aligned}
v_0^s|_{\partial D} &= -v^i(1,\theta) + \text{constant}, \\
v^s|_{\partial D_\epsilon} &= -v^i(1,\theta) - \epsilon \partial_r v^i(1,\theta) f(\theta) + \text{constant} + O(\epsilon^2).
\end{aligned}$$

Note that

$$\partial_r v^i(1,\theta) = -\mathcal{N}_0(v^i|_{\partial D}).$$

Here we have the minus sign on the right hand side because N_0 is the Dirichlet-to-Neumann operator for the exterior harmonic functions. Applying (3.4) and (3.5) with letting $\Psi = -v^i|_{\partial D}$ and $\Psi = \mathcal{N}_0(v^i|_{\partial D}) f$, respectively, we obtain for $r \gg 1$ that

$$(v^s - v_0^s)(r,\theta) \sim 2\frac{\epsilon}{r}\Big[\hat{a}_1\Big(\mathcal{N}_0(v^i|_{\partial D})f\Big)\cos\theta + \hat{b}_1\Big(\mathcal{N}_0(v^i|_{\partial D})f\Big)\sin\theta\Big]. \tag{3.11}$$

Now define entire harmonic functions $v^{n,i}$ and $w^{n,i}$, for $n \in \mathbb{N}$, by

$$v^{n,i}(r,\theta) = -\frac{1}{n} r^n \sin n\theta,\ w^{n,i}(r,\theta) = \frac{1}{n} r^n \cos n\theta.$$

Let $v^{n,s}$ and $w^{n,s}$ be the solution to (2.2) with the initial potential $v^{n,i}$ and $w^{n,i}$, respectively. In the same way, define $v_0^{n,s}$ and $w_0^{n,s}$ as the solution to (2.3). Let

$$c_1(v^{n,i}) := \frac{1}{\epsilon}\hat{a}_1\Big(r\cdot(v^{n,s}-v_0^{n,s})\Big),\ d_1(v^{n,i}) := \frac{1}{\epsilon}\hat{b}_1\Big(r\cdot(v^{n,s}-v_0^{n,s})\Big),$$

$$c_1(w^{n,i}) := \frac{1}{\epsilon}\hat{a}_1\Big(r\cdot(w^{n,s}-w_0^{n,s})\Big),\ d_1(w^{n,i}) := \frac{1}{\epsilon}\hat{b}_1\Big(r\cdot(w^{n,s}-w_0^{n,s})\Big).$$

From (3.11), it follows that

$$c_1(v^{n,i}) \sim 2\hat{a}_1\Big(f\cdot\mathcal{N}_0(-\frac{1}{n}\sin n\theta)\Big) = 2\hat{a}_1\Big(f\cdot\sin n\theta\Big),$$
$$d_1(v^{n,i}) \sim 2\hat{b}_1\Big(f\cdot\mathcal{N}_0(-\frac{1}{n}\sin n\theta)\Big) = 2\hat{b}_1\Big(f\cdot\sin n\theta\Big).$$

By the same way, we obtain

$$c_1(w^{n,i}) \sim -2\hat{a}_1\Big(f\cdot\cos n\theta\Big),$$
$$d_1(w^{n,i}) \sim -2\hat{b}_1\Big(f\cdot\cos n\theta\Big).$$

Thus we obtain that

$$\begin{aligned} c_1(v^{n,i}) \pm d_1(w^{n,i}) &= \frac{1}{\pi}\int_0^{2\pi} 2f(\theta)(\sin n\theta\cos\theta \mp \cos n\theta\sin\theta)\ d\theta \\ &= \frac{2}{\pi}\int_0^{2\pi} f(\theta)\sin(n\mp 1)\theta\ d\theta = 2\hat{b}_{n\mp 1}(f), \end{aligned}$$

$$\begin{aligned} \pm d_1(v^{n,i}) - c_1(w^{n,i}) &= \frac{1}{\pi}\int_0^{2\pi} 2f(\theta)(\pm\sin n\theta\sin\theta + \cos n\theta\cos\theta)\ d\theta \\ &= \frac{2}{\pi}\int_0^{2\pi} f(\theta)\cos(n\mp 1)\theta\ d\theta = 2\hat{a}_{n\mp 1}(f). \end{aligned}$$

Therefore, we arrive at

$$\hat{b}_{n-1}(f) = \frac{c_1(v^{n,i}) + d_1(w^{n,i})}{2}, \qquad \hat{b}_{n+1}(f) = \frac{c_1(v^{n,i}) - d_1(w^{n,i})}{2},$$

and

$$\hat{a}_{n-1}(f) = \frac{d_1(v^{n,i}) - c_1(w^{n,i})}{2}, \qquad \hat{a}_{n+1}(f) = \frac{-d_1(v^{n,i}) - c_1(w^{n,i})}{2}, \quad n \geq 1.$$

This simple calculation shows that in order to detect a perturbation that has oscillations of order $1/n$, one needs to use the first n eigenvectors ($e^{il\theta}, l = 1, \dots, n,$) of the Dirichlet-to-Neumann operator $\mathcal{N}_0$ as Dirichlet boundary data. This is a relatively simple but quite deep observation. We conjecture that this result holds for general domains. Another observation is that our asymptotic formula is in fact a low-frequency expansion which holds for fixed n as ϵ goes to zero. It would be interesting to derive an expansion which is valid for high-frequencies, not just for finite n.

3.3 Proofs of Theorem 3.1 and Lemma 3.3

We modify u_0 and u_1 to the solutions $u_0^{\epsilon M}$ and $u_1^{\epsilon M}$ of

$$\begin{cases} \Delta u_0^{\epsilon M} = 0, & \text{in } \mathbb{R}^2 \setminus \overline{B(1-\epsilon M, 0)}, \\ u_0^{\epsilon M}(1-\epsilon M, \theta) = \Psi(\theta), & \text{for } \theta \in [0, 2\pi] \\ u_0(r,\theta) = \text{constant} + O(1/r), & \text{as } r \to +\infty, \end{cases}$$

and

$$\begin{cases} \Delta u_1^{\epsilon M} = 0, & \text{in } \mathbb{R}^2 \setminus \overline{B(1-\epsilon M, 0)}, \\ u_1^{\epsilon M}(1-\epsilon M, \theta) = -[f(\theta)+M]\partial_r u_0(1,\theta), & \text{for } \theta \in [0, 2\pi] \\ u_1(r,\theta) = \text{constant} + O(1/r), & \text{as } r \to +\infty, \end{cases}$$

where

$$M := \max(\|f\|_{L^\infty}, \|\dot{f}\|_{L^\infty}, 1). \tag{3.12}$$

From the fourier expansion of Ψ, we obtain

$$u_0^{\epsilon M}(r,\theta) = \sum_{n=0}^{+\infty} \Big(\frac{1-\epsilon M}{r}\Big)^n \Big[\hat{a}_n(\Psi)\cos n\theta + \hat{b}_n(\Psi)\sin n\theta\Big], \quad \text{for } r \geq 1-\epsilon M, \tag{3.13}$$

and

$$u_1^{\epsilon M}(r,\theta) = -\sum_{n=0}^{+\infty} \Big(\frac{1-\epsilon M}{r}\Big)^n \Big[\hat{a}_n([f+M]\mathcal{N}_0(\Psi))\cos n\theta + \hat{b}_n([f+M]\mathcal{N}_0(\Psi))\sin n\theta\Big]. \tag{3.14}$$

The following is the key lemma to obtain the asymptotic expansion of $(u - u_0)$.

Lemma 3.4 *For a 2π-periodic function Ψ, we let u and u_0 be the solution to* (2.5) *and* (2.6)*, respectively.*

1. *For $\Psi \in C^4([0, 2\pi])$, we have the following asymptotic expansion holds uniformly on ∂D_ϵ:*
$$u = u_0^{\epsilon M} + \epsilon u_1^{\epsilon M} + C + O(\epsilon^{\frac{3}{2}}), \tag{3.15}$$
where C is a constant, and $O(\epsilon^{\frac{3}{2}})$ depends on the Lipschitz constant of f and $\|\Psi\|_{C^4}$.

2. *For a Lipschitz function Ψ, we have the following asymptotic expansion holds uniformly on ∂D_ϵ:*
$$u = u_0^{\epsilon M} + O(\epsilon^{\frac{1}{2}}), \tag{3.16}$$
where $O(\epsilon^{\frac{1}{2}})$ depends on the Lipschitz constant of f and Ψ.

Proof. Note that the Dirichlet value of u on ∂D_ϵ is Ψ. Using (3.13) and (3.14), we obtain

$$\begin{aligned} &(u - u_0^{\epsilon M} - \epsilon u_1^{\epsilon M})(1+\epsilon f, \theta) \\ &= C + \sum_{n=1}^{+\infty} \Big[1 - \Big(\frac{1-\epsilon M}{1+\epsilon f}\Big)^n - \epsilon n(M+f)\Big] \Big(\hat{a}_n(\Psi)\cos n\theta + \hat{b}_n(\Psi)\sin n\theta\Big) \\ &\quad + \epsilon \sum_{n=1}^{+\infty} \Big[\Big(\frac{1-\epsilon M}{1+\epsilon f}\Big)^n - 1\Big] \Big(\hat{a}_n([f+M]\mathcal{N}_0(\Psi))\cos n\theta + \hat{b}_n([f+M]\mathcal{N}_0(\Psi))\sin n\theta\Big) \\ &=: C + I + II, \end{aligned}$$

where C is a constant.

Note that

$$|1-(1-t)^n - nt| \leq n^2 t^2, \tag{3.17}$$
$$|1-(1-t)^n| \leq \max\{1,\ 2nt\}. \tag{3.18}$$

For $\Psi \in C^4([0,2\pi])$, we have

$$|\hat{a}_n(\Psi)|, |\hat{b}_n(\Psi)| \leq C\frac{\|\Psi\|_{C^4}}{n^4}, \qquad \text{for each } n \in \mathbb{N}, \tag{3.19}$$

and from (3.17), it follows that

$$I = O(\epsilon^2).$$

Now, applying Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}
|II|^2 &\leq \epsilon^2 \sum_{n=1}^{+\infty} \frac{1}{n^2}\Big[\Big(\frac{1-\epsilon M}{1+\epsilon f}\Big)^n - 1\Big]^2 \\
&\quad \times \sum_{n=1}^{+\infty} n^2 \Big[\hat{a}_n([f+M]\mathcal{N}_0(\Psi))\cos n\theta + \hat{b}_n([f+M]\mathcal{N}_0(\Psi))\sin n\theta\Big]^2 \\
&\leq \epsilon^2 \Big\|\frac{d}{d\theta}\Big([f+M]\mathcal{N}_0(\Psi)\Big)\Big\|^2_{L^2([0,2\pi])} \sum_{n=1}^{+\infty}\frac{1}{n^2}\Big[1 - \Big(\frac{1-\epsilon M}{1+\epsilon f}\Big)^n\Big]^2.
\end{aligned}$$

From (3.18), it follows

$$\sum_{n=1}^{+\infty}\frac{1}{n^2}\Big[1-\Big(\frac{1-\epsilon M}{1+\epsilon f}\Big)^n\Big]^2 = \sum_{n\leq 1/\epsilon}\frac{1}{n^2}(C\epsilon n)^2 + \sum_{n>1/\epsilon}\frac{1}{n^2} \leq C\epsilon. \tag{3.20}$$

Therefore we have

$$|II|^2 \leq C\epsilon^3, \tag{3.21}$$

where C depends on the Lipshitz constant of f and $\|\Psi\|_{C^4}$.

When Ψ is a Lipschtz function, from (3.20) we have

$$\begin{aligned}
\Big|(u-u_0^{\epsilon M})(1+\epsilon f,\theta)\Big| &= \Big|\sum_{n=1}^{+\infty}\Big[1-\Big(\frac{1-\epsilon M}{1+\epsilon f}\Big)^n\Big]\Big(\hat{a}_n(\Psi)\cos n\theta + \hat{b}_n(\Psi)\sin n\theta\Big)\Big| \\
&\leq \|\dot{\Psi}\|_{L^2([0,2\pi])}\Big(\sum_{n=1}^{+\infty}\frac{1}{n^2}\Big[1-\Big(\frac{1-\epsilon M}{1+\epsilon f}\Big)^n\Big]^2\Big)^{\frac{1}{2}} \\
&\leq C\epsilon^{\frac{1}{2}}
\end{aligned}$$

□

Proof of Theorem 3.1 For $\Psi \in C^4([0,2\pi])$, from (3.15) and the decaying condition of u, $u_0^{\epsilon M}$ and $u_1^{\epsilon M}$ at infinity,

$$u(r,\theta) = (u_0^{\epsilon M} + \epsilon u_1^{\epsilon M})(r,\theta) + \text{constant} + O(\epsilon^{\frac{3}{2}}), \quad r \gg 1.$$

Let Ω be a ball containing D_ϵ, then from the invertibility of the Double layer potential in $L_0^2(\partial\Omega)$, it follows that

$$u(r,\theta) = (u_0^{\epsilon M} + \epsilon u_1^{\epsilon M})(r,\theta) + \text{constant} + O(\epsilon^{\frac{3}{2}}/r), \quad r \gg 1.$$

We calculate that

$$
\begin{aligned}
&(u_0^{\epsilon M}+\epsilon u_1^{\epsilon M}-u_0)(r,\theta)\\
&=\sum_{n=0}^{+\infty}\frac{(1-\epsilon M)^n-1}{r^n}\Big[\hat{a}_n(\Psi)\cos n\theta+\hat{b}_n(\Psi)\sin n\theta\Big]\\
&\quad-\epsilon\sum_{n=0}^{+\infty}\Big(\frac{1-\epsilon M}{r}\Big)^n\Big[\hat{a}_n([f+M]\mathcal{N}_0(\Psi))\cos n\theta+\hat{b}_n([f+M]\mathcal{N}_0(\Psi))\sin n\theta\Big]\\
&=-\frac{\epsilon}{r}\Big[\hat{a}_1(\mathcal{N}_0(\Psi)f)\cos\theta+\hat{b}_1(\mathcal{N}_0(\Psi)f)\sin\theta\Big]+C+O(\frac{\epsilon}{r^2}+\frac{\epsilon^2}{r}),\qquad \text{for } r\gg 1,
\end{aligned}
$$

where C is a constant, and $O(\frac{\epsilon}{r^2}+\frac{\epsilon^2}{r})$ depends on the Lipschtz constant of f and $\|\Psi\|_{C^4}$. Therefore we prove (3.4).

By the same way, we can prove (3.5) □

Proof of Lemma 3.3 Note that

$$
\begin{aligned}
&|\hat{a}_n([f+M]\mathcal{N}_0(\Psi))|,\ |\hat{b}_n([f+M]\mathcal{N}_0(\Psi))|\le C\frac{1}{n^2},\qquad \text{for } n\in\mathbb{N},\\
&\sum_{n=1}^{+\infty}n^4\Big(\hat{a}_n([f+M]\mathcal{N}_0(\Psi))\cos n\theta+\hat{b}_n([f+M]\mathcal{N}_0(\Psi))\sin n\theta\Big)^2\le C,
\end{aligned}
\tag{3.22}
$$

where C depends on $\|f\|_{C^2}$ and $\|\Psi\|_{C^4}$.

Thus we obtain that

$$
\begin{aligned}
&\frac{\partial}{\partial\theta}(u-u_0^{\epsilon M}-\epsilon u_1^{\epsilon M})|_{\partial D_\epsilon}\\
&=\sum_{n=1}^{+\infty}n\Big[1-\Big(\frac{1-\epsilon M}{1+\epsilon f}\Big)^n-\epsilon n(M+f)\Big]\Big(-\hat{a}_n(\Psi)\sin n\theta+\hat{b}_n(\Psi)\cos n\theta\Big)\\
&\quad+\epsilon\sum_{n=1}^{+\infty}n\Big[\Big(\frac{1-\epsilon M}{1+\epsilon f}\Big)^n-1\Big]\Big(-\hat{a}_n([f+M]\mathcal{N}_0(\Psi))\sin n\theta+\hat{b}_n([f+M]\mathcal{N}_0(\Psi))\cos n\theta\Big)\\
&\quad+\epsilon\dot{f}(\theta)\sum_{n=1}^{+\infty}n\Big[\Big(\frac{1-\epsilon M}{1+\epsilon f}\Big)^n\frac{1}{1+\epsilon f}-1\Big]\Big(\hat{a}_n(\Psi)\cos n\theta+\hat{b}_n(\Psi)\sin n\theta\Big)\\
&\quad-\epsilon^2\dot{f}(\theta)\sum_{n=1}^{+\infty}n\Big(\frac{1-\epsilon M}{1+\epsilon f}\Big)^n\frac{1}{1+\epsilon f}\Big(\hat{a}_n([f+M]\mathcal{N}_0(\Psi))\cos n\theta+\hat{b}_n([f+M]\mathcal{N}_0(\Psi))\sin n\theta\Big)\\
&=O(\epsilon^{\frac{3}{2}}).
\end{aligned}
\tag{3.23}
$$

There exists a constant C which depends on the Lipschitz character of ∂D_ϵ, see [1], such that

$$
\Big\|\frac{\partial}{\partial\nu}(u-u_0^{\epsilon M}-\epsilon u_1^{\epsilon M})\Big\|_{L^2(\partial D_\epsilon)}\le C\Big\|\frac{\partial}{\partial T}(u-u_0^{\epsilon M}-\epsilon u_1^{\epsilon M})\Big\|_{L^2(\partial D_\epsilon)},
$$

where T is the unit tangent vector on ∂D_ϵ. From (3.23) and the fact that $\frac{\partial}{\partial\theta}=(1+\epsilon f+O(\epsilon^2))\frac{\partial}{\partial T}$, it follows that

$$
\Big\|\frac{\partial}{\partial\nu}(u-u_0^{\epsilon M}-\epsilon u_1^{\epsilon M})\Big\|_{L^2(\partial D_\epsilon)}=O(\epsilon^{\frac{3}{2}}).
\tag{3.24}
$$

From (3.6), (3.7) and (3.24), we have

$$\begin{aligned}\mathcal{N}_{\epsilon f}(\Psi)(\theta) &= \langle \nabla(u_0^{\epsilon M} + \epsilon u_1^{\epsilon M}),\ \nu_y\rangle + O(\epsilon^{\frac{3}{2}}) \\ &= (1-\epsilon f)\langle \nabla(u_0^{\epsilon M} + \epsilon u_1^{\epsilon M}), N_\theta\rangle + O(\epsilon^{\frac{3}{2}}) \\ &= \frac{\partial}{\partial r}(u_0^{\epsilon M} + \epsilon u_1^{\epsilon M})\Big|_{r=1+\epsilon f(\theta)} - \epsilon \dot{f}\frac{\partial}{\partial \theta}(u_0^{\epsilon M} + \epsilon u_1^{\epsilon M})\Big|_{r=1+\epsilon f(\theta)} + O(\epsilon^{\frac{3}{2}}) \\ &= \frac{\partial}{\partial r}(u_0^{\epsilon M} + \epsilon u_1^{\epsilon M})\Big|_{r=1+\epsilon f(\theta)} - \epsilon \dot{f}\dot{\Psi} + O(\epsilon^{\frac{3}{2}}). \end{aligned} \tag{3.25}$$

We compute

$$\begin{aligned}&\frac{\partial}{\partial r}(u_0^{\epsilon M} + \epsilon u_1^{\epsilon M})\Big|_{r=1+\epsilon f(\theta)} \\ &= \sum_{n=1}^{+\infty} \frac{1}{1+\epsilon f}\Big(\frac{1-\epsilon M}{1+\epsilon f}\Big)^n (-n)\Big(\hat{a}_n(\Psi)\sin n\theta + \hat{b}_n(\Psi)\cos n\theta\Big) \\ &\quad + \epsilon \sum_{n=1}^{+\infty} \frac{1}{1+\epsilon f}\Big(\frac{1-\epsilon M}{1+\epsilon f}\Big)^n n\Big(\hat{a}_n([f+M]\mathcal{N}_0(\Psi))\cos n\theta + \hat{b}_n([f+M]\mathcal{N}_0(\Psi))\sin n\theta\Big).\end{aligned}$$

Since

$$\begin{aligned}-n\frac{1}{1+\epsilon f}\Big(\frac{1-\epsilon M}{1+\epsilon f}\Big)^n &= -n + \epsilon n(n+1)f + \epsilon n^2 M + O(\epsilon^2 n^3), \\ \epsilon n\frac{1}{1+\epsilon f}\Big(\frac{1-\epsilon M}{1+\epsilon f}\Big)^n &= \epsilon n + \epsilon n\Big[\Big(\frac{1-\epsilon M}{1+\epsilon f}\Big)^n - 1\Big] + O(\epsilon^2 n),\end{aligned}$$

we have

$$\frac{\partial}{\partial r}(u_0^{\epsilon M} + \epsilon u_1^{\epsilon M})\Big|_{r=1+\epsilon f(\theta)} = \mathcal{N}_0(\Psi) + \epsilon\Big(\mathcal{D}_0\mathcal{N}_0(\Psi)f - \mathcal{N}_0(\mathcal{N}_0(\Psi)f)\Big) + O(\epsilon^{\frac{3}{2}}).$$

From (3.25), we prove the lemma.

□

4 Formulation of the Acoustic Problem

Analogously to the Laplacian one, we study the inverse scattering problem of reconstructing a sound-soft obstacle, call it D_ϵ, whose boundary is the perturbation of the unit circle and is given as (2.1).

4.1 Inverse Scattering Problem

For a incident field v^i, we denote v^s and v_0^s as the scattered field from D_ϵ and D, respectively.

In other words, v^s and v_0^s are the solutions to

$$\begin{cases} \Delta v^s + k^2 v^s = 0, & \text{in } \mathbb{R}^2 \setminus \overline{D}_\epsilon, \\ v^s = -v^i, & \text{on } \partial D_\epsilon, \\ \dfrac{\partial}{\partial r} v^s(r,\theta) - ikv^s(r,\theta) = o(r^{-\frac{1}{2}}), & r \longrightarrow +\infty. \end{cases} \tag{4.1}$$

and

$$\begin{cases} \Delta v_0^s + k^2 v_0^s = 0, & \text{in } \mathbb{R}^2 \setminus \overline{D}, \\ v_0^s = -v^i, & \text{on } \quad \partial D, \\ \dfrac{\partial}{\partial r} v_0^s(r,\theta) - ikv_0^s(r,\theta) = o(r^{-\frac{1}{2}}), & r \longrightarrow +\infty, \end{cases} \tag{4.2}$$

Here we used the polar coordinates $x(r,\theta) = (r\cos\theta, r\sin\theta)$, and the wave number k is given by a positive constant.

4.2 Fixed Dirichlet boundary value problem

For a 2π-periodic continuous function Ψ, we let u be the solution to the Helmholtz problem with the prescribed boundary data Ψ on ∂D_ϵ, i.e.,

$$\begin{cases} \Delta u + k^2 u = 0, & \text{in } \mathbb{R}^2 \setminus \overline{D}_\epsilon, \\ u(1+\epsilon f(\theta),\theta) = \Psi(\theta), & \text{for } \theta \in [0,2\pi], \\ \dfrac{\partial}{\partial r} u(r,\theta) - iku(r,\theta) = o(r^{-\frac{1}{2}}), & r \longrightarrow +\infty. \end{cases} \tag{4.3}$$

The solution u corresponding to the unit disk D satisfies that

$$\begin{cases} \Delta u_0 + k^2 u_0 = 0, & \text{in } \mathbb{R}^2 \setminus \overline{D}, \\ u_0(1,\theta) = \Psi(\theta), & \text{for } \theta \in [0,2\pi], \\ \dfrac{\partial}{\partial r} u_0(r,\theta) - iku_0(r,\theta) = o(r^{-\frac{1}{2}}), & r \longrightarrow +\infty. \end{cases} \tag{4.4}$$

We investigate the Far-Field difference between u and u_0, especially when Ψ is a C^4-function or a Lipschitz function.

5 Acoustic far-field formula and inversion algorithm

5.1 Asymptotic Far-field expansion for the Dirichlet problem

We parametrize the unit circle ∂D by $\theta \in [0,2\pi]$ and expand Ψ as

$$\Psi(\theta) = \sum_{n\in\mathbb{Z}} \hat{c}_n(\Psi) e^{in\theta},$$

where $\hat{c}_n(\Psi)$ is the fourier coefficient with respect to $e^{in\theta}$. By the uniqueness of the exterior Dirichlet problem, it follows that

$$u_0(r,\theta) = \sum_{n\in\mathbb{Z}} \frac{H^{(1)}_{|n|}(kr)}{H^{(1)}_{|n|}(k)} \; \hat{c}_n(\Psi) e^{in\theta}. \tag{5.1}$$

Define the Dirichlet-to-Neumann operator $\mathcal{N}_0$ with respect to D by

$$\mathcal{N}_0 : u_0|_S \to \partial_r u_0|_S,$$

then, in a pseudodifferential fashion, $\mathcal{N}_0$ can be written as follows (see [7]):

$$\mathcal{N}_0(\Psi)(\theta) = \sum_{n\in\mathbb{Z}} \sigma_1(n,k) \hat{c}_n(\Psi) e^{in\theta}, \tag{5.2}$$

where the so-called discrete symbol σ_1 is given by

$$\sigma_1(n,k) = k\frac{H^{(1)'}_{|n|}(k)}{H^{(1)}_{|n|}(k)} = -k\frac{H^{(1)}_{|n+1|}(k)}{H^{(1)}_{|n|}(k)} + |n|.$$

Thus, for fixed k, we have

$$\sigma_1(n,k) \sim |n|, \quad \text{as } |n| \to \infty. \tag{5.3}$$

By the same way as the electric problem, we can consider u_1 as the solution to (4.4) with the boundary value $(-\mathcal{N}_0(\Psi)f)$ on ∂D instead of Ψ, and it follows

$$u_1(r,\theta) = -\sum_{n\in\mathbb{Z}} \frac{H^{(1)}_{|n|}(kr)}{H^{(1)}_{|n|}(k)}\hat{c}_n(\mathcal{N}_0(\Psi)f)e^{in\theta}.$$

It is known that, for a fixed n, the Hankel function of the first kind satisfies

$$H^{(1)}_{|n|}(x) = \sqrt{\frac{2}{\pi x}}e^{i(x-\frac{\pi}{4}-|n|\frac{\pi}{2})} + O(|x|^{-1}), \quad x \gg |n|. \tag{5.4}$$

We refer to [3] for more properties of the Hankel function.

Choose $N \in \mathbb{N}$ satisfying that

$$\sum_{|n|>N} \left|\hat{c}_n(f\mathcal{N}_0(\Psi))\right| = O(\epsilon^{\frac{1}{2}}), \tag{5.5}$$

then we have the following lemma. More precise proof is given in the Subsection 5.3.

Theorem 5.1 *1. Let $\Psi \in C^4([0,2\pi])$ and u be the solution to (4.3). For $r \gg 1$, we have*

$$(u-u_0)(r,\theta) = -\epsilon\sqrt{\frac{2}{\pi r}}e^{ikr}\sum_{|n|\le N}\frac{\hat{c}_n(\mathcal{N}_0(\Psi)f)}{H^{(1)}_{|n|}(k)}e^{-i(\frac{\pi}{4}+\frac{|n|\pi}{2})}e^{in\theta} + O(\epsilon^{\frac{3}{2}}/\sqrt{r}+\epsilon/r). \tag{5.6}$$

where N is defined by (5.5), and $O(\epsilon^{\frac{3}{2}}/\sqrt{r}+\epsilon/r)$ depends on the Lipschitz constant of f and $\|\Psi\|_{C^4}$.

2. Let Ψ be a Lipschitz function and u be the solution to (4.3). We have that

$$(u-u_0)(r,\theta) = O(\epsilon^{\frac{1}{2}}/\sqrt{r}), \quad \textit{for } r \gg 1, \tag{5.7}$$

where $O(\epsilon^{\frac{1}{2}}/\sqrt{r})$ depends on the Lipschitz constant of f and Ψ.

We define

$$\mathcal{D}_0(\Psi)(\theta) := \sum_{n\in\mathbb{Z}} \sigma_2(n,k)\hat{c}_n(\Psi)e^{in\theta}, \tag{5.8}$$

with

$$\sigma_2(n,k) = k\frac{H^{(1)''}_{|n|}(k)}{H^{(1)'}_{|n|}(k)}.$$

Lemma 5.2 *For $f \in C^2([0,2\pi])$ and $\Psi \in C^4([0,2\pi])$, we have that*

$$\mathcal{N}_{\epsilon f}(\Psi) = \mathcal{N}_0(\Psi) + \epsilon\mathcal{N}^1_f(\Psi) + O(\epsilon^{\frac{3}{2}}),$$

where

$$\mathcal{N}^1_f(\Psi) = \mathcal{D}_0\mathcal{N}_0(\Psi)f - \mathcal{N}_0(\mathcal{N}_0(\Psi)f) - \dot{f}\dot{\Psi}.$$

5.2 Algorithm for the Inverse Shape Problem

Let v^i be the incoming wave, and define v^s and v_0^s as the solution to (4.1) and (4.2), respectively. Note that

$$v^i(1+\epsilon f(\theta),\theta) = v^i(1,\theta) + \epsilon f(\theta)\partial_r v^i(1,\theta) + O(\epsilon^2).$$

Applying Theorem 5.1 by letting $\Psi = -v^i(1,\theta)$ and $\Psi = -\epsilon f(\theta)\partial_r v^i(1,\theta)$, we have for $r \gg 1$ that

$$(v^s - v_0^s)(r,\theta) \sim \epsilon\sqrt{\frac{2}{\pi r}}e^{ikr}\sum_{|n|\leq N}\frac{\hat{c}_n\Big(f\mathcal{N}_0(v^i|_{\partial D}) - f\partial_r v^i|_{\partial D}\Big)}{H^{(1)}_{|n|}(k)}e^{-i(\frac{\pi}{4}+\frac{n\pi}{2})}e^{in\theta},$$

where N is defined by $\sum_{|n|>N}\left|\hat{c}_n\Big(f\mathcal{N}_0(v^i|_{\partial D}) - f\partial_r v^i|_{\partial D}\Big)\right| = O(\epsilon^{\frac{1}{2}})$. This yields to stable reconstruction of the Fourier coefficients $\hat{c}_n\Big(f\mathcal{N}_0(v^i|_{\partial D}) - f\partial_r v^i|_{\partial D}\Big)$ for n such that $H^{(1)}_{|n|}(k)$ is not too big.

Suppose now that v^i satisfies

$$\mathcal{N}_0(v^i|_{\partial D}) - \partial_r v^i|_{\partial D} = e^{-i(m-1)\theta},$$

then by measuring $\hat{c}_1(v^s - v_0^s)$, then we can reconstruct $\hat{c}_m(f)$.

5.3 Proofs of Theorem 5.1 and Lemma 5.2

We modify u_0 and u_1 as $u_0^{\epsilon M}$ and $u_1^{\epsilon M}$ which satisfy

$$\begin{cases} (\Delta + k^2)u_0^{\epsilon M} = 0, & \text{in } \mathbb{R}^2 \setminus \overline{B(1-\epsilon M,0)}, \\ u_0^{\epsilon M}(1-\epsilon M,\theta) = \Psi(\theta), & \text{for } \theta \in [0,2\pi] \\ \dfrac{\partial}{\partial r}u_0^{\epsilon M}(r,\theta) - iku_0^{\epsilon M}(r,\theta) = o(r^{-\frac{1}{2}}), & r \longrightarrow +\infty. \end{cases}$$

and

$$\begin{cases} (\Delta + k^2)u_1^{\epsilon M} = 0, & \text{in } \mathbb{R}^2 \setminus \overline{B(1-\epsilon M,0)}, \\ u_1^{\epsilon M}(1-\epsilon M,\theta) = -[f(\theta)+M]\mathcal{N}_0(\Psi), & \text{for } \theta \in [0,2\pi], \\ \dfrac{\partial}{\partial r}u_1^{\epsilon M}(r,\theta) - iku_1^{\epsilon M}(r,\theta) = o(r^{-\frac{1}{2}}), & r \longrightarrow +\infty, \end{cases}$$

where M is the constant defined by (3.12). From the fourier expansion of Ψ and the uniqueness of the exterior Dirichlet problem, $u_0^{\epsilon M}$ and $u_1^{\epsilon M}$ have the expansion as follows:

$$u_0^{\epsilon M}(r,\theta) = \sum_{n\in\mathbb{Z}}\frac{H^{(1)}_{|n|}(kr)}{H^{(1)}_{|n|}(k-\epsilon kM)}\ \hat{c}_n(\Psi)e^{in\theta}, \tag{5.9}$$

$$u_1^{\epsilon M}(r,\theta) = -\sum_{n\in\mathbb{Z}}\frac{H^{(1)}_{|n|}(kr)}{H^{(1)}_{|n|}(k-\epsilon kM)}\ \hat{c}_n([f+M]\mathcal{N}_0(\Psi))e^{in\theta}. \tag{5.10}$$

We have the following key lemma to prove Theorem 5.1 and Lemma 5.2.

Lemma 5.3 *1. For $\Psi \in C^4([0,2\pi])$, we have the following asymptotic expansion for the solution u to (4.3) holds uniformly on ∂D_ϵ:*

$$u = u_0^{\epsilon M} + \epsilon u_1^{\epsilon M} + O(\epsilon^{\frac{3}{2}}), \tag{5.11}$$

where $O(\epsilon^{\frac{3}{2}})$ depends on the Lipschitz constant of f and $\|\Psi\|_{C^4}$.

2. For a Lipschitz function Ψ,we have the following asymptotic expansion for the solution u to (4.3) holds uniformly on ∂D_ϵ:

$$u = u_0^{\epsilon M} + O(\epsilon^{\frac{1}{2}}), \tag{5.12}$$

where $O(\epsilon^{\frac{1}{2}})$ depends on the Lipschitz constant of f and Ψ.

Proof. From (5.9), (5.10) and the boundary condition of u on ∂D_ϵ, we have that

$$\begin{aligned}(u - u_0^{\epsilon M} - \epsilon u_1^{\epsilon M})(1+\epsilon f(\theta),\theta) = \sum_{n\in\mathbb{Z}} \Big[1 - \frac{H^{(1)}_{|n|}(k+\epsilon k f)}{H^{(1)}_{|n|}(k-\epsilon k M)} + \epsilon[f+M]\sigma_1(n,k)\Big]\ \hat{c}_n(\Psi) e^{in\theta} \\ + \epsilon \sum_{n\in\mathbb{Z}} \Big[\frac{H^{(1)}_{|n|}(k+\epsilon k f)}{H^{(1)}_{|n|}(k-\epsilon k M)} - 1\Big]\ \hat{c}_n([f+M]\mathcal{N}_0(\Psi)) e^{in\theta},\end{aligned}$$

and

$$(u - u_0^{\epsilon M})(1+\epsilon f(\theta),\theta) = \sum_{n\in\mathbb{Z}} \Big[1 - \frac{H^{(1)}_{|n|}(k+\epsilon k f)}{H^{(1)}_{|n|}(k-\epsilon k M)}\Big]\ \hat{c}_n(\Psi) e^{in\theta}.$$

Note that

$$\Big|H^{(1)}_{|n|}(k+\epsilon t) - H^{(1)}_{|n|}(k)\Big| \le \epsilon|t|\ \|H^{(1)'}_{|n|}\|_{L^\infty([k-\epsilon|t|,\ k+\epsilon|t|])},$$

$$\Big|H^{(1)}_{|n|}(k+\epsilon t) - H^{(1)}_{|n|}(k) - \epsilon t H^{(1)'}_{|n|}(k)\Big| \le \frac{\epsilon^2 t^2}{2} \|H^{(1)''}_{|n|}\|_{L^\infty([k-\epsilon|t|,\ k+\epsilon|t|])}.$$

From the fact that

$$H^{(1)'}_{|n|}(z) = -H^{(1)}_{|n+1|}(z) + \frac{|n|}{z} H^{(1)}_{|n|}(z),$$

we can show that

$$1 - \frac{H^{(1)}_{|n|}(k+\epsilon k f)}{H^{(1)}_{|n|}(k-\epsilon k M)} = O(\epsilon n), \tag{5.13}$$

$$1 - \frac{H^{(1)}_{|n|}(k+\epsilon k f)}{H^{(1)}_{|n|}(k-\epsilon k M)} + \epsilon[f+M]\sigma_1(n,k) = O(\epsilon^2 n^2), \tag{5.14}$$

where $O(\epsilon n)$ and $O(\epsilon^2 n^2)$ depend on M and k. Moreover, $|H^{(1)}_{|n|}|(z)$ is decreasing function for $z > 0$, and

$$1 - \frac{H^{(1)}_{|n|}(k+\epsilon k f)}{H^{(1)}_{|n|}(k-\epsilon k M)} = O(1). \tag{5.15}$$

Using (5.13), (5.14) and (5.15), we can prove the lemma by the same way to prove Lemma 3.4. □

Proof of Theorem 5.1 At first, we assume $\Psi \in C^4$. From the solution expression using boundary integral methods (for example, see [5]), we can show that

$$(u - u_0^{\epsilon M} - \epsilon u_1^{\epsilon M})(r,\theta) = \frac{1}{\sqrt{r}} \|u - u_0^{\epsilon M} - \epsilon u_1^{\epsilon M}\|_{L^2(\partial D_\epsilon)}, \quad \text{as } r \to +\infty. \tag{5.16}$$

Therefore

$$(u - u_0)(r,\theta) = (u_0^{\epsilon M} + \epsilon u_1^{\epsilon M} - u_0)(r,\theta) + O(\epsilon^{\frac{3}{2}}/\sqrt{r}), \quad \text{as } r \to +\infty.$$

From (5.1), (5.9) and (5.10), we compute that

$$\begin{aligned}
&(u_0^{\epsilon M} + \epsilon u_1^{\epsilon M} - u_0)(r,\theta) \\
&= \sum_{n\in\mathbb{Z}} \frac{H_{|n|}^{(1)}(kr)}{H_{|n|}^{(1)}(k)} \Big[\frac{H_{|n|}^{(1)}(k)}{H_{|n|}^{(1)}(k - \epsilon k M)} - 1 \Big] \hat{c}_n(\Psi) e^{in\theta} \\
&+ \epsilon \sum_{n\in\mathbb{Z}} \frac{H_{|n|}^{(1)}(kr)}{H_{|n|}^{(1)}(k)} \Big[- \frac{H_{|n|}^{(1)}(k)}{H_{|n|}^{(1)}(k - \epsilon k M)} \Big] \Big(\hat{c}_n(f\mathcal{N}_0(\Psi)) + \hat{c}_n(M\mathcal{N}_0(\Psi)) \Big) e^{in\theta} \\
&= \sum_{n\in\mathbb{Z}} \frac{H_{|n|}^{(1)}(kr)}{H_{|n|}^{(1)}(k)} \Big[\frac{H_{|n|}^{(1)}(k)}{H_{|n|}^{(1)}(k - \epsilon k M)} - 1 - \epsilon M \sigma_1(n,k) \Big] \hat{c}_n(\Psi) e^{in\theta} \\
&+ \epsilon \sum_{n\in\mathbb{Z}} \frac{H_{|n|}^{(1)}(kr)}{H_{|n|}^{(1)}(k)} \Big[- \frac{H_{|n|}^{(1)}(k)}{H_{|n|}^{(1)}(k - \epsilon k M)} \Big] \hat{c}_n(f\mathcal{N}_0(\Psi)) e^{in\theta} \\
&+ \epsilon \sum_{n\in\mathbb{Z}} \frac{H_{|n|}^{(1)}(kr)}{H_{|n|}^{(1)}(k)} \Big[1 - \frac{H_{|n|}^{(1)}(k)}{H_{|n|}^{(1)}(k - \epsilon k M)} \Big] M \hat{c}_n(\mathcal{N}_0(\Psi)) e^{in\theta} \\
&=: I + II + III.
\end{aligned}$$

From (5.13) and (5.14) with replacing f by 0 and (5.16), it follows

$$I + III = \frac{1}{\sqrt{r}} O(\epsilon^2). \tag{5.17}$$

Using (5.4) and (5.16), we obtain

$$II = -\epsilon \sqrt{\frac{2}{\pi r}} e^{ikr} \sum_{|n|\le N} \frac{\hat{c}_n(f\mathcal{N}_0(\Psi))}{H_{|n|}^{(1)}(k)} e^{-i(\frac{\pi}{4} + \frac{|n|\pi}{2})} e^{in\theta} + \frac{1}{r} O(\epsilon) + O(\epsilon^{\frac{3}{2}}/\sqrt{r}). \tag{5.18}$$

When Ψ is a Lipschitz function, we have

$$(u - u_0)(r,\theta) = (u_0^{\epsilon M} - u_0)(r,\theta) + O(\epsilon^{\frac{1}{2}}/\sqrt{r}) = O(\epsilon^{\frac{1}{2}}/\sqrt{r}), \quad \text{as } r \to +\infty.$$

□

Proof of Lemma 5.2 Note that ∂D is a C^2-domain, and using boundary integral methods, we have that (see [5])

$$\Big\| \frac{\partial}{\partial \nu} (u - u_0^{\epsilon M} - \epsilon u_1^{\epsilon M}) \Big\|_{C^{0,\alpha}(\partial D_\epsilon)} \le C \Big\| u - u_0^{\epsilon M} - \epsilon u_1^{\epsilon M} \Big\|_{C^{1,\alpha}(\partial D_\epsilon)}. \tag{5.19}$$

By the same way as the conductivity case, we can prove the lemma by calculating $\frac{\partial}{\partial \nu}(u_0^{\epsilon M} + \epsilon u_1^{\epsilon M})$. □

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References

[1] H. Ammari and H. Kang, *Reconstruction of Small Inhomogeneities from Boundary Measurements*, Lecture Notes in Mathematics, Volume 1846, Springer-Verlag, Berlin, 2004.

[2] H. Ammari, H. Kang, M. Lim, and H. Zribi, Conductivity interface problems. Part I: Small perturbations of an interface, preprint.

[3] M. Abrohmwitz and I. A. Stegun, *Handbook of Mathematical Functions*, New York: Dover, 1974.

[4] R. Coifman, M. Goldberg, T. Hrycak, M. Israeli, and V. Rokhlin, An improved operator expansion algorithm for direct and inverse scattering computations, Waves Random Media 9 (1999), 441-457.

[5] D. Colton and R. Kress, *Inverse Acoustic and Electromagnetic Scattering Theory*, 2nd ed. Berlin, Germany: Springer-Verlag, 1998.

[6] P.G. Kaup and F. Santosa, Nondestructive evaluation of corrosion damage using electrostatic measurements, J. Nondestr. Eval. 14 (1995), 127-136.

[7] M. F. Kondratieva and S. Yu. Sadov, Symbol of the Dirichlet-to-Neumann operator in 2D diffraction problems with large wavenumber, Day on Diffraction, 2003 Proceedings. International Seminar, 88 - 98.

[8] D.M. Milder, An improved formalism for wave scattering from rough surfaces, J. Acoust. Soc. Am., 89 (1991), 529541.

[9] D.P. Nicholls and F. Reitich, Analytic continuation of Dirichlet-Neumann operators, Numer. Math. 94 (2003), 107-146.

[10] C.F. Tolmasky and A. Wiegmann, Recovery of small perturbations of an interface for an elliptic inverse problem via linearization, Inverse Problems 15 (1999), 465-487.