Finite-Dimensional Calculus

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Abstract. We develop finite-dimensional calculus using matrices. The truncated Heisenberg-Weyl algebra is called a TAA algebra after Tekin, Aydin, and Arik who formulated it in terms of orthofermions. The matrix approach is used to implement our method of polynomial inversion in one-variable and multivariable settings. Here we establish notations, present some algebraic developments, and discuss the univariate case in detail.

Keywords: inversion formulas, orthofermions, Heisenberg-Weyl algebra, polynomial systems

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1 Introduction

The TAA algebra is generated by an operator a and its adjoint a^*. Let $N = a^*a$. Then the defining commutation rule is $[a, N] = a$. Taking adjoints gives the complementary rule $[N, a^*] = a^*$. In other words, we assume that N and a generate the two-dimensional Lie algebra of the affine group. Writing it out

$$aa^*a - a^*aa = a \tag{1}$$

we see that the commutation rule $aa^* - a^*a = \mathbf{1}$ of the Heisenberg-Weyl algebra has been multiplied by a on the right. This modification is enough to yield finite-dimensional representations, including the truncated HW-algebra given by the operators X =multiplication by x and $D = d/dx$ acting on polynomials of a given bounded degree. Writing matrices for these operators we will see that they obey equation (1).

This article may be thought of as realizing Rota's idea of "Finite Operator Calculus". There is work of P. R. Vein along similar lines [7, 8]. The main feature here is that, in fact, the operator calculus is done on finite-dimensional spaces and can be carried out explicitly using matrices. The approach in this paper is based on algebraic properties of the operators and includes indications for the multivariable case. The one-variable case is dual to that presented in [2].

2 Orthofermion formulation

Orthofermions with regard to connections with the HW algebra have been studied in [6]. Start with a set of operators $\{c_1, \ldots, c_p\}$, with p a positive integer and form the star-algebra generated by the $\{c_i\}$ modulo the following relations

$$\begin{aligned} c_i c_j &= 0 \\ c_i c_j^* + \delta_{ij} \sum_{k=1}^{p} c_k^* c_k &= \delta_{ij}\,\mathbf{1} \end{aligned} \tag{2}$$

where $\mathbf{1}$ is the identity operator. Setting $\Pi = \mathbf{1} - \sum_{k=1}^{p} c_k^* c_k$ (as in [5]) we can write this last relation as

$$c_i c_j^* = \delta_{ij}\, \Pi$$

where one readily shows from the defining relations (2) that $\Pi^2 = \Pi$, i.e., Π is a projection as suggested by the notation.

From the defining relations, we see that $\Pi c_k = c_k$ and from the second relation of eq. (2) follows the useful relation

$$c_i c_j^* c_k = \delta_{ij}\, c_k \tag{3}$$

Within the orthofermion algebra, following [6], modifying slightly their formulation, we set

$$\begin{aligned} a &= c_1 + \sum_{k=2}^{p} k\, c_{k-1}^* c_k \\ a^\dagger &= c_1^* + \sum_{k=2}^{p} c_k^* c_{k-1} \end{aligned}$$

Using equation (3), we get

$$a a^\dagger - a^\dagger a = \mathbf{1} - (p+1)\, c_p^* c_p$$

which then yields the relation corresponding to equation (1) of the TAA algebra.

3 One-variable calculus with matrices

Restricting the differentiation operator to the finite-dimensional space of polynomials of degree less than or equal to p is no problem. Use the standard basis $\{1, x, x^2, \ldots, x^p\}$. For $p = 4$, we have

$$\hat{D} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

with the extension to general p following the same pattern. However, multiplication by x must be cut off. If we define $X\,x^i = x^{i+1}$ for $i < p$ and $X\,x^p = 0$, we no longer have the relation $DX - XD = \mathbf{1}$. But we still have $DXD - XDD = D$ as for the TAA algebra. The matrix of X has the form, for $p = 4$, e.g.,

$$\hat{X} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

Note that $\hat{X}^{p+1} = 0$. To keep in line with the powers of x, we label the basis elements starting from 0. So let $\mathbf{e}_k$ denote the column vector with the only nonzero entry equal to 1 in the $(k+1)^{\text{st}}$ position. The vacuum state is $\Omega = \mathbf{e}_0$, satisfying $\hat{D}\Omega = 0$. And $\hat{X}^k\Omega = \mathbf{e}_k$, for $1 \le k \le p$. As expected, these are raising and lowering operators satisfying

$$\hat{X}\mathbf{e}_k = \mathbf{e}_{k+1}\,\theta_{kp}\,, \qquad \hat{D}\mathbf{e}_k = k\,\mathbf{e}_{k-1}$$

where $\theta_{ij} = 1$ if $i < j$, zero otherwise.

With the inner product $\langle \mathbf{e}_n, \mathbf{e}_m \rangle = \delta_{nm}\, n!\,$, we indeed have $\hat{D}^* = \hat{X}$.

Let E_{ij} denote the standard unit matrices with all but one entry equal to zero, $(E_{ij})_{kl} = \delta_{ik}\delta_{jl}$, $1 \le i, j, k, l \le p+1$. The connection with orthofermions is given by the $(p+1) \times (p+1)$ matrix realization

$$\hat{c}_i = E_{1\,i+1}$$

for $1 \le i \le p$. We have the orthofermion relations and particularly for this realization

$$\hat{c}_i^*\hat{c}_j = E_{i+1\,j+1}$$

Note that $\hat{\Pi} = E_{11}$ and that the star-algebra generated by the $\hat{c}_i$ is the full matrix algebra.

As long as X never multiplies the power x^p, the matrix implementation agrees with usual calculus. The TAA relation formulates this algebraically.

The following theorem shows that $\hat{D}$ and $\hat{X}$ not only do not generate a Heisenberg algebra, but, in fact, are as far as possible from doing so.

Theorem 3.1 *For* $p > 0$*, let* $\hat{D}$ *and* $\hat{X}$ *be* $(p+1)\times(p+1)$ *matrices defined by* $\hat{D} = \sum_{k=1}^{p} k\, E_{k\,k+1}$*,* $\hat{X} = \sum_{k=1}^{p} E_{k+1\,k}$*. Then the Lie algebra generated by* $\{\hat{X}, \hat{D}\}$ *is* $\mathfrak{sl}(p+1)$ *.*

Proof: For convenience set $n = p + 1$. First we have

$$H = [\hat{D}, \hat{X}] = -p\, E_{nn} + \sum_{k=1}^{p} E_{kk}$$

Set $\xi_1 = \hat{X}$, $\eta_1 = \hat{D}$, and $H_1 = H$. For $2 \le k \le n$, let $\xi_k = H(\overleftarrow{\mathrm{ad}\,\hat{X}})^k$, and $\eta_k = (\mathrm{ad}\,\hat{D})^k H$, where $(\mathrm{ad}\,A)B = [A, B]$ and $A(\overleftarrow{\mathrm{ad}\,B}) = [A, B]$. Then it is easily checked by induction that

$$\xi_k = -n\, E_{n\,n-k+1} \qquad \text{and} \qquad \eta_k = a_k\, \xi_k^{\dagger}$$

for nonzero constants a_k, the $\dagger$ denoting matrix transpose. Thus, we have E_{in} and E_{ni} for $1 \le i \le p$. Noting that $[E_{in}, E_{nj}] = E_{ij}$ if $i \ne j$, we have all of the off-diagonal $E's$. And

$$H_k = [\eta_k, \xi_k] = -na_k\, (E_{n-k+1\,n-k+1} - E_{nn})$$

fill out the Cartan elements of $\mathfrak{sl}(n)$. □

3.1 Examples

Here we look at some important operators for the case $p = 4$.

Example 3.2 The *number operator* is XD. We have

$$\hat{X}\hat{D} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 4 \end{pmatrix}$$

And, in general, we calculate

$$\hat{X}\hat{D} = \sum_{k=1}^{p} k\, E_{k+1\,k+1}$$

as we want.

Example 3.3 The *Ornstein-Uhlenbeck operator*, $XD - D^2$, has the form

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -2 & 0 & 0 \\ 0 & 0 & 2 & -6 & 0 \\ 0 & 0 & 0 & 3 & -12 \\ 0 & 0 & 0 & 0 & 4 \end{pmatrix}$$

Example 3.4 The *translation operator*, e^D looks like

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 3 & 6 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

with columns given by binomial coefficients, corresponding to the action $x \to x + 1$ on the basis polynomials x^i.

Example 3.5 The *Gegenbauer operator*, see, e.g. [1], $(XD + \alpha)^2 - D^2$, has the matrix form

$$\begin{pmatrix} \alpha^2 & 0 & -2 & 0 & 0 \\ 0 & (1+\alpha)^2 & 0 & -6 & 0 \\ 0 & 0 & (2+\alpha)^2 & 0 & -12 \\ 0 & 0 & 0 & (3+\alpha)^2 & 0 \\ 0 & 0 & 0 & 0 & (4+\alpha)^2 \end{pmatrix}$$

with corresponding Gegenbauer polynomials as eigenfunctions.

4 Canonical variables and polynomials

Basic to our approach is the use of *canonical variables* which are functions of X and D obeying the HW relations on an infinite-dimensional space, which restricts to the TAA relation on spaces of polynomials in x of a given bounded degree.

Let us review the basic construction and notations. We will discuss the general, multivariable, case.

Note. We use our usual convention of summing over repeated Greek indices, irrespective of position.

Given $V: \mathbf{C}^N \to \mathbf{C}^N$, $V(z) = (V_1(z_1,\ldots,z_N),\ldots,V_N(z_1,\ldots,z_N))$ holomorphic in a neighborhood of the origin, satisfying $V(0) = 0$, we construct a corresponding abelian family of dual vector fields. Corresponding to the operators X_i of multiplication by x_i, we have the partial differentiation operators, D_i. In this context, a function of $x = (x_1,\ldots,x_N)$, $f(x)$, is identified with $f(X)1$, the operator of multiplication by $f(X)$ acting on the *vacuum state* 1, with $D_i 1 = 0$, for all $1 \le i \le N$. We define operators $V(D) = (V_1(D_1,\ldots,D_N),\ldots,V_N(D_1,\ldots,D_N))$. These are our canonical lowering operators, corresponding to differentiation.

Denoting the Jacobian $\left(\dfrac{\partial V_i}{\partial z_j}\right)$ by $V'(z)$, let $W(z) = (V'(z))^{-1}$, be the inverse (matrix inverse) Jacobian. Then the boson commutation relations extend to $[V(D), X_i] = \dfrac{\partial V}{\partial D_i}$. Now define the operators

$$Y_i = X_\mu W_{\mu i}(D)$$

These are our canonical raising operators, corresponding to multiplication by X_i. The canonical system of raising and lowering operators $\{Y_j\}$, $\{V_i\}$, $1 \le i,j \le N$ indeed satisfy $[V_i, Y_j] = \delta_{ij}\mathbf{1}$. The essential feature, which has to be checked, is that, $[Y_i, Y_j] = [V_i, V_j] = 0$. Notice that exchanging D with X is a formal Fourier transformation and turns the variables Y_i into the vector fields $\tilde{Y}_i = W(x)_{\mu i}\partial/\partial x_\mu$. Thus, the Y_i are *dual vector fields.*

Notation. We complement the standard notations used along with V and W, letting U denote the inverse function to V. I.e., $U \circ V = V \circ U = \text{id}$. Explicitly: $U(V(z)) = z$.

Observe that since $W = V'^{-1}$, we have $W(z) = U'(V(z))$. In other words, converting from z to V acting on functions of the canonical variables Y_i, we have $X = Y\,U'(V)^{-1}$.
The main formula (cf. [3, p. 185, eq. (1)]) is

$$\exp(\alpha_\mu Y_\mu)\,1 = \exp(x_\mu U_\mu(\alpha)) = \sum_{n\geq 0} \frac{\alpha^n}{n!}\,y_n(x)$$

the n denoting a multi-index $(n_1, \ldots, n_N)$.
This expansion defines the *canonical polynomials*: $y_n(x) = Y^n\,1$.

5 Multivariable calculus with matrices

Here we extend Section 3 to N variables. For matrices, A, B, the tensor product $A \otimes B$ denotes the *Kronecker product* of the two matrices. That is, if A is $n \times n$, and B is $m \times m$, then $A \otimes B$ is $nm \times nm$ with entries formed by replacing each entry a_{ij} in A with the block matrix $a_{ij}B$. For products of more than two matrices, we conventionally associate to the left.

For a fixed p, we have $(p+1) \times (p+1)$ matrices $\hat{D}$ and $\hat{X}$. Let I denote the $(p+1) \times (p+1)$ identity matrix. Then we set

$$\begin{aligned} \hat{D}_j &= I \otimes I \otimes \cdots \otimes \hat{D} \otimes I \cdots \otimes I && (\hat{D} \text{ in the } j^{\text{th}} \text{ spot}) \\ \hat{X}_j &= I \otimes I \otimes \cdots \otimes \hat{X} \otimes I \cdots \otimes I && (\hat{X} \text{ in the } j^{\text{th}} \text{ spot}) \end{aligned}$$

Then $\hat{D}_j$ and $\hat{X}_j$ will satisfy the TAA relations while $[\hat{D}_j, \hat{X}_i] = [\hat{X}_j, \hat{X}_i] = [\hat{D}_j, \hat{D}_i] = 0$ for $i \neq j$.

6 Canonical calculus with matrices

First consider the case $N = 1$. We have a function $V(z)$ analytic in a neighborhood of the origin in $\mathbb{C}$, normalized to $V(0) = 0$, $V'(0) \neq 0$. Let

$W(z) = 1/V'(z)$ have the Taylor expansion

$$W(z) = w_0 + w_1 z + \cdots + w_k z^k + \cdots$$

The corresponding canonical variable is $Y = XW(D)$, satisfying $[V(D), Y] = \mathbf{1}$. The canonical basis polynomials are $y_n(x) = Y^n 1$, $n \geq 0$. Fix the order p. Let $\hat{W} = W(\hat{D})$. Then we employ the algebra generated by the operators $\hat{V} = V(\hat{D})$ and $\hat{Y} = \hat{X}\hat{W}$. Note, e.g., that since $\hat{D}^{p+1} = 0$, the operators $\hat{V}$ and $\hat{W}$ are polynomials in $\hat{D}$. Similarly, since $\hat{X}^{p+1} = 0$, the polynomials $y_n(\hat{X})$ are truncated if $n > p$. However, for $n \leq p$, the correspondence between the polynomials $y_n(x)$ and vectors $\hat{y}_n = y_n(\hat{X})\mathbf{e}_0$ is exact. Namely, the vector $\hat{y}_n$ gives the coefficients of the polynomial $y_n(x)$. The reason this works is that up to order p, the operator $\hat{X}$ never acts on a power of x greater than p.

6.1 Examples

Example 6.1 A basic example is given by

$$V(z) = e^z - 1\,, \qquad U(v) = \log(1+v)$$

so $W(z) = e^{-z}$, $Y = Xe^{-D}$. We can easily calculate

$$y_n(x) = x(x-1)\cdots(x-n+1)\ .$$

For $p = 4$, we get $\hat{Y} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 1 & -1 & 1 \\ 0 & 1 & -2 & 3 & -4 \\ 0 & 0 & 1 & -3 & 6 \\ 0 & 0 & 0 & 1 & -4 \end{pmatrix}$ and

$$\hat{Y}^2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -4 & 8 & -15 \\ 1 & -3 & 8 & -20 & 43 \\ 0 & 1 & -5 & 18 & -46 \\ 0 & 0 & 1 & -7 & 22 \end{pmatrix}, \quad \hat{Y}^3 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 2 & -6 & 18 & -53 & 126 \\ -3 & 11 & -39 & 130 & -327 \\ 1 & -6 & 29 & -116 & 313 \\ 0 & 1 & -9 & 46 & -134 \end{pmatrix}$$

and

$$\hat{Y}^4 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ -6 & 24 & -95 & 345 & -900 \\ 11 & -50 & 219 & -845 & 2255 \\ -6 & 35 & -180 & 754 & -2070 \\ 1 & -10 & 65 & -300 & 849 \end{pmatrix}, \quad \hat{Y}^5 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 24 & -119 & 559 & -2244 & 6074 \\ -50 & 269 & -1333 & 5497 & -15016 \\ 35 & -215 & 1149 & -4907 & 13559 \\ -10 & 75 & -440 & 1954 & -5466 \end{pmatrix}$$

with the first column giving the coefficients of the corresponding polynomial y_n, where we can see the truncation beginning in this last.

With $U(v) = \log(1+v)$, the relation $X = YU'(V)^{-1}$ reads $X = Y + YV$ or $xy_n = y_{n+1} + ny_n$ yielding the recurrence

$$y_{n+1} = (x-n)y_n \ .$$

Example 6.2 Another interesting example is the Gaussian with drift,

$$V(z) = \alpha z - z^2/2 \,, \qquad U(v) = \alpha - \sqrt{\alpha^2 - 2v}$$

the minus sign taken in $U(v)$ to have $U(0) = 0$. Then $W(z) = \dfrac{1}{\alpha - z}$, and

$$\hat{Y} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ \alpha^{-1} & \alpha^{-2} & 2\,\alpha^{-3} & 6\,\alpha^{-4} & 24\,\alpha^{-5} \\ 0 & \alpha^{-1} & 2\,\alpha^{-2} & 6\,\alpha^{-3} & 24\,\alpha^{-4} \\ 0 & 0 & \alpha^{-1} & 3\,\alpha^{-2} & 12\,\alpha^{-3} \\ 0 & 0 & 0 & \alpha^{-1} & 4\,\alpha^{-2} \end{pmatrix}$$

Powers of $\hat{Y}$ yield the canonical polynomials, the first few of which are

$$\begin{aligned} y_1 &= \frac{x}{\alpha} \\ y_2 &= \frac{x}{\alpha^3} + \frac{x^2}{\alpha^2} \\ y_3 &= 3\,\frac{x}{\alpha^5} + 3\,\frac{x^2}{\alpha^4} + \frac{x^3}{\alpha^3} \end{aligned}$$

$$
\begin{aligned}
y_4 &= 15\,\frac{x}{\alpha^7} + 15\,\frac{x^2}{\alpha^6} + 6\,\frac{x^3}{\alpha^5} + \frac{x^4}{\alpha^4} \\
y_5 &= 105\,\frac{x}{\alpha^9} + 105\,\frac{x^2}{\alpha^8} + 45\,\frac{x^3}{\alpha^7} + 10\,\frac{x^4}{\alpha^6} + \frac{x^5}{\alpha^5}
\end{aligned}
$$

These are a scaled variation of Bessel polynomials.
In this case $U'(V)^{-1} = \alpha\left(1 - \frac{2V}{\alpha^2}\right)^{1/2}$. Thus, expanding and rearranging the relation $X = YU'(V)^{-1}$,

$$
\alpha Y = X + \alpha Y\left(\frac{V}{\alpha^2} + \frac{1}{2}\,\frac{V^2}{\alpha^4} + \frac{1}{2}\,\frac{V^3}{\alpha^6} + \frac{5}{8}\,\frac{V^4}{\alpha^8} + \frac{7}{8}\,\frac{V^5}{\alpha^{10}} + \frac{21}{16}\,\frac{V^6}{\alpha^{12}} + \frac{33}{16}\,\frac{V^7}{\alpha^{14}} + \dots\right)
$$

which translates to

$$
\begin{aligned}
\alpha\, y_{n+1} &= x y_n + \frac{n}{\alpha}\, y_n + \frac{n(n-1)}{2\alpha^3}\, y_{n-1} + \frac{n(n-1)(n-2)}{2\alpha^5}\, y_{n-2} + \dots \\
&= x y_n + \frac{n}{\alpha}\, y_n + \sum_{k=2}^{n} \binom{n}{k} \frac{(2k-3)!!}{\alpha^{2k-1}}\, y_{n-k+1}
\end{aligned}
$$

Example 6.3 For our final example in this section, we mention the reflected LambertW function, which we denote $\mathcal{W}$ to avoid confusion with our W. Take $V(z) = ze^{-z}$, see [4, p. 110]. Then $U(v) = -\mathcal{W}(-v)$. We find $Y = Xe^D(I-D)^{-1}$ and with $p = 7$ the corresponding matrix

$$
\hat{Y} = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 2 & 5 & 16 & 65 & 326 & 1957 & 13700 \\
0 & 1 & 4 & 15 & 64 & 325 & 1956 & 13699 \\
0 & 0 & 1 & 6 & 30 & 160 & 975 & 6846 \\
0 & 0 & 0 & 1 & 8 & 50 & 320 & 2275 \\
0 & 0 & 0 & 0 & 1 & 10 & 75 & 560 \\
0 & 0 & 0 & 0 & 0 & 1 & 12 & 105 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 14
\end{pmatrix}
$$

One can show that $y_n = x(x+n)^{n-1}$ and that the relation $X = YU'(V)^{-1}$ leads to the recurrence

$$
y_{n+1} = (x+2n)y_n + \sum_{k=2}^{n} \binom{n}{k} (k-1)^{k-1}\, y_{n-k+1}\ .
$$

6.2 Matrix expansions

Using the unit matrices E_{ij} of size $(p+1)\times(p+1)$, we can write formulas for the basic operators. We have used in Theorem 3.1 the expressions

$$\hat{D} = \sum_{k=1}^{p} k\, E_{k\,k+1} \qquad \text{and} \qquad \hat{X} = \sum_{k=1}^{p} E_{k+1\,k}$$

Then induction yields

$$\hat{D}^j = \sum_{\substack{1\le k\le p\\ 1\le k+j\le p+1}} (k)_j\, E_{k\,k+j}$$

with $(k)_j = k(k+1)\cdots(k+j-1)$ denoting the rising factorial. Multiplying by $\hat{X}$ gives

$$\hat{Y} = \sum_{\substack{1\le k\le p\\ 1\le k+j\le p+1}} E_{k+1\,k+j}\,(k)_j\, w_j$$

For $N > 1$, the matrices for D_j and X_j provide the operators $\hat{Y}_j = \hat{X}_\mu W_{\mu i}(\hat{D})$ as matrices. Repeated multiplication on the vacuum vector $\mathbf{e}_0$ yields exactly the coefficients of the polynomials y_n up to order p. Here, order p means that no variable x_i appears to a power higher than p.

7 Summary and prospects

In this article, the one-variable case of the matrix approach has been presented in some detail, along with the basic theory for the multivariate case. The TAA algebra conveniently replaces the HW algebra in the finite-dimensional setting. The connection with orthofermions is interesting and clarifies the underlying structure.

In Part II of this series of articles, we will look at multivariate systems from the point of view of dual vector fields. We will present and use corresponding matrix formulations. In Part III, we expect to have some C and/or Java code implementing the techniques of Parts I and II. For Part IV, the plan is to develop useful rigorous estimates for establishing the order of truncation required for applications.

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