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On sets represented by partitions

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Abstract: We prove a lemma that is useful to get upper bounds for the number of partitions without a given subsum. From this we can deduce an improved upper bound for the number of sets represented by the (unrestricted or into unequal parts) partitions of an integer n.

1 Introduction

Let n be an integer and let

$$n = n_1 + n_2 + ... + n_j,\ n_i \in \mathbb{N}^*,\ 1 \leq n_1 \leq n_2 \leq ... \leq n_j$$

be a partition Π of n. We shall say that this partition represents an integer a if there exist $\epsilon_1, \epsilon_2, ..., \epsilon_j \in \{0, 1\}$ such that $a = \sum_{i=1}^{j} \epsilon_i n_i$. Let $\mathcal{E}(\Pi)$ denote the set of these integers; we shall call it the set represented by Π. One can easily see that $\mathcal{E}(\Pi)$ is included in $[0, n]$ and symmetric (if it contains a, it also contains $n - a$). For fixed n, let us introduce $p(n)$ the number of partitions of n and $\hat{p}(n)$ the number of different sets amongst the sets $\mathcal{E}(\Pi)$ (where Π runs over the $p(n)$ partitions of n).

Let k be a positive integer. We shall say that a partition is k-reduced if and only if each summand appears at most k times; for instance the 1-reduced partitions are the partitions into unequal parts. We shall use $q(n, k)$ the number of k-reduced partitions of n and $\hat{q}(n, k)$ the number of different $\mathcal{E}(\Pi)$ where Π runs over the $q(n, k)$ k-reduced partitions of n. When k equals 1, we shall note: $q(n) = q(n, 1)$ and $\hat{q}(n) = \hat{q}(n, 1)$.

Following an idea due to P. Erdös, the sets represented by the partitions of an integer n were first studied at the end of the 80's. P. Erdös, J.-L. Nicolas and A. Sárközy (cf. [3]) obtained upper bounds for the number of partitions without a given subsum. P. Erdös then proposed to study the asymptotic behaviour of $\hat{p}(n)$ and $\hat{q}(n)$. In [1] and [5], M. Delglise, P. Erdös, J.-L. Nicolas and A. Sárközy proved the following estimates:

Theorem 1: *For n large enough, one has*

$$q(n)^{0.51} \leq \hat{q}(n) \leq q(n)^{0.96}$$

and

$$p(n)^{0.361} \leq \hat{p}(n) \leq p(n)^{0.773}.$$

We shall obtain the following improved upper bounds:

Theorem 2: *For n large enough, one has*

$$\hat{q}(n) \leq q(n)^{0.955} \quad \textit{and} \quad \hat{p}(n) \leq p(n)^{0.768}.$$

To get these new exponents, we shall prove in part 2 a lemma improving a result due to J. Dixmier [2], whose application in part 3 gives the announced improvements.

2 The main lemma

Let a be an integer, $a \leq n$. We introduce $\mathcal{R}(n,a)$, the set of partitions of n that do not represent a, and $R(n,a)$ shall denote its cardinality. In the case of partitions into unequal parts, we shall need the same notions, with the similar notations $\mathcal{Q}(n,a)$ and $Q(n,a)$. We shall also define $Q(n,a,2)$ as the number of 2-reduced partitions Π of n such that a is not represented by Π.

Lemma 1: *Let $\epsilon > 0$. Assume there exists $\delta \in]0,1[$ such that, for any integer n and for any integer a, the following property holds*

$$\epsilon\sqrt{n} - 1 \leq a \leq 2\epsilon\sqrt{n} \Rightarrow R(n,a) \leq p(n)^{\delta}. \tag{1}$$

Then, for n large enough, one has

$$\frac{j}{2}\epsilon\sqrt{n} \leq a \leq \frac{(j+1)}{2}\epsilon\sqrt{n} \Rightarrow R(n,a) \leq (2p(\epsilon\sqrt{n}))^{j-2}p(n)^{\delta}$$

- *for $j = 2,3,...,2[\sqrt{n}/2]$ if $\epsilon < 1$,*
- *for $j = 2,3,...,\tau(n)$ with $\tau(n) = o(\sqrt{n})$ for every ϵ.*

Remark 1: To obtain a similar conclusion, J. Dixmier [2] assumed that hypothesis (1) is true for $\epsilon\sqrt{n} \leq a \leq 3\epsilon\sqrt{n}$.

Proof: We shall prove Lemma 1 by induction on j. It is true for $j = 2,3$ by (1). Let us suppose that $j \geq 4$ and that the result is true up to $j-1$. Let a be such that $\frac{j}{2}\epsilon\sqrt{n} \leq a \leq \frac{(j+1)}{2}\epsilon\sqrt{n}$. Let $\Pi \in \mathcal{R}(n,a)$ and $b = [\epsilon\sqrt{n}]$.

If b is not represented by Π, then Π belongs to a set $\mathcal{E}$ such that $|\mathcal{E}| \leq p(n)^\delta$.

If b is represented by Π, then we can write $\Pi = (\Pi', \Pi'')$, where $S(\Pi') = b$, $S(\Pi'') = n - b$ and Π'' does not represent $a - b$. We get

$$a - b \geq \frac{j}{2}\epsilon\sqrt{n} - \epsilon\sqrt{n} = \frac{j-2}{2}\epsilon\sqrt{n} \geq \epsilon\sqrt{n-b}$$

since $j \geq 4$, and

$$a - b \leq \frac{j+1}{2}\epsilon\sqrt{n} - \epsilon\sqrt{n} + 1 = \frac{j-1}{2}\epsilon\sqrt{n} + 1.$$

Moreover we have

$$\frac{j}{2}\epsilon\sqrt{n-b} \geq \frac{j}{2}\epsilon\sqrt{n}\Big(1 - \frac{b}{n}\Big) \geq \frac{j}{2}\epsilon\sqrt{n} - \epsilon\frac{j}{2}\frac{\epsilon\sqrt{n}}{\sqrt{n}}.$$

We still have to show (at least for n large enough)

$$\frac{j-1}{2}\epsilon\sqrt{n} + 1 \leq \frac{j}{2}\epsilon\sqrt{n} - \frac{j}{2}\epsilon^2.$$

- If $\epsilon < 1$, since $j/2 \leq \sqrt{n}/2$, we have to check the inequality

  $$-1/2\epsilon\sqrt{n} + 1 \leq -\epsilon^2\sqrt{n}/2,$$

  which is true when n is large enough.

- In the second case, we want to show

  $$-1/2\epsilon\sqrt{n} + 1 \leq -1/2\epsilon^2\frac{\tau(n)}{\sqrt{n}}.$$

  This is true when n is large enough by using the hypothesis on $\tau(n)$.

We finally get

$$\frac{j}{2}\epsilon\sqrt{n-b} \geq a - b.$$

We deduce from the induction hypothesis that Π'' belongs to a set $\mathcal{F}$ such that

$$|\mathcal{F}| \leq (2p(\epsilon\sqrt{n}))^{j-3}p(n)^\delta.$$

This implies that Π belongs to a set $\mathcal{G}$ such that

$$|\mathcal{G}| \leq p((\epsilon\sqrt{n}))(2p(\epsilon\sqrt{n}))^{j-3}p(n)^\delta.$$

Hence we have

$$R(n,a) \le p(n)^\delta + p((\epsilon\sqrt{n}))(2p(\epsilon\sqrt{n}))^{j-3}p(n)^\delta \le (2p(\epsilon\sqrt{n}))^{j-2}p(n)^\delta$$

which completes the proof of the lemma.

Remark 2: It is easy to see that the result remains true when we replace all the $R(n,a)$'s by $Q(n,a)$'s or by $Q(n,a,2)$'s, i.e. when we deal with partitions into unequal parts or with 2-reduced partitions (in the proof, if Π is into unequal parts, then Π' and Π'' are also into unequal parts; the same phenomenon occurs when we are dealing with 2-reduced partitions).

3 Applications

This lemma is useful to get upper bounds for $\hat{p}(n)$ and $\hat{q}(n)$ improving those obtained in [1]. Lemma 1 allows us to prove the following lemma:

Lemma 2: *When $n \to \infty$ we have:*

1. *for* $1.07\sqrt{n} \le a \le n - 1.07\sqrt{n}$,
$$Q(n,a) \le \exp((1+o(1))1.732\sqrt{n}),$$
2. *for* $0.81\sqrt{n} \le a \le n - 0.81\sqrt{n}$,
$$Q(n,a,2) \le \exp((1+o(1))1.969\sqrt{n}).$$

To get Lemma 2 (the method is developped in [1]), we find upper bounds for $Q(n,a)$ and $Q(n,a,2)$ when a ranges over the interval $[\epsilon\sqrt{n}, 2\epsilon\sqrt{n}]$ and we choose the best ϵ; then we use Lemma 1 and the results in [3].

From Lemma 2, we get Theorem 2 as in [1]. For instance, when studying $\hat{q}(n)$, we distinguish two cases according to whether the partition represents all integers between $1.07\sqrt{n}$ and $n - 1.07\sqrt{n}$ or not. We get this way

$$\hat{q}(n) \le n\exp((1+o(1))1.732\sqrt{n}) + 2^{1.07\sqrt{n}} \le q(n)^{0.955}$$

since $q(n) = \exp((1+o(1))\pi\sqrt{n/3})$ (cf. [4]).

The method is the same for $\hat{p}(n)$, since $\hat{p}(n) = \hat{q}(n,2)$ [1, Thorme 1].

Remark 3: The improvement on the exponents in the Theorem 2 is small (5.10^{-3}). This comes from the fact that the functions (cf. [1]) we bound on an interval $[x, 2x]$ (and not $[x, 3x]$, see Remark 1) have slow variations around their minimum value. Indeed, even replacing $[x, 2x]$ by $[x, (1+\eta)x]$ with η decreasing to 0 would only lead to another small improvement (4.10^{-3} less than our results). To make the exponents in the upper bounds really smaller, we need to find another method.

4 References

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