Restrictions of continuous functions

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Introduction

Given a continuous real-valued function on $[0,1]$, and a closed subset $E \subset [0,1]$ we denote by $f|_E$ the restriction of f to E, that is, the function defined only on E that takes the same values as f at every point of E. The restriction $f|_E$ will typically be "better behaved" than f. It may have bounded variation when f doesn't, it may have a better modulus of continuity than f, it may be monotone when f is not, etc. All this clearly depends on f and on E, and the questions that we discuss here are about the existence, for every f, or every f in some class, of "substantial" sets E such that $f|_E$ has bounded total variation, is monotone, or satisfies a given modulus of continuity. The notion of "substantial" that we use is that of either Hausdorff or Minkowski dimensions, both are defined below.

Here is an outline of the paper. We refer to theorems by the subsection in which they are stated.

Section 2 deals with restrictions of bounded variation. Theorem 2.1, part ***I*** states that every continuous real-valued function on $[0,1]$ has bounded variation on some set of Hausdorff dimension 1/2. Part ***II*** of the theorem shows that this is optimal by constructing an appropriate lacunary series whose sum has unbounded variation on every closed set of Minkowski dimension bigger than $1/2$ (and hence on every set of Hausdorff dimension bigger than $1/2$). Analogous results for $\mathbb{R}^d$-valued functions are proved in subsection 2.6.

Section 3 deals with restrictions that satisfy a Hölder condition with parameter $\alpha \in (0,1)$. It was known, though never stated in this form, that for every continuous function f on $[0,1]$ and every $\alpha \in (0,1)$ there exists sets E of Hausdorff dimension $1-\alpha$ such that $f|_E$ satisfies a Hölder α condition (see subsection 3.1). Extending the methods used in the proof of theorem 2.1, we give an elementary proof of the result (theorem 3.1 part ***I***) and show, in part ***II***, that it is optimal by constructing, as in the proof of part ***II*** of theorem 2.1, an approriate lacunary series whose sum is a function for which nothing better can be done.

In section 4, theorem 4.1, we construct continuos functions f that satisfy a Hölder-α condition for all $\alpha < 1$ and yet if $f|_E$ is Lipschitz or monotone, then E is “arbitrarily thin”. Theorem 4.2 deals with monotone restrictions of continuous functions.

In section 5 we consider the relative advantage of restrictions of functions that satisfy various Hölder smoothness conditions, give partial results and point out some open problems.

By including the short section 1, we try to make the exposition elementary and self-contained, requiring no background material beyond what should be “commonly known”.

Notations and terminology.

A modulus of continuity is a monotone increasing continuous concave function $\omega(t)$ on $[0,1]$, such that $\omega(0) = 0$.

Given a real-valued function f on $[0,1]$, a closed set E, and a modulus of continuity ω, we write $f|_E \in C_\omega$ if for all $t \in E$ there exist $\delta = \delta(t) > 0$ and $C = C(t)$ such that if $\tau \in E$ and $|t-\tau| \le \delta(t)$ then $|f(t) - f(\tau)| \le C(t)\omega(t-\tau)$.

For $\omega(t) = t^\alpha$, $0 < \alpha \le 1$ we write Lip_α instead of C_ω. Lip_1 is usually referred to as the *Lipschitz class*, while Lip_α, $0 < \alpha < 1$, as the Hölder α class.[1]

The *(total) variation*, $\mathrm{var}(E, f)$, of a function f on a closed set E, is defined by

$$\mathrm{var}(E, f) = \sup \sum |f(x_{j+1}) - f(x_j)|,$$

the supremum is for all finite monotone increasing sequences $\{x_j\} \subset E$. We write $f \in BV(E)$ if $\mathrm{var}(E, f) < \infty$.

The oscillation of g on a set E is

$$\mathrm{osc}(g, E) = \max_{x\in E} g(x) - \min_{x\in E} g(x). \tag{1}$$

Finally, if $E \subset [0,1]$ is closed, we denote by $|E|$ the (Lebesgue) measure of E.

1 Dimensions

1.1 (Lower) Minkowski dimension.

DEFINITION. *Let* $s > 0$. *An* s-separated set of length m *is a set* $J = \{x_j\}_{j=1}^m$ *in* $[0,1]$ *such that* $|x_k - x_j| > s$ *for* $j \ne k$.

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[1] Some classics refer to the Hölder classes as *the Lipschitz α classes* —hence the notation.

For a subset $E \subset [0,1]$, denote by $L_n(E)$ the smallest number of intervals of length n^{-1} needed to cover E. Denote by L_n^* the largest number L such that E contains some n^{-1}-separated sequence of length L.

Lemma.

$$L_n(E) \le L_{2n}^*(E) \le L_{2n}(E). \tag{2}$$

PROOF: A pair of points whose distance is $> (2n)^{-1}$ cannot belong to the same interval of length $(2n)^{-1}$. Conversely, if $\{x_j\}_{j=1}^{L_n^*}$ is a maximal $(2n)^{-1}$ separated subset of E, then the intervals of length n^{-1} centered at x_j cover E. ◀

The Minkowski dimension, $\mathscr{M}$-dim(E) of E is defined as the limit, if it exists,

$$\mathscr{M}\text{-dim}(E) = \lim_{n\to\infty} \frac{\log L_n(E)}{\log n} = \lim_{n\to\infty} \frac{\log L_n^*(E)}{\log n}. \tag{3}$$

The lower Minkowski dimension $\mathscr{L}\mathscr{M}$-dim(E) of E is well defined for all sets by

$$\mathscr{L}\mathscr{M}\text{-dim}(E) = \liminf \frac{\log L_n(E)}{\log n} = \liminf \frac{\log L_n^*(E)}{\log n}. \tag{4}$$

Example. If $E = \{\frac{1}{j}\}_{j=1}^{\infty}$, the subset $\{\frac{1}{j}\}_{j=1}^{n}$ is n^{-2} separated and $L_{n^2}^*(E) \ge n$. On the other hand the intervals $[jn^{-2}, (j+1)n^{-2}]$, $j = 1, \dots, n$ cover $\{\frac{1}{j}\}_{j=n}^{\infty}$, and n additional intervals of the same size cover $\{\frac{1}{j}\}_{j=1}^{n}$, so that $L_{n^2}(E) \le 2n$. By (2) $L_{n^2}(E) \sim n$, the limit in (3) exists, and $\mathscr{M}$-dim$(E) = \frac{1}{2}$.

1.2 Hausdorff dimension. The Hausdorff dimension $\mathscr{H}$-dim(E) of a set $E \subset \mathbb{R}$ is the infimum of the numbers c for which there is a constant C such that, for every $\varepsilon > 0$, there exists a covering of E by intervals I_n satisfying:

$$\sup_n |I_n| < \varepsilon \quad \text{and} \quad \sum |I_n|^c < C. \tag{5}$$

Since covering by intervals of arbitrary lengths $\le \varepsilon$ can be more efficient than covering by intervals of a fixed length,

$$\mathscr{H}\text{-dim}(E) \le \mathscr{L}\mathscr{M}\text{-dim}(E); \tag{6}$$

the Hausdorff dimension of a set E is bounded above by its lower Minkowski dimension. The inequality can be strict: for example, if E is countable then $\mathscr{H}$-dim$(E) = 0$, while $\mathscr{L}\mathscr{M}$-dim(E) can be as high as 1.

A useful criterion for a lower bound on the Hausdorff dimension of a closed set E is the following:

Lemma. *Assume that E carries a probability measure* μ *such that* $\mu(I) \leq C|I|^{\delta}$ *for every interval I then* $\mathscr{H}\text{-dim}(E) \geq \delta$.

PROOF: If $c < \delta$, and I_n are intervals such that $|I_n| < \varepsilon$ and $\cup I_n \supset E$, then

$$1 \leq \sum \mu(I_n) \leq C \sum |I_n|^{\delta} \leq C\varepsilon^{\delta - c} \sum |I_n|^{c}. \tag{7}$$

That means $\sum |I_n|^c > C^{-1}\varepsilon^{c-\delta}$ which is unbounded as $\varepsilon \to 0$. ◀

1.3 Determining functions. A *Hausdorff determining function* is a continuous nondecreasing function h on $[0,1]$ satisfying $h(0) = 0$. The Hausdorff dimension introduced in the previous subsection uses explicitly, in (5), the functions $h_c(t) = t^c$, with $0 < c \leq 1$ as does (implicitly) the definition of the Minkowski dimension.

A set $E \subset [0,1]$ has zero h-meassure if, for every $\varepsilon > 0$, there exist intervals I_n such that $\sum h(|I_n|) < \varepsilon$ and $E \subset \cup I_n$.

A set $E \subset [0,1]$ is Minkowski-h-null if $\liminf L_n h(1/n) = 0$.

A set that is Minkowski h-null has zero h-measure. The converse is false.

2 Restrictions of Bounded Variation

2.1 The total variation of restrictions. Given a function f on $\mathbb{R}$ and a closed set E, we denotes the total variation of the restriction $f|_E$ of f to E by $\text{var}(E, f)$, and write $f \in BV(E)$ if $\text{var}(E, f) < \infty$.

Theorem. I*: For every real-valued* $f \in C([0, 1])$, *there are closed sets* $G \subset [0, 1]$, *such that* $\mathscr{H}\text{-dim}(G) \geq \frac{1}{2}$ *and* $f \in BV(G)$.

II*: There exists real-valued functions* $F \in C([0,1])$ *such that* $var(E,F) = \infty$ *for every closed set* $E \subset [0,1]$ *such that* $\mathscr{L}\mathscr{M}\text{-dim}(E) > \frac{1}{2}$, *(and, in particular, for closed sets E such that* $\mathscr{H}\text{-dim}(E) > \frac{1}{2}$*).*

2.2 The proof of part ***I*** of the theorem uses the following lemma.

Lemma. *Let I be an interval and* $E \subset I$ *a closed set,* $\varphi \in C(E)$ *and* $\mathit{osc}(\varphi, E) = a$. *Then there are subsets* $E_j \subset E$, $j = 1, 2$, *carried by disjoint intervals* I_j, *such that* $|E_j| \geq \frac{1}{4}|E|$ *and* $\mathit{osc}(\varphi, E_j) \leq \frac{a}{2}$.

PROOF: If $I = [t_1, t_2]$ let t_3 be such that $|E \cap [t_1, t_3]| = \frac{1}{2}|E|$. Set $I_1 = [t_1, t_3]$ and $I_2 = [t_3, t_2]$.

Define $E_1 \subset I_1$ as follows: Let $[c, c+a]$ be the smallest interval containing $\varphi(E \cap I_1)$. Write $G_1 = E \cap \varphi^{-1}([c, c+\frac{1}{2}a])$ and $G_2 = E \cap \varphi^{-1}([c+\frac{1}{2}a, c+a])$, and observe that either $|G_1| \geq \frac{1}{2}|E|$ or $|G_2| \geq \frac{1}{2}|E|$ (or both). Set E_1 as G_1 in the first case, and as G_2 otherwise. Define $E_2 \subset I_2$ in the same way. ◀

We call the sets E_j *descendants* of E, and refer to the replacement of each E by its two descendants as the *standard procedure*. We sometime use the *alternate procedure* in which we replace each E by only one of the two descendants.

PROOF OF THE THEOREM, PART ***I***: Let $f \in C([0,1])$ be real-valued. We apply the lemma, with $\varphi = f$, repeatedly. We use the standard procedure most steps and the alternate procedure occasionally, $c(k) \sim 2\log_2 k$ times out of k. After k iterations we have a set $\mathscr{E}_k$ which is the union of $2^{k-c(k)} \sim 2^k k^{-2}$ sets $E_{k,\alpha}$, each of Lebesgue measure $\geq 2^{-2k}$, carried by disjoint intervals $I_{k,\alpha}$, and such that $\operatorname{osc}(g, E_{k,\alpha}) \leq 2^{-k}$. Write $G = \bigcap_k \mathscr{E}_k$.

For $x, y \in G$ let $k(x,y)$ be the last k such that x and y are in the same component $E_{k,\alpha}$. Remember that $|f(x) - f(y)| \leq 2^{-k}$.

In a monotone sequence $\{x_j\}_{j=1}^N \subset G$ and any $k \in \mathbb{N}$, there are at most $2^{k-c(k)} \sim 2^k k^{-2}$ values of j for which $k(x_j, x_{j+1}) = k$; so that

$$\sum |f(x_{j+1}) - f(x_j)| \leq \sum 2^{k-c(k)} 2^{-k} \sim \sum 2^k k^{-2} 2^{-k} = \sum k^{-2}. \tag{8}$$

It follows that the total variation of $f|_G$ is bounded by $\sum k^{-2}$.

Let μ_k a probability measure carried by $\mathscr{E}_k$ that puts the same mass $2^{c(k)-k}$ on every $E_{k,\alpha}$. Observe that, for all $l \in \mathbb{N}$, $\mu_{k+l}(E_{k,\alpha}) = \mu_k(E_{k,\alpha})$.

Let μ be a weak-star limit of μ_k as $k \to \infty$. Since every interval I of length 2^{-2k} intersects at most two sets of the form $E_{k,\alpha}$ we have $\mu(I) \leq C|I|^{\frac{k-c(k)}{2k}}$ and, by lemma 1.2 $\mathscr{H}$-dim $G \geq 1/2$. ◀

2.3 The proof of part ***II*** of the theorem is a construction that uses as a building block the 2-periodic function φ, defined by:

$$\varphi(2m + x) = 1 - |x| \qquad \text{for } |x| \leq 1 \text{ and } m \in \mathbb{Z}. \tag{9}$$

We write $\varphi_n(x) = \varphi(2nx)$.

Lemma. *Let $J = \{x_j\} \subset [0,1]$ be an s-separated monotone sequence of length m. If $m > 2n$, then, for $a > 0$,*

$$var(J, a\varphi_n) = \sum |a\varphi_n(x_{j+1}) - a\varphi_n(x_j)| \geq (m - 2n)2nas. \tag{10}$$

PROOF: There are at most $2n$ values of j for which x_j and x_{j+1} are separated by some $\frac{\ell}{2n}$, ($\ell = 1,\dots,2n$). For all other j we have $a\varphi_n$ linear and $|a\varphi_n'| = 2an$ in $[x_j, x_{j+1}]$ so that

$$|a\varphi_n(x_{j+1}) - a\varphi_n(x_j)| = 2na(x_{j+1} - x_j) \geq 2nas, \tag{11}$$

and there are at least $m - 2n$ such values of j. ◀

2.4 We can modify $a\varphi_n$ somewhat without affecting (10) materially.

Lemma. *Let $g \in C([0,1])$, $\|g\|_\infty < nsa/10$, and $G \in C([0,1])$ with Lipschitz constant bounded by $\frac{na}{10}$, then*

$$var(J, G + a\varphi_n + g) \geq (m - 2n)nsa. \tag{12}$$

PROOF: For the values of j for which x_j and x_{j+1} are *not* separated by some $\frac{\ell}{2n}$ we have

$$\begin{aligned} |a\varphi_n(x_{j+1}) - a\varphi_n(x_j)| &= 2na(x_{j+1} - x_j), \\ |G(x_{j+1}) - G(x_j)| &\leq \frac{na}{10}(x_{j+1} - x_j), \\ |g(x_{j+1}) - g(x_j)| &\leq \frac{nas}{5} \leq \frac{na}{5}(x_{j+1} - x_j), \end{aligned} \tag{13}$$

so that

$$|G + a\varphi_n + g)(x_{j+1}) - (G + a\varphi_n + g)(x_j)| \geq (2na - \frac{na}{10})(x_{j+1} - x_j) - \frac{nsa}{5} > nsa$$

which implies (12) ◀

We use the lemma with $m = 20n$ and the right-hand sides of (10) and (12) will be (wastefully) written simply as n^2as.

2.5

PROOF OF THEOREM 2.1, PART ***II*** : For sequences $\{a_l\}$, $a_l > 0$, and $\{n_l\} \subset \mathbb{N}$ write: $m_l = 20n_l$, $s_l = n_l^{-2}\log n_l$, and

$$F = \sum_{l=1}^{\infty} a_l\varphi_{n_l}, \qquad G_k = \sum_{l=1}^{k-1} a_l\varphi_{n_l}, \qquad g_k = \sum_{l=k+1}^{\infty} a_l\varphi_{n_l}, \tag{14}$$

The sequences $\{a_l\}$, $a_l > 0$ and $\{n_l\} \subset \mathbb{N}$ are chosen (below) so that

a. $a_k \log n_k > k$,

b. $\sum_{l=1}^{k-1} a_l n_l < \frac{1}{10} a_k n_k$

c. $\sum_{l>k} a_l < \frac{1}{10} n_k a_k s_k$.

These conditions guarantee that the lemma applies with $n = n_k$, $G = G_k$ and $g = g_k$ so that if J is s_k separated of length m_k, then

$$\mathrm{var}(J,F) \geq n_k^2 a_k s_k = a_k \log n_k > k. \tag{15}$$

If $\mathscr{L}\mathscr{M}$-$\dim(E) > 1/2$ then, for all $k > k(E)$, E contains s_k-separated sequences $J_E(n_k)$ of length m_k, so that

$$\mathrm{var}(E,F) \geq \mathrm{var}(J_E(n_k),F) > k, \tag{16}$$

and the function $F = \sum_{l=1}^{\infty} a_l \varphi_{n_l}$ has infinite variation on every closed E such that $\mathscr{L}\mathscr{M}$-$\dim(E) > \frac{1}{2}$.

The sequences $\{a_l\}$ and $\{n_l\}$ are defined recursively:

Take $a_1 = 1/2$ and $n_1 = 100$.

If a_l and n_l defined for $l \leq k$, set $a_{k+1} = \frac{1}{20} a_k n_k^{-1}$, and observe that this rule guarantees that $\sum_{j>k} a_j < 2a_k$, so that ***c.*** is satisfied.

Now take n_{k+1} big enough to satisfy conditions ***a.*** and ***b.*** ◀

2.6 $\mathbb{R}^d$-valued functions.

The generalization of Theorem 2.1 to $\mathbb{R}^d$-valued functions is the following statement:

Theorem. I*: For every continuous $\mathbb{R}^d$-valued function g, there are closed sets $E \subset [0,1]$, such that $\mathscr{H}$-$\dim(E) \geq \frac{1}{d+1}$ and $g \in BV(E)$.*

II*: There exists continuous $\mathbb{R}^d$-valued functions F such that if $E \subset [0,1]$ is closed and $\mathscr{L}\mathscr{M}$-$\dim(E) > \frac{1}{d+1}$ then $var(E,F) = \infty$.*

The proofs of both parts are the obvious variations on the proofs for $d = 1$.

The proof of part **I** differs from that of the corresponding part of Theorem 2.1 only in the estimate of the measures of the sets $E_{k,\alpha}$ defined at the k'th stage, carried, as before, by disjoint intervals $I_{k,\alpha}$, and such that $\mathrm{osc}(g, E_{k,\alpha}) \leq 2^{-k}$, but now of Lebesgue measure $\geq 2^{-(d+1)k}$. This guarantees that the Hausdorff dimension of the set, constructed as before, is $\geq \frac{1}{d+1}$.

For part ***II*** we replace the function φ_n by $\psi_n = \psi_{n,d}(mx)$ where $m = [n^{1/d}]$ (the integer part of $n^{1/d}$) and $\psi_{n,d}$ is an even 2-periodic $\mathbb{R}^d$-valued function satisfying: $\|\psi_{n,d}\| \leq 1$ and, for x,y such that $[x] = [y]$ and $|x-y| \geq 1/n$:

$$\|\psi_{n,d}(x) - \psi_{n,d}(y)\| \geq n^{-\frac{1}{d}} \tag{17}$$

so that

$$\|\psi_n(x) - \psi_n(y)\| \geq n^{-\frac{1}{d}} \quad \text{if} \quad [mx] = [my] \quad \text{and} \quad |x-y| \geq n^{-\frac{d+1}{d}}. \tag{18}$$

A set E such that $\mathscr{L}\mathscr{M}$-dim$(E) > \frac{1}{d+1}$, E contains, when n is large, $n^{-\frac{d+1}{d}}$-separated sequences $\{x_j\}_1^L$ of length $L >> n^{\frac{1}{d}}$ and for all, but at most $m \sim n^{\frac{1}{d}}$ values of j, we have $\|\psi_n(x_{j+1}) - \psi_n(x_j)\| \geq n^{-\frac{1}{d}}$ so that the variation of ψ_n on E is large.

One can construct the functions $\psi_{n,d}$ as follows. Let $A_m = A_{m,d}$ be the set of $(m+1)^d$ points $v_l = (k_1, \dots k_d)$ satisfying $0 \leq k_j \leq m$ in $\mathbb{N}^d$, enumerated in a way that $\|v_{l+1} - v_l\| = 1$, i.e., v_l and v_{l+1} have the same entries except for one, on which they differ by 1. The function $\psi_{n,d}$ is defined on $[-1,1]$ by stipulating that it is 2-periodic, even, and it maps $[\frac{l}{(m+1)^d}, \frac{l+1}{(m+1)^d}]$ linearly onto $[\frac{v_l}{m}, \frac{v_{l+1}}{m}]$.

3 Hölder restrictions

3.1 ***Theorem. I****: Assume* $0 < \alpha < 1$. *Given a continuous function* f*, there exists a closed set E such that* $\mathscr{H}$-dim$E = 1-\alpha$, *and* $f|_E \in Lip_\alpha$.

II*: For* $0 < \alpha < 1$ *there exist continuous functions* f *such that if* $f|_E \in Lip_\alpha$ *for a closed set* E*, then* $\mathscr{H}$-dim$E \leq 1-\alpha$.

Part ***I*** of the theorem derives easily from properties of Gaussian stationary processes on the circle, established in [1]. The proof reads:

"Take a Gaussian stationary process X on the circle (Fourier series with independent Gaussian coefficients) such that $X \in \text{Lip}_\alpha$ and $\mathscr{H}$-dim$X^{-1}(0) = \alpha$ a.s. Then write $E = (X-f)^{-1}(0)$ and apply remark 2 in Chapter 14, section 5, page 206 of [1]."

Part ***II*** of the theorem shows that part ***I*** is optimal. We give here an elementary proof of both parts.

3.2 We prove part ***I*** of the theorem by the method used in the proof of part ***I*** of theorem 2.1. The following is an extension of the procedures introduced in 2.2.

Lemma. *Let $E \subset I \subset [0,1]$ be a closed set, $f \in C_{\mathbb{R}}(E)$ and $\mathit{osc}(f,E) = a$. Given $\varepsilon > 0$, integers $k \geq 2$ and $l \geq 2$, there are subsets $E_m \subset E$, $m = 1,2,\dots,k$, carried by disjoint intervals I_m, such that*

a. *The distance between any two $E_m{}'s$ is at least $|E|\varepsilon/k$;*

b. $|E_m| \geq \frac{1-\varepsilon}{kl}|E|$;

c. $\mathit{osc}(f,E_m) \leq \frac{a}{l}$.

PROOF: Choose the increasing sequence $\{x_m\}$, $m = 0,\dots,k$ so that

$$|E \cap [0,x_m]| = |E|\tfrac{m}{k},$$

and let $y_m = x_m + |E|\varepsilon/k$. Write $I_m = [y_m, x_{m+1}]$ and $\tilde{E}_m = E \cap I_m$.

Then $|\tilde{E}_m| \geq |E|\frac{1-\varepsilon}{k}$.

Let $J = [\min_{x\in E} f(x), \max_{x\in E} f(x)]$ (so that $|J| = a$). Divide J into l equal intervals, J_s, $s = 1,\dots,l$, and write $E_{m,s} = \tilde{E}_m \cap f^{-1}J_s$. For every m let $s(m)$ be such that $|E_{m,s(m)}| \geq |E|\frac{1-\varepsilon}{kl}$, and set $E_m = \tilde{E}_{m,s(m)}$. ◄

We refer to this as the *k,l,ε procedure on $(I;E)$*, call the pairs $(I_m;E_m)$ the (first generation) descendants of $(I;E)$ and rename them as $(I_{1,m};E_{1,m})$.

We rename the parameters k,l,ε as k_1,l_1,ε_1, and repeat the procedure on each $(I_{1,m};E_{1,m})$ with parameters k_2,l_2,ε_2. We have the second generation, with k_1k_2 descendants named $(I_{2,m};E_{2,m})$, $m = 1,\dots,k_1k_2$.

We iterate the procedure repeatedly with parameters k_j,l_j,ε_j for the j'th round, and denote

$$K_n = \prod_{j=1}^{n} k_j, \quad L_n = \prod_{j=1}^{n} l_j \quad \tilde{\eta}_n = \prod_{1}^{n}(1-\varepsilon_j). \tag{19}$$

After n iterations we have K_n intervals $I_{n,m}$, each carrying a subset $E_{n,m}$ of E such that $|E_{n,m}| \geq \tilde{\eta}_n K_n^{-1} L_n^{-1}|E|$, and any two are separated by intervals of length $\geq \varepsilon_n \tilde{\eta}_{n-1} K_n^{-1} L_{n-1}^{-1}|E|$.

Given $\alpha \in (0,1)$, we choose the parameters k_j, l_j uniformly bounded, and $\varepsilon_j \to 0$ so that

$$\alpha_n = \frac{\log L_n}{\log K_n + \log L_n - \log(\varepsilon_n\tilde{\eta}_n)} > \alpha, \quad \beta_n = \frac{\log K_n}{\log K_n + \log L_n - \log\tilde{\eta}_n} < 1-\alpha, \tag{20}$$

and $\alpha_n \to \alpha$, $\beta_n \to 1-\alpha$.

Denote $E_n^* = \cup_{m=1}^{K_n} E_{n,m}$, observe that $E_n^* \subset E_{n-1}^*$, and set $E^* = \cap E_n^*$.

We claim that E^* satisfies the requirements of part ***I*** of the theorem. To prove the claim we need to show

a. $\mathscr{H}\text{-}\dim E^* \geq (1-\alpha)$.

b. $f|_{E^*} \in \mathrm{Lip}_\alpha$.

PROOF: For claim ***a.*** we construct a probability measure μ^* on E^*, such that for every $\alpha' > \alpha$, there exists a constant $C = C(\alpha')$ such that $\mu^*(I) \leq C|I|^{\alpha'}$ for all intervals I. By lemma 1.2 this proves $\mathscr{H}\text{-}\dim E^* \geq (1-\alpha)$.

Denote by μ_n the probability measure obtained by normalizing the Lebesgue measure on E_n^* by multiplying it, on each $E_{n,m}$, by a constant $c_{n,m} = K_n^{-1}|E_{n,m}|^{-1}$, so that $\mu_n(E_{n,m}) = K_n^{-1}$. The sequence $\{\mu_n\}$ converges in the weak-star topology to a measure μ^* carried by E^*. Observe that $\mu^*(E_{n,m}) = \mu_n(E_{n,m}) = K_n^{-1}$.

We evaluate the modulus of continuity of the primitive of μ^* by estimating the size of intervals A such that $\mu^*(A) \geq 2K_n^{-1}$. Such interval must contain an interval $I_{n,m}$, and hence $E_{n,m}$, and it follows that

$$|A| \geq |I_{n,m}| \geq |E_{n,m}| \geq \tilde{\eta}_n K_n^{-1} L_n^{-1} |E| \tag{21}$$

which means that for every $\alpha' > \alpha$ we have for n large enough and every interval $I_{n,m}$

$$\mu^*(I_{n,m}) \leq |I_{n,m}|^{\frac{\log K_n}{\log K_n + \log L_n - \log \tilde{\eta}_n}} = |I_{n,m}|^{\beta_n} \leq |I_{n,m}|^{1-\alpha'} \tag{22}$$

and it follows that for arbitrary intervals I and any $\alpha' > \alpha$, as $|I| \to 0$

$$\mu^*(I) = O\left(|I|^{1-\alpha'}\right) \tag{23}$$

which means that the Hausdorff dimension of E^* is at least $1-\alpha$.

The modulus of continuity ϑ of $f|_{E^*}$ is determined by:

"Let $x, y \in E^*$. Let n be the smallest index such that x, y are not in the same $E_{n,m}$. Then $|x-y| \geq \varepsilon_n \tilde{\tilde{\eta}}_n K_n^{-1} L_n^{-1} |E|$ and $|f(x) - f(y)| \leq L_{n-1}^{-1}$." which translates to $\vartheta(\varepsilon_n \tilde{\eta}_n K_n^{-1} L_{n-1}^{-1} |E|) \leq L_{n-1}^{-1}$, or, for t in this range $\vartheta(t) = O(t^{\alpha_n})$, and for all t

$$\vartheta(t) = O(t^\alpha) \tag{24}$$

◀

Remark: Reversing the inequalities in (20) by an appropriate choice of the parameters we obtain a set E^* that has positive measure in dimension $1-\alpha$, such that the modulus of continuity of $f|_{E^*}$ is bounded by $t^\alpha |\log t|^{\alpha+\varepsilon}$ as $t \to 0$.

3.3 Proof of theorem 3.1, part *II*. As in section 2, we write

$$f(x) = \sum_1^\infty a_j \varphi(\lambda_j x), \quad \text{and} \quad f_n(x) = \sum_1^n a_j \varphi(\lambda_j x) \tag{25}$$

where φ is the 2-periodic function defined by (9), a_j is fast decreasing, λ_j fast increasing. Both a_j and λ_j depend on α, and will be defined inductively.

Choose (arbitrarily) $a_1 = \frac{1}{2}$, and $\lambda_1 = 10$.

Assuming a_j and λ_j have been chosen for $j \le n$, we shall choose a_{n+1} small (see below) and then λ_{n+1} a large enough integral multiple of λ_n so that:

$$\lambda_n \mid \lambda_{n+1}, \quad \text{and} \quad a_{n+1}\lambda_{n+1} \ge 2\sum_1^n a_j \lambda_j, \tag{26}$$

The divisibility guarantees that that f_n is linear in each of the intervals $(\frac{j}{\lambda_n}, \frac{j+1}{\lambda_n})$ and the successive inequalities in (26) that $|\frac{d}{dt} f_n| \ge \frac{1}{2} a_n \lambda_n > 2^n$.

Let E be closed, and assume that $f|_E \in \text{Lip}_\alpha$. Denote

$$E_n = \{x : x \in E, \ \ |f(x) - f(y)| \le n|x-y|^\alpha \text{ for all } y \in E \text{ such that } |x-y| \le \lambda_n^{-1}\}.$$

Clearly $E_n \subset E_{n+1}$, and $E^* = \lim E_n \supset E$. It suffices, therefore, to show that E_n can be covered by intervals $I_{j,n}$ such that $\sum_j |I_{j,n}|^\beta < \varepsilon_{n,\beta}$, with $\varepsilon_{n,\beta} \to 0$ for every $\beta > 1-\alpha$.

Write $E_{n,j} = E_n \cap [\frac{j}{\lambda_n}, \frac{j+1}{\lambda_n}]$. For $x,y \in E_{n,j}$, and in particular the pair x,y such that $E_{n,j} \subset [t,y]$ we have

$$n|x-y|^\alpha \ge |f(x) - f(y)| \ge \frac{1}{2} a_n \lambda_n |x-y| - 2a_{n+1}. \tag{27}$$

If a_{n+1} is small enough, this implies $|x-y|^{1-\alpha} \le \frac{2n}{a_n \lambda_n}$, and E_n can be covered by λ_n intervals $I_{j,n}$ of length $|I_{j,n}| \le (\frac{2n}{a_n \lambda_n})^{\frac{1}{1-\alpha}}$.

For any β,

$$|I_{j,n}|^\beta \le \left(\frac{2n}{a_n \lambda_n}\right)^{\frac{\beta}{1-\alpha}}, \quad \text{and} \quad \sum |I_{j,n}|^\beta \le \left(\frac{2n}{a_n}\right)^{\frac{\beta}{1-\alpha}} \lambda_n^{1-\frac{\beta}{1-\alpha}}. \tag{28}$$

For $\beta > 1-\alpha$ the exponent of λ_n is negative, and we take λ_n big enough (after choosing a_n).

This concludes the proof of theorem 3. ◀

4 Lipschitz and monotone restrictions

4.1 Lipschitz restrictions. Part ***II*** of theorem 3 indicates that there are continuous functions f such that if $f|_E \in \mathrm{Lip}_1$ then $\mathscr{H}$-dim$E = 0$. The following refinement shows that even if f is "almost" Lip_1, the set E can be "arbitrarily" thin.

Theorem. *Given a Hausdorff determining function h, and a modulus of continuity ω such that $\lim_{s\to 0}\omega(s)/s = \infty$, there exist functions $f \in C_\omega$ such that if $f|_E \in Lip_1$, then E has zero h-measure.*

Notice that the assumption $\lim_{s\to 0}\omega(s)/s = \infty$, allows $\omega(s) = O(s^\alpha)$ for all $\alpha < 1$. The corresponding $f \in C_\omega$ belongs to Lip_α for all $\alpha < 1$.

PROOF: We use again the series (25), namely

$$f = \sum_1^\infty a_j\varphi(\lambda_j x),$$

and adapt the parameters a_n and λ_n to the current context. Both a_j and λ_j will be defined inductively, a_j will be fast decreasing, λ_j fast increasing.

Denote by $\omega_n(s) = \max_{x,\,|\tau|\le s} a_n|\varphi(\lambda_n(x+\tau)) - \varphi(\lambda_n(x))|$, the modulus of continuity of $a_n\varphi(\lambda_n x)$. The condition $\sum_n \omega_n(s) = O(\omega(s))$, as $s \to 0$, guarantees that $f \in C_\omega$. Observe that

$$\omega_n(s) = \min(a_n, a_n\lambda_n s) = \begin{cases} a_n & \text{if } s > \lambda_n^{-1} \\ a_n\lambda_n s & \text{if } 0 \le s \le \lambda_n^{-1}. \end{cases} \tag{29}$$

i. The first condition we impose on a_n, λ_n is: $a_n \le \omega(1/\lambda_n)$. It implies that $\omega_n(s) \le \min(a_n, \omega(s))$ for all s. As $\omega(1/\lambda) >> 1/\lambda$, the condition is consistent with having $a_n\lambda_n$ arbitrarily large.

ii. Given a_n and λ_n, define c_n by the condition $\omega(c_n) = 2^n a_n\lambda_n c_n = 2^n\omega_n(c_n)$. This implies that for $s \le c_n$ we have $\omega(s) \ge 2^n a_n\lambda_n s$ and

$$\omega_n(s) \le \begin{cases} a_n & \text{if } s > c_n \\ 2^{-n}\omega(s) & \text{if } s \le c_n. \end{cases} \tag{30}$$

so that for $c_{n+1} \le s \le c_n$ we have $\sum \omega_j(s) \le \omega(s) + \sum_{j=n+1}^\infty a_j$. It follows that if a_n decreases fast enough (while λ_n increases, allowing $a_n\lambda_n$ to be as large as is needed), we have indeed $f \in C_\omega$.

iii. Assuming a_j and λ_j have been chosen for $j \le n$, we shall choose a_{n+1} small (see below) and then λ_{n+1} a large enough integral multiple of λ_n so that:

$$\lambda_n \mid \lambda_{n+1}, \quad \text{and} \quad a_{n+1}\lambda_{n+1} \ge 2\sum_1^n a_j\lambda_j, \tag{31}$$

The divisibility guarantees that that f_n is linear in each of the intervals $(\frac{j}{\lambda_n}, \frac{j+1}{\lambda_n})$ and the successive inequalities in (31) that $|\frac{d}{dt} f_n| \ge \frac{1}{2}a_n\lambda_n >> 2^n$.

Let E be closed, and assume that $f|_E \in \mathrm{Lip}_1$. Denote

$$E_n = \{x : x \in E, \quad |f(x) - f(y)| \le n|x-y| \text{ for all } y \in E \text{ such that } |x-y| \le \lambda_n^{-1}\}.$$

Clearly $E_n \subset E_{n+1}$, and $E^* = \lim E_n \supset E$. It suffices, therefore, to show that E_n can be covered by intervals $I_{j,n}$ such that $\sum_j h(|I_{j,n}|) < \varepsilon_n$, with $\varepsilon_n \to 0$.

Write $E_{n,j} = E_n \cap [\frac{j}{\lambda_n}, \frac{j+1}{\lambda_n}]$. If $x, y \in E_{n,j}$ then

$$n|x-y| \ge |f(x) - f(y)| \ge \frac{1}{2}a_n\lambda_n|x-y| - 2a_{n+1} \tag{32}$$

which implies $|x-y| \le 4a_{n+1}/(a_n\lambda_n - 2n)$. It follows that E_n can be covered by λ_n arcs of length bounded by $l_n = 4a_{n+1}/(a_n\lambda_n - 2n) < 5a_{n+1}/a_n\lambda_n$.

Choose a_{n+1} small enough so that $\lambda_n h(l_n) < n^{-n}$, and then λ_{n+1} appropriate to guarantee (31).

Remark: The proof shows, in fact, that E is Minkowski h-null. ◀

4.2 Monotone restrictions.

Does there exist a function $f \in C([0, 1)$ such that if $f|_E$ is monotone then E has Hausdorff dimension 0?

Theorem. *Given a Hausdorff determining function h, there exists $f \in C([0, 1])$ such that if $f|_E$ is monotone, then E has zero h-measure.*

PROOF: Now we have to give up the building block φ defined in (9) and the corresponding functions φ_n. Let us denote by $\psi_m(x)$ the 1-periodic function satisfying: $\psi_m(0) = \psi(1) = 0$, $\psi_m(m^{-1}) = 1$ and $\psi_m(x)$ linear on $[0, m^{-1}]$ and on $[m^{-1}, 1]$.

Write $f = \sum_1^\infty a_j\psi_{m_j}((-1)^j\lambda_j x)$ and $f_n = \sum_1^n a_j\psi_{m_j}((-1)^j\lambda_j x)$, where a_j, m_j, and λ_j will be defined inductively.

The first conditions are

$$m_{j-1}\lambda_{j-1} \mid m_j\lambda_j, \quad \text{and} \quad a_n\lambda_n \ge 2\sum_1^{n-1} a_j m_j\lambda_j, \tag{33}$$

so that f_n is linear in each of the intervals (n-intervals) $(\frac{j}{m_n\lambda_n}, \frac{j+1}{m_n\lambda_n})$. Each such interval is divided in the next generation into one "fast" interval on which $|\frac{d}{dt} f_{n+1}| \sim a_{n+1}m_{n+1}\lambda_{n+1}$ and the union of the remaining "slow" intervals on which $|\frac{d}{dt} f_{n+1}| \sim a_{n+1}\lambda_{n+1}$.

For even n (resp. odd n) f_n is increasing (resp. decreasing) on the fast intervals and decreasing (resp. increasing) on the unions of the slow ones contained in an $(n-1)$-interval.

Let E be closed, $f|_E$ monotone increasing. Let n be even. Then, if J is the slow part of an n-interval, the diameter of $J \cap E$ is bounded by $a_{n+1}/a_n\lambda_n$. The number of such J's is λ_n. Choose a_{n+1} such that $\lambda_n h(a_{n+1}/a_n\lambda_n) \to 0$.

$E \setminus \bigcup J$ is covered by the union of the fast n-intervals that is λ_n intervals of length m_n^{-1}. Choose m_n (after choosing λ_n) so that $\lambda_n h(m_n^{-1}) \to 0$. ◀

5 Restrictions of Hölder functions

5.1 Smoothness.

Theorem. *Assume that $0 < \beta < \alpha < 1$. There exist functions $f \in Lip_\beta$ such that if $f|_E \in Lip_\alpha$, then E has Hausdorff dimension bounded by $\frac{1-\alpha}{1-\beta}$.*

PROOF: We keep the notations used in the proof of theorem 4.1. As observed there, the condition $f \in \mathrm{Lip}\,\beta$ is equivalent to $a_n = O\left(\lambda_n^{-\beta}\right)$ (if λ_n grows fast enough). Now $a_n^{-\frac{\alpha'}{1-\alpha}}\lambda_n^{1-\frac{\alpha'}{1-\alpha}} = O\left(\lambda_n^{-\beta\frac{\alpha'}{1-\alpha}+1-\frac{\alpha'}{1-\alpha}}\right)$ and the exponent is negative if $\alpha' > \frac{1-\alpha}{1-\beta}$. ◀

Question. Is the following statement valid?

Assume $0 < \beta < \alpha < 1$. If $f \in Lip_\beta$ there exists a set E such that $\mathscr{H}\text{-}\dim E = \frac{1-\alpha}{1-\beta}$, and $f|_E \in Lip_\alpha$.

5.2 Bounded variation.

For $\alpha \in (0,1)$, denote by $\|\ \|_\alpha$ the Lip_α norm. It is easy to see that $\|a\varphi_n\|_\alpha \sim an^\alpha$ and if n_k increases fast enough, say $n_{k+1} > 2n_k$, then $\sum a_k\varphi_{n_k} \in \mathrm{Lip}_\alpha$ if, and only if, $a_k = O\left(n_k^{-\alpha}\right)$.

Theorem. *There exists real-valued functions $F \in Lip_\alpha$ such that if $E \subset [0,1]$ is closed and $\mathscr{L}\mathscr{M}\text{-}\dim(E) > \frac{1}{2-\alpha}$ then $var(E,F) = \infty$.*

PROOF: As in the example above define $F = \sum a_k \varphi_{n_k}$ where now $n_k = a_k^{-1/\alpha}$. If $\mathscr{L}\mathscr{M}$-dim$(E) > \frac{1}{2-\alpha}$, and we set $s_k = n_k^{\alpha-2} \log n_k$, then E contains s_k-separated sequences J'_k of length $m_k > 20n_k$, and $\operatorname{var}(E,F) = \infty$ since for every k,

$$\operatorname{var}(E,F) \geq \operatorname{var}(J'_k,F) \geq n_k^2 a_k s_k = \log n_k. \tag{34}$$

◀

Question: Is the result best possible: does every $f \in \mathrm{Lip}_\alpha$ have bounded variation on some set of dimension $c = \frac{1}{2-\alpha}$?

References

[1] Kahane, Jean-Pierre, *Some random series of functions,* Cambridge Studies in Advanced Mathematics, Vol 5, second edition, Cambridge University Press, 1985.