

Large time existence for $3D$ water-waves and asymptotics

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Abstract. We rigorously justify in $3D$ the main asymptotic models used in coastal oceanography, including: shallow-water equations, Boussinesq systems, Kadomtsev-Petviashvili (KP) approximation, Green-Naghdi equations, Serre approximation and full-dispersion model. We first introduce a “variable” nondimensionalized version of the water-waves equations which vary from shallow to deep water, and which involves four dimensionless parameters. Using a nonlocal energy adapted to the equations, we can prove a well-posedness theorem, uniformly with respect to all the parameters. Its validity ranges therefore from shallow to deep-water, from small to large surface and bottom variations, and from fully to weakly transverse waves. The physical regimes corresponding to the aforementioned models can therefore be studied as particular cases; it turns out that the existence time and the energy bounds given by the theorem are always those needed to justify the asymptotic models. We can therefore derive and justify them in a systematic way.

1. Introduction

1.1. General setting

The motion of a perfect, incompressible and irrotational fluid under the influence of gravity is described by the free surface Euler (or water-waves) equations. Their complexity led physicists and mathematicians to derive simpler sets of equations likely to describe the dynamics of the water-waves equations in some specific physical regimes. In fact, many of the most famous equations of mathematical physics

were historically obtained as formal asymptotic limits of the water-waves equations: the shallow-water equations, the Korteweg-de Vries (KdV) and Kadomtsev-Petviashvili (KP) equations, the Boussinesq systems, etc. Each of these asymptotic limits corresponds to a very specific physical regime whose range of validity is determined in terms of the characteristics of the flow (amplitude, wavelength, anisotropy, bottom topography, depth, ...).

The derivation of these models goes back to the XIXth century, but the rigorous analysis of their relevance as approximate models for the water-waves equations only began three decades ago with the works of Ovsjannikov [40,41], Craig [13], and Kano and Nishida [27,28,26] who first addressed the problem of justifying the formal asymptotics. For all the different asymptotic models, the problem can be formulated as follows: 1) do the water-waves equations have a solution on the time scale relevant for the asymptotic model and 2) does this model furnish a good approximation of the solution? Answering the first question requires a large-time existence theorem for the water-waves equations, while the second one requires a rigorous derivation of the asymptotic models and a precise control of the approximation error.

Following the pioneer works for one-dimensional surfaces ($1DH$) of Ovsjannikov [41] and Nalimov [38] (see also Yosihara [50,51]), Craig [13], and Kano and Nishida [27] provided the first justification of the KdV and $1DH$ Boussinesq and shallow water approximations. However, the comprehension of the well-posedness theory for the water-waves equations hindered the perspective of justifying the other asymptotic regimes until the breakthroughs of S. Wu ([48] and [49] respectively for the $1DH$ and $2DH$ case, in infinite depth, and without restrictive assumptions). Since then, the literature on free surface Euler equations has been very active: the case of finite depth was proved in [29], and in the related case of the study of the free surface of a liquid in vacuum with zero gravity, Lindblad [34,35]. More recently Coutand and Shkoller [12] and Shatah and Zeng [45] managed to remove the irrotationality condition and/or took into account surface tension effects (see also [3] for $1DH$ water-waves with surface tension).

In order to review the existing results of the rigorous justification of asymptotic models for water-waves, it is suitable to classify the different physical regimes using two dimensionless numbers: the amplitude parameter ε and the shallowness parameter μ (defined below in (1.2)):

- Shallow-water, large amplitude ($\mu \ll 1$, $\varepsilon \sim 1$). Formally, this regime leads at first order to the well-known “shallow-water equations” (or Saint-Venant) and at second order to the so-called “Green-Naghdi” model, often used in coastal oceanography because it takes into account the dispersive effects neglected by

the shallow-water equations. The first rigorous justification of the shallow-water model goes back to Ovsjannikov [40,41] and Kano and Nishida [27] who proved the convergence of the solutions of the shallow-water equations to solutions of the water-waves equations as $\mu \rightarrow 0$ in $1DH$, and under some restrictive assumptions (small and analytic data). More recently, Y. A. Li [33] removed these assumptions and rigorously justified the shallow-water and Green-Naghdi equations, in $1DH$ for flat bottoms. Finally, the first and so far only rigorous work on a $2DH$ asymptotic model is due to a very recent work by T. Iguchi [24] in which he justified the $2DH$ shallow-water equations, also allowing non-flat bottoms, but under a restrictive zero mass assumption on the velocity.

- Shallow water, medium amplitude ($\mu \ll 1$, $\varepsilon \sim \sqrt{\mu}$). This regime leads to the so-called Serre equations, which are quite similar to the aforementioned Green-Naghdi equations and are also often used in coastal oceanography. To our knowledge, no rigorous result exists on that model.
- Shallow water, small amplitude ($\mu \ll 1$, $\varepsilon \sim \mu$). This regime (also called long-waves regime) leads to many mathematically interesting models due to the balance of nonlinear and dispersive effects:
 - Boussinesq systems: since the first derivation by Boussinesq, many formally equivalent models (also named after Boussinesq) have been derived. W. Craig [13] and Kano and Nishida [28] were the first to give a full justification of these models, in $1DH$ (and for flat bottoms and small data). Note, however, that the convergence result given in [28] is given on a time scale too short to capture the nonlinear and dispersive effects specific to the Boussinesq systems; in [13], the correct *large time* existence (and convergence) results for the water-waves equations are given. The proof, of such a large time well-posedness result for the water-waves equations, is the most delicate point in the justification process. Furthermore, it is the last step needed to fully justify the Boussinesq systems in $2DH$, owing to [5] (flat bottoms) and [9] (general bottom topography), where the convergence property is proved *assuming* that the large-time well-posedness theorem holds.
 - Uncoupled models: at first order, the Boussinesq systems reduce to a simple wave equation and, in $1DH$, the motion of the free surface can be described as the sum of two uncoupled counter-propagating waves, slightly modulated by a Korteweg-de Vries (KdV) equation. In $2DH$ and for weakly transverse waves, a similar phenomenon occurs, but with the Kadomtsev-Petviashvili (KP) equation replacing the KdV equation. Many papers addressed the problem of validating the KdV model (e.g. [13,28,43,5,47,23]) and its justification is now complete. For the KP model, a first attempt was done in [26], under re-

strictive assumption (small and analytic data), but as in [27], the time scale considered is unfortunately too small for the relevant dynamics. A series of works then proved the KP limit for simplified systems and toy models [20, 4, 42], while a different approach was used in [32] where the KP limit is proved for the full water-waves equations, *assuming* a large-time well-posedness theorem and a specific control of the solutions.

- Deep-water, small steepness ($\mu \geq 1$, $\varepsilon\sqrt{\mu} \ll 1$). This regime leads to the full-dispersion (or Matsuno) equations; to our knowledge, no rigorous result exists on this point.

Instead of developing an existence/convergence theory for each physical scaling, we hereby propose a global method which allows one to justify all the asymptotics mentioned above at once. In order to do that, we nondimensionalize the water-waves equations, and keep track of the five physical quantities which characterize the dynamics of the water-waves: amplitude, depth, wavelength in the longitudinal direction, wavelength in the transverse direction and amplitude of the bottom variations.

Our main theorem gives an estimate of the existence time of the solution of the water-waves equations which is *uniform* with respect to *all* these parameters. In order to prove this theorem, we introduce an energy which involves the aforementioned parameters and use it to construct our solution by an iterative scheme. Moreover, this energy provides some bounds on the solutions which appear to be exactly those needed in the justification of the asymptotics regimes mentioned above.

1.2. Presentation of the results

Parameterizing the free surface by $z = \zeta(t, X)$ (with $X = (x, y) \in \mathbb{R}^2$) and the bottom by $z = -d + b(X)$ (with $d > 0$ constant), one can use the incompressibility and irrotationality conditions to write the water-waves equations under Bernoulli's formulation, in terms of a velocity potential ϕ (i.e., the velocity field is given by $\mathbf{v} = \nabla_{X,z}\phi$):

$$\begin{cases} \partial_x^2 \phi + \partial_y^2 \phi + \partial_z^2 \phi = 0, & -d + b \leq z \leq \zeta, \\ \partial_n \phi = 0, & z = -d + b, \\ \partial_t \zeta + \nabla \zeta \cdot \nabla \phi = \partial_z \phi, & z = \zeta, \\ \partial_t \phi + \frac{1}{2}(|\nabla \phi|^2 + (\partial_z \phi)^2) + \zeta = 0, & z = \zeta, \end{cases} \quad (1.1)$$

where $\nabla = (\partial_x, \partial_y)^T$ and $\partial_n \phi$ is the outward normal derivative at the boundary of the fluid domain.

The qualitative study of the water-waves equations is made easier by the introduction of dimensionless variables and unknowns. This

requires the introduction of various orders of magnitude linked to the physical regime under consideration. More precisely, let us introduce the following quantities: a is the order of amplitude of the waves; λ is the wave-length of the waves in the x direction; λ/γ is the wave-length of the waves in the y direction; B is the order of amplitude of the variations of the bottom topography.

We also introduce the following dimensionless parameters

$$\frac{a}{d} = \varepsilon, \quad \frac{d^2}{\lambda^2} = \mu, \quad \frac{B}{d} = \beta; \quad (1.2)$$

the parameter ε is often called *nonlinearity* parameter, while μ is the *shallowness* parameter. In total generality, one has

$$(\varepsilon, \mu, \gamma, \beta) \in (0, 1] \times (0, \infty) \times (0, 1] \times [0, 1] \quad (1.3)$$

(the conditions $\varepsilon \in (0, 1]$ and $\beta \in [0, 1]$ mean that the the surface and bottom variations are at most of the order of depth — $\beta = 0$ corresponding to flat bottoms— and the condition $\gamma \in (0, 1]$ says that the x axis is chosen to be the longitudinal direction for weakly transverse waves).

Zakharov [52] remarked that the system (1.1) could be written in Hamiltonian form in terms of the free surface elevation ζ and of the trace of the velocity potential at the surface $\psi = \phi|_{z=\zeta}$ and Craig, Sulem and Sulem [17] and Craig, Schanz and Sulem [16] used the fact that (1.1) could be reduced to a system of two evolution equations on ζ and ψ ; this formulation has commonly been used since then. The dimensionless form of this formulation involves the parameters introduced in (1.2), the transversity γ , and a parameter $\nu = (1 + \sqrt{\mu})^{-1}$ whose presence is due to the fact that the nondimensionalization is not the same in deep and shallow water. It is derived in Appendix A:

$$\begin{cases} \partial_t \zeta - \frac{1}{\mu\nu} \mathcal{G}_{\mu,\gamma}[\varepsilon\zeta, \beta b] \psi = 0, \\ \partial_t \psi + \zeta + \frac{\varepsilon}{2\nu} |\nabla^\gamma \psi|^2 - \frac{\varepsilon\mu}{\nu} \frac{(\frac{1}{\mu} \mathcal{G}_{\mu,\gamma}[\varepsilon\zeta, \beta b] \psi + \varepsilon \nabla^\gamma \zeta \cdot \nabla^\gamma \psi)^2}{2(1 + \varepsilon^2 \mu |\nabla^\gamma \zeta|^2)} = 0, \end{cases} \quad (1.4)$$

where $\nabla^\gamma = (\partial_x, \gamma \partial_y)^T$ and $\mathcal{G}_{\mu,\gamma}[\varepsilon\zeta, \beta b]$ is the Dirichlet-Neumann operator defined by $\mathcal{G}_{\mu,\gamma}[\varepsilon\zeta, \beta b] \psi = (1 + \varepsilon^2 |\nabla^\gamma \zeta|^2)^{1/2} \partial_n \Phi|_{z=\varepsilon\zeta}$, with Φ solving

$$\begin{cases} \partial_z^2 \Phi + \mu \partial_x^2 \Phi + \gamma^2 \mu \partial_y^2 \Phi = 0, & -1 + \beta b < z < \varepsilon\zeta, \\ \Phi|_{z=\varepsilon\zeta} = \psi, & \partial_n \Phi|_{z=-1+\beta b} = 0. \end{cases} \quad (1.5)$$

In Section 2, we give some preliminary results which will be used throughout the paper: a few technical results (such as commutator

estimates) are given in §2.1 and elliptic boundary value problems directly linked to (1.5) are studied in §2.2.

Section 3 is devoted to the study of various aspects of the Dirichlet-Neumann operator $\mathcal{G}_{\mu,\gamma}[\varepsilon\zeta, \beta b]$. It is well-known that the Dirichlet-Neumann operator is a pseudo-differential operator of order one; in particular, it acts continuously on Sobolev spaces and its operator norm, commutators with derivatives, etc., have been extensively studied. The task here is more delicate because of the presence of four parameters $(\varepsilon, \mu, \gamma, \beta)$ in the operator $\mathcal{G}_{\mu,\gamma}[\varepsilon\zeta, \beta b]$. Indeed, some of the classical estimates on the Dirichlet-Neumann are not uniform with respect to the parameters and must be modified. But the main difficulty is due to the fact that the energy introduced in this paper is not of Sobolev type; namely, it is given by

$$\forall s \geq 0, \quad \forall U = (\zeta, \psi), \quad |U|_{\tilde{X}^s} = |\zeta|_{H^s} + \left| \frac{\nu^{-1/2} |D^\gamma|}{(1 + \sqrt{\mu} |D^\gamma|)^{1/2}} \psi \right|_{H^s}, \quad (1.6)$$

where $|D^\gamma| := (-\partial_x^2 - \gamma^2 \partial_y^2)^{1/2}$. For high frequencies, this energy is equivalent to the $H^s \times H^{s+1/2}$ -norm specific to the non-strictly hyperbolic nature of the water-waves equations (see [14] for a detailed comment on this point), but the equivalence is not uniform with respect to the parameters, and the $H^s \times H^{s+1/2}$ estimates of [48, 49, 29, 3] for instance, are useless for our purposes here. We thus have to work with estimates in $|\cdot|_{\tilde{X}^s}$ -type norms and the classical results on Sobolev estimates of pseudodifferential operators cannot be used. Consequently, we must rely on the structural properties of the water-waves equations much more heavily than in the previous works quoted above.

Fundamental properties of the DN operators are given in §3.1, while commutator estimates and further properties are investigated in §3.2 and §3.3. We then give asymptotic expansions of $\mathcal{G}_{\mu,\gamma}[\varepsilon\zeta, \beta b]\psi$ in terms of the parameters in §3.4.

Using the results of the previous sections, we study the Cauchy problem associated to the linearization of (1.4) in Section 4; the main energy estimate is given in Proposition 4.1.

The full nonlinear equations are addressed in Section 5 and our main result is stated in Theorem 5.1; it gives a “large-time” (of order $O(\varepsilon/\nu)$) existence result for the water-waves equations (1.4) and a bound on its energy (defined in (1.6)). The most important point is that this result is uniform with respect to *all* the parameters $(\varepsilon, \mu, \gamma, \beta)$ satisfying (1.3) and such that the steepness $\varepsilon\sqrt{\mu}$ and the ratio β/ε remain bounded. The theorem also requires a classical *Taylor sign condition* on the initial data; we give in Proposition 5.1 very simple sufficient conditions (involving in particular the “anisotropic Hessian” of the bottom parameterization b), which imply that the

Taylor sign condition is satisfied.

Both Theorem 5.1 and Proposition 5.1 can be used for all the physical regimes given in the previous section, and the solution they provide exists over a time scale relevant with respect to the dynamics of the asymptotic models. We can therefore study the asymptotic limits, which is done in Section 6. It is convenient to use the classification introduced previously to present our results (we also refer to [31] for an overview of the methods developed here):

- Shallow-water, large amplitude ($\mu \ll 1, \varepsilon \sim 1$). We justify in §6.1.1 the shallow-water equations without the restrictive assumptions of [24] and previous works. For the Green-Naghdi model, we extend in §6.1.2 the result of [33] to non-flat bottoms, and to two dimensional surfaces.
- Shallow water, medium amplitude ($\mu \ll 1, \varepsilon \sim \sqrt{\mu}$). We rigorously justify the Serre approximation over the relevant $O(1/\sqrt{\mu})$ time scale in §6.1.2.
- Shallow water, small amplitude ($\mu \ll 1, \varepsilon \sim \mu$).
 - Boussinesq systems: In §6.2, we fully justify all the Boussinesq systems in the open case of two-dimensional surfaces (flat or non-flat bottoms).
 - Uncoupled models: We complete the full justification of the KP approximation in §6.3.
- Deep-water, small steepness ($\mu \geq 1, \varepsilon\sqrt{\mu} \ll 1$). We show in §6.4.1 that the solutions of the full-dispersion model converge to exact solutions of the water-waves equations as the steepness goes to zero and give accurate error estimates.

We also give in §6.4.2 an estimate on the precision of a model used for the numerical computation of the water-waves equations (see [15] for instance).

1.3. Notations

- We use the generic notation $C(\lambda_1, \lambda_2, \dots)$ to denote a nondecreasing function of the parameters $\lambda_1, \lambda_2, \dots$.
- The notation $a \lesssim b$ means that $a \leq Cb$, for some nonnegative constant C whose exact expression is of no importance (*in particular, it is independent of the small parameters involved*).
- For all tempered distribution $u \in \mathcal{S}'(\mathbb{R}^2)$, we denote by \widehat{u} its Fourier transform.
- Fourier multipliers: For all rapidly decaying $u \in \mathcal{S}(\mathbb{R}^2)$ and all $f \in C(\mathbb{R}^2)$ with tempered growth, $f(D)$ is the distribution defined by

$$\forall \xi \in \mathbb{R}^2, \quad \widehat{f(D)u}(\xi) = f(\xi)\widehat{u}(\xi); \quad (1.7)$$

(this definition can be extended to wider spaces of functions).

- We write $\langle \xi \rangle = (1 + |\xi|^2)^{1/2}$ and $\Lambda = \langle D \rangle$.

- For all $1 \leq p \leq \infty$, $|\cdot|_p$ denotes the classical norm of $L^p(\mathbb{R}^2)$ while $\|\cdot\|_p$ stands for the canonical norm of $L^p(\mathcal{S})$, with $\mathcal{S} = \mathbb{R}^2 \times (-1, 0)$.
- For all $s \in \mathbb{R}$, $H^s(\mathbb{R}^2)$ is the classical Sobolev space defined as

$$H^s(\mathbb{R}^2) = \{u \in \mathcal{S}'(\mathbb{R}^2), |u|_{H^s} := |A^s u|_2 < \infty\}.$$

- For all $s \in \mathbb{R}$, $\|\cdot\|_{L^\infty H^s}$ denotes the canonical norm of $L^\infty([-1, 0]; H^s(\mathbb{R}^2))$.
- If B is a Banach space, then $|\cdot|_{B,T}$ stands for the canonical norm of $L^\infty([0, T]; B)$.
- For all $\gamma > 0$, we write $\nabla^\gamma = (\partial_x, \gamma \partial_y)^T$, so that ∇^γ coincides with the usual gradient when $\gamma = 1$. We also use the Fourier multiplier $|D^\gamma|$ defined as

$$|D^\gamma| = \sqrt{D_x^2 + \gamma^2 D_y^2},$$

as well as the anisotropic divergence operator

$$\operatorname{div}_\gamma = (\nabla^\gamma)^T.$$

- We denote by $\mathfrak{P}_{\mu, \gamma}$ (or simply \mathfrak{P} when no confusion is possible) the Fourier multiplier of order $1/2$

$$\mathfrak{P}_{\mu, \gamma} (= \mathfrak{P}) := \frac{\nu^{-1/2} |D^\gamma|}{(1 + \sqrt{\mu} |D^\gamma|)^{1/2}}. \quad (1.8)$$

- We write $X = (x, y)$ and $\nabla_{X,z} = (\partial_x, \partial_y, \partial_z)^T$; we also write

$$\nabla^{\mu, \gamma} = (\sqrt{\mu} \partial_x, \gamma \sqrt{\mu} \partial_y, \partial_z)^T.$$

- We use the condensed notation

$$A_s = B_s + \langle C_s \rangle_{s > \underline{s}} \quad (1.9)$$

to say that $A_s = B_s$ if $s \leq \underline{s}$ and $A_s = B_s + C_s$ if $s > \underline{s}$.

- By convention, we take

$$\prod_{k=1}^0 p_k = 1 \quad \text{and} \quad \sum_{k=1}^0 p_k = 0. \quad (1.10)$$

- When the notation $\partial_n u|_{\partial\Omega}$ is used for boundary conditions of an elliptic equation of the form $\nabla_{X,z} \cdot P \nabla_{X,z} u = h$ in some open set Ω , it stands for the *outward conormal derivative* associated to this operator, namely,

$$\partial_n u|_{\partial\Omega} = \mathbf{n} \cdot P \nabla_{X,z} u|_{\partial\Omega}, \quad (1.11)$$

\mathbf{n} standing for the *outward* unit normal vector to $\partial\Omega$.

2. Preliminary results

2.1. Commutator estimates and anisotropic Poisson regularization

We recall first the tame product and Moser estimates in Sobolev spaces: if $t_0 > 1$ and $s \geq 0$, then $\forall f \in H^s \cap H^{t_0}(\mathbb{R}^2), \forall g \in H^s(\mathbb{R}^2)$,

$$|fg|_{H^s} \lesssim |f|_{H^{t_0}} |g|_{H^s} + \langle |f|_{H^s} |g|_{H^{t_0}} \rangle_{s > t_0} \quad (2.1)$$

and, for all $F \in C^\infty(\mathbb{R}^n; \mathbb{R}^m)$ such that $F(0) = 0$,

$$\forall u \in H^s(\mathbb{R}^2)^n, \quad F(u) \in H^s(\mathbb{R}^2)^m \quad \text{and} \quad |F(u)|_{H^s} \leq C(|u|_\infty) |u|_{H^s}. \quad (2.2)$$

In the next proposition, we give tame commutator estimates.

Proposition 2.1 (Ths. 3 and 6 of [30]). Let $t_0 > 1$ and $-t_0 < r \leq t_0 + 1$. Then, for all $s \geq 0$, $f \in H^{t_0+1} \cap H^{s+r}(\mathbb{R}^2)$ and $u \in H^{s+r-1}(\mathbb{R}^2)$,

$$|[A^s, f]u|_{H^r} \lesssim |\nabla f|_{H^{t_0}} |u|_{H^{s+r-1}} + \langle |\nabla f|_{H^{s+r-1}} |u|_{H^{t_0}} \rangle_{s > t_0+1-r},$$

where we used the notation (1.9).

One can deduce from the above proposition some commutator estimates useful in the present study.

Corollary 2.1. Let $t_0 > 1$, $s \geq 0$ and $\gamma \in (0, 1]$. Then:

i. For all $\mathbf{v} \in H^{t_0+2} \cap H^{s+1}(\mathbb{R}^2)^2$ and $u \in H^s(\mathbb{R}^2)$, one has

$$|[A^s, \operatorname{div}_\gamma(\mathbf{v} \cdot)]u|_2 \leq |\mathbf{v}|_{H^{t_0+2}} |u|_{H^s} + \langle |u|_{H^{t_0+1}} |\mathbf{v}|_{H^{s+1}} \rangle_{s > t_0+1}.$$

ii. For all $0 \leq r \leq t_0 + 1$, $f \in L^\infty((-1, 0); H^{s+r} \cap H^{t_0+1}(\mathbb{R}^2))$ and $u \in L^2((-1, 0); H^{s+r-1}(\mathbb{R}^2))$,

$$\begin{aligned} \|A^r [A^s, f]u\|_2 &\lesssim \|f\|_{L^\infty H^{t_0+1}} \|A^{s+r-1}u\|_2 \\ &\quad + \langle \|f\|_{L^\infty H^{s+r}} \|A^{t_0}u\|_2 \rangle_{s > t_0+1-r}. \end{aligned}$$

Proof. For the first point, just remark that

$$[A^s, \operatorname{div}_\gamma(\mathbf{v} \cdot)]u = [A^s, \operatorname{div}_\gamma(\mathbf{v})]u + [A^s, \mathbf{v}] \cdot \nabla^\gamma u,$$

and use Proposition 2.1 to obtain the result (recall that $\gamma \leq 1$).

For the second point of the corollary, remark that for all $z \in [-1, 0]$,

$$|[A^s, f]u(z)|_2 \lesssim |f(z)|_{H^{t_0+1}} |u(z)|_{H^s} + \langle |f(z)|_{H^{s+1}} |u(z)|_{H^{t_0}} \rangle_{s > t_0},$$

as a consequence of Proposition 2.1 (with $r = 1$). The corollary then follows easily. \square

Let us end this section with a result on anisotropic Poisson regularization (when $\gamma = \mu = 1$, the result below is just the standard gain of half a derivative of the Poisson regularization).

Proposition 2.2. Let $\gamma \in (0, 1]$, $\mu > 0$ and χ be a smooth, compactly supported function and $u \in \mathfrak{S}'(\mathbb{R}^2)$. Define also $u^\dagger := \chi(\sqrt{\mu}z|D^\gamma|)u$. For all $s \in \mathbb{R}$, if $u \in H^{s-1/2}(\mathbb{R}^2)$, one has $\Lambda^s u^\dagger \in L^2(\mathcal{S})$ and

$$c_1 \left| \frac{1}{(1 + \sqrt{\mu}|D^\gamma|)^{1/2}} u \right|_{H^s} \leq \| \Lambda^s u^\dagger \|_2 \leq c_2 \left| \frac{1}{(1 + \sqrt{\mu}|D^\gamma|)^{1/2}} u \right|_{H^s}.$$

Moreover, for all $s \in \mathbb{R}$, if $u \in H^{s+1/2}(\mathbb{R}^2)$, one has $\Lambda^s \nabla^{\mu, \gamma} u^\dagger \in L^2(\mathcal{S})^3$ and

$$c'_1 \left| \frac{\sqrt{\mu}|D^\gamma|}{(1 + \sqrt{\mu}|D^\gamma|)^{1/2}} u \right|_{H^s} \leq \| \Lambda^s \nabla^{\mu, \gamma} u^\dagger \|_2 \leq c'_2 \left| \frac{\sqrt{\mu}|D^\gamma|}{(1 + \sqrt{\mu}|D^\gamma|)^{1/2}} u \right|_{H^s}.$$

In the above estimates, c_1 , c_2 , c'_1 and c'_2 are nonnegative constants which depend only on χ .

Proof. Write classically (with $|\xi^\gamma| = \sqrt{\xi_1^2 + \gamma^2 \xi_2^2}$),

$$\begin{aligned} \| \chi(\sqrt{\mu}z|D^\gamma|)u \|_{s,0}^2 &= \int_{\mathbb{R}^2} \int_{-1}^0 \langle \xi \rangle^{2s} \chi(\sqrt{\mu}z|\xi^\gamma|)^2 |\widehat{u}(\xi)|^2 dz d\xi \\ &= \int_{\mathbb{R}^2} \langle \xi \rangle^{2s} \frac{F(0) - F(-\sqrt{\mu}|\xi^\gamma|)}{\sqrt{\mu}|\xi^\gamma|} |\widehat{u}(\xi)|^2 d\xi, \end{aligned}$$

where F denotes a primitive of χ^2 . The first estimate of the proposition then follows from the observation that

$$c_1^2 \frac{1}{1 + \sqrt{\mu}|\xi^\gamma|} \leq \frac{F(0) - F(-\sqrt{\mu}|\xi^\gamma|)}{\sqrt{\mu}|\xi^\gamma|} \leq c_2^2 \frac{1}{1 + \sqrt{\mu}|\xi^\gamma|},$$

where the constants depend only on χ .

For the second estimate of the proposition, remark that

$$\| \nabla^{\mu, \gamma} u^\dagger \|_2 \sim \sqrt{\mu} \| \chi(\sqrt{\mu}z|D^\gamma|) |D^\gamma| u \|_2 + \sqrt{\mu} \| \chi'(z\sqrt{\mu}|D^\gamma|) |D^\gamma| u \|,$$

and use the first part of the proposition. \square

2.2. Elliptic estimates on a strip

We recall that the velocity potential Φ within the fluid domain solves the boundary value elliptic problem

$$\begin{cases} \partial_z^2 \Phi + \mu \partial_x^2 \Phi + \gamma^2 \mu \partial_y^2 \Phi = 0, & -1 + \beta b < z < \varepsilon \zeta, \\ \Phi|_{z=\varepsilon \zeta} = \psi, & \partial_n \Phi|_{z=-1+\beta b} = 0, \end{cases} \quad (2.3)$$

with $(\varepsilon, \mu, \gamma, \beta) \in (0, 1] \times (0, \infty) \times (0, 1] \times [0, 1]$.

Denote by \mathcal{S} the flat strip $\mathcal{S} = \mathbb{R}^2 \times (-1, 0)$, and assume that the following assumption is satisfied:

$$\text{There exists } h_0 > 0 \text{ such that } 1 + \varepsilon \zeta - \beta b \geq h_0. \quad (2.4)$$

Under this assumption, one can define a diffeomorphism S mapping \mathcal{S} onto the fluid domain Ω :

$$S: \begin{array}{c} \mathcal{S} \\ (X, z) \end{array} \rightarrow \begin{array}{c} \Omega \\ S(X, z) := (X, z + \sigma(X, z)) \end{array},$$

with

$$\sigma(X, z) = -\beta z b(X) + \varepsilon(z + 1)\zeta(X). \quad (2.5)$$

Remark 2.1. The mapping σ used in (2.5) to define the diffeomorphism S is the most simple one can think of. If one wanted to have optimal estimates with respect to the fluid or bottom parameterization (but unfortunately not uniform with respect to the parameters), one should use instead *regularizing diffeomorphisms* as in Prop. 2.13 of [29].

From Proposition 2.7 of [29], we know that the BVP (2.3) is equivalent to the BVP (recall that we use the convention (1.11) for normal derivatives),

$$\begin{cases} \nabla_{X,z} \cdot P[\sigma] \nabla_{X,z} \phi = 0, & \text{in } \mathcal{S}, \\ \phi|_{z=0} = \psi, & \partial_n \phi|_{z=-1} = 0, \end{cases} \quad (2.6)$$

with $\phi = \Phi \circ S$ and with the $(2+1) \times (2+1)$ matrix $P[\sigma]$ given by

$$P[\sigma] := P_{\mu,\gamma}[\sigma] = \begin{pmatrix} \mu(1 + \partial_z \sigma) & 0 & -\mu \partial_x \sigma \\ 0 & \gamma^2 \mu(1 + \partial_z \sigma) & -\gamma^2 \mu \partial_y \sigma \\ -\mu \partial_x \sigma & -\gamma^2 \mu \partial_y \sigma & \frac{1 + \mu(\partial_x \sigma)^2 + \gamma^2 \mu(\partial_y \sigma)^2}{1 + \partial_z \sigma} \end{pmatrix}.$$

Remark also that it follows from the expression of $P[\sigma]$ that

$$\nabla_{X,z} \cdot P[\sigma] \nabla_{X,z} = \nabla^{\mu,\gamma} \cdot (1 + Q[\sigma]) \nabla^{\mu,\gamma},$$

where

$$Q[\sigma] := Q_{\mu,\gamma}[\sigma] = \begin{pmatrix} \partial_z \sigma & 0 & -\sqrt{\mu} \partial_x \sigma \\ 0 & \partial_z \sigma & -\gamma \sqrt{\mu} \partial_y \sigma \\ -\sqrt{\mu} \partial_x \sigma & -\gamma \sqrt{\mu} \partial_y \sigma & \frac{-\partial_z \sigma + \mu(\partial_x \sigma)^2 + \gamma^2 \mu(\partial_y \sigma)^2}{1 + \partial_z \sigma} \end{pmatrix}. \quad (2.7)$$

Below we provide two important properties satisfied by $Q[\sigma]$.

Proposition 2.3. Let $t_0 > 1$, $s \geq 0$, and $\zeta, b \in H^{t_0+1} \cap H^{s+1}(\mathbb{R}^2)$ be such that (2.4) is satisfied. Assume also that σ is as defined in (2.5). Then:

i. One has

$$\|Q[\sigma]\|_{L^\infty H^s} \leq C \left(\frac{1}{h_0}, \|\nabla^{\mu,\gamma} \sigma\|_{L^\infty H^{t_0}} \right) \|\nabla^{\mu,\gamma} \sigma\|_{L^\infty H^s}$$

and, when σ is also time dependent,

$$\|\partial_t Q[\sigma]\|_{\infty, T} \leq C \left(\frac{1}{h_0}, \|\nabla^{\mu, \gamma} \sigma\|_{\infty, T} \right) \|\nabla^{\mu, \gamma} \partial_t \sigma\|_{\infty, T}.$$

ii. For all $j \geq 1$ and $\mathbf{h} \in H^{t_0+1} \cap H^{s+1}(\mathbb{R}^2)^j$, and denoting by $Q^{(j)}[\sigma] \cdot \mathbf{h}$ the j -th derivative of $\zeta \mapsto Q[\zeta]$ in the direction \mathbf{h} , one has

$$\begin{aligned} \|Q^{(j)}[\sigma] \cdot \mathbf{h}\|_{L^\infty H^s} &\leq \left(\frac{\varepsilon}{\nu} \right)^j C \left(\frac{1}{h_0}, \varepsilon \sqrt{\mu}, \|\nabla^{\mu, \gamma} \sigma\|_{L^\infty H^{t_0}} \right) \\ &\times \left(\sum_{k=1}^j |h_k|_{H^{s+1}} \prod_{l \neq k} |h_l|_{H^{t_0+1}} + \langle (1 + \|\nabla^{\mu, \gamma} \sigma\|_{L^\infty H^s}) \prod_{k=1}^j |h_k|_{H^{t_0+1}} \rangle_{s > t_0} \right). \end{aligned}$$

iii. The matrix $1 + Q[\sigma]$ is coercive in the sense that

$$\forall \Theta \in \mathbb{R}^{2+1}, \quad |\Theta|^2 \lesssim k[\sigma](1 + Q[\sigma])\Theta \cdot \Theta,$$

with

$$k[\sigma] := k_{\mu, \gamma}[\sigma] = 1 + \|\partial_z \sigma\|_{\infty} + \frac{1}{h_0} \left(1 + \sqrt{\mu} \|\nabla^\gamma \sigma\|_{\infty} \right)^2.$$

Proof. The first two points follow directly from the tame product and Moser's estimate (2.1) and (2.2), and the explicit expression of $Q[\sigma]$. It is not difficult to see that $(1 + Q[\sigma])\Theta \cdot \Theta = \frac{1}{1 + \partial_z \sigma} |B\Theta|^2$, where

$$B = \begin{pmatrix} 1 + \partial_z \sigma & 0 & -\sqrt{\mu} \partial_x \sigma \\ 0 & 1 + \partial_z \sigma & -\gamma \sqrt{\mu} \partial_y \sigma \\ 0 & 0 & 1 \end{pmatrix}.$$

The matrix B is invertible and its inverse is given by

$$B^{-1} = \frac{1}{1 + \partial_z \sigma} \begin{pmatrix} 1 & 0 & \sqrt{\mu} \partial_x \sigma \\ 0 & 1 & \gamma \sqrt{\mu} \partial_y \sigma \\ 0 & 0 & 1 + \partial_z \sigma \end{pmatrix}.$$

Remark now that owing to (2.4), the mapping σ , as given by (2.5), satisfies $(1 + \partial_z \sigma)^{-1} \leq h_0^{-1}$, so that

$$\sqrt{1 + \partial_z \sigma} |B^{-1}|_{\mathbb{R}^3 \rightarrow \mathbb{R}^3} \lesssim \sqrt{1 + \|\partial_z \sigma\|_{\infty}} + \frac{1}{\sqrt{h_0}} \left(1 + \sqrt{\mu} \|\nabla^\gamma \sigma\|_{\infty} \right).$$

Since $|B\Theta| |B^{-1}|_{\mathbb{R}^3 \rightarrow \mathbb{R}^3} \geq |\Theta|$, the third claim of the proposition follows. \square

Since the Dirichlet condition in (2.6) can be 'lifted' in order to take homogeneous Dirichlet boundary condition, we are led to study the following class of elliptic BVPs:

$$\begin{cases} \nabla^{\mu,\gamma} \cdot (1 + Q[\sigma]) \nabla^{\mu,\gamma} u = \nabla^{\mu,\gamma} \cdot \mathbf{g}, & \text{in } \mathcal{S}, \\ u|_{z=0} = 0, \quad \partial_n u|_{z=-1} = -\mathbf{e}_z \cdot \mathbf{g}|_{z=-1}, \end{cases} \quad (2.8)$$

where, according to the notation (1.11), $\partial_n u|_{z=-1}$ stands for

$$\partial_n u|_{z=-1} = -\mathbf{e}_z \cdot (1 + Q[\sigma]) \nabla^{\mu,\gamma} u|_{z=-1}.$$

Before stating the main result of this section let us introduce a notation:

Notation 2.1. We generically write

$$M[\sigma] := C\left(\varepsilon\sqrt{\mu}, \frac{1}{h_0}, \|\nabla^{\mu,\gamma}\sigma\|_{L^\infty H^{t_0+1}}\right), \quad (2.9)$$

where, as usual, $C(\cdot)$ is a nondecreasing function of its arguments.

Proposition 2.4. Let $t_0 > 1$, $s \geq 0$ and $\zeta, b \in H^{t_0+2} \cap H^{s+1}(\mathbb{R}^2)$ be such that (2.4) is satisfied, and let σ be given by (2.5). Then for all $\mathbf{g} \in C([-1, 0]; H^s(\mathbb{R}^2)^3)$, there exists a unique variational solution $u \in H^1(\mathcal{S})$ to the BVP (2.8) and

$$\|A^s \nabla^{\mu,\gamma} u\|_2 \leq M[\sigma] (\|A^s \mathbf{g}\|_2 + \langle \|\nabla^{\mu,\gamma}\sigma\|_{L^\infty H^s} \|A^{t_0} \mathbf{g}\|_2 \rangle_{s>t_0+1}),$$

where $M[\sigma]$ is defined in (2.9).

Proof. The existence of the solution can be obtained with very classical tools and we therefore omit it. We thus focus our attention on the proof of the estimate.

Let $\chi(\cdot)$ be a smooth, compactly supported function such that $\chi(\xi) = 1$ in a neighborhood of $\xi = 0$, and define $\Lambda_h := \Lambda * \chi(hD)$. Using $\Lambda_h^{2s} u$ as test function in the variational formulation of (2.8), one gets

$$\int_{\mathcal{S}} (1 + Q[\sigma]) \nabla^{\mu,\gamma} u \cdot \nabla^{\mu,\gamma} \Lambda_h^{2s} u = \int_{\mathcal{S}} \mathbf{g} \cdot \nabla^{\mu,\gamma} \Lambda_h^{2s} u,$$

so that using the fact that Λ_h^s is L^2 -self-adjoint, one gets, with $v_h = \Lambda_h^s u$,

$$\int_{\mathcal{S}} \Lambda_h^s (1 + Q[\sigma]) \nabla^{\mu,\gamma} u \cdot \nabla^{\mu,\gamma} v_h = \int_{\mathcal{S}} \Lambda_h^s \mathbf{g} \cdot \nabla^{\mu,\gamma} v_h,$$

and thus

$$\int_{\mathcal{S}} (1 + Q[\sigma]) \nabla^{\mu,\gamma} v_h \cdot \nabla^{\mu,\gamma} v_h = \int_{\mathcal{S}} (\Lambda_h^s \mathbf{g} \cdot \nabla^{\mu,\gamma} v_h - [\Lambda_h^s, Q[\sigma]] \nabla^{\mu,\gamma} u \cdot \nabla^{\mu,\gamma} v_h).$$

Thanks to the coercitivity property of Proposition 2.3, one gets

$$k[\sigma]^{-1} \| \Lambda_h^s \nabla^{\mu, \gamma} u \|_2 \lesssim \| [\Lambda_h^s, Q[\sigma]] \nabla^{\mu, \gamma} u \|_2 + \| \Lambda_h^s \mathbf{g} \|_2; \quad (2.10)$$

since $[\Lambda^s, Q[\sigma]]$ is of order $s - 1$, the above estimates allows one to conclude, after letting h go to zero, that $\Lambda^s \nabla^{\mu, \gamma} u \in L^2(\mathcal{S})$; more precisely, thanks to Corollary 2.1, one deduces

$$k[\sigma]^{-1} \| \Lambda^s \nabla^{\mu, \gamma} u \|_2 \lesssim \| \Lambda^s \mathbf{g} \|_2 + \| Q[\sigma] \|_{L^\infty H^{t_0+1}} \| \Lambda^{s-1} \nabla^{\mu, \gamma} u \|_2 \\ + \langle \| Q[\sigma] \|_{L^\infty H^s} \| \Lambda^{t_0} \nabla^{\mu, \gamma} u \|_2 \rangle_{s > t_0+1},$$

and thus

$$\| \Lambda^s \nabla^{\mu, \gamma} u \|_2 \leq C(k[\sigma], \| Q[\sigma] \|_{L^\infty H^{t_0+1}}) \quad (2.11) \\ \times (\| \Lambda^s \mathbf{g} \|_2 + \| \nabla^{\mu, \gamma} u \|_2 + \langle \| Q[\sigma] \|_{L^\infty H^s} \| \Lambda^{t_0} \nabla^{\mu, \gamma} u \|_2 \rangle_{s > t_0+1}).$$

One also gets $\| \nabla^{\mu, \gamma} u \|_2 \leq k[\sigma] \| \mathbf{g} \|_2$ from (2.10) after remarking that the commutator in the r.h.s. vanishes when $s = h = 0$; taking $s = t_0$ in (2.11) then gives $\| \Lambda^{t_0} \nabla^{\mu, \gamma} u \|_2 \leq C(k[\sigma], \| Q[\sigma] \|_{L^\infty H^{t_0+1}}) \| \Lambda^{t_0} \mathbf{g} \|_2$, so that the r.h.s. of (2.11) is bounded from above by

$$C(k[\sigma], \| Q[\sigma] \|_{L^\infty H^{t_0+1}}) (\| \Lambda^s \mathbf{g} \|_2 + \langle \| Q[\sigma] \|_{L^\infty H^s} \| \Lambda^{t_0} \mathbf{g} \|_2 \rangle_{s > t_0+1}).$$

The proposition follows therefore from Proposition 2.3. \square

Before stating a corollary to Proposition 2.4, let us introduce a few notations:

Notation 2.2. i. For all $u \in H^{3/2}(\mathbb{R}^2)$, we define u^b as the solution to the BVP

$$\begin{cases} \nabla^{\mu, \gamma} \cdot (1 + Q[\sigma]) \nabla^{\mu, \gamma} u^b = 0 \\ u^b|_{z=0} = u, \quad \partial_n u^b|_{z=-1} = 0. \end{cases} \quad (2.12)$$

ii. For all $u \in \mathfrak{S}'(\mathbb{R}^2)$, one defines u^\dagger as

$$\forall z \in [-1, 0], u^\dagger(\cdot, z) = \chi(\sqrt{\mu}z |D^\gamma|)u,$$

where χ is a smooth, compactly supported function such that $\chi(0) = 1$.

The following corollary gives some control on the extension mapping $u \mapsto u^b$.

Corollary 2.2. Let $t_0 > 1$ and $s \geq 0$. Let also $\zeta, b \in H^{t_0+2} \cap H^{s+1}(\mathbb{R}^2)$ be such that (2.4) is satisfied, and σ be given by (2.5).

Then for all $u \in H^{s+1/2}(\mathbb{R}^2)$, there exists a unique solution $u^b \in H^1(\mathcal{S})$ and

$$\| \Lambda^s \nabla^{\mu, \gamma} u^b \|_2 \leq \sqrt{\mu\nu} M[\sigma] (\| \mathfrak{P}u \|_{H^s} + \langle \| \nabla^{\mu, \gamma} \sigma \|_{L^\infty H^s} \| \mathfrak{P}u \|_{H^{t_0}} \rangle_{s > t_0+1}),$$

with \mathfrak{P} as defined in (1.8).

Proof. Looking for u^b under the form $u^b = v + u^\dagger$, with u^\dagger given by Notation 2.2, one must solve

$$\begin{cases} \nabla^{\mu,\gamma} \cdot (1 + Q[\sigma]) \nabla^{\mu,\gamma} v = -\nabla^{\mu,\gamma} \cdot (1 + Q[\sigma]) \nabla^{\mu,\gamma} u^\dagger, \\ v|_{z=0} = 0, \quad \partial_n v|_{z=-1} = \mathbf{e}_z \cdot (1 + Q[\sigma]) \nabla^{\mu,\gamma} u^\dagger|_{z=-1}. \end{cases} \quad (2.13)$$

Applying Proposition 2.4 (with $\mathbf{g} = -(1 + Q[\sigma]) \nabla^{\mu,\gamma} u^\dagger$), one gets

$$\|A^s \nabla^{\mu,\gamma} v\|_2 \leq M[\sigma] (\|A^s \nabla^{\mu,\gamma} u^\dagger\|_2 + \langle \|\nabla^{\mu,\gamma} \sigma\|_{L^\infty H^s} \|A^{t_0} \nabla^{\mu,\gamma} u^\dagger\|_2 \rangle_{s > t_0 + 1}),$$

and since $u^b = u^\dagger + v$, the corollary follows from Proposition 2.2. \square

Remark 2.2. From the variational formulation of (2.13), one gets easily

$$\|(1 + Q[\sigma])^{1/2} \nabla^{\mu,\gamma} v\|_2 \leq \|(1 + Q[\sigma])^{1/2} \nabla^{\mu,\gamma} u^\dagger\|_2.$$

3. The Dirichlet-Neumann operator

As seen in the introduction, we define the Dirichlet-Neumann operator $\mathcal{G}_{\mu,\gamma}[\varepsilon\zeta, \beta b]$ as

$$\mathcal{G}_{\mu,\gamma}[\varepsilon\zeta, \beta b]\psi = \sqrt{1 + |\varepsilon \nabla \zeta|^2} \partial_n \Phi|_{z=\varepsilon\zeta},$$

where Φ solves (2.3). Using Notation 2.2, one can give an alternate definition of $\mathcal{G}_{\mu,\gamma}[\varepsilon\zeta, \beta b]$ (see Proposition 3.4 of [29]), namely,

$$\mathcal{G}_{\mu,\gamma}[\varepsilon\zeta, \beta b]\psi = \partial_n \psi^b|_{z=0} \quad (= \mathbf{e}_z \cdot P[\sigma] \nabla_{X,z} \psi^b|_{z=0}).$$

More, precisely one has:

Proposition 3.1. Let $t_0 > 1$, $s \geq 0$ and $\zeta, b \in H^{t_0+2} \cap H^{s+1}(\mathbb{R}^2)$ be such that (2.4) is satisfied, and let σ be given by (2.5).

Then one can define the mapping $\mathcal{G}_{\mu,\gamma}[\varepsilon\zeta, \beta b]$ (or simply $\mathcal{G}[\varepsilon\zeta]$ when no confusion is possible) as

$$\mathcal{G}_{\mu,\gamma}[\varepsilon\zeta, \beta b] (= \mathcal{G}[\varepsilon\zeta]) : \begin{array}{ccc} H^{s+1/2}(\mathbb{R}^2) & \rightarrow & H^{s-1/2}(\mathbb{R}^2) \\ u & \mapsto & \partial_n u^b|_{z=0} \end{array}.$$

Proof. The extension u^b is well-defined owing to Corollary 2.2. Moreover, we can use the definition of $P[\sigma]$ and $Q[\sigma]$ to see that

$$\mathbf{e}_z \cdot P[\sigma] \nabla_{X,z} u^b = \mathbf{e}_z \cdot (1 + Q[\sigma]) \nabla^{\mu,\gamma} u^b.$$

We will now show that it makes sense to take the trace of the above expression at $z = 0$. This is trivially true for $Q[\sigma]$, so that we are left with u^b . After a brief look at the proof of Corollary 2.2, and using the same notations, one gets $u^b = v + u^\dagger$. Since one obviously has

$\nabla^{\mu,\gamma}u^\dagger \in C([-1, 0]; H^{s-1/2}(\mathbb{R}^2)^3)$, the trace $\nabla^{\mu,\gamma}u^\dagger|_{z=0}$ makes sense. In order to prove that $\nabla^{\mu,\gamma}v|_{z=0}$ is also defined, remark that v , which solves (2.13) satisfies $\nabla^{\mu,\gamma}v \in L^2((-1, 0); H^s(\mathbb{R}^2)^3)$ and, using the equation, $\partial_z \nabla^{\mu,\gamma}v \in L^2((-1, 0); H^{s-1}(\mathbb{R}^2)^3)$. By the trace theorem, these two properties show that $\nabla^{\mu,\gamma}v|_{z=0} \in H^{s-1/2}(\mathbb{R}^2)^3$. \square

3.1. Fundamental properties

We begin this section with two basic properties of the Dirichlet-Neumann operator which play a key role in the energy estimates.

Proposition 3.2. Let $t_0 > 1$ and $\zeta, b \in H^{t_0+2}(\mathbb{R}^2)$ be such that (2.4) is satisfied. Then

i. The Dirichlet-Neumann operator is self-adjoint:

$$\forall u, v \in H^{1/2}(\mathbb{R}^2), \quad (u, \mathcal{G}[\varepsilon\zeta]v) = (v, \mathcal{G}[\varepsilon\zeta]u).$$

ii. One has

$$\forall u, v \in H^{1/2}(\mathbb{R}^2), \quad |(u, \mathcal{G}[\varepsilon\zeta]v)| \leq (u, \mathcal{G}[\varepsilon\zeta]u)^{1/2} (v, \mathcal{G}[\varepsilon\zeta]v)^{1/2}.$$

Proof. Using Notation 2.2, one gets by Green's identity that

$$(u, \mathcal{G}[\varepsilon\zeta]v) = \int_S (1 + Q[\sigma]) \nabla^{\mu,\gamma}u^b \cdot \nabla^{\mu,\gamma}v^b \quad (3.1)$$

$$= \int_S (1 + Q[\sigma])^{1/2} \nabla^{\mu,\gamma}u^b \cdot (1 + Q[\sigma])^{1/2} \nabla^{\mu,\gamma}v^b, \quad (3.2)$$

where $(1 + Q[\sigma])^{1/2}$ stands for the square root of the positive definite matrix $(1 + Q[\sigma])$ (note that the symmetry in u and v of the above expression proves the –very classical– first point of the proposition). It follows therefore from Cauchy-Schwartz inequality that

$$(u, \mathcal{G}[\varepsilon\zeta]v) \leq \|(1 + Q[\sigma])^{1/2} \nabla^{\mu,\gamma}u^b\|_2 \|(1 + Q[\sigma])^{1/2} \nabla^{\mu,\gamma}v^b\|_2,$$

which yields the second point of the proposition, since one has

$$(u, \mathcal{G}[\varepsilon\zeta]u) = \|(1 + Q[\sigma])^{1/2} \nabla^{\mu,\gamma}u^b\|_2^2 \quad (3.3)$$

(just take $u = v$ in (3.2)). \square

The next proposition is related to the variational formula of Hadamard and gives a uniform control of the operator norm of the DN operator and its derivatives (recall that we use the convention (1.10) and that $d_\zeta^j \mathcal{G}[\varepsilon \cdot]u \cdot \mathbf{h} = \mathcal{G}[\varepsilon\zeta]u$ when $j = 0$).

Proposition 3.3. Let $t_0 > 1$, $s \geq 0$ and $\zeta, b \in H^{t_0+2} \cap H^{s+1}(\mathbb{R}^2)$ be such that (2.4) is satisfied, and let σ be given by (2.5).

For all $u \in H^{s+1/2}(\mathbb{R}^2)$, $j \in \mathbb{N}$ and $\mathbf{h} \in H^{t_0+2} \cap H^{s+1}(\mathbb{R}^2)^j$, one has

$$\begin{aligned} \left| \frac{1}{\sqrt{\mu}} d_\zeta^j \mathcal{G}[\varepsilon \cdot] u \cdot \mathbf{h} \right|_{H^{s-1/2}} &\leq \left(\frac{\varepsilon}{\nu} \right)^j M[\sigma] \left(|\mathfrak{P}u|_{H^s} \prod_{k=1}^j |h_k|_{H^{t_0+1}} \right. \\ &\quad \left. + \langle (1 + \|\nabla^{\mu, \gamma} \sigma\|_{L^\infty H^s}) |\mathfrak{P}u|_{H^{t_0}} \prod_{k=1}^j |h_k|_{H^{t_0+1}} \rangle_{s > t_0} \right. \\ &\quad \left. + \langle |\mathfrak{P}u|_{H^{t_0}} \sum_{k=1}^j |h_k|_{H^{s+1}} \prod_{l \neq k} |h_l|_{H^{t_0+1}} \rangle_{s > t_0} \right), \end{aligned}$$

with $M[\sigma]$ as in (2.9) while \mathfrak{P} is defined in (1.8).

Remark 3.1. When $j = 0$, the proposition gives a much more precise estimate on $|\mathcal{G}[\varepsilon \zeta]u|_{H^{s-1/2}}$ than Theorem 3.6 of [29], but requires $\zeta \in H^{s+1}(\mathbb{R}^2)$ while $\zeta \in H^{s+1/2}(\mathbb{R}^2)$ is enough, as shown in [29] through the use of regularizing diffeomorphisms. This lack of optimality in the ζ -dependence is the price to pay to obtain uniform estimates in terms of a $\mathcal{E}^s(\cdot)$ rather than Sobolev-type norm.

Remark 3.2. The r.h.s. of the estimate given in the proposition (when $j = 0$) is itself bounded from above by

$$M[\sigma] \left(|u|_{H^{s+1}} + \langle \|\nabla^{\mu, \gamma} \sigma\|_{L^\infty H^s} |u|_{H^{t_0+1}} \rangle_{s > t_0} \right).$$

Proof. First remark that one has $\Lambda^{s-1/2} v^\dagger|_{z=0} = \Lambda^{s-1/2} v$ (with v^\dagger as in Notation 2.2), so that one gets by Green's identity,

$$\begin{aligned} (\Lambda^{s-1/2} \mathcal{G}[\varepsilon \zeta]u, v) &= (\mathcal{G}[\varepsilon \zeta]u, \Lambda^{s-1/2} v) \\ &= \int_{\mathcal{S}} (1 + Q[\sigma]) \nabla^{\mu, \gamma} u^b \cdot \Lambda^{s-1/2} \nabla^{\mu, \gamma} v^\dagger \\ &= \int_{\mathcal{S}} \Lambda^s (1 + Q[\sigma]) \nabla^{\mu, \gamma} u^b \cdot \Lambda^{-1/2} \nabla^{\mu, \gamma} v^\dagger. \end{aligned} \quad (3.4)$$

A Cauchy-Schwartz inequality then yields,

$$(\Lambda^{s-1/2} \mathcal{G}[\varepsilon \zeta]u, v) \leq \|\Lambda^s (1 + Q[\sigma]) \nabla^{\mu, \gamma} u^b\|_2 \|\Lambda^{-1/2} \nabla^{\mu, \gamma} v^\dagger\|_2, \quad (3.5)$$

and since it follows from the product estimate (2.1) that $\|\Lambda^s (1 + Q[\sigma]) \nabla^{\mu, \gamma} u^b\|_2$ is bounded from above by

$$(1 + \|Q[\sigma]\|_{L^\infty H^{t_0}}) \|\Lambda^s \nabla^{\mu, \gamma} u^b\|_2 + \langle \|Q[\sigma]\|_{L^\infty H^s} \|\Lambda^{t_0} \nabla^{\mu, \gamma} u^b\|_2 \rangle_{s > t_0},$$

one can deduce from Propositions 2.2 and 2.3 that (recall that $\nu = \frac{1}{1+\sqrt{\mu}}$),

$$\begin{aligned} (\Lambda^{s-1/2} \mathcal{G}[\varepsilon \zeta] u, v) &\leq \nu^{-1/2} M[\sigma] |v|_2 \\ &\times \left(\|\Lambda^s \nabla^{\mu, \gamma} u^b\|_2 + \langle \|\nabla^{\mu, \gamma} \sigma\|_{L^\infty H^s} \|\Lambda^{t_0} \nabla^{\mu, \gamma} u^b\|_2 \rangle_{s > t_0} \right), \end{aligned}$$

and the proposition thus follows directly from Corollary 2.2 and a duality argument in the case $j = 0$.

In the case $j \neq 0$, after differentiating (3.4), and using the same notation as in Proposition 2.3, one gets

$$\begin{aligned} (\Lambda^{s-1/2} d_\zeta^j \mathcal{G}[\varepsilon \cdot] u \cdot \mathbf{h}, v) &= \int_{\mathcal{S}} \Lambda^s (Q^{(j)}[\sigma] \cdot \mathbf{h}) \nabla^{\mu, \gamma} u^b \cdot \Lambda^{-1/2} \nabla^{\mu, \gamma} v^\dagger \\ &+ \sum_{k=1}^j \sum_{\mathbf{h}_k, \mathbf{h}_{j-k}} \int_{\mathcal{S}} \Lambda^s B(\mathbf{h}_k, \mathbf{h}_{j-k}) \cdot \Lambda^{-1/2} \nabla^{\mu, \gamma} v^\dagger, \quad (3.6) \end{aligned}$$

where the second summation is over all the k -uplets \mathbf{h}_k and $(j-k)$ -uplets \mathbf{h}_{j-k} such that $(\mathbf{h}_k, \mathbf{h}_{j-k})$ is a permutation of \mathbf{h} , and where $B(\mathbf{h}_k, \mathbf{h}_{j-k})$ is given by

$$B(\mathbf{h}_k, \mathbf{h}_{j-k}) = (Q^{(j-k)}[\sigma] \cdot \mathbf{h}_{j-k}) \nabla^{\mu, \gamma} (u^{b, k} \cdot \mathbf{h}_k)$$

($u^{b, k} \cdot \mathbf{h}_k$ standing for the k -th order derivative of $\zeta \mapsto u^b$ at ζ and in the direction \mathbf{h}_k).

Proceeding as for the case $j = 0$ and using the estimates on $\|Q^{(j)}[\sigma] \cdot \mathbf{h}\|_{L^\infty H^s}$ provided by Proposition 2.3, one arrives at the desired estimate for the first term of the r.h.s. of (3.6). For the other terms, one has to remark first that $u^{b, k} \cdot \mathbf{h}_k$ solves a bvp like (2.8) with

$$\mathbf{g} = - \sum_{l=0}^{k-1} \sum_{\mathbf{h}_{k,l}, \mathbf{h}_{k,k-l}} B(\mathbf{h}_{k,l}, \mathbf{h}_{k,k-l}),$$

where the second summation is taken over all the l and $k-l$ -uplets such that $(\mathbf{h}_{k,l}, \mathbf{h}_{k,k-l})$ is a permutation of \mathbf{h}_k . A control of $\|\nabla^{\mu, \gamma} u^{b, k} \cdot \mathbf{h}_k\|_{L^\infty H^s}$ in terms of $\|Q^{(k-l)}[\sigma] \cdot \mathbf{h}_{k,k-l}\|_{L^\infty H^s}$ and $\|\nabla^{\mu, \gamma} u^{b, l} \cdot \mathbf{h}_{k,l}\|_{L^\infty H^s}$ is therefore provided by Proposition 2.4. It is then easy to conclude by a simple induction. \square

Remark 3.3. Instead of (3.5), one can easily get

$$(\Lambda^{s-1/2} \mathcal{G}[\varepsilon \zeta] u, v) \leq \|\Lambda^{s+1/2} (1 + Q[\sigma]) \nabla^{\mu, \gamma} u^b\|_2 \|\Lambda^{-1} \nabla^{\mu, \gamma} v^\dagger\|_2,$$

and since $\|\Lambda^{-1}\nabla^{\mu,\gamma}v^\dagger\|_2 \lesssim \sqrt{\mu}|v|_2$ one also has the estimate (with $\zeta = 0$ for the sake of simplicity):

$$\left| \frac{1}{\mu} \mathcal{G}[0]\psi \right|_{H^{s-1/2}} \leq C \left(\frac{1}{h_0}, \beta\sqrt{\mu}, |b|_{H^{s+3/2}} \right) \frac{|D^\gamma|}{(1 + \sqrt{\mu}|D^\gamma|)^{1/2}} \psi \Big|_{H^{s+1/2}},$$

showing that $\frac{1}{\mu} \mathcal{G}[0] \cdot$ can be uniformly controlled when μ goes to zero.

The proposition below show that controls in terms of $|\mathfrak{P}u|_2$ or $(u, \frac{1}{\mu\nu} \mathcal{G}[\varepsilon\zeta]u)^{1/2}$ are equivalent. This result can be seen as a version of the Garding inequality for the DN operator.

Proposition 3.4. Let $t_0 > 1$ and $\zeta, b \in H^{t_0+2}(\mathbb{R}^2)$ be such that (2.4) is satisfied, and let σ be given by (2.5), $k[\sigma]$ be as defined in Proposition 2.3 and \mathfrak{P} be given by (1.8). For all $u \in H^{1/2}(\mathbb{R}^2)$, one has

$$\left(u, \frac{1}{\mu\nu} \mathcal{G}[\varepsilon\zeta]u \right) \leq M[\sigma] |\mathfrak{P}u|_2^2 \quad \text{and} \quad k[\sigma]^{-1} |\mathfrak{P}u|_2^2 \lesssim \left(u, \frac{1}{\mu\nu} \mathcal{G}[\varepsilon\zeta]u \right).$$

Proof. The first estimate of the proposition follows directly from (3.3) and Corollary 2.2.

The second estimate is more delicate. Let φ be a smooth function, with compact support in $(-1, 0]$ and such that $\varphi(0) = 1$; define also $v(X, z) = \varphi(z)u^b$ (with u^b defined as in Notation 2.2). Since $v|_{z=-1} = 0$, one can get, after taking the Fourier transform with respect to the horizontal variables,

$$\frac{|\xi^\gamma|^2}{1 + \sqrt{\mu}|\xi^\gamma|} |\widehat{u}(\xi)|^2 \leq 2 \int_{-1}^0 \frac{|\xi^\gamma|^2}{1 + \sqrt{\mu}|\xi^\gamma|} |\widehat{v}(\xi, z)| |\partial_z \widehat{v}(\xi, z)| dz.$$

Remarking that

$$|\widehat{v}| \leq |\varphi|_\infty |\widehat{u}^b| \quad \text{and} \quad |\partial_z \widehat{v}| \leq |\partial_z \varphi|_\infty |\widehat{u}^b| + |\varphi|_\infty |\partial_z \widehat{u}^b|,$$

one gets

$$\begin{aligned} \frac{|\xi^\gamma|^2}{1 + \sqrt{\mu}|\xi^\gamma|} |\widehat{u}(\xi)|^2 &\leq 2|\varphi|_\infty |\partial_z \varphi|_\infty \int_{-1}^0 \frac{|\xi^\gamma|^2}{1 + \sqrt{\mu}|\xi^\gamma|} |\widehat{u}^b(\xi, z)|^2 dz \\ &\quad + 2|\varphi|_\infty^2 \int_{-1}^0 \frac{|\xi^\gamma|^2}{1 + \sqrt{\mu}|\xi^\gamma|} |\widehat{u}^b(\xi, z)| |\partial_z \widehat{u}^b(\xi, z)| dz, \\ &\leq 2|\varphi|_\infty |\partial_z \varphi|_\infty \int_{-1}^0 |\xi^\gamma|^2 |\widehat{u}^b(\xi, z)|^2 dz \\ &\quad + |\varphi|_\infty^2 \int_{-1}^0 \frac{\mu |\xi^\gamma|^4}{(1 + \sqrt{\mu}|\xi^\gamma|)^2} |\widehat{u}^b(\xi, z)|^2 dz + |\varphi|_\infty^2 \int_{-1}^0 \frac{1}{\mu} |\partial_z \widehat{u}^b(\xi, z)|^2 dz, \end{aligned}$$

where Young's inequality has been used to obtain the last line. Remarking now that $\frac{\mu|\xi^\gamma|^4}{(1+\sqrt{\mu}|\xi^\gamma|)^2} \leq |\xi^\gamma|^2$, one has

$$\frac{|\xi^\gamma|^2}{1+\sqrt{\mu}|\xi^\gamma|} |\widehat{u}(\xi)|^2 \lesssim \int_{-1}^0 |\xi^\gamma|^2 |\widehat{u}^b(\xi, z)|^2 dz + \int_{-1}^0 \frac{1}{\mu} |\partial_z \widehat{u}^b(\xi, z)|^2 dz,$$

so that, integrating with respect to ξ , one gets

$$\left| \frac{|D^\gamma|}{(1+\sqrt{\mu}|D^\gamma|)^{1/2}} u \right|_2^2 \lesssim \frac{1}{\mu} \|\nabla^{\mu, \gamma} u^b\|_2^2.$$

Owing to Proposition 2.3 and (3.1) (with $v = u$), one has

$$\|\nabla^{\mu, \gamma} u^b\|_2^2 \lesssim k[\sigma](u, \mathcal{G}[\varepsilon\zeta]u),$$

and the proposition follows. \square

3.2. Commutator estimates

In the following proposition, we show how to control in the energy estimates, the terms involving commutators between the Dirichlet-Neumann operator and spatial or time derivatives and in terms of $\mathcal{E}^s(\cdot)$ rather than Sobolev-type norms.

Proposition 3.5. Let $t_0 > 1$, $s \geq 0$ and $\zeta, b \in H^{t_0+2} \cap H^{s+2}(\mathbb{R}^2)$ be such that (2.4) is satisfied, and let σ be given by (2.5). Then, for all $v \in H^{s+1/2}(\mathbb{R}^2)$,

$$\begin{aligned} |[\Lambda^s, \frac{1}{\mu\nu} \mathcal{G}[\varepsilon\zeta]]v|_2 \leq M[\sigma] & \left(\|\nabla^{\mu, \gamma} \sigma\|_{L^\infty H^{t_0+1}} |\mathfrak{P}v|_{H^s} \right. \\ & \left. + \langle \|\nabla^{\mu, \gamma} \sigma\|_{L^\infty H^{s+1}} |\mathfrak{P}v|_{H^{t_0}} \rangle_{s>t_0} \right), \end{aligned}$$

where $M[\sigma]$ is as in (2.9) while \mathfrak{P} is defined in (1.8).

Proof. First remark that for all $u \in \mathfrak{S}(\mathbb{R}^2)$,

$$(u, [\mathcal{G}[\varepsilon\zeta], \Lambda^s]v) = (u, \mathcal{G}[\varepsilon\zeta]\Lambda^s v) - (\Lambda^s u, \mathcal{G}[\varepsilon\zeta]v).$$

Since $u^\dagger|_{z=0} = u$ and $\Lambda^s u^\dagger|_{z=0} = \Lambda^s u$ (we use here Notation 2.2), it follows from Green's identity that

$$\begin{aligned} (u, [\mathcal{G}[\varepsilon\zeta], \Lambda^s]v) &= \int_{\mathcal{S}} \nabla^{\mu, \gamma} u^\dagger \cdot (1 + Q[\sigma]) \nabla^{\mu, \gamma} (\Lambda^s v)^b \\ &\quad - \int_{\mathcal{S}} (1 + Q[\sigma]) \nabla^{\mu, \gamma} v^b \cdot \nabla^{\mu, \gamma} \Lambda^s u^\dagger \\ &= \int_{\mathcal{S}} \nabla^{\mu, \gamma} u^\dagger \cdot ((1 + Q[\sigma]) \nabla^{\mu, \gamma} ((\Lambda^s v)^b - \Lambda^s v^b) - [\Lambda^s, Q[\sigma]] \nabla^{\mu, \gamma} v^b). \end{aligned} \tag{3.7}$$

Let us now prove the following lemma:

Lemma 3.1. For all $f \in L^2(\mathbb{R}^2)$ and $\mathbf{g} \in H^1(\mathcal{S})^3$, one has

$$\int_{\mathcal{S}} \nabla^{\mu, \gamma} f^\dagger \cdot \mathbf{g} \lesssim \sqrt{\mu} \sqrt{\nu} |f|_2 \|\Lambda \mathbf{g}\|_2.$$

Proof. By definition of f^\dagger , one has

$$\nabla^{\mu, \gamma} f^\dagger = \sqrt{\mu} \begin{pmatrix} \chi(z\sqrt{\mu}|D^\gamma|) \partial_x f \\ \gamma \chi(z\sqrt{\mu}|D^\gamma|) \partial_y f \\ \chi'(z\sqrt{\mu}|D^\gamma|) |D^\gamma| f \end{pmatrix}.$$

Replacing $\nabla^{\mu, \gamma} f^\dagger$ in the integral to control by this expression, and using the self-adjointness of Λ , one gets easily from Proposition 2.2 that

$$\int_{\mathcal{S}} \nabla^{\mu, \gamma} f^\dagger \cdot \mathbf{g} \lesssim \sqrt{\mu} \sqrt{\nu} |\mathfrak{P} \Lambda^{-1} f|_2 \|\Lambda \mathbf{g}\|_2;$$

recalling that $\nu = \frac{1}{1+\sqrt{\mu}}$ and $\gamma \leq 1$, one can check that $|\mathfrak{P} \Lambda^{-1} f|_2 \lesssim |f|_2$, uniformly with respect to μ and γ , and the lemma follows. \square

It is then a simple consequence of the lemma, (3.7) and Proposition 2.3 that

$$\begin{aligned} (u, [\mathcal{G}[\varepsilon\zeta], \Lambda^s] v) &\lesssim \sqrt{\mu} \sqrt{\nu} |u|_2 \left(\|A[\Lambda^s, Q[\sigma]] \nabla^{\mu, \gamma} v^b\|_2 \right. \\ &\quad \left. + (1 + \|\nabla^{\mu, \gamma} \sigma\|_{L^\infty H^{t_0+1}}) \|A \nabla^{\mu, \gamma} ((\Lambda^s v)^b - \Lambda^s v^b)\|_2 \right), \end{aligned} \quad (3.8)$$

which motivates the following lemma:

Lemma 3.2. One has

$$\|A \nabla^{\mu, \gamma} ((\Lambda^s v)^b - \Lambda^s v^b)\|_2 \leq M[\sigma] \|A[\Lambda^s, Q[\sigma]] \nabla^{\mu, \gamma} v^b\|_2.$$

Proof. Just remark that $w := (\Lambda^s v)^b - \Lambda^s v^b$ solves

$$\begin{cases} \nabla^{\mu, \gamma} \cdot (1 + Q[\sigma]) \nabla^{\mu, \gamma} w = \nabla^{\mu, \gamma} \cdot \mathbf{g}, \\ w|_{z=0} = 0, \quad \partial_n w|_{z=-1} = -\mathbf{e}_z \cdot \mathbf{g}|_{z=-1}, \end{cases}$$

with $\mathbf{g} = [A^s, Q[\sigma]] \nabla^{\mu, \gamma} v^b$, and use Proposition 2.4. \square

With the help of the lemma, one deduces from (3.8) that

$$(u, [\mathcal{G}[\varepsilon\zeta], \Lambda^s] v) \leq \sqrt{\mu} \sqrt{\nu} M[\sigma] \|A[\Lambda^s, Q[\sigma]] \nabla^{\mu, \gamma} v^b\|_2 |u|_2,$$

and thus, owing to Corollary 2.1 and Proposition 2.3

$$\begin{aligned} (u, [\mathcal{G}[\varepsilon\zeta], \Lambda^s] v) &\leq \sqrt{\mu} \sqrt{\nu} M[\sigma] |u|_2 \\ &\times \left(\|\nabla^{\mu, \gamma} \sigma\|_{L^\infty H^{t_0+1}} \|\Lambda^s \nabla^{\mu, \gamma} v^b\|_2 + \langle \|\nabla^{\mu, \gamma} \sigma\|_{L^\infty H^{s+1}} \|\Lambda^{t_0} \nabla^{\mu, \gamma} v^b\|_2 \rangle_{s>t_0} \right), \end{aligned}$$

and the result follows therefore from Corollary 2.2 and a duality argument. \square

The next proposition gives control of the commutator between the Dirichlet-Neumann operator and a time derivative.

Proposition 3.6. Let $t_0 > 1$, $T > 0$ and $\zeta, b \in C^1([0, T]; H^{t_0+2}(\mathbb{R}^2))$ be such that (2.4) is satisfied (uniformly with respect to t), and let σ be given by (2.5). Then, for all $u \in C^1([0, T]; H^{1/2}(\mathbb{R}^2))$ and $t \in [0, T]$,

$$\left| \left([\partial_t, \frac{1}{\mu\nu} \mathcal{G}[\varepsilon\zeta]] u(t), u(t) \right) \right| \leq M[\sigma(t)] \|\nabla^{\mu, \gamma} \partial_t \sigma\|_{\infty, T} |\mathfrak{P}u(t)|_2^2,$$

where $M[\sigma(t)]$ is as in (2.9) while \mathfrak{P} is defined in (1.8).

Proof. First remark that

$$(u, [\partial_t, \mathcal{G}[\varepsilon\zeta]]u) = \partial_t(u, \mathcal{G}[\varepsilon\zeta]u) - 2(u, \mathcal{G}[\varepsilon\zeta]\partial_t u),$$

so that using Green's identity, one gets

$$\begin{aligned} (u, [\partial_t, \mathcal{G}[\varepsilon\zeta]]u) &= \partial_t \int_S (1 + Q[\sigma]) \nabla^{\mu, \gamma} u^b \cdot \nabla^{\mu, \gamma} u^b \\ &\quad - 2 \int_S (1 + Q[\sigma]) \nabla^{\mu, \gamma} (\partial_t u)^b \cdot \nabla^{\mu, \gamma} u^b \\ &= \int_S (\partial_t Q[\sigma]) \nabla^{\mu, \gamma} u^b \cdot \nabla^{\mu, \gamma} u^b \\ &\quad - 2 \int_S (1 + Q[\sigma]) \nabla^{\mu, \gamma} ((\partial_t u)^b - \partial_t u^b) \cdot \nabla^{\mu, \gamma} u^b. \end{aligned}$$

It follows directly that

$$\begin{aligned} (u, [\partial_t, \mathcal{G}[\varepsilon\zeta]]u) &\lesssim \|\partial_t Q[\sigma]\|_{\infty} \|\nabla^{\mu, \gamma} u^b\|_2^2 \\ &\quad + (1 + \|Q[\sigma]\|_{\infty}) \|\nabla^{\mu, \gamma} ((\partial_t u)^b - \partial_t u^b)\|_2 \|\nabla^{\mu, \gamma} u^b\|_2. \end{aligned}$$

Proceeding exactly as in the proof of Lemma 3.2, one gets

$$\|\nabla^{\mu, \gamma} ((\partial_t u)^b - \partial_t u^b)\|_2 \lesssim \|\partial_t Q[\sigma]\|_{\infty} \|\nabla^{\mu, \gamma} u^b\|_2,$$

and the result follows therefore from Corollary 2.2 and Proposition 2.3. \square

3.3. Other properties

Propositions 3.2 and 3.4 allow one to control $(u, \mathcal{G}[\varepsilon\zeta]v)$ in general. However, it is sometimes necessary to have more precise estimates, when u and v have some special structure that can be exploited.

Proposition 3.7. Let $t_0 > 1$, $s \geq 0$ and $\zeta, b \in H^{t_0+2} \cap H^{s+2}(\mathbb{R}^2)$ be such that (2.4) is satisfied, and let σ be given by (2.5).

i. For all $\underline{\mathbf{v}} \in H^{s+1} \cap H^{t_0+2}(\mathbb{R}^2)^2$ and $u \in H^{s+1/2}(\mathbb{R}^2)$, one has

$$\begin{aligned} & \left([A^s, \underline{\mathbf{v}}] \cdot \nabla^\gamma u, \frac{1}{\mu\nu} \mathcal{G}[\varepsilon\zeta]([A^s, \underline{\mathbf{v}}] \cdot \nabla^\gamma u) \right)^{1/2} \leq M[\sigma] \\ & \times \left(|\underline{\mathbf{v}}|_{H^{t_0+2}} |\mathfrak{P}u|_{H^s} + \langle |\underline{\mathbf{v}}|_{H^{s+1}} |\mathfrak{P}u|_{H^{t_0+1}} \rangle_{s>t_0+1} \right). \end{aligned}$$

ii. For all $\underline{\mathbf{v}} \in H^{t_0+1}(\mathbb{R}^2)^2$ and $u \in H^{1/2}(\mathbb{R}^2)$, one has

$$\left((\underline{\mathbf{v}} \cdot \nabla^\gamma u), \frac{1}{\mu\nu} \mathcal{G}[\varepsilon\zeta]u \right) \leq M[\sigma] |\underline{\mathbf{v}}|_{W^{1,\infty}} |\mathfrak{P}u|_2^2.$$

Proof. In order to prove the first point of the proposition, define $U = [A^s, \underline{\mathbf{v}}] \cdot \nabla^\gamma u$ and recall that we saw in (3.3) that

$$(U, \mathcal{G}[\varepsilon\zeta]U) = \|(1 + Q[\sigma])^{1/2} \nabla^{\mu,\gamma} U^b\|_2^2.$$

Using Notation 2.2, we define $U^\natural = [A^s, \underline{\mathbf{v}}] \cdot \nabla^\gamma u^\dagger$; as in Remark 2.2 (with U^\natural instead of U^\dagger), one deduces

$$(U, \mathcal{G}[\varepsilon\zeta]U) \leq \|(1 + Q[\sigma])^{1/2} \nabla^{\mu,\gamma} U^\natural\|_2^2.$$

Since $\gamma \leq 1$, one has $\|\nabla^\gamma U^\natural\|_2 \lesssim \|AU^\natural\|_2$ and one gets with Proposition 2.1,

$$\|\nabla^\gamma U^\natural\|_2 \lesssim |\underline{\mathbf{v}}|_{H^{t_0+1}} \|A^s \nabla^\gamma u^\dagger\|_2 + \langle |\underline{\mathbf{v}}|_{H^{s+1}} \|A^{t_0} \nabla^\gamma u^\dagger\|_2 \rangle_{s>t_0};$$

similarly, since $\partial_z U^\natural = [A^s, \underline{\mathbf{v}}] \cdot \nabla^\gamma \partial_z u^\dagger$, one gets

$$\|\partial_z U^\natural\|_2 \lesssim |\underline{\mathbf{v}}|_{H^{t_0+1}} \|A^s \partial_z u^\dagger\|_2 + \langle |\underline{\mathbf{v}}|_{H^s} \|A^{t_0+1} \partial_z u^\dagger\|_2 \rangle_{s>t_0+1},$$

so that, finally,

$$\|\nabla^{\mu,\gamma} U^\natural\|_2 \lesssim |\underline{\mathbf{v}}|_{H^{t_0+2}} \|A^s \nabla^{\mu,\gamma} u^\dagger\|_2 + \langle |\underline{\mathbf{v}}|_{H^{s+1}} \|A^{t_0+1} \nabla^{\mu,\gamma} u^\dagger\|_2 \rangle_{s>t_0+1}.$$

It follows therefore from Proposition 2.2 that

$$\begin{aligned} \|\nabla^{\mu,\gamma} U^\natural\|_2 & \lesssim \sqrt{\mu} |\underline{\mathbf{v}}|_{H^{t_0+2}} \left| \frac{|D^\gamma|}{(1 + \sqrt{\mu}|D^\gamma|)^{1/2}} u \right|_{H^s} \\ & + \langle \sqrt{\mu} |\underline{\mathbf{v}}|_{H^{s+1}} \left| \frac{|D^\gamma|}{(1 + \sqrt{\mu}|D^\gamma|)^{1/2}} u \right|_{H^{t_0+1}} \rangle_{s>t_0+1}, \end{aligned} \quad (3.9)$$

and the result follows.

To establish the second point of the proposition, first remark that, owing to Green's identity,

$$\left((\underline{\mathbf{v}} \cdot \nabla^\gamma u), \mathcal{G}[\varepsilon\zeta]u \right) = \int_{\mathcal{S}} (1 + Q[\sigma]) \nabla^{\mu,\gamma} u^b \cdot \nabla^{\mu,\gamma} (\underline{\mathbf{v}} \cdot \nabla^\gamma u^b), \quad (3.10)$$

so that,

$$\begin{aligned}
((\underline{\mathbf{v}} \cdot \nabla^\gamma u), \mathcal{G}[\varepsilon\zeta]u) &= \int_S (1 + Q[\sigma]) \nabla^{\mu,\gamma} u^b \cdot [\nabla^{\mu,\gamma}, \underline{\mathbf{v}} \cdot \nabla^\gamma] u^b \\
&+ \int_S \nabla^{\mu,\gamma} u^b \cdot [Q[\sigma], (\underline{\mathbf{v}} \cdot \nabla^\gamma)] \nabla^{\mu,\gamma} u^b \\
&+ \int_S \nabla^{\mu,\gamma} u^b \cdot (\underline{\mathbf{v}} \cdot \nabla^\gamma) (1 + Q[\sigma]) \nabla^{\mu,\gamma} u^b. \tag{3.11}
\end{aligned}$$

Integrating by parts, one finds

$$\begin{aligned}
&\int_S \nabla^{\mu,\gamma} u^b \cdot (\underline{\mathbf{v}} \cdot \nabla^\gamma) (1 + Q[\sigma]) \nabla^{\mu,\gamma} u^b \\
&= - \int_S ((\operatorname{div}_\gamma \underline{\mathbf{v}}) + \underline{\mathbf{v}} \cdot \nabla^\gamma) \nabla^{\mu,\gamma} u^b \cdot (1 + Q[\sigma]) \nabla^{\mu,\gamma} u^b \tag{3.12}
\end{aligned}$$

$$\begin{aligned}
&= - \int_S (\operatorname{div}_\gamma \underline{\mathbf{v}}) \nabla^{\mu,\gamma} u^b \cdot (1 + Q[\sigma]) \nabla^{\mu,\gamma} u^b \\
&\quad - \int_S [\underline{\mathbf{v}} \cdot \nabla^\gamma, \nabla^{\mu,\gamma}] u^b \cdot (1 + Q[\sigma]) \nabla^{\mu,\gamma} u^b \\
&\quad - \int_S \nabla^{\mu,\gamma} (\underline{\mathbf{v}} \cdot \nabla^\gamma u^b) \cdot (1 + Q[\sigma]) \nabla^{\mu,\gamma} u^b. \tag{3.13}
\end{aligned}$$

From (3.11), (3.12) and (3.13), one gets therefore

$$\begin{aligned}
((\underline{\mathbf{v}} \cdot \nabla^\gamma u), \mathcal{G}[\varepsilon\zeta]u) &= \int_S (1 + Q[\sigma]) \nabla^{\mu,\gamma} u^b \cdot [\nabla^{\mu,\gamma}, \underline{\mathbf{v}} \cdot \nabla^\gamma] u^b \\
&\quad + \frac{1}{2} \int_S \nabla^{\mu,\gamma} u^b \cdot [Q[\sigma], (\underline{\mathbf{v}} \cdot \nabla^\gamma)] \nabla^{\mu,\gamma} u^b \\
&\quad - \frac{1}{2} \int_S (\operatorname{div}_\gamma \underline{\mathbf{v}}) \nabla^{\mu,\gamma} u^b \cdot (1 + Q[\sigma]) \nabla^{\mu,\gamma} u^b.
\end{aligned}$$

Remarking that $[\nabla^{\mu,\gamma}, \underline{\mathbf{v}} \cdot \nabla^\gamma] = \begin{pmatrix} \nabla^\gamma \underline{\mathbf{v}}_1 \sqrt{\mu} \partial_x + \nabla^\gamma \underline{\mathbf{v}}_2 \gamma \sqrt{\mu} \partial_y \\ 0 \end{pmatrix}$, one deduces easily that

$$((\underline{\mathbf{v}} \cdot \nabla^\gamma u), \mathcal{G}[\varepsilon\zeta]u) \lesssim \|\underline{\mathbf{v}}\|_{W^{1,\infty}} (1 + \|Q[\sigma]\|_{W^{1,\infty}}) \|\nabla^{\mu,\gamma} u^b\|_2^2$$

and the results follows from Corollary 2.2. \square

We finally state the following theorem, which gives an explicit formula for the shape derivative of the Dirichlet-Neumann operator. This theorem is a particular case of Theorem 3.20 of [29].

Theorem 3.1. Let $t_0 > 1$, $s \geq t_0$ and $\underline{\zeta}, b \in H^{s+3/2}(\mathbb{R}^2)$ be such that (2.4) is satisfied. For all $\underline{\psi} \in H^{s+3/2}(\mathbb{R}^2)$, the mapping

$$\zeta \mapsto \mathcal{G}[\varepsilon \underline{\zeta}] \underline{\psi} \in H^{s+1/2}(\mathbb{R}^2)$$

is well defined and differentiable in a neighborhood of $\underline{\zeta}$ in $H^{s+3/2}(\mathbb{R}^2)$, and

$$\forall h \in H^{s+3/2}(\mathbb{R}^2), \quad d_{\underline{\zeta}} \mathcal{G}[\varepsilon \cdot] \underline{\psi} \cdot h = -\varepsilon \mathcal{G}[\varepsilon \underline{\zeta}](h \underline{Z}) - \varepsilon \mu \nabla^\gamma \cdot (h \underline{\mathbf{v}}),$$

with $\underline{Z} := \mathcal{Z}[\varepsilon \underline{\zeta}] \underline{\psi}$ and $\underline{\mathbf{v}} := \nabla^\gamma \underline{\psi} - \varepsilon \underline{Z} \nabla^\gamma \underline{\zeta}$, and where

$$\mathcal{Z}[\varepsilon \underline{\zeta}] := \frac{1}{1 + \varepsilon^2 \mu |\nabla^\gamma \underline{\zeta}|^2} (\mathcal{G}[\varepsilon \underline{\zeta}] + \varepsilon \mu \nabla^\gamma \underline{\zeta} \cdot \nabla^\gamma).$$

Remark 3.4. We take this opportunity to correct a harmless misprint in the statement of Theorem 3.20 of [29]. It should read

$$d_{\underline{a}} G(\cdot, b) f \cdot h = -G(\underline{a}, b)(h \underline{Z}) - \begin{pmatrix} \nabla^X \\ 0 \end{pmatrix} \cdot \left[h P \begin{pmatrix} \underline{\mathbf{v}} \\ \underline{Z} \end{pmatrix} \right],$$

and $\tilde{P}_{\underline{a}}$ should be replaced by P on the right hand side of the equation in the statement of Lemma 3.24.

3.4. Asymptotic expansions

This subsection is devoted to the asymptotic expansion of the DN operator $\mathcal{G}[\varepsilon \underline{\zeta}] \underline{\psi} (= \mathcal{G}_{\mu, \gamma}[\varepsilon \underline{\zeta}, \beta b] \underline{\psi})$ in terms of one or several of the parameters ε , μ , γ and β . We consider two cases which cover all the physical regimes described in the introduction.

3.4.1. Expansions in shallow-water ($\mu \ll 1$) In shallow water, that is when $\mu \ll 1$, the Laplace equation (2.3) –or its straightened version (2.6)– reduces at first order to the ODE $\partial_z^2 \Phi = 0$. This fact can be exploited to find an approximate solution $\tilde{\Phi}_{app}$ of the Laplace equation by a standard BKW expansion. This method has been used in the long-waves regime in [5, 32, 9] (see also [39]) where the corresponding expansions of the DN operator can be found. We prove here that it can be used uniformly with respect to ε and β , which allows one to consider at once the shallow-water/Green-Naghdi and Serre scalings. The difference between both regimes is that $\varepsilon = \beta = 1$ in the former (large amplitude for the surface and bottom variations), while $\varepsilon = \beta = \sqrt{\mu}$ in the latter (medium amplitude variations for the

surface and bottom variations).

Let us first define the first order linear operator $\mathcal{T}[h, b]$ as

$$\mathcal{T}[h, b]V := -\frac{1}{3}\nabla(h^3\nabla \cdot V) + \frac{1}{2}[\nabla(h^2\nabla b \cdot V) - h^2\nabla b \nabla \cdot V] + h\nabla b \nabla b \cdot V. \quad (3.14)$$

Proposition 3.8 (Shallow-water and Serre scalings). Let $\gamma = 1$, $s \geq t_0 > 1$, $\nabla\psi \in H^{s+11/2}(\mathbb{R}^2)$, $b \in H^{s+11/2}(\mathbb{R}^2)$ and $\zeta \in H^{s+9/2}(\mathbb{R}^2)$ and assume that (2.4) is satisfied.

With $h := 1 + \varepsilon\zeta - \beta b$, one then has

$$\begin{aligned} |\mathcal{G}[\varepsilon\zeta]\psi - \nabla \cdot (-\mu h \nabla \psi)|_{H^s} &\leq \mu^2 C_0 \\ |\mathcal{G}[\varepsilon\zeta]\psi - \nabla \cdot (-\mu h \nabla \psi + \mu^2 \mathcal{T}[h, \beta b] \nabla \psi)|_{H^s} &\leq \mu^3 C_1, \end{aligned}$$

with $C_j = C(\frac{1}{h_0}, |\zeta|_{H^{s+5/2+2j}}, |b|_{H^{s+7/2+2j}}, |\nabla^\gamma \psi|_{H^{s+7/2+2j}})$ ($j = 0, 1$), and uniformly with respect to $\varepsilon, \beta \in [0, 1]$.

Proof. We look for an approximate solution ϕ_{app} to the exact solution ϕ of the potential equation (2.6) under the form

$$\phi_{app}(X, z) = \psi(X) + \mu\phi_1(X, z).$$

Plugging this ansatz into (2.6), and expanding the result into powers of μ , one can cancel the leading term by a good choice of ϕ_1 , namely,

$$\phi_1(X, z) = -h\left(h\left(\frac{z^2}{2} + z\right)\Delta\psi - z\beta\nabla b \cdot \nabla\psi\right).$$

One can then check that

$$\begin{cases} \nabla_{X,z} \cdot P[\sigma] \nabla_{X,z} \phi_{app} = \mu^2 R_\mu, & \text{in } \mathcal{S}, \\ \phi_{app}|_{z=0} = \psi, & \partial_n \phi_{app}|_{z=-1} = \mu^2 r_\mu, \end{cases}$$

with (R_μ, r_μ) satisfying, uniformly with respect to $\mu \in (0, 1)$,

$$\|A^{s+1/2} R_\mu\|_2 + |r_\mu|_{H^{s+1/2}} \leq C(|\zeta|_{H^{s+5/2}}, |b|_{H^{s+7/2}}, |\nabla^\gamma \psi|_{H^{s+7/2}}). \quad (3.15)$$

Since $\mathcal{G}[\varepsilon\zeta]\psi - \partial_n \phi_{app}|_{z=0} = \partial_n(\phi - \phi_{app})|_{z=0}$, the truncation error can be estimated using the trace theorem and an elliptic estimate on the BVP solved by $\phi - \phi_{app}$; this is exactly what is done in Theorem 1.6 of [9] for instance, which gives here:

$$|\mathcal{G}[\varepsilon\zeta]\psi - \partial_n \phi_{app}|_{z=0}|_{H^s} \leq \mu^2 C_s (\|A^{s+1/2} R_\mu\|_2 + |r_\mu|_{H^{s+1/2}}),$$

with $C_s = C(|\zeta|_{H^{s+5/2}}, |b|_{H^{s+5/2}})$. Together with (3.15), this gives the result.

In order to prove the second estimate of the proposition, one must look for a higher order approximate solution of (2.6), namely $\phi_{app} = \psi + \mu\phi_1 + \mu^2\phi_2$. The computations can be performed by any software of symbolic calculus and the estimates are exactly the same as above; we thus omit this technical step \square

3.4.2. The case of small amplitude waves ($\varepsilon \ll 1$) Expansions of the Dirichlet-Neumann operator for small amplitude waves has been developed in [17,16]. This method is very efficient to compute the formal expansion, but instead of adapting it in the present case to give uniform estimates on the truncation error, we rather propose a very simple method based on Theorem 3.1.

Proposition 3.9. Let $s \geq t_0 > 1$, $\mathfrak{P}\psi \in H^{s+1/2}(\mathbb{R}^2)$ and $\zeta \in H^{s+3/2}(\mathbb{R}^2)$ be such that (2.4) is satisfied for some $h_0 > 0$. Then one has

$$\begin{aligned} & \left| \mathcal{G}[\varepsilon\zeta]\psi - [\mathcal{G}[0]\psi - \varepsilon\mathcal{G}[0](\zeta(\mathcal{G}[0]\psi)) - \varepsilon\mu\nabla^\gamma \cdot (\zeta\nabla^\gamma\psi)] \right|_{H^s} \\ & \leq \left(\frac{\varepsilon}{\nu}\right)^2 \sqrt{\mu} C \left(\frac{1}{h_0}, \varepsilon\sqrt{\mu}, |\zeta|_{H^{s+3/2}}, |\mathfrak{P}\psi|_{H^{s+1/2}}\right). \end{aligned}$$

Proof. A second order Taylor expansion of $\mathcal{G}[\varepsilon\zeta]\psi$ gives

$$\mathcal{G}[\varepsilon\zeta]\psi = \mathcal{G}[0]\psi + d_0\mathcal{G}[\varepsilon\cdot]\psi \cdot \zeta + \int_0^1 (1-z)d_{z\zeta}^2\mathcal{G}[\varepsilon\cdot]\psi \cdot (\zeta, \zeta)dz.$$

Using Theorem 3.1, one computes

$$d_0\mathcal{G}[\varepsilon\cdot]\psi \cdot \zeta = -\varepsilon\mathcal{G}[0](\zeta(\mathcal{G}[0]\psi)) - \varepsilon\mu\nabla^\gamma \cdot (\zeta\nabla^\gamma\psi),$$

while for all $z \in [-1, 0]$, Proposition 3.3 controls $d_{z\zeta}^2\mathcal{G}[\varepsilon\cdot]\psi \cdot (\zeta, \zeta)$ in H^s by the r.h.s. of the estimate given in the statement. \square

4. Linear analysis

The water-waves equations (1.4) can be written in condensed form as

$$\partial_t U + \mathcal{L}U + \frac{\varepsilon}{\nu}\mathcal{A}[U] = 0,$$

with $U = (\zeta, \psi)^T$, $\mathcal{A}[U] = (\mathcal{A}_1[U], \mathcal{A}_2[U])^T$ and where

$$\mathcal{L} := \begin{pmatrix} 0 & -\frac{1}{\mu\nu}\mathcal{G}[0]\cdot \\ 1 & 0 \end{pmatrix} \quad (4.1)$$

and

$$\begin{aligned} \mathcal{A}_1[U] &= -\frac{1}{\varepsilon\mu}(\mathcal{G}[\varepsilon\zeta]\psi - \mathcal{G}[0]\psi), \\ \mathcal{A}_2[U] &= \frac{1}{2}|\nabla^\gamma\psi|^2 - \frac{(\frac{1}{\sqrt{\mu}}\mathcal{G}[\varepsilon\zeta]\psi + \varepsilon\sqrt{\mu}\nabla^\gamma\zeta \cdot \nabla^\gamma\psi)^2}{2(1 + \varepsilon^2\mu|\nabla^\gamma\zeta|^2)}. \end{aligned} \quad (4.2)$$

By definition, the linearized operator $\mathfrak{L}_{(\underline{\zeta}, \underline{\psi})}$ around some reference state $\underline{U} = (\underline{\zeta}, \underline{\psi})^T$ is given by

$$\mathfrak{L}_{(\underline{\zeta}, \underline{\psi})} = \partial_t + \mathcal{L} + \frac{\varepsilon}{\nu}d_{\underline{U}}\mathcal{A};$$

assuming that \underline{U} is such that the assumptions of Theorem 3.1 are satisfied, one computes that $\mathfrak{L}_{(\zeta, \psi)}$ is equal to

$$\partial_t + \begin{pmatrix} \frac{\varepsilon}{\mu\nu} \mathcal{G}[\varepsilon \underline{\zeta}](\underline{Z}\cdot) + \frac{\varepsilon}{\nu} \nabla^\gamma \cdot (\cdot \underline{\mathbf{v}}) & -\frac{1}{\mu\nu} \mathcal{G}[\varepsilon \underline{\zeta}] \cdot \\ \frac{\varepsilon^2}{\mu\nu} \underline{Z} \mathcal{G}[\varepsilon \underline{\zeta}](\underline{Z}\cdot) + (1 + \frac{\varepsilon^2}{\nu} \underline{Z} \nabla^\gamma \cdot \underline{\mathbf{v}}) & \frac{\varepsilon}{\nu} \underline{\mathbf{v}} \cdot \nabla^\gamma \cdot - \frac{\varepsilon}{\mu\nu} \underline{Z} \mathcal{G}[\varepsilon \underline{\zeta}] \cdot \end{pmatrix}, \quad (4.3)$$

where $\underline{\mathbf{v}}$ and \underline{Z} are as in the statement of Theorem 3.1.

This section is devoted to the proof of energy estimates for the associated initial value problem,

$$\begin{cases} \mathfrak{L}_{(\zeta, \psi)} U = \frac{\varepsilon}{\nu} G \\ U|_{t=0} = U^0. \end{cases} \quad (4.4)$$

Defining

$$\underline{\mathbf{a}} = 1 + \frac{\varepsilon}{\nu} \underline{\mathbf{b}}, \quad \text{and} \quad \underline{\mathbf{b}} = \varepsilon \underline{\mathbf{v}} \cdot \nabla^\gamma \underline{Z} + \nu \partial_t \underline{Z}, \quad (4.5)$$

we first introduce the notion of *admissible* reference state:

Definition 4.1. Let $t_0 > 1$, $T > 0$ and $b \in H^{t_0+2}(\mathbb{R}^2)$. We say that $\underline{U} = (\underline{\zeta}, \underline{\psi})$ is *admissible* on $[0, \frac{\nu T}{\varepsilon}]$ if

- The surface and bottom parameterizations $\underline{\zeta}$ and b satisfy (2.4) for some $h_0 > 0$, uniformly on $[0, \frac{\nu T}{\varepsilon}]$;
- There exists $c_0 > 0$ such that $\underline{\mathbf{a}} \geq c_0$, uniformly on $[0, \frac{\nu T}{\varepsilon}]$.

We also need to define some functional spaces and notations linked to the energy (1.6) mentioned in the introduction.

Definition 4.2. For all $s \in \mathbb{R}$ and $T > 0$,

i. We denote by X^s the vector space $H^s(\mathbb{R}^2) \times H^{s+1/2}(\mathbb{R}^2)$ endowed with the norm

$$\forall U = (\zeta, \psi)^T \in X^s, \quad |U|_{X^s} := |\zeta|_{H^s} + \frac{\varepsilon}{\nu} |\psi|_{H^s} + |\mathfrak{P}\psi|_{H^s},$$

while X_T^s stands for $C([0, \frac{\nu T}{\varepsilon}]; X^s)$ endowed with its canonical norm.

ii. We define the space \tilde{X}^s as

$$\tilde{X}^s := \{U = (\zeta, \psi)^T, \zeta \in H^s(\mathbb{R}^2), \nabla \psi \in H^{s-1/2}(\mathbb{R}^2)^2\},$$

and endow it with the semi-norm $|U|_{\tilde{X}^s} := |\zeta|_{H^s} + |\mathfrak{P}\psi|_{H^s}$.

iii. We define the semi-normed space $(Y_T^s, |\cdot|_{Y_T^s})$ as

$$Y_T^s := \bigcap_{k=0}^2 C^k([0, \frac{\nu T}{\varepsilon}]; \tilde{X}^{s-\frac{3}{2}k}) \quad \text{and} \quad |U|_{Y_T^s} = \sum_{k=0}^2 \sup_{[0, \frac{\nu T}{\varepsilon}]} |\partial_t^k U|_{\tilde{X}^{s-\frac{3}{2}k}}.$$

iv. For all $(G, U^0) \in X_T^s \times X^s$, we define

$$\mathcal{I}^s(t, U^0, G) := |U^0|_{X^s} + \frac{\varepsilon}{\nu} \int_0^t \sup_{0 \leq t'' \leq t'} |G(t'')|_{X^s} dt'.$$

We can now state the energy estimate associated to (4.4), and whose proof is given in the next two subsections.

Proposition 4.1. Let $s \geq t_0 > 1$, $T > 0$, $b \in H^{s+9/2}(\mathbb{R}^2)$, and $\underline{U} = (\underline{\zeta}, \underline{\psi}) \in Y_T^{s+9/2}$ be admissible on $[0, \frac{\nu T}{\varepsilon}]$ for some $h_0 > 0$ and $c_0 > 0$. Let also $(G, U^0) \in X_T^{s+2} \times X^{s+2}$.

There exists a unique solution $U \in X_T^s$ to (4.4); moreover, for all $0 \leq t \leq \frac{\nu T}{\varepsilon}$, one has

$$|U(t)|_{X^s} \leq \underline{C}(\mathcal{I}^{s+2}(t, U^0, G) + |\underline{U}|_{Y_T^{s+9/2}} \mathcal{I}^{t_0+2}(t, U^0, G)),$$

where $\underline{C} = C(T, \frac{1}{h_0}, \frac{1}{c_0}, \frac{\varepsilon}{\nu}, \frac{\beta}{\varepsilon}, |b|_{H^{s+9/2}}, |\underline{U}|_{Y_T^{t_0+9/2}})$.

4.1. Energy estimates for the trigonalized linearized operator

As shown in [29], the operator $\mathfrak{L}_{(\underline{\zeta}, \underline{\psi})}$ is non-strictly hyperbolic, in the sense that its principal symbol has a double purely imaginary eigenvalue, with a nontrivial Jordan block. It was shown in Prop. 4.2 of [29] that a simple change of basis can be used to put the principal symbol of $\mathfrak{L}_{(\underline{\zeta}, \underline{\psi})}$ under a canonical trigonal form. This result is generalized to the present case. More precisely, with $\underline{\mathbf{a}}$ as defined in (4.5) and defining the operator $\mathfrak{M}_{(\underline{\zeta}, \underline{\psi})} = \partial_t + M_{(\underline{\zeta}, \underline{\psi})}$ with

$$M_{(\underline{\zeta}, \underline{\psi})} = \begin{pmatrix} \frac{\varepsilon}{\nu} \nabla^\gamma \cdot (\cdot \underline{\mathbf{v}}) - \frac{1}{\nu} \mathcal{G}[\varepsilon \underline{\zeta}] \cdot & \\ \underline{\mathbf{a}} & \frac{\varepsilon}{\nu} \underline{\mathbf{v}} \cdot \nabla^\gamma \cdot \end{pmatrix}, \quad (4.6)$$

one reduces the study of (4.4) to the study of the initial value problem

$$\begin{cases} \mathfrak{M}_{(\underline{\zeta}, \underline{\psi})} V = \frac{\varepsilon}{\nu} H \\ V|_{t=0} = V^0, \end{cases} \quad (4.7)$$

as shown in the following proposition (whose proof relies on simple computations and is omitted).

Proposition 4.2. The following two assertions are equivalent:

- The pair $U = (\zeta, \psi)^T$ solves (4.4);
- The pair $V = (\zeta, \psi - \varepsilon \underline{Z} \zeta)^T$ solves (4.7), with $H = (G_1, G_2 - \varepsilon \underline{Z} G_1)^T$ and $V^0 = (\zeta^0, \psi^0 - \varepsilon \underline{Z}|_{t=0} \zeta^0)^T$.

In view of this proposition, it is a key step to understand (4.7), and the rest of this subsection is thus devoted to the proof of energy estimates for this initial value problem.

First remark that a symmetrizer for $\mathfrak{M}_{(\underline{\zeta}, \underline{\psi})}$ is given by

$$S = \begin{pmatrix} \underline{\mathbf{a}} & 0 \\ 0 & \frac{\varepsilon^2}{\nu^2} + \frac{1}{\mu\nu} \mathcal{G}[\varepsilon \underline{\zeta}] \cdot \end{pmatrix}, \quad (4.8)$$

so that (provided that $\underline{\mathbf{a}}$ is nonnegative), a natural energy for the IVP (4.7) is given by

$$\begin{aligned} E^s(V)^2 &= (\Lambda^s V, S \Lambda^s V) \\ &= |\sqrt{\underline{\mathbf{a}}} \Lambda^s V_1|_2^2 + \frac{\varepsilon^2}{\nu^2} |V_2|_{H^s}^2 + (\Lambda^s V_2, \frac{1}{\mu\nu} \mathcal{G}[\varepsilon \underline{\zeta}] \Lambda^s V_2). \end{aligned} \quad (4.9)$$

Remark 4.1. The introduction of the term ε^2/ν^2 in (4.8) –and thus of $\varepsilon^2/\nu^2 |V_2|_{H^s}^2$ in (4.9)– is not necessary to the energy estimate below. But this constant term plays a crucial role in the iterative scheme used to solve the nonlinear problem because it controls the low frequencies. It also turns out that the order $O(\varepsilon^2/\nu^2)$ of this constant term is the only one which allows uniform estimates.

We can now give the energy estimate associated to (4.7); in the statement below, we use the notation

$$I^s(t, V^0, H) := E^s(V^0) + \frac{\varepsilon}{\nu} \int_0^t \sup_{0 \leq t'' \leq t'} E^s(H(t'')) dt',$$

while $s \vee t_0 := \max\{s, t_0\}$ and $\underline{\mathcal{C}}$ is as defined in Proposition 4.1.

Proposition 4.3. Let $s \geq 0$, $t_0 > 1$, $T > 0$, $b \in H^{s \vee t_0 + 9/2}(\mathbb{R}^2)$, and $\underline{U} = (\underline{\zeta}, \underline{\psi}) \in Y_T^{s \vee t_0 + 9/2}$ be admissible on $[0, \frac{\nu T}{\varepsilon}]$ for some $h_0 > 0$ and $c_0 > 0$.

Then, for all $(H, V^0) \in X_T^s \times X^s$, there exists a unique solution $V \in X_T^s$ to (4.7) and for all $0 \leq t \leq \frac{\nu T}{\varepsilon}$,

$$E^s(V(t)) \leq \underline{\mathcal{C}}(I^s(t, V^0, H) + \langle \underline{U} \rangle_{Y_T^{s+7/2}} I^{t_0+1}(t, V^0, H) \rangle_{s > t_0+1}).$$

Proof. Throughout this proof, $\underline{\mathcal{C}}_0$ denotes a nondecreasing function of $\frac{1}{c_0}$, $\frac{\varepsilon}{\nu}$, $M[\underline{\sigma}]$, $|\underline{\mathbf{v}}|_{H^{t_0+2}}$, $|\underline{\mathbf{h}}|_{H^{t_0+2}}$, and $|\partial_t \underline{\mathbf{h}}|_\infty$ which may vary from one line to another, and $\underline{\sigma}$ is given by (2.5) with $\zeta = \underline{\zeta}$.

Existence of a solution to the IVP (4.7) is achieved by classical means,

and we thus focus our attention on the proof of the energy estimate. For any given $\kappa \in \mathbb{R}$, we compute

$$\begin{aligned} e^{\frac{\varepsilon\kappa}{\nu}t} \frac{d}{dt} (e^{-\frac{\varepsilon\kappa}{\nu}t} E^s(V)^2) &= -\frac{\varepsilon\kappa}{\nu} E^s(V)^2 + 2\frac{\varepsilon}{\nu} (\Lambda^s H, S\Lambda^s V) \\ &\quad - 2(\Lambda^s M_{(\underline{\zeta}, \underline{\psi})} V, S\Lambda^s V) + (\Lambda^s V, [\partial_t, S]\Lambda^s V). \end{aligned} \quad (4.10)$$

We now turn to bound from above the different components of the r.h.s. of (4.10).

- Estimate of $(\Lambda^s H, S\Lambda^s V)$. We can rewrite this term as

$$(\sqrt{\underline{\mathbf{a}}}\Lambda^s H_1, \sqrt{\underline{\mathbf{a}}}\Lambda^s V_1) + \left(\frac{\varepsilon}{\nu}\Lambda^s H_2, \frac{\varepsilon}{\nu}\Lambda^s V_2\right) + (\Lambda^s H_2, \frac{1}{\mu\nu}\mathcal{G}[\varepsilon\underline{\zeta}]\Lambda^s V_2),$$

so that Cauchy-Schwartz inequality and Proposition 3.2 yield

$$(\Lambda^s H, S\Lambda^s V) \leq E^s(H)E^s(V). \quad (4.11)$$

- Estimate of $(\Lambda^s M_{(\underline{\zeta}, \underline{\psi})} V, S\Lambda^s V)$. One computes

$$\begin{aligned} (\Lambda^s M_{(\underline{\zeta}, \underline{\psi})} V, S\Lambda^s V) &= \left(\Lambda^s \left(\frac{\varepsilon}{\nu} \operatorname{div}_\gamma(\underline{\mathbf{v}}V_1) - \frac{1}{\mu\nu}\mathcal{G}[\varepsilon\underline{\zeta}]V_2\right), \underline{\mathbf{a}}\Lambda^s V_1\right) \\ &\quad + \left(\Lambda^s (\underline{\mathbf{a}}V_1 + \frac{\varepsilon}{\nu}\underline{\mathbf{v}} \cdot \nabla^\gamma V_2), \left(\frac{\varepsilon^2}{\nu^2} + \frac{1}{\mu\nu}\mathcal{G}[\varepsilon\underline{\zeta}]\right)\Lambda^s V_2\right), \end{aligned}$$

so that one can write

$$(\Lambda^s M_{(\underline{\zeta}, \underline{\psi})} V, S\Lambda^s V) = A_1 + A_2 + A_3 + A_4 + A_5,$$

with

$$\begin{aligned} A_1 &= \frac{\varepsilon}{\nu} (\Lambda^s \operatorname{div}_\gamma(\underline{\mathbf{v}}V_1), \underline{\mathbf{a}}\Lambda^s V_1), \\ A_2 &= \frac{\varepsilon}{\nu} (\Lambda^s (\underline{\mathbf{v}} \cdot \nabla^\gamma V_2), \frac{\varepsilon^2}{\nu^2}\Lambda^s V_2), \\ A_3 &= \frac{\varepsilon}{\nu} (\Lambda^s (\underline{\mathbf{v}} \cdot \nabla^\gamma V_2), \frac{1}{\mu\nu}\mathcal{G}[\varepsilon\underline{\zeta}]\Lambda^s V_2), \\ A_4 &= (\Lambda^s (\underline{\mathbf{a}}V_1), \frac{1}{\mu\nu}\mathcal{G}[\varepsilon\underline{\zeta}]\Lambda^s V_2) - (\underline{\mathbf{a}}\Lambda^s V_1, \Lambda^s (\frac{1}{\mu\nu}\mathcal{G}[\varepsilon\underline{\zeta}]V_2)), \\ A_5 &= (\Lambda^s (\underline{\mathbf{a}}V_1), \frac{\varepsilon^2}{\nu^2}\Lambda^s V_2). \end{aligned}$$

We now turn to prove the following estimates:

$$\begin{aligned} A_j &\leq \frac{\varepsilon}{\nu} \underline{C}_0 E^s(V) \left(\left(1 + \frac{\nu}{\varepsilon} \|\nabla^{\mu, \gamma} \underline{\sigma}\|_{L^\infty H^{t_0+2}}\right) E^s(V) \right. \\ &\quad \left. + \left\langle (|\underline{\mathbf{v}}|_{H^{s+1}} + |\underline{\mathbf{b}}|_{H^{s+1}} + \frac{\nu}{\varepsilon} \|\nabla^{\mu, \gamma} \underline{\sigma}\|_{L^\infty H^{s+1}}) E^{t_0+1}(V) \right\rangle_{s > t_0+1} \right), \end{aligned} \quad (4.12)$$

for $j = 1, \dots, 5$.

– Control of A_1 and A_2 . Integrating by parts, one obtains

$$\begin{aligned} \frac{\nu}{\varepsilon} A_1 &= ([A^s, \operatorname{div}_\gamma(\underline{\mathbf{v}})] V_1, \underline{\mathbf{a}} A^s V_1) - \frac{1}{2} (A^s V_1, (\underline{\mathbf{v}} \cdot \nabla^\gamma \underline{\mathbf{a}}) A^s V_1) \\ &\quad + \frac{1}{2} (\underline{\mathbf{a}} A^s V_1, (\operatorname{div}_\gamma \underline{\mathbf{v}}) A^s V_1) \end{aligned}$$

and

$$\frac{\nu}{\varepsilon} A_2 = ([A^s, \underline{\mathbf{v}}] \nabla^\gamma V_2, \frac{\varepsilon^2}{\nu^2} A^s V_2) - \frac{\varepsilon^2}{2\nu^2} (A^s V_2, (\operatorname{div}_\gamma \underline{\mathbf{v}}) A^s V_2).$$

Recalling that $\underline{\mathbf{a}} = 1 + \frac{\varepsilon}{\nu} \underline{\mathbf{h}}$, one can then deduce easily (with the help of Proposition 2.1 and Corollary 2.1 to control the commutators in the above expressions) that (4.12) holds for $j = 1, 2$.

– Control of A_3 . First write $A_3 = A_{31} + A_{32}$ with

$$\begin{aligned} A_{31} &= \frac{\varepsilon}{\nu} \left(\frac{1}{\mu\nu} \mathcal{G}[\varepsilon \underline{\zeta}] A^s V_2, [A^s, \underline{\mathbf{v}}] \cdot \nabla^\gamma V_2 \right) \\ A_{32} &= \frac{\varepsilon}{\nu} \left(\underline{\mathbf{v}} \cdot \nabla^\gamma A^s V_2, \frac{1}{\mu\nu} \mathcal{G}[\varepsilon \underline{\zeta}] A^s V_2 \right). \end{aligned}$$

Thanks to Proposition 3.2, one gets

$$A_{31} \leq \frac{\varepsilon}{\nu} \left(\frac{1}{\mu\nu} \mathcal{G}[\varepsilon \underline{\zeta}] [A^s, \underline{\mathbf{v}}] \cdot \nabla^\gamma V_2, [A^s, \underline{\mathbf{v}}] \cdot \nabla^\gamma V_2 \right)^{1/2} E^s(V),$$

and Propositions 3.7.i and 3.4 can then be used to show that A_{31} is bounded from above by the r.h.s. of (4.12). This is also the case of A_{32} , as a direct consequence of Propositions 3.7.ii and 3.4. It follows that (4.12) holds for $j = 3$.

– Control of A_4 . One computes, remarking that $[A^s, \underline{\mathbf{a}}] = \frac{\varepsilon}{\nu} [A^s, \underline{\mathbf{h}}]$,

$$\begin{aligned} A_4 &= (\underline{\mathbf{a}} A^s V_1, [\frac{1}{\mu\nu} \mathcal{G}[\varepsilon \underline{\zeta}], A^s] V_2) + \frac{\varepsilon}{\nu} ([A^s, \underline{\mathbf{h}}] V_1, \frac{1}{\mu\nu} \mathcal{G}[\varepsilon \underline{\zeta}] A^s V_2) \\ &:= A_{41} + A_{42}. \end{aligned}$$

Using successively Cauchy-Schwartz inequality, Proposition 3.5, and Proposition 3.4, one obtains directly that A_{41} is bounded from above by the r.h.s. of (4.12). In order to control A_{42} , first remark that using Propositions 3.2 and 3.4, one gets

$$\begin{aligned} \frac{\nu}{\varepsilon} A_{42} &\leq ([A^s, \underline{\mathbf{h}}] V_1, \frac{1}{\mu\nu} \mathcal{G}[\varepsilon \underline{\zeta}] [A^s, \underline{\mathbf{h}}] V_1)^{1/2} E^s(V) \\ &\leq M[\underline{\sigma}] |\mathfrak{P}[A^s, \underline{\mathbf{h}}] V_1|_2 E^s(V). \end{aligned}$$

Recalling that $\nu = \frac{1}{1+\sqrt{\mu}}$ one can check that for all $\xi \in \mathbb{R}^2$, $\frac{\nu^{-1/2}|\xi^\gamma|}{(1+\sqrt{\mu}|\xi^\gamma|)^{1/2}} \lesssim \langle \xi \rangle$, uniformly with respect to μ and γ , so that one deduces

$$\frac{\nu}{\varepsilon} A_{42} \leq M[\underline{\sigma}] |[A^s, \underline{\mathbf{b}}]V_1|_{H^1} E^s(V).$$

Remarking that owing to Proposition 2.1, one has

$$\begin{aligned} |[A^s, \underline{\mathbf{b}}]V_1|_{H^1} &\leq |\underline{\mathbf{b}}|_{H^{t_0+1}} |V_1|_{H^s} + \langle |\underline{\mathbf{b}}|_{H^{s+1}} |V_1|_{H^{t_0}} \rangle_{s>t_0} \\ &\leq \frac{1}{\sqrt{C_0}} (|\underline{\mathbf{b}}|_{H^{t_0+2}} E^s(V) + \langle |\underline{\mathbf{b}}|_{H^{s+1}} E^{t_0+1}(V) \rangle_{s>t_0+1}), \end{aligned}$$

and A_{42} is thus bounded from above by the r.h.s. of (4.12). This shows that (4.12) also holds for $j = 4$.

– Control of A_5 . First remark that

$$A_5 = (A^s V_1, \frac{\varepsilon^2}{\nu^2} A^s V_2) + \frac{\varepsilon}{\nu} (A^s(\underline{\mathbf{b}}V_1), \frac{\varepsilon^2}{\nu^2} A^s V_2),$$

so that Cauchy-Schwartz inequality and the tame product estimate (2.1) yield

$$\begin{aligned} A_5 &\leq \frac{\varepsilon}{\nu} ((1 + |\frac{\varepsilon}{\nu} \underline{\mathbf{b}}|_{H^{t_0}}) |V_1|_{H^s} + \langle |\frac{\varepsilon}{\nu} \underline{\mathbf{b}}|_{H^s} |V_1|_{H^{t_0}} \rangle_{s>t_0}) \frac{\varepsilon}{\nu} |V_2|_{H^s} \\ &\leq \frac{\varepsilon}{\nu} \frac{1}{\sqrt{C_0}} ((1 + |\frac{\varepsilon}{\nu} \underline{\mathbf{b}}|_{H^{t_0+1}}) E^s(V) + \langle |\frac{\varepsilon}{\nu} \underline{\mathbf{b}}|_{H^s} E^{t_0+1}(V) \rangle_{s>t_0+1}) E^s(V), \end{aligned}$$

and (4.12) thus holds for $j = 5$.

From (4.12), we obtain directly

$$\begin{aligned} (A^s M_{(\underline{\zeta}, \underline{\psi})} V, S A^s V) &\leq \frac{\varepsilon}{\nu} \underline{C}_0 E^s(V) ((1 + \frac{\nu}{\varepsilon} \|\nabla^{\mu, \gamma} \underline{\sigma}\|_{L^\infty H^{t_0+2}}) E^s(V) \\ &\quad + \langle (|\underline{\mathbf{v}}|_{H^{s+1}} + |\underline{\mathbf{b}}|_{H^{s+1}} + \frac{\nu}{\varepsilon} \|\nabla^{\mu, \gamma} \underline{\sigma}\|_{L^\infty H^{s+1}}) E^{t_0+1}(V) \rangle_{s>t_0+1}). \end{aligned} \quad (4.13)$$

• Estimate of $(A^s V, [\partial_t, S] A^s V)$. One has

$$(A^s V, [\partial_t, S] A^s V) = \frac{\varepsilon}{\nu} (A^s V_1, \partial_t \underline{\mathbf{b}} A^s V_1) + (A^s V_2, [\partial_t, \frac{1}{\mu\nu} \mathcal{G}[\varepsilon \underline{\zeta}]] A^s V_2),$$

so that, using Proposition 3.6 to control the second component of the r.h.s., one gets easily

$$(A^s V, [\partial_t, S] A^s V) \leq \frac{\varepsilon}{\nu} \underline{C}_0 (1 + \frac{\nu}{\varepsilon} \|\nabla^{\mu, \gamma} \partial_t \underline{\sigma}\|_\infty) E^s(V)^2. \quad (4.14)$$

According to (4.10), (4.11), (4.13) and (4.14), we have

$$e^{\frac{\varepsilon\kappa}{\nu}t} \frac{d}{dt} (e^{-\frac{\varepsilon\kappa}{\nu}t} E^s(V)^2) \leq \frac{\varepsilon}{\nu} E^s(V) (2E^s(H) + \underline{C}_0 \langle D_s E^{t_0+1}(V) \rangle_{s>t_0+1}), \quad (4.15)$$

with $D_s := (|\underline{\mathbf{v}}|_{H^{s+1}} + \frac{\nu}{\varepsilon} \|\nabla^{\mu,\gamma} \underline{\sigma}\|_{L^\infty H^{s+1}} + |\underline{\mathbf{b}}|_{H^{s+1}})$, provided that κ is large enough, how large depending only on

$$\sup_{t \in [0, \frac{\nu T}{\varepsilon}]} \left[\underline{C}_0(t) \left(1 + \frac{\nu}{\varepsilon} \|\nabla^{\mu,\gamma} \underline{\sigma}(t)\|_{L^\infty H^{t_0+2}} + \frac{\nu}{\varepsilon} \|\nabla^{\mu,\gamma} \partial_t \underline{\sigma}(t)\|_\infty \right) \right].$$

It follows from (4.15) that,

$$\begin{aligned} E^s(V(t)) &\leq e^{\frac{\varepsilon\kappa}{\nu}t} E^s(V^0) + \frac{\varepsilon}{\nu} \int_0^t e^{\frac{\varepsilon\kappa}{\nu}(t-t')} E^s(H(t')) dt' \\ &\quad + \left\langle \frac{\varepsilon}{\nu} \underline{C}_0 \left(\sup_{[0, \nu T/\varepsilon]} D_s \right) \int_0^t e^{\frac{\varepsilon\kappa}{\nu}(t-t')} E^{t_0+1}(V(t')) dt' \right\rangle_{s>t_0+1}; \end{aligned} \quad (4.16)$$

using (4.16) with $s = t_0 + 1$ gives

$$E^{t_0+1}(V(t)) \leq e^{\frac{\varepsilon\kappa}{\nu}t} E^{t_0+1}(V^0) + \frac{\varepsilon}{\nu} t e^{\frac{\varepsilon\kappa}{\nu}t} \sup_{0 \leq t' \leq t} E^{t_0+1}(H(t')),$$

and plugging this expression back into (4.16) gives therefore

$$E^s(V(t)) \leq \underline{C}_1 (I^s(t, V^0, H) + \langle (\sup_{t \in [0, \nu T/\varepsilon]} D_s) I^{t_0+1}(t, V^0, H) \rangle_{s>t_0+1}),$$

where \underline{C}_1 is a nondecreasing function of $T, \frac{1}{c_0}, \frac{1}{h_0}, \frac{\varepsilon}{\nu}$ and of the supremum on the time interval $[0, \frac{\nu T}{\varepsilon}]$ of $\frac{\nu}{\varepsilon} \|\nabla^{\mu,\gamma} \underline{\sigma}\|_{L^\infty H^{t_0+2}}, \frac{\nu}{\varepsilon} \|\nabla^{\mu,\gamma} \partial_t \underline{\sigma}\|_\infty, |\underline{\mathbf{v}}|_{H^{t_0+2}}, |\underline{\mathbf{b}}|_{H^{t_0+2}}$ and $|\partial_t \underline{\mathbf{b}}|_{L^\infty}$. The proposition follows therefore from the following lemma:

Lemma 4.1. With \underline{C} and $|\cdot|_{Y_T^s}$ as defined in the statement of Proposition 4.1 and Definition 4.2, one has,

$$\forall s \geq t_0 + 1, \quad \sup_{t \in [0, \nu T/\varepsilon]} D_s(t) \leq \underline{C} |\underline{U}|_{Y_T^{s+7/2}} \quad \text{and} \quad \underline{C}_1 \leq \underline{C}.$$

Proof. Remark first that, as a consequence of Proposition 3.3, one has for all $r \geq t_0 + 1$,

$$\left| \frac{1}{\sqrt{\mu}} \mathcal{G}[\varepsilon \underline{\zeta}] \underline{\psi} \right|_{H^r} \leq C \left(\frac{1}{h_0}, \frac{1}{c_0}, \frac{\varepsilon}{\nu}, \frac{\beta}{\varepsilon}, |\underline{U}|_{\tilde{X}^{t_0+2}}, |b|_{H^{r+3/2}} \right) |\underline{U}|_{\tilde{X}^{r+3/2}}; \quad (4.17)$$

since $|\xi^\gamma| \leq \frac{\nu^{-1/2}|\xi^\gamma|(1+|\xi|)^{1/2}}{(1+\sqrt{\mu}|\xi^\gamma|)^{1/2}}$, uniformly with respect to $\gamma \in (0, 1]$ and $\mu > 0$, one also has

$$|\nabla^\gamma \underline{\psi}|_{H^r} \leq |\mathfrak{P}\underline{\psi}|_{H^{r+1/2}} \leq |\underline{U}|_{\tilde{X}^{r+1/2}}. \quad (4.18)$$

It follows from the explicit expression of \underline{Z} given in Theorem 3.1 that $\varepsilon \underline{Z}$ is a smooth function of $\varepsilon \sqrt{\mu} \leq \varepsilon/\nu$, $\nabla^\gamma \underline{\zeta}$, $\nabla^\gamma \underline{\psi}$ and $\frac{1}{\sqrt{\mu}} \mathcal{G}[\varepsilon \underline{\zeta}] \underline{\psi}$. Moser's type estimates then imply that for all $r \geq t_0 + 1$, $|\varepsilon \underline{Z}|_{H^r}$ – and hence $|\underline{\mathbf{v}}|_{H^{s+1}}$ – is bounded from above by $\underline{C} |\underline{U}|_{Y_T^{s+7/2}}$. This is also the case of the second component of D_s , as a direct consequence of (2.5), and because $\nu \sqrt{\mu} \leq 1$.

To control the third component of D_s , namely, $\sup_{[0, \frac{\nu T}{\varepsilon}]} |\underline{\mathbf{b}}|_{H^{s+1}}$ (with $\underline{\mathbf{b}}$ given by (4.5)), we need to bound $|\nu \partial_t \underline{Z}|_{H^{s+1}}$ from above. Using Theorem 3.1 to compute explicitly $\partial_t \underline{Z}$, one finds

$$\begin{aligned} \nu \partial_t \underline{Z} &= \frac{\sqrt{\mu} \nu}{1 + \varepsilon^2 \mu |\nabla^\gamma \underline{\zeta}|^2} \left(\frac{1}{\sqrt{\mu}} \mathcal{G}[\varepsilon \underline{\zeta}] \partial_t \underline{\psi} + \varepsilon \sqrt{\mu} \nabla^\gamma \underline{\zeta} \cdot \nabla^\gamma \partial_t \underline{\psi} \right. \\ &\quad \left. - \varepsilon \sqrt{\mu} \nabla^\gamma \partial_t \underline{\zeta} \cdot (\varepsilon \underline{Z}) \nabla^\gamma \underline{\zeta} - \varepsilon \sqrt{\mu} \partial_t \underline{\zeta} \operatorname{div}_{\gamma} \underline{\mathbf{v}} - \frac{1}{\sqrt{\mu}} \mathcal{G}[\varepsilon \underline{\zeta}] (\partial_t \underline{\zeta} (\varepsilon \underline{Z})) \right), \end{aligned}$$

which is a smooth function of $\varepsilon \sqrt{\mu} \leq \varepsilon/\nu$, $\nabla^\gamma \underline{\zeta}$, $\partial_t \underline{\zeta}$, $\nabla^\gamma \partial_t \underline{\zeta}$, $\nabla^\gamma \partial_t \underline{\psi}$, $\varepsilon \underline{Z}$, $\frac{1}{\sqrt{\mu}} \mathcal{G}[\varepsilon \underline{\zeta}] \partial_t \underline{\psi}$ and $\frac{1}{\sqrt{\mu}} \mathcal{G}[\varepsilon \underline{\zeta}] (\partial_t \underline{\zeta} (\varepsilon \underline{Z}))$. The sought after estimate on $|\underline{\mathbf{b}}|_{H^{s+1}}$ thus follows from Moser's type estimates (note that Remark 3.2 is used to control $\frac{1}{\sqrt{\mu}} \mathcal{G}[\varepsilon \underline{\zeta}] (\partial_t \underline{\zeta} (\varepsilon \underline{Z}))$ in terms of Sobolev norms of $\partial_t \underline{\zeta}$ and $\varepsilon \underline{Z}$).

The estimate on \underline{C}_1 is obtained exactly in the same way and we omit the proof. \square

\square

4.2. Proof of Proposition 4.1

Deducing Proposition 4.1 from Proposition 4.3 is only a technical step, essentially based on the equivalence of the norms E^s and $|\cdot|_{X^s}$ stemming from Proposition 3.4. We only give the main steps of the proof.

Step 1. Since $U = (V_1, V_2 + \varepsilon \underline{Z} V_1)$, one can expand $|U|_{X^s}$ in terms of V_1 and V_2 and control the different components using the norm E^s to obtain:

$$|U|_{X^s} \leq \underline{C} \times (E^{s+1}(V) + \langle |\underline{U}|_{\tilde{X}^{s+5/2}} E^{t_0+1}(V) \rangle_{s>t_0}). \quad (4.19)$$

Step 2. Using Proposition 4.3 to control $E^{s+1}(V)$ and $E^{t_0+1}(V)$ in terms of $V^0 = (U_1^0, U_2^0 - \varepsilon \underline{Z}|_{t=0} U_1^0)$ and $H = (G_1, G_2 - \varepsilon \underline{Z} G_1)$ in (4.19), one gets

$$|U(t)|_{X^s} \leq \underline{C} (\mathcal{I}^{s+1}(t, V^0, H) + \langle |\underline{U}|_{Y_T^{s+9/2}} (\mathcal{I}^{t_0+1}(t, V^0, H)) \rangle_{s>t_0}).$$

Step 3. Replacing H by $(G_1, G_2 - \varepsilon \underline{Z}G_1)$ and V^0 by $(U_0^1, U_2^0 - \varepsilon \underline{Z}|_{t=0}U_1^0)$ one obtains the following control on $\mathcal{I}^{r+1}(t, V^0, H)$ ($r = s, t_0$):

$$\mathcal{I}^{r+1}(t, V^0, H) \leq \underline{C}(\mathcal{I}^{r+2}(t, U^0, G) + \langle |\underline{U}|_{\tilde{X}^{r+7/2}} \mathcal{I}^{t_0+2}(t, U^0, G) \rangle_{r>t_0}).$$

Step 4. The proposition follows from Steps 2 and 3.

5. Main results

5.1. Large time existence for the water-waves equations

In this section we prove the main result of this paper, which proves the well-posedness of the water-waves equations over large times and provides a uniform energy control which will allow us to justify all the asymptotic regimes evoked in the introduction. Recall first that the semi-normed spaces $(\tilde{X}^s, |\cdot|_{\tilde{X}^s})$ have been defined in Definition 4.2 as

$$\tilde{X}^s := \{(\zeta, \psi), \zeta \in H^s(\mathbb{R}^2), \nabla \psi \in H^{s-1/2}(\mathbb{R}^2)^2\},$$

and $|(\zeta, \psi)|_{\tilde{X}^s} := |\zeta|_{H^s} + |\mathfrak{P}\psi|_{H^s}$, and define also the mapping \mathfrak{a} by

$$\begin{aligned} \mathfrak{a}(\zeta, \psi) &:= \frac{\varepsilon^2}{\nu} (\nabla^\gamma \psi - \varepsilon \mathcal{Z}[\varepsilon \zeta] \nabla^\gamma \zeta) \cdot \nabla^\gamma \mathcal{Z}[\varepsilon \zeta] \psi \\ &\quad - \varepsilon \mathcal{Z}[\varepsilon \zeta] (\zeta + \mathcal{A}_2[(\zeta, \psi)]) + \varepsilon d_\zeta \mathcal{Z}[\varepsilon \cdot] \psi \cdot \mathcal{G}[\varepsilon \zeta] \psi + 1, \end{aligned}$$

where $\mathcal{Z}[\varepsilon \zeta]$ is as defined in Theorem 3.1 and \mathcal{A}_2 is defined in (4.2), and where $\nu = (1 + \sqrt{\mu})^{-1}$. The only condition we impose on the parameters is that the steepness $\varepsilon \sqrt{\mu}$ and the ratio β/ε remain bounded. More precisely, $(\varepsilon, \mu, \gamma, \beta) \in \mathcal{P}_M$ ($M > 0$) with

$$\mathcal{P}_M = \{(\varepsilon, \mu, \gamma, \beta) \in (0, 1] \times (0, \infty) \times (0, 1] \times [0, 1], \varepsilon \sqrt{\mu} \leq M \text{ and } \frac{\beta}{\varepsilon} \leq M\}.$$

We can now state the theorem:

Theorem 5.1. Let $t_0 > 1$, $M > 0$ and $\mathcal{P} \subset \mathcal{P}_M$.

There exists $P > D > 0$ such that for all $s \geq s_0$, $b \in H^{s+P}(\mathbb{R}^2)$, and all family $(\zeta_p^0, \psi_p^0)_{p \in \mathcal{P}}$ bounded in \tilde{X}^{s+P} satisfying

$$\inf_{\mathbb{R}^s} 1 + \varepsilon \zeta_p^0 - \beta b > 0 \quad \text{and} \quad \inf_{\mathbb{R}^2} \mathfrak{a}(\zeta_p^0, \psi_p^0) > 0$$

(uniformly with respect to $p = (\varepsilon, \mu, \gamma, \beta) \in \mathcal{P}$), there exist $T > 0$ and a unique family $(\zeta_p, \psi_p)_{p \in \mathcal{P}}$ bounded in $C([0, \frac{\nu T}{\varepsilon}]; \tilde{X}^{s+D})$ solving (1.4) with initial condition $(\zeta_p^0, \psi_p^0)_{p \in \mathcal{P}}$.

Remark 5.1. The time interval of the solution varies with $p \in \mathcal{P}$ (through ε and ν); when we say that $(\zeta_p, \psi_p)_{p \in \mathcal{P}}$ is bounded in $C([0, \frac{\nu T}{\varepsilon}]; \tilde{X}^{s+D})$, we mean that there exists C such that

$$\forall p \in \mathcal{P}, \quad \forall t \in [0, \frac{\nu T}{\varepsilon}], \quad |\zeta_p(t)|_{H^{s+D}} + |\mathfrak{P}\psi_p(t)|_{H^{s+D}} \leq C.$$

Remark 5.2. For the shallow water regime for instance, one has $\varepsilon = \beta = \gamma = 1$ and μ is small (say, $\mu < 1$); thus, we can take $\mathcal{P} = \{1\} \times (0, 1) \times \{1\} \times \{1\}$; for the KP regime (with flat bottom), one takes $\mathcal{P} = \{(\varepsilon, \varepsilon, \sqrt{\varepsilon}), \varepsilon \in (0, 1)\} \times \{0\}$, etc.

Remark 5.3. The condition $\inf_{\mathbb{R}^2} \mathfrak{a}(\zeta_p^0, \psi_p^0) > 0$ is the classical Taylor sign condition proper to the water-wave equations ([48, 49, 29, 34, 35, 3, 12, 45], among others). It is obviously true for small data and we give in Proposition 5.1 some simple sufficient conditions on the initial data and the bottom parameterization b which ensure that it is satisfied.

Remark 5.4. One also has the following stability property (see Corollary 1 in [2]): let $\underline{T} > 0$ and $(U_p^{app})_{p \in \mathcal{P}} = (\zeta_p^{app}, \psi_p^{app})_{p \in \mathcal{P}}$, bounded in $Y_{\underline{T}}^{s+P}$ (see Definition 4.2), be an approximate solution of (1.4) in the sense that

$$\partial_t U_p^{app} + \mathcal{L}U_p^{app} + \frac{\varepsilon}{\nu} \mathcal{A}[U_p^{app}] = \frac{\varepsilon}{\nu} \delta_p R_p, \quad U_p^{app}|_{t=0} = (\zeta_p^0, \psi_p^0) + \delta_p r_p,$$

with $(R_p, r_p)_p$ bounded in $C([0, \frac{\nu T}{\varepsilon}]; X^{s+P}) \cap C^1([0, \frac{\nu T}{\varepsilon}]; X^{s+P-5/2}) \times X^{s+P}$ (and $\delta_p \geq 0$). If moreover the U_p^{app} are admissible, then one has

$$\forall t \in [0, \frac{\nu}{\varepsilon} \inf\{T, \underline{T}\}], \quad |U_p(t) - U_p^{app}(t)|_{\tilde{X}^{s+D}} \leq \text{Cst } \delta_p,$$

where $U_p \in C([0, \frac{\nu T}{\varepsilon}]; \tilde{X}^{s+D})$ is the solution furnished by the theorem. For δ_p small enough, one can take $T = \underline{T}$.

Remark 5.5. The numbers P and D could be explicitated in the above theorem (as in Theorem 1 of [2] for instance), but since the focus here is not on the regularity of the solutions, we chose to alleviate the proof as much as possible. For the same reason, we use a Nash-Moser iterative scheme which allows us to deal with all the different regimes at once, though it is possible in some cases to push further the analysis of the linearized operator and use a standard iterative scheme (as shown in [24] for the shallow-water regime).

Proof. Let us denote in this proof $\epsilon = \varepsilon/\nu$ and omit the index p for the sake of clarity. Rescaling the time by $t \rightsquigarrow t/\epsilon$ and using the same

notations as in (4.1) and (4.2), the theorem reduces to proving the well-posedness of the IVP

$$\begin{cases} \partial_t U + \frac{1}{\epsilon} \mathcal{L}U + \mathcal{A}[U] = 0, \\ U|_{t=0} = U^0, \end{cases}$$

on a time interval $[0, T]$, with $T > 0$ independent of all the parameters. Define first the evolution operator $S^\epsilon(\cdot)$ associated to the linear part of the above IVP. The following lemma shows that the definition

$$S^\epsilon(t)U^0 := U(t), \quad \text{with} \quad \partial_t U + \frac{1}{\epsilon} \mathcal{L}U = 0 \quad \text{and} \quad U|_{t=0} = U^0 \quad (5.1)$$

makes sense for all data $U^0 \in \tilde{X}^s$.

Lemma 5.1. For all $U^0 \in \tilde{X}^s$, $S^\epsilon(\cdot)U^0$ is well defined in $C([0, T]; \tilde{X}^s)$ by (5.1). Moreover, for all $0 \leq t \leq T$,

$$|S^\epsilon(t)U^0|_{\tilde{X}^s} \leq C(T, \frac{1}{h_0}, |b|_{H^{s+7/2}}, \frac{\beta}{\epsilon}, \frac{\varepsilon}{\nu}) |U^0|_{\tilde{X}^s}.$$

Proof. Proceeding as in the proof of Proposition 4.1 (in the very simple case $\underline{U} = (0, 0)^T$), one checks that $S^\epsilon(t)U^0$ makes sense and that the estimate of the lemma holds if $U^0 \in X^s$.

Now, let us extend this result to data $U^0 \in \tilde{X}^s$. Let ι be a smooth function vanishing in a neighborhood of the origin and being constant equal to one outside the unit disc, and define, for all $\delta > 0$, $\iota^\delta = \iota(|D|/\delta)$. The couple $U^{0,\delta} := (\zeta^0, \iota^\delta \psi^0)^T = (\zeta^0, \psi^{0,\delta})^T$ then belongs to X^s and $U^\delta(t) := S^\epsilon(t)U^{0,\delta} = (\zeta^\delta(t), \psi^\delta(t))^T$ is well defined in X^s . Since

$$|U^\delta(t) - U^{\delta'}(t)|_{\tilde{X}^s} \leq C(T, \frac{1}{h_0}, |b|_{H^{s+7/2}}, \frac{\beta}{\epsilon}, \frac{\varepsilon}{\nu}) |U^{0,\delta} - U^{0,\delta'}|_{\tilde{X}^s},$$

it follows by dominated convergence that $(\zeta^\delta)_{\delta \rightarrow 0}$ and $(\mathfrak{P}\psi^\delta)_{\delta \rightarrow 0}$ are Cauchy sequences in $C([0, T]; H^s(\mathbb{R}^2))$. Therefore, $(\zeta^\delta) \rightarrow \zeta$ and $(\mathfrak{P}\psi^\delta) \rightarrow \omega$ in $C([0, T]; H^s(\mathbb{R}^2))$, as δ goes to 0. Defining $\psi(t) = \psi^0 - \frac{1}{\epsilon} \int_0^t \zeta(t') dt'$ and using $\psi^\delta(t) = \psi^{0,\delta} - \frac{1}{\epsilon} \int_0^t \zeta^\delta(t') dt'$, one deduces $\omega = \mathfrak{P}\psi$, from which one infers that $\nabla \psi \in C([0, T]; H^{s-1/2}(\mathbb{R}^2)^2)$.

From the convergence $\mathfrak{P}\psi^\delta \rightarrow \omega = \mathfrak{P}\psi$ in $H^s(\mathbb{R}^2)$ and Proposition 3.3 one deduces also that $\mathcal{G}[0]\psi^\delta \rightarrow \mathcal{G}[0]\psi$ in $H^{s-1/2}(\mathbb{R}^2)$. One can thus take the limit as $\delta \rightarrow 0$ in the relation $\partial_t \zeta^\delta(t) - \frac{1}{\epsilon} \frac{1}{\mu\nu} \mathcal{G}[0]\psi^\delta(t) = 0$, thus proving that $(\zeta, \psi) \in \tilde{X}^s$ solves the IVP (5.1). Since the solution to this IVP is obviously unique, this shows that $S^\epsilon(\cdot)U^0$ makes sense in \tilde{X}^s when $U^0 \in \tilde{X}^s$.

The last assertion of the lemma follows by taking the limit when $\delta \rightarrow 0$ in the following expression

$$|S^\epsilon(t)U^{0,\delta}|_{\tilde{X}^s} \leq C(T, \frac{1}{h_0}, |b|_{H^{s+7/2}}, \frac{\beta}{\epsilon}, \frac{\epsilon}{\nu})|U^{0,\delta}|_{\tilde{X}^s}.$$

□

We now look for the exact solution under the form $U = S^\epsilon(t)U^0 + V$, which is equivalent to solving

$$\begin{cases} \partial_t V + \frac{1}{\epsilon}\mathcal{L}V + \mathcal{F}[t, V] = h \\ V|_{t=0} = (0, 0)^T, \end{cases} \quad (5.2)$$

with $\mathcal{F}[t, V] := \mathcal{A}[S^\epsilon(t)U^0 + V] - \mathcal{A}[S^\epsilon(t)U^0]$ and $h := -\mathcal{A}[S^\epsilon(t)U^0]$. We can now state two important properties satisfied by \mathcal{L} and \mathcal{F} (in the statement below, the notation $\mathcal{F}_{(j)}^{(i)}$ means that \mathcal{F} has been differentiated i times with respect to time and j with respect to its second argument).

Lemma 5.2. Let $T > 0$, $p = 1$ and $m = 5/2$. Then:

i. For all $s \geq t_0$, the mapping $\mathcal{L} : X^{s+m} \rightarrow X^s$ is well defined and continuous; moreover, the family of evolution operators $(S^\epsilon(\cdot))_{0 < \epsilon < \epsilon_0}$ is uniformly bounded in $C([-T, T]; Lin(X^{s+m}, X^s))$.

ii. For all $0 \leq i \leq p$ and $0 \leq i + j \leq p + 2$, and for all $s \geq t_0 + im$, one has

$$\begin{aligned} & \sup_{t \in [0, T]} |\epsilon^i \mathcal{F}_{(j)}^{(i)}[t, U](V_1, \dots, V_j)|_{s-im} \leq C(s, T, |U|_{t_0+(i+1)m}) \\ & \times \left(\sum_{k=1}^j |V_k|_{s+m} \prod_{l \neq k} |V_l|_{t_0+(i+1)m} + |U|_{s+m} \prod_{k=1}^j |V_k|_{t_0+(i+1)m} \right), \end{aligned}$$

for all $U \in H^{s+m}(\mathbb{R}^2)$ and $(V_1, \dots, V_j) \in H^{s+m}(\mathbb{R}^2)^j$.

Proof. **i.** The property on $S^\epsilon(\cdot)$ follows from Proposition 4.3 with $\underline{U} = (0, 0)$ (recall that we rescaled the time variable). In order to prove the continuity of \mathcal{L} , let us write, for all $W = (\zeta, \psi)^T$,

$$\begin{aligned} |\mathcal{L}W|_{X^s} & \leq \left| \frac{1}{\mu\nu} \mathcal{G}[0]\psi \right|_{H^s} + \frac{\epsilon}{\nu} |\zeta|_{H^s} + |\mathfrak{P}\zeta|_{H^s} \\ & \leq \left| \frac{1}{\mu\nu} \mathcal{G}[0]\psi \right|_{H^s} + C(\epsilon) |\zeta|_{H^{s+1}}. \end{aligned}$$

One therefore deduces the continuity property on \mathcal{L} from the following inequality:

$$\left| \frac{1}{\mu\nu} \mathcal{G}[0]\psi \right|_{H^s} \leq C\left(\frac{1}{h_0}, \epsilon\sqrt{\mu}, \frac{\beta}{\epsilon}, |b|_{H^{s+2}}\right) |\mathfrak{P}\psi|_{H^{s+1}};$$

for $\mu \geq 1$, one has the uniform bound $\frac{1}{\mu\nu} \lesssim 1/\sqrt{\mu}$, and the inequality is a direct consequence of Proposition 3.3; for $\mu \leq 1$, one has $\nu \sim 1$ and we rather use Remark 3.3.

ii. Since by definition $\mathcal{F}[t, U] = \mathcal{A}[S^\varepsilon(t)U^0 + U] - \mathcal{A}(S^\varepsilon(t)U^0)$, it follows from the first point that it suffices to prove the estimates in the case $i = 0$ and with \mathcal{F} replaced by \mathcal{A} . Recall that \mathcal{A} is explicitly given by (4.2) and remark that

$$\begin{aligned} \mathcal{A}_1[U] &= -\frac{1}{\varepsilon\mu} \int_0^1 d_z \zeta \mathcal{G}[\varepsilon \cdot] \psi \cdot \zeta dz, \\ &= \int_0^1 \frac{1}{\sqrt{\mu}} \mathcal{G}[\varepsilon z \zeta] \left(\zeta \frac{1}{\sqrt{\mu}} \underline{Z} \right) + z \nabla^\gamma \cdot (\zeta \underline{\mathbf{v}}) dz, \end{aligned}$$

where \underline{Z} and $\underline{\mathbf{v}}$ are as in Theorem 3.1 (with $\zeta = \zeta$ and $\psi = \psi$).

The estimates on \mathcal{A} are therefore a direct consequence of Proposition 3.3. \square

The well-posedness of (5.2) is deduced from the general Nash-Moser theorem for singular evolution equations of [2] (Theorem 1'), provided that the three assumptions (Assumptions 1', 2' and 3' in [2]) it requires are satisfied. The first two, which concern the linear operator \mathcal{L} and the nonlinearity $\mathcal{F}[t, \cdot]$, are exactly the results stated in Lemma 5.2. The third assumption concerns the linearized operator around \underline{U} associated to (5.2); after remarking that

$$\partial_t + \frac{1}{\varepsilon} \mathcal{L} + d_{\underline{V}} \mathcal{F}[t, \cdot] = \mathfrak{L}_{(\underline{\zeta}, \underline{\psi})},$$

with $\underline{U} = (\underline{\zeta}, \underline{\psi})^T = S^\varepsilon(t)U^0 + \underline{V}$, one can check that this last assumption is exactly the result stated in Proposition 4.1, provided that the following quantity (which is the first iterate of the Nash-Moser scheme, see Remark 3.2.2 of [2])

$$U_0 := t \mapsto S^\varepsilon(t)U^0 + \int_0^t S^\varepsilon(t-t') \mathcal{F}[t', U^0] dt' \quad (5.3)$$

is an admissible reference state in the sense of Definition 4.1 on the time interval $[0, T]$ (recall that we rescaled the time variable). Taking a smaller T if necessary, it is sufficient to check the admissibility at $t = 0$, which is equivalent to the two assumptions made in the statement of the theorem (after remarking that $\mathfrak{a}(\zeta^0, \psi^0) = \mathfrak{a}(t = 0)$, with \mathfrak{a} as defined in (4.5) and $\underline{U} = U_0$). The proof is thus complete. \square

We end this section with a proposition showing that the Taylor sign condition

$$\inf_{\mathbb{R}^2} \mathfrak{a}(\zeta_p^0, \psi_p^0)_{p \in \mathcal{P}} > 0, \quad \text{uniformly with respect to } p \in \mathcal{P} \quad (5.4)$$

can be replaced in Theorem 5.1 by a much simpler condition. We need to introduce first the ‘‘anisotropic Hessian’’ \mathcal{H}_b^γ associated to the bottom parameterization b ,

$$\mathcal{H}_b^\gamma := \begin{pmatrix} \partial_x^2 b & \gamma^2 \partial_{xy}^2 b \\ \gamma^2 \partial_{xy}^2 b & \gamma^4 \partial_y^2 b \end{pmatrix}$$

and the initial velocity potential Φ_p^0 given by the BVP

$$\begin{cases} \mu \partial_x^2 \Phi_p^0 + \gamma^2 \mu \partial_y^2 \Phi_p^0 + \partial_z^2 \Phi_p^0 = 0, & -1 + \beta b \leq z \leq \varepsilon \zeta_p^0, \\ \Phi_p^0|_{z=\varepsilon \zeta_p^0} = \psi_p^0, & \partial_n \Phi_p^0|_{z=-1+\beta b} = 0. \end{cases} \quad (5.5)$$

Proposition 5.1. Let $t_0 > 1$, $M > 0$ and $\mathcal{P} \subset \mathcal{P}_M$; let also $b \in H^{t_0+2}(\mathbb{R}^2)$, $(\zeta_p^0, \psi_p^0)_{p \in \mathcal{P}}$ be bounded in \tilde{X}^{t_0+1} and $(\Phi_p^0)_{p \in \mathcal{P}}$ solve the BVPs (5.5). Then,

- i. There exists $\varepsilon_0 > 0$ such that (5.4) is satisfied if one replaces \mathcal{P} by $\mathcal{P}_{\varepsilon_0} := \{p = (\varepsilon, \mu, \gamma, \beta) \in \mathcal{P}, \varepsilon \nu^{-1} \leq \varepsilon_0\}$;
- ii. If there exist $\mu_1 > 0$ and $\underline{\gamma} \in C((0, 1] \times (0, \mu_1])$ such that for all $p = (\varepsilon, \mu, \gamma, \beta) \in \mathcal{P}$ one has $\mu \leq \mu_1$ and $\gamma = \underline{\gamma}(\varepsilon, \mu)$, and if

$$-\varepsilon^2 \beta \mu \mathcal{H}_b^\gamma(\nabla \Phi_p^0|_{z=-1+\beta b}) \leq 1, \quad \text{for all } p \in \mathcal{P},$$

then the Taylor sign condition (5.4) is satisfied.

Remark 5.6. The first point of the proposition is used to check the Taylor condition in deep water regime; in this latter case, one has indeed $\varepsilon/\nu \sim \varepsilon\sqrt{\mu}$ which is the *steepness* of the wave, the small parameter with respect to which asymptotic models are derived.

The second point of the proposition is essential in the shallow-water regime (since ε/ν does not go to zero as $\mu \rightarrow 0$). It is important to notice that it implies that the Taylor condition *is automatically satisfied for flat bottoms*.

Remark 5.7. S. Wu proved in [48, 49] that the Taylor sign condition (5.4) is automatically satisfied in infinite depth; this result was extended in [29] to finite depth with flat bottoms. The result needed here is stronger, since we want (5.4) to be satisfied uniformly with respect to the parameters. In the 1DH-case, for flat bottoms, and in the particular case of the shallow-water regime, such a result was established in [33].

Proof. As in the proof of Theorem 5.1, we omit the index p to alleviate the notations.

i. As seen in the proof of Theorem 5.1, one has $\mathbf{a}(\zeta^0, \psi^0) = \underline{\mathbf{a}}(t=0)$, where $\underline{\mathbf{a}}$ is as defined in (4.5) (with $\underline{U} = U_0$ and U_0 given by (5.3)). Thus, $\mathbf{a}(\zeta^0, \psi^0) = 1 + \frac{\varepsilon}{\nu} \underline{\mathbf{b}}$ and $|\underline{\mathbf{a}}|_{L^\infty} \geq 1 - \varepsilon_0 |\underline{\mathbf{b}}|_{L^\infty}$. It follows from

Lemma 4.1 that for the range of parameters considered here, $|\underline{\mathbf{b}}|_{L^\infty}$ is uniformly bounded on $[0, T]$, so that the result follows when ϵ_0 is small enough.

ii. Step 1: *There exists $\mu_0 > 0$ such that (5.4) is satisfied for all $p = (\epsilon, \mu, \gamma, \beta) \in \mathcal{P}$ such that $\mu \leq \mu_0$.* It is indeed a consequence of Remark 3.3 that $|\underline{\mathbf{b}}|_{L^\infty} = O(\sqrt{\mu})$ as $\mu \rightarrow 0$; since moreover $\epsilon = \frac{\epsilon}{\nu}$ remains bounded, one can conclude as in the first step.

Step 2. *The case $\mu \geq \mu_0$.* For all time t , let $\Phi(t)$ denote the solution of the BVP (5.5), with the Dirichlet condition at the surface replaced by $\Phi^0|_{z=\epsilon\zeta^0} = \psi_0(t)$, where $U_0(t) = (\zeta_0(t), \psi_0(t))$ is given by (5.3). Let us also define the ‘‘pressure’’ P as

$$-\frac{1}{\epsilon}P := \partial_t \Phi + \frac{1}{2} \left(\frac{\epsilon}{\nu} |\nabla^\gamma \Phi|^2 + \frac{\epsilon}{\mu\nu} (\partial_z \Phi)^2 \right) + \frac{1}{\epsilon} z.$$

Since $U_0 = (\zeta_0, \psi_0)$ solves (1.4) at $t = 0$, one can check as in Proposition 4.4 of [29] that $P(t = 0, X, \epsilon\zeta^0(X)) = 0$. Differentiating this relation with respect to X shows that $-\nabla^\gamma \zeta^0 \cdot \nabla^\gamma P = \epsilon |\nabla^\gamma \zeta^0|^2 \partial_z P$ on the surface, from which one deduces the identity,

$$(1 + \epsilon^2 |\nabla \zeta^0|^2)^{1/2} \partial_n P|_{z=\epsilon\zeta^0} = (1 + \epsilon^2 \mu |\nabla^\gamma \zeta^0|^2) \partial_z P|_{z=\epsilon\zeta^0},$$

where $\partial_n P|_{z=\epsilon\zeta^0}$ stands for the outwards conormal derivative associated to the elliptic operator $\mu \partial_x^2 + \gamma^2 \mu \partial_y^2 + \partial_z^2$. Expressing Φ and its derivatives evaluated at the surface in terms of Ψ , one can then check that

$$\underline{\mathbf{a}}(t = 0) = -\frac{1}{1 + \epsilon^2 \mu^2 |\nabla^\gamma \zeta^0|^2} (1 + \epsilon^2 |\nabla \zeta^0|^2)^{1/2} \partial_n P|_{z=\epsilon\zeta^0}. \quad (5.6)$$

Let us now remark that P solves the BVP

$$\begin{cases} (\mu \partial_x^2 + \gamma^2 \mu \partial_y^2 + \partial_z^2) P = h, & -1 + \beta b \leq z \leq \epsilon \zeta^0, \\ P|_{z=\epsilon\zeta^0} = 0, & \partial_n P|_{z=-1+\beta b} = g, \end{cases}$$

with

$$\begin{aligned} h &:= -\frac{1}{2} (\mu \partial_x^2 + \gamma^2 \mu \partial_y^2 + \partial_z^2) \left(\frac{\epsilon^2}{\nu} |\nabla^\gamma \Phi|^2 + \frac{\epsilon^2}{\mu\nu} (\partial_z \Phi)^2 \right) \\ g &:= -\frac{1}{2} \partial_n \left(\frac{\epsilon^2}{\nu} |\nabla^\gamma \Phi|^2 + \frac{\epsilon^2}{\mu\nu} (\partial_z \Phi)^2 \right)|_{z=-1+\beta b} - \partial_n(z)|_{z=-1+\beta b}. \end{aligned}$$

Exactly as in the proof of Proposition 4.15 of [29], one can check that $h \leq 0$ and use a maximum principle (using the fact that (5.6) links $\underline{\mathbf{a}}$ to the normal derivative of P at the surface) to show that if $g \leq 0$ then there exists a constant $c(\epsilon, \mu, \beta) > 0$ such that $\underline{\mathbf{a}}(t_0) \geq c(\epsilon, \mu, \beta)$.

We thus turn to prove that $g \leq 0$.
Recall that by construction of Φ ,

$$(1 + \beta^2 |\nabla b|^2)^{1/2} \partial_n \Phi|_{z=-1+\beta b} (= \beta \mu \nabla^\gamma b \cdot \nabla^\gamma \Phi|_{z=-1+\beta b} - \partial_z \Phi|_{z=-1+\beta b}) = 0.$$

Differentiating this relation with respect to j ($j = x, y$), one gets

$$\begin{aligned} (1 + \beta^2 |\nabla b|^2)^{1/2} \partial_n (\partial_j \Phi)|_{z=-1+\beta b} &= -\beta \mu \nabla^\gamma \partial_j b \cdot \nabla^\gamma \Phi|_{z=-1+\beta b} \\ &+ \beta \partial_j b (1 + \beta^2 |\nabla b|^2)^{1/2} \partial_n (\partial_z \Phi)|_{z=-1+\beta b}, \end{aligned}$$

and using this formula one computes

$$\begin{aligned} &\frac{1}{2} (1 + \beta^2 |\nabla b|^2)^{1/2} \partial_n \left(\frac{\varepsilon^2}{\nu} |\nabla^\gamma \Phi|^2 + \frac{\varepsilon^2}{\mu \nu} (\partial_z \Phi)^2 \right)|_{z=-1+\beta b} \\ &= -\frac{\varepsilon^2 \beta \mu}{\nu} \left((\partial_x \Phi)^2 \partial_x^2 b + 2\gamma^2 \partial_x \Phi \partial_y \Phi \partial_{xy}^2 b + \gamma^4 (\partial_y \Phi)^2 \partial_y^2 b \right) \\ &= -\frac{\varepsilon^2 \beta \mu}{\nu} \mathcal{H}_b^\gamma (\nabla \Phi|_{z=-1+\beta b}); \end{aligned}$$

since moreover $(1 + \beta^2 |\nabla b|^2)^{1/2} \partial_n(z)|_{z=-1+\beta b} = 1$, one gets $g \geq 0$ if the condition given in the statement of the proposition is fulfilled.

As detailed above, we therefore have $\underline{a}(t=0) \geq c(\varepsilon, \mu, \beta) > 0$; moreover, there exists by assumption μ_1 such that for all $p = (\varepsilon, \mu, \gamma, \beta) \in \mathcal{P}$, one has $\mu \leq \mu_1$; due to the first point of the proposition, Step 1 and the fact the $\gamma = \underline{\gamma}(\varepsilon, \mu)$, it is sufficient to prove the proposition for all the parameters $p \in \mathcal{P}_1$ with

$$\mathcal{P}_1 := [\varepsilon_0, 1] \times [\mu_0, \mu_1] \times \underline{\gamma}([\varepsilon_0, 1] \times [\mu_0, \mu_1]) \times [0, 1] \quad (\varepsilon_0 := (1 + \sqrt{\mu_1})^{-1} \varepsilon_0).$$

The dependence of $c(\varepsilon, \mu, \beta) > 0$ on ε , μ and β is continuous and therefore, $\inf_{[\varepsilon_0, 1] \times [\mu_0, \mu_1] \times [0, 1]} c(\varepsilon, \mu, \beta) > 0$. \square

6. Asymptotics for 3D water-waves

We will now provide a rigorous justification of the main asymptotic models used in coastal oceanography.

Remark 6.1. Throughout this section, we assume the following:

- P and D are as in the statement of Theorem 5.1;
- Φ^0 stands for the initial velocity potential as in Proposition 5.1;
- The bottom parameterization satisfies $b \in H^{s+P}(\mathbb{R}^2)$;
- Except for the KP equations, we always consider fully transverse waves ($\gamma = 1$), but one could easily use the methods set in this paper to derive and justify weakly transverse models in the other regimes.

6.1. Shallow-water and Serre regimes

We recall that the so-called “shallow-water” regime corresponds to the conditions $\mu \ll 1$ (so that $\nu \sim 1$) and $\varepsilon = \gamma = 1$; we also consider bottom variations which can be of large amplitude ($\beta = 1$). Without restriction, we can assume that $\nu = 1$ (which corresponds to the nondimensionalization (1.4)). The shallow-water model – which goes back to Airy [1] and Friedrichs [19]– consists of neglecting the $O(\mu)$ terms in the water-waves equations, while the Green-Naghdi equations [21, 22, 46] is a more precise approximation, which neglects only the $O(\mu^2)$ quantities. The Serre equations [44, 46] are quite similar to the Green-Naghdi equations, but assume that the bottom and surface variations are of medium amplitude: $\varepsilon = \beta = \sqrt{\mu}$.

6.1.1. The shallow-water equations The shallow water equations are

$$\begin{cases} \partial_t V + \nabla \zeta + \frac{1}{2} \nabla |V|^2 = 0, \\ \partial_t \zeta + \nabla \cdot ((1 + \zeta - b)V) = 0, \end{cases} \quad (6.1)$$

and the following theorem shows that they provide a good approximation to the exact solution of the water-waves equations.

Theorem 6.1 (Shallow-water equations). Let $s \geq t_0 > 1$ and $(\zeta_\mu^0, \psi_\mu^0)_{0 < \mu < 1}$ be bounded in \tilde{X}^{s+P} . Assume moreover that there exist $h_0 > 0$ and $\mu_0 > 0$ such that for all $\mu \in (0, \mu_0)$,

$$\inf_{\mathbb{R}^2} (1 + \zeta_\mu^0 - b) \geq h_0 \quad \text{and} \quad -\mu \mathcal{H}_b^\gamma (\nabla \Phi_\mu^0|_{z=-1+b}) \leq 1.$$

Then there exists $T > 0$ and:

1. a unique family $(\zeta_\mu, \psi_\mu)_{0 < \mu < \mu_0}$ bounded in $C([0, T]; \tilde{X}^{s+D})$ and solving (1.4) with initial conditions $(\zeta_\mu^0, \psi_\mu^0)_{0 < \mu < \mu_0}$;
2. a unique family $(V_\mu^{SW}, \zeta_\mu^{SW})_{0 < \mu < \mu_0}$ bounded in $C([0, T]; H^{s+P-1/2}(\mathbb{R}^2)^3)$ and solving (6.1) with initial conditions $(\zeta_\mu^0, \nabla \psi_\mu^0)_{0 < \mu < \mu_0}$.

Moreover, one has, for some $C > 0$,

$$\forall 0 < \mu < \mu_0, \quad |\zeta_\mu - \zeta_\mu^{SW}|_{L^\infty([0, T] \times \mathbb{R}^2)} + |\nabla \psi_\mu - V_\mu^{SW}|_{L^\infty([0, T] \times \mathbb{R}^2)} \leq C\mu.$$

Remark 6.2. The existence time provided by Theorem 5.1 is $O(1)$, but is large in the sense that it does not shrink to zero when $\mu \rightarrow 0$.

Remark 6.3. Instead of assuming that the initial data $(\zeta_\mu^0, \psi_\mu^0)_{0 < \mu < 1}$ are bounded in \tilde{X}^{s+P} , we could assume that $(\zeta_\mu^0, \nabla \psi_\mu^0)_{0 < \mu < 1}$ is bounded in $H^{s+P}(\mathbb{R}^2)^3$ (because $|\mathfrak{F}\psi|_{H^{s+P}} \lesssim |\nabla \psi|_{H^{s+P}}$, uniformly with respect to $\mu \in (0, 1)$).

Remark 6.4. The 2DH shallow-water model has been justified rigorously by Iguchi in a recent work [24], but under two restrictions: a) The velocity potential is assumed to have Sobolev regularity which implies that the velocity must satisfy some restrictive zero mass assumptions and b) the theorem holds only for very small values of μ . These assumptions are removed in the above result.

Proof. The assumptions allow us to use Theorem 5.1 and Proposition 5.1 with $\mathcal{P} = \{1\} \times (0, \mu_0) \times \{1\} \times \{1\}$, which proves the first part of the theorem.

The second point of the theorem is straightforward since $|\nabla \psi_\mu^0|_{H^{s+P-1/2}} \leq |\mathfrak{P}\psi_\mu^0|_{H^{s+P}}$ (recall that $\mu < 1$ here), and because (6.1) is a quasilinear hyperbolic system (since $\inf_{\mathbb{R}^2}(1 + \zeta - b) > 0$). In order to prove the error estimate, plug the expansion furnished by Proposition 3.8 into (1.4) and take the gradient of the second equation in order to obtain a system of equations on ζ_μ and $V_\mu = \nabla \psi_\mu$. One gets

$$\begin{cases} \partial_t V_\mu + \nabla \zeta_\mu + \frac{1}{2} \nabla |V_\mu|^2 = \mu R_\mu^1, \\ \partial_t \zeta_\mu + \nabla \cdot ((1 + \zeta_\mu - b)V_\mu) = \mu R_\mu^2, \end{cases} \quad (6.2)$$

with (R_μ^1, R_μ^2) uniformly bounded in $L^\infty([0, T]; H^{t_0}(\mathbb{R}^2)^{2+1})$. An energy estimate on (6.2) thus gives a Sobolev error estimate from which one deduces the L^∞ estimate of the theorem using the classical continuous embedding $H^{t_0} \subset L^\infty$. \square

6.1.2. The Green-Naghdi and Serre equations Though corresponding to two different physical regimes, the Green-Naghdi and Serre equations can both be written at the same time if one assumes that $\varepsilon = 1$ for the Green-Naghdi equations and $\varepsilon = \sqrt{\mu}$ for the Serre equations in the formulation below:

$$\begin{cases} (h + \mu \mathcal{T}[h, \varepsilon b]) \partial_t V + h \nabla \zeta + \varepsilon h (V \cdot \nabla) V \\ \quad + \mu \varepsilon \left[\frac{1}{3} \nabla (h^3 \mathcal{D}_V \operatorname{div}(V)) + \mathcal{Q}[h, \varepsilon b](V) \right] = 0 \\ \partial_t \zeta + \nabla \cdot (hV) = 0, \end{cases} \quad (6.3)$$

where $h := 1 + \varepsilon(\zeta - b)$ while the linear operators $\mathcal{T}[h, b]$ and \mathcal{D}_V and the quadratic form $\mathcal{Q}[h, b](\cdot)$ are defined as

$$\begin{aligned} \mathcal{T}[h, b]V &:= -\frac{1}{3} \nabla (h^3 \nabla \cdot V) + \frac{1}{2} [\nabla (h^2 \nabla b \cdot V) - h^2 \nabla b \nabla \cdot V] \\ &\quad + h \nabla b \nabla b \cdot V, \\ \mathcal{D}_V &:= -(V \cdot \nabla) + \operatorname{div}(V), \\ \mathcal{Q}[h, b](V) &:= \frac{1}{2} \nabla (h^2 (V \cdot \nabla)^2 b) + h \left(\frac{h}{2} \mathcal{D}_V \operatorname{div}(V) + (V \cdot \nabla)^2 b \right) \nabla b. \end{aligned}$$

Both the Green-Naghdi and Serre models are rigorously justified in the theorem below:

Theorem 6.2 (Green-Naghdi and Serre equations). Let $s \geq t_0 > 1$ and $(\zeta_\mu^0, \psi_\mu^0)_{0 < \mu < 1}$ be bounded in \tilde{X}^{s+P} . Let $\varepsilon = 1$ (Green-Naghdi) or $\varepsilon = \sqrt{\mu}$ (Serre) and assume that for some $h_0 > 0$, $\mu_0 > 0$ and for all $\mu \in (0, \mu_0)$,

$$\inf_{\mathbb{R}^2} (1 + \varepsilon(\zeta_\mu^0 - b)) \geq h_0 \quad \text{and} \quad -\mu\varepsilon^3 \mathcal{H}_b^\gamma(\nabla \Phi_\mu^0|_{z=-1+\varepsilon b}) \leq 1.$$

Then there exists $T > 0$ and:

1. a unique family $(\zeta_\mu, \psi_\mu)_{0 < \mu < \mu_0}$ bounded in $C([0, \frac{T}{\varepsilon}]; \tilde{X}^{s+D})$ and solving (1.4) with initial conditions $(\zeta_\mu^0, \psi_\mu^0)_{0 < \mu < \mu_0}$;
2. a unique family $(V_\mu^{GN}, \zeta_\mu^{GN})_{0 < \mu < \mu_0}$ bounded in $C([0, \frac{T}{\varepsilon}]; H^s(\mathbb{R}^2)^3)$ and solving (6.3) with initial conditions $(\zeta_\mu^0, (1 - \frac{\mu}{h^0} \mathcal{T}[h^0, \varepsilon b]) \nabla \psi_\mu^0)$ (with $h^0 = 1 + \varepsilon(\zeta^0 - b)$).

Moreover, one has for some $C > 0$ independent of $\mu \in (0, \mu_0)$,

$$|\zeta_\mu - \zeta_\mu^{GN}|_{L^\infty([0, \frac{T}{\varepsilon}] \times \mathbb{R}^2)} + |\nabla \psi_\mu - (1 + \frac{\mu}{h} \mathcal{T}[h, \varepsilon b]) V_\mu^{GN}|_{L^\infty([0, \frac{T}{\varepsilon}] \times \mathbb{R}^2)} \leq C \frac{\mu^2}{\varepsilon}.$$

Remark 6.5. The precision of the GN approximation ($\varepsilon = \beta = 1$) is therefore one order better than the shallow-water equations. This model had been justified in *1DH* and for flat bottoms by Y. A. Li [33]. The theorem above is stated in *2DH* but one can cover the open case of *1DH* non-flat bottoms with a straightforward adaptation.

Remark 6.6. In the Serre scaling, one has $\varepsilon = \sqrt{\mu}$, and the precision of the theorem is therefore $O(\mu^{3/2})$, which is worse than the $O(\mu^2)$ precision of the GN model, but the approximation remains valid over a larger time scale (namely, $O(\mu^{-1/2})$ versus $O(1)$ for GN). Notice also that at first order in μ , the Serre equations reduce to a simple wave equation (speed ± 1) on ζ and V , which is not the case for GN where the shallow-water equations (6.1) are found at first order.

Proof. The first assertion of the theorem is exactly the same as in Theorem 6.1 in the GN case. For the Serre equations, it is also a direct consequence of Theorem 5.1 and Proposition 5.1, with $\mathcal{P} = \{(\sqrt{\mu}, \mu, 1, \sqrt{\mu}), \mu \in (0, \mu_0)\}$.

For the second assertion, we replace $\mathcal{G}_{\mu, \gamma}[\varepsilon \zeta, \beta b]$ in (1.4) by the expansion given in Proposition 3.8 and take the gradient of the equation on ψ to obtain

$$\begin{cases} \partial_t V_\mu + \nabla \zeta_\mu + \frac{\varepsilon}{2} \nabla |V_\mu|^2 - \frac{\varepsilon \mu}{2} \nabla (h \nabla \cdot V_\mu - \varepsilon \nabla b \cdot V_\mu)^2 = \mu^2 R_\mu^1, \\ \partial_t \zeta_\mu + \nabla \cdot (hV) = \mu^2 R_\mu^2, \end{cases} \quad (6.4)$$

with $(R_\mu^1, R_\mu^2)_\mu$ bounded in $L^\infty([0, \frac{T}{\varepsilon}]; H^{t_0}(\mathbb{R}^2)^{2+1})$ while V is defined as $V := V_\mu - \frac{\mu}{h} \mathcal{T}[h, b] V_\mu$, so that $V_\mu = V + \frac{\mu}{h} \mathcal{T}[h, b] V + O(\mu^2)$.

Replacing V_μ by this expression in (6.4) and neglecting the $O(\mu^2)$ terms then gives (6.3). The theorem is then a direct consequence of the well-posedness theorem for the Green-Naghdi and Serre equations proved in [2] and of the error estimates given in Theorem 3 of that reference. \square

6.2. Long-waves regime: the Boussinesq approximation

The long-wave regime is characterized by the scaling $\gamma = 1$, $\mu = \varepsilon \ll 1$, so that one has $\nu \sim 1$. As for the shallow-water equations, we take $\nu = 1$ for notational convenience. When the bottom is non-flat, it is assumed that its variations are of the order of the size of the waves, that is, $\beta = \varepsilon$. Since the pioneer work of Boussinesq [8], many formally equivalent systems, generically called Boussinesq systems, have been derived to model the dynamics of the waves under this scaling. Following [7], these systems were derived in a systematic way in [6, 5, 10, 9]. In [5, 9] some interesting *symmetric* systems were introduced:

$$S'_{\theta, p_1, p_2} \begin{cases} (1 - \varepsilon a_2 \Delta) \partial_t V + \nabla \zeta + \varepsilon \left(\frac{1}{4} \nabla |V|^2 + \frac{1}{2} (V \cdot \nabla) V + \frac{1}{2} V \nabla \cdot V \right. \\ \quad \left. + \frac{1}{4} \nabla |\zeta|^2 - \frac{1}{2} b \nabla \zeta + a_1 \Delta \nabla \zeta \right) = 0, \\ (1 - \varepsilon a_4 \Delta) \partial_t \zeta + \nabla \cdot V + \frac{\varepsilon}{2} (\nabla \cdot ((\zeta - b) V) + a_3 \Delta \nabla \cdot V) = 0, \end{cases}$$

where the coefficients a_j ($j = 1, \dots, 4$) depend on $p_1, p_2 \in \mathbb{R}$ and $\theta \in [0, 1]$ through the relations $a_1 = (\frac{\theta^2}{2} - \frac{1}{6})p_1$, $a_2 = (\frac{\theta^2}{2} - \frac{1}{6})(1 - p_1)$, $a_3 = \frac{1 - \theta^2}{2} p_2$, and $a_4 = \frac{1 - \theta^2}{2} (1 - p_2)$; some choices of parameters yield $a_1 = a_3$ and $a_2 \geq 0$, $a_4 \geq 0$, and the corresponding systems S'_{θ, p_1, p_2} are the completely symmetric systems mentioned above. The so-called Boussinesq approximation associated to a family of initial data $(\zeta_\varepsilon^0, \psi_\varepsilon^0)_{0 < \varepsilon < 1}$ is given by

$$\zeta_\varepsilon^{app} = \zeta_\varepsilon^B \quad \text{and} \quad V_\varepsilon^{app} = (1 - \frac{\varepsilon}{2} (1 - \theta^2) \Delta) (1 - \frac{\varepsilon}{2} (\zeta_\varepsilon^B - b) V_\varepsilon^B), \quad (6.5)$$

where $(V_\varepsilon^B, \zeta_\varepsilon^B)_{0 < \varepsilon < 1}$ solves S'_{θ, p_1, p_2} with initial data

$$V_\varepsilon^{B,0} = (1 + \frac{\varepsilon}{2} (\zeta_\varepsilon^0 - b)) (1 - \frac{\varepsilon}{2} (1 - \theta^2) \Delta)^{-1} \nabla \psi_\varepsilon^0 \quad \text{and} \quad \zeta_\varepsilon^{B,0} = \zeta_\varepsilon^0. \quad (6.6)$$

The following theorem fully justifies this approximation.

Theorem 6.3 (Boussinesq systems). Let $s \geq t_0 > 1$ and $(\zeta_\varepsilon^0, \psi_\varepsilon^0)_{0 < \varepsilon < 1}$ be bounded in \tilde{X}^{s+P} and assume that there exist $h_0 > 0$ and $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$,

$$\inf_{\mathbb{R}^2} (1 + \varepsilon (\zeta_\varepsilon^0 - b)) \geq h_0 \quad \text{and} \quad -\varepsilon^4 \mathcal{H}_b^\gamma (\nabla \Phi_\mu^0 |_{z=-1+\varepsilon b}) \leq 1.$$

Then there exists $T > 0$ and:

1. a unique family $(\zeta_\varepsilon, \psi_\varepsilon)_{0 < \varepsilon < \varepsilon_0}$ bounded in $C([0, \frac{T}{\varepsilon}]; \tilde{X}^{s+D})$ and solving (1.4) with initial conditions $(\zeta_\varepsilon^0, \psi_\varepsilon^0)_{0 < \varepsilon < \varepsilon_0}$;
2. a unique family $(V_\varepsilon^B, \zeta_\varepsilon^B)_{0 < \varepsilon < \varepsilon_0}$ bounded in $C([0, \frac{T}{\varepsilon}]; H^{s+P-\frac{1}{2}}(\mathbb{R}^2)^3)$ and solving S'_{θ, p_1, p_2} with initial conditions (6.6).

Moreover, for some $C > 0$ independent of $\varepsilon \in (0, \varepsilon_0)$, one has

$$\forall 0 \leq t \leq \frac{T}{\varepsilon}, \quad |\zeta_\varepsilon(t) - \zeta_\varepsilon^{app}(t)|_\infty + |\nabla \psi_\varepsilon(t) - V_\varepsilon^{app}(t)|_\infty \leq C\varepsilon^2 t,$$

where $(V_\varepsilon^{app}, \zeta_\varepsilon^{app})$ is given by (6.5).

Remark 6.7. The above theorem justifies *all* the Boussinesq systems and not only the completely symmetric Boussinesq systems considered here: it is proved in [5, 9] that the justification of *all* the Boussinesq systems follows directly from the justification of *one* of them, in the sense that their solutions (if they exist!) provide an approximation of order $O(\varepsilon^2 t)$ to the water-waves equations.

Proof. The “if-theorems” of [5, 9] prove the result assuming that the first statement of the theorem holds, which is a direct consequence of Theorem 5.1 and Proposition 5.1 with $\mathcal{P} = \{(\varepsilon, \varepsilon, 1, \varepsilon), \varepsilon \in (0, \varepsilon_0)\}$. \square

6.3. Weakly transverse long-waves: the KP approximation

We recall that the KP regime is the same as the long-waves regime, but with $\gamma = \sqrt{\varepsilon}$. Moreover, we assume here that the bottom is flat, for the sake of simplicity. The KP approximation [25] consists in replacing the exact water elevation ζ_ε by the sum of two counter propagating waves, slowly modulated by a KP equation; more precisely, one defines ζ_ε^{KP} as

$$\zeta_\varepsilon^{KP}(t, x) = \frac{1}{2}(\zeta_+(\varepsilon t, \sqrt{\varepsilon}y, x - t) + \zeta_-(\varepsilon t, \sqrt{\varepsilon}y, x + t)) \quad (6.7)$$

where $\zeta_\pm(\tau, Y, X)$ solve the KP equation

$$\partial_\tau \zeta_\pm \pm \frac{1}{2} \partial_X^{-1} \partial_Y^2 \zeta_\pm \pm \frac{1}{6} \partial_X^3 \zeta_\pm \pm \frac{3}{2} \zeta_\pm \partial_X \zeta_\pm = 0. \quad (KP)_\pm$$

This approximation is rigorously justified in the theorem below:

Theorem 6.4 (KP equation). Let $s \geq t_0 > 1$ and $(\zeta^0, \psi^0) \in \tilde{X}^{s+P}$ and assume that there exist $h_0 > 0$ and $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$,

$$\inf_{\mathbb{R}^2} (1 + \varepsilon \zeta^0) \geq h_0,$$

and assume also that $(\partial_y^2 \partial_x \psi^0, \partial_y^2 \zeta^0) \in \partial_x^2 H^{s+P}(\mathbb{R}^2)^2$.

Then there exists $T > 0$ and:

1. a unique family $(\zeta_\varepsilon, \psi_\varepsilon)_{0 < \varepsilon < \varepsilon_0}$ solving (1.4) with initial conditions (ζ^0, ψ^0) and such that $(\zeta_\varepsilon)_{0 < \varepsilon < \varepsilon_0}$, $(\partial_x \psi_\varepsilon)_{0 < \varepsilon < \varepsilon_0}$ and $(\sqrt{\varepsilon} \partial_y \psi_\varepsilon)_{0 < \varepsilon < \varepsilon_0}$ are bounded in $C([0, \frac{T}{\varepsilon}]; H^{s+D-1/2})$;
2. a unique solution $\zeta_\pm \in C([0, T]; H^{s+P-1/2}(\mathbb{R}^2))$ to $(\text{KP})_\pm$ with initial condition $(\zeta^0 \pm \partial_x \psi^0)/2$.

Moreover, one has the following error estimate for the approximation (6.7):

$$\lim_{\varepsilon \rightarrow 0} |\zeta_\varepsilon - \zeta_\varepsilon^{KP}|_{L^\infty([0, \frac{T}{\varepsilon}] \times \mathbb{R}^2)} = 0.$$

Remark 6.8. The very restrictive “zero mass” assumptions that $\partial_y^2 \partial_x \psi^0$ and $\partial_y^2 \zeta^0$ are twice the derivative of a Sobolev function comes from the singular component $\partial_X^{-1} \partial_Y^2$ of the KP equations $(\text{KP})_\pm$. Furthermore, the error estimate is much worse than for the Boussinesq approximations. These two drawbacks are removed if one replaces the KP approximation by the approximation furnished by the *weakly transverse Boussinesq systems* introduced in [32]. As shown in [32], the first assertion of the above theorem rigorously justify these systems: they provide an approximation of order $O(\varepsilon^2 t)$ on the time interval $[0, T/\varepsilon]$, and do not require the “zero mass” assumptions.

Proof. As for the Boussinesq systems, we only have to prove the first assertion of the theorem, and the whole result then follows from the “if-theorem” of [32]. Taking $\mathcal{P} = \{(\varepsilon, \varepsilon, \sqrt{\varepsilon}, 0), \varepsilon \in (0, \varepsilon_0)\}$, Theorem 5.1 and Proposition 5.1 give a family of solutions $(\zeta_\varepsilon, \psi_\varepsilon)_{0 < \varepsilon < \varepsilon_0}$ bounded in $C([0, \frac{T}{\varepsilon}]; \tilde{X}^{s+D})$. In particular, $(|\mathfrak{P}\psi_\varepsilon|_{H^{s+D}})_\varepsilon$ is bounded, and thus $(|\nabla^\gamma \psi_\varepsilon|_{H^{s+D-1/2}})_\varepsilon$ is also bounded. Since $\gamma = \sqrt{\varepsilon}$, one has $|\partial_x \psi|_{H^{s+D-1/2}} + \sqrt{\varepsilon} |\partial_y \psi_\varepsilon|_{H^{s+D-1/2}} \lesssim |\nabla^\gamma \psi_\varepsilon|_{H^{s+D-1/2}}$ and the claim follows. \square

6.4. Deep water

6.4.1. Full dispersion model We present here the so-called full dispersion (or Matsuno) model for deep water-waves. Contrary to all the asymptotic models seen above, the shallowness parameter μ is allowed to take large values (deep water) provided that the *steepness* of the waves $\varepsilon \sqrt{\mu}$ remains small; without restriction, we can therefore take $\nu = \mu^{-1/2}$ here (i.e., we use the nondimensionalization (A.2)). Introducing $\varepsilon = \varepsilon \sqrt{\mu}$ the full dispersion model derived in [36, 37, 11] can be written in the case of flat bottoms ($\beta = 0$):

$$\begin{cases} \partial_t \zeta - \mathcal{T}_\mu V + \varepsilon (\mathcal{T}_\mu (\zeta \nabla \mathcal{T}_\mu V) + \nabla \cdot (\zeta V)) = 0, \\ \partial_t V + \nabla \zeta + \varepsilon (\frac{1}{2} \nabla |V|^2 - \nabla \zeta \mathcal{T}_\mu \nabla \zeta) = 0, \end{cases} \quad (6.8)$$

where \mathcal{T}_μ is a Fourier multiplier defined as

$$\forall V \in \mathfrak{G}(\mathbb{R}^2)^2, \quad \widehat{\mathcal{T}_\mu V}(\xi) = -\frac{\tanh(\sqrt{\mu}|\xi|)}{|\xi|}(i\xi) \cdot \widehat{V}(\xi).$$

Since $\beta = 0$ (flat bottom) and $\gamma = 1$ (fully transverse), the full dispersion model depends on two parameters (ε, μ) which are linked by a small steepness assumption:

$$\exists \epsilon_0 > 0, \quad (\varepsilon, \mu) \in \mathcal{P}_{\epsilon_0} \subset \{(\varepsilon, \mu) \in (0, 1] \times [1, \infty), \epsilon := \varepsilon\sqrt{\mu} \leq \epsilon_0\}.$$

The well-posedness of the full-dispersion model has not been investigated yet, but we can prove that if a solution exists on $[0, \frac{T}{\epsilon}]$ ($\epsilon > 0$ small enough), then the solution of the water-waves equations exists over the same time interval and is well approximated by the solution of the full-dispersion model:

Theorem 6.5 (Full-dispersion model). Let $\epsilon_0 > 0$, $Q \geq P$ large enough, $s \geq t_0 > 1$ and $(\zeta^0, \psi^0) \in \widetilde{X}^{s+P}$, and assume that

$$\forall \varepsilon \in (0, 1], \quad \inf_{\mathbb{R}^2} (1 + \varepsilon(\zeta^0 - b)) \geq h_0 > 0.$$

Let also $\underline{T} > 0$ and let $(\zeta_{\varepsilon, \mu}^{FD}, V_{\varepsilon, \mu}^{FD})_{(\varepsilon, \mu) \in \mathcal{P}_{\epsilon_0}}$ be bounded in $C([0, \frac{T}{\epsilon}], H^{s+Q}(\mathbb{R}^2)^3)$ and solving (6.8) with initial condition $(\zeta^0, \nabla \psi^0 - \epsilon(\mathcal{T}_\mu \nabla \psi^0) \nabla \zeta^0)$.

Then, if ϵ_0 is small enough, there is a unique family $(\zeta_{\varepsilon, \mu}, \psi_{\varepsilon, \mu})_{(\varepsilon, \mu) \in \mathcal{P}_{\epsilon_0}}$ bounded in $C([0, \frac{T}{\epsilon}]; \widetilde{X}^{s+D})$ and solving (1.4) with initial conditions (ζ^0, ψ^0) . In addition, for some $C > 0$ independent of $(\varepsilon, \mu) \in \mathcal{P}_{\epsilon_0}$, one has

$$|\zeta_{\varepsilon, \mu} - \zeta_{\varepsilon, \mu}^{FD}|_{L^\infty([0, \frac{T}{\epsilon}] \times \mathbb{R}^2)} + |\nabla \psi_\varepsilon - V_\varepsilon^{FD}|_{L^\infty([0, \frac{T}{\epsilon}] \times \mathbb{R}^2)} \leq C\epsilon \quad (\epsilon := \varepsilon\sqrt{\mu}).$$

Proof. Since $\mathcal{T}_\mu : H^r(\mathbb{R}^2)^2 \mapsto H^r(\mathbb{R}^2)$ is continuous with operator norm bounded from above by 1, the mapping $V \in H^r(\mathbb{R}^2)^2 \mapsto V - \epsilon(\mathcal{T}_\mu V) \nabla \zeta \in H^r(\mathbb{R}^2)^2$ is continuous for all $r \geq t_0$ and $\zeta \in H^{r+1}(\mathbb{R}^2)$. Moreover, this mapping is invertible for ϵ small enough, and one can accordingly define $\widetilde{V} := (1 - \epsilon \nabla \zeta \mathcal{T}_\mu)^{-1} V$, so that $V = \widetilde{V} - \epsilon(\mathcal{T}_\mu \widetilde{V}) \nabla \zeta$. Replacing V by this expression in (6.8) gives

$$\begin{cases} \partial_t \zeta - \mathcal{T}_\mu \widetilde{V} + \epsilon(\nabla \cdot (\zeta \widetilde{V}) + \mathcal{T}_\mu \nabla (\zeta \mathcal{T}_\mu \widetilde{V})) = \epsilon^2 r_\epsilon^1, \\ \partial_t \widetilde{V} + \nabla \zeta + \epsilon \frac{1}{2} (\nabla |\widetilde{V}|^2 - \nabla (\mathcal{T}_\mu \widetilde{V})^2) = \epsilon^2 \nabla r_\epsilon^2, \end{cases} \quad (6.9)$$

and where the exact expression of $R_\epsilon := (r_\epsilon^1, \nabla r_\epsilon^2)_\epsilon$ is of no importance. Now, let $\partial_t + L$ denote the linear part of the above system and $S(t)$ its evolution operator: $L := \begin{pmatrix} 0 & -\mathcal{T}_\mu \\ \nabla & 0 \end{pmatrix}$, and for all $U = (\zeta, V)$,

$S(t)U := u(t)$, where u solves $(\partial_t + L)u = 0$, with initial condition $u|_{t=0} = U$. Since \mathcal{T}_μ is a Fourier multiplier, one can find an explicit expression for $S(t)$, but we only need the following property: $S(t)$ is unitary on Z^r ($r \in \mathbb{R}$) defined as

$$Z^r := \{U = (\zeta, V) \in H^r(\mathbb{R}^2)^3, \\ |U|_{Z^r} := |\zeta|_{H^r} + \left| \left(\frac{\tanh(\sqrt{\mu}|\xi|)}{|\xi|} \right)^{1/2} V \right|_{H^r} < \infty\}.$$

Writing $\tilde{u} := (\zeta, \tilde{V})$, we define $w := (\tilde{\zeta}, W)$ as

$$w := \tilde{u} - \epsilon^2 \int_0^t S(t-t') R_\epsilon(t') dt';$$

remarking that $|V|_{H^{r-1/2}} \lesssim \left| \left(\frac{\tanh(\sqrt{\mu}|\xi|)}{|\xi|} \right)^{1/2} V \right|_{H^r}$ and that

$$\forall f \in H^{r+1}(\mathbb{R}^2), \quad \left| \left(\frac{\tanh(\sqrt{\mu}|\xi|)}{|\xi|} \right)^{1/2} \nabla f \right|_{H^{r+1/2}} \lesssim |f|_{H^{r+1}},$$

uniformly with respect to $\mu \geq 1$, and since $S(t)$ is unitary on Z^r , one gets

$$\forall r \geq 0, \quad \sup_{[0, \frac{T}{\epsilon}]} |w(t) - \tilde{u}(t)|_{H^r} \lesssim \epsilon \underline{T} \sup_{[0, \frac{T}{\epsilon}]} (|r_\epsilon^1|_{H^{r+1/2}} + |r_\epsilon^2|_{H^{r+1}}). \quad (6.10)$$

Furthermore, one immediately checks that w solves

$$\begin{cases} \partial_t \tilde{\zeta} - \mathcal{T}_\mu W + \epsilon f^1(\tilde{\zeta}, W) = \epsilon^2 k_\epsilon^1, \\ \partial_t W + \nabla \tilde{\zeta} + \epsilon \nabla f^2(\tilde{\zeta}, W) = \epsilon^2 \nabla k_\epsilon^2, \end{cases}$$

with initial condition $w|_{t=0} = (\zeta^0, \nabla \psi^0)^T$, and where $k_\epsilon^1 := \frac{1}{\epsilon}(f^1(w) - f^1(\tilde{u}))$, $k_\epsilon^2 := \frac{1}{\epsilon}(f^2(w) - f^2(\tilde{u}))$, and

$$f^1(\zeta, V) := \nabla \cdot (\zeta V) + \mathcal{T}_\mu \nabla (\zeta \mathcal{T}_\mu V), \quad f^2(\zeta, V) := \frac{1}{2}(|V|^2 - (\mathcal{T}_\mu V)^2);$$

from (6.10), one gets in particular that

$$|(k_\epsilon^1, k_\epsilon^2)|_{X_{\underline{T}}^{s+P}} + |(\partial_t k_\epsilon^1, \partial_t k_\epsilon^2)|_{X_{\underline{T}}^{s+P-5/2}} \lesssim C(\underline{T}, |(\zeta, V)|_{X_{\underline{T}}^{s+Q}}), \quad (6.11)$$

provided that Q is large enough.

Using the fact that all the terms in the equation on W are a gradient of a scalar expression, as well as $W|_{t=0}$, it is possible to write $w = (\tilde{\zeta}, \nabla \psi)^T$, and $(\tilde{\zeta}, \psi)$ solves

$$\begin{cases} \partial_t \tilde{\zeta} - \mathcal{T}_\mu \nabla \psi + \epsilon f^1(\tilde{\zeta}, \nabla \psi) = \epsilon^2 k_\epsilon^1, \\ \partial_t \psi + \tilde{\zeta} + \epsilon f^2(\tilde{\zeta}, \nabla \psi) = \epsilon^2 k_\epsilon^2, \end{cases} \quad (6.12)$$

with initial condition $(\tilde{\zeta}, \psi)|_{t=0} = (\zeta^0, \psi^0)$.

Remarking now that $\mathcal{G}[0]\psi = \sqrt{\mu}\mathcal{T}_\mu\nabla\psi$ and writing $U = (\tilde{\zeta}, \psi)$, one can check that (6.12) can be written

$$\partial_t U + \mathcal{L}U + \epsilon\mathcal{A}^{(1)}[U] = \epsilon^2(k_\epsilon^1, k_\epsilon^2)^T, \quad (6.13)$$

where $\mathcal{A}^{(1)}[U]$ is given by the same formula (4.2) as $\mathcal{A}[U]$, but with the Dirichlet-Neumann operator $\mathcal{G}[\varepsilon\tilde{\zeta}]\psi$ replaced by the first order expansion given in Proposition 3.9, and with the $O(\epsilon^2)$ terms neglected. One thus gets

$$\partial_t U + \mathcal{L}U + \epsilon\mathcal{A}[U] = \epsilon^2 H_\epsilon,$$

with $H_\epsilon = (k_\epsilon^1, k_\epsilon^2)^T + \frac{1}{\epsilon}(\mathcal{A}^{(1)}[U] - \mathcal{A}[U])$. From (6.10), Proposition 3.9, and (6.11), one has $(H_\epsilon)_\epsilon$ is uniformly bounded in $C([0, \frac{T}{\epsilon}], X^{s+P}) \cap C^1([0, \frac{T}{\epsilon}], X^{s+P-5/2})$ and we can therefore conclude with Remark 5.4 and Proposition 5.1. \square

6.4.2. A remark on a model used for numerical computations The Dirichlet-Neumann operator is one of the main difficulties in the numerical computation of solutions to the water-waves equations (1.4) because it requires to solve a $d+1$ ($d=1, 2$ is the surface dimension) Laplace equation on a domain which changes at each time step. A common strategy is to replace the full Dirichlet-Neumann operator by an approximation which requires less computations. An efficient method, set forth in [15], consists in replacing the Dirichlet-Neumann operator by its n -th order expansion with respect to the surface elevation ζ . When $n=1$, it turns out that the model thus obtained is exactly the same as the system (6.13) used in the proof of Theorem 6.5.

We can therefore use this theorem to state that: *the precision of the modelization in the numerical computations of [15] is of the same order as the steepness of the wave.*

One will easily check that when the n -th order expansion is used, then the precision is of the same order as the n -th power of the steepness.

A. Nondimensionalization(s) of the equations

Depending on the value of μ , two distinct nondimensionalizations are commonly used in oceanography (see for instance [18]). Namely, with dimensionless quantities denoted with a prime:

– Shallow-water, ie $\mu \ll 1$, one writes

$$\begin{aligned} x &= \lambda x', & y &= \frac{\lambda}{\gamma} y', & z &= dz', & t &= \frac{\lambda}{\sqrt{gd}} t', \\ \zeta &= a\zeta', & \Phi &= \frac{a}{d}\lambda\sqrt{gd}\Phi', & b &= Bb'. \end{aligned} \quad (\text{A.1})$$

– Deep-water, ie $\mu \gg 1$, one writes

$$\begin{aligned} x &= \lambda x', y = \frac{\lambda}{\gamma} y', & z &= \lambda z', t = \frac{\lambda}{\sqrt{g\lambda}} t', \\ \zeta &= a\zeta', \Phi = a\sqrt{g\lambda}\Phi', b = Bb'. \end{aligned} \quad (\text{A.2})$$

Remark that when $\mu \sim 1$, that is when $\lambda \sim d$, both nondimensionalizations are equivalent, we introduce the following general nondimensionalization, which is valid for all $\mu > 0$:

$$\begin{aligned} x &= \lambda x', y = \frac{\lambda}{\gamma} y', & z &= d\nu z', t = \frac{\lambda}{\sqrt{gd\nu}} t', \\ \zeta &= a\zeta', \Phi = \frac{a}{d}\lambda\sqrt{\frac{gd}{\nu}}\Phi', b = Bb', \end{aligned}$$

where ν is a smooth function of μ such that $\nu \sim 1$ when $\mu \ll 1$ and $\nu \sim \mu^{-1/2} (= \lambda/d)$ when $\mu \gg 1$ (say, $\nu = (1 + \sqrt{\mu})^{-1}$).

The equations of motion (1.1) then become (after dropping the primes for the sake of clarity):

$$\begin{cases} \nu^2 \mu \partial_x^2 \Phi + \nu^2 \gamma^2 \mu \partial_y^2 \Phi + \partial_z^2 \Phi = 0, & \frac{1}{\nu}(-1 + \beta b) \leq z \leq \frac{\varepsilon}{\nu} \zeta, \\ -\nu^2 \mu \nabla^\gamma \left(\frac{\beta b}{\nu} \right) \cdot \nabla^\gamma \Phi + \partial_z \Phi = 0, & z = \frac{1}{\nu}(-1 + \beta b), \\ \partial_t \zeta - \frac{1}{\mu \nu^2} \left(-\nu^2 \mu \nabla^\gamma \left(\frac{\varepsilon}{\nu} \zeta \right) \cdot \nabla^\gamma \Phi + \partial_z \Phi \right) = 0, & z = \frac{\varepsilon}{\nu} \zeta, \\ \partial_t \Phi + \frac{1}{2} \left(\frac{\varepsilon}{\nu} |\nabla^\gamma \Phi|^2 + \frac{\varepsilon}{\mu \nu^3} (\partial_z \Phi)^2 \right) + \zeta = 0, & z = \frac{\varepsilon}{\nu} \zeta. \end{cases} \quad (\text{A.3})$$

In order to reduce this set of equations into a system of two evolution equations, define the Dirichlet-Neumann operator $\mathcal{G}_{\mu, \gamma}^\nu \left[\frac{\varepsilon}{\nu} \zeta, \beta b \right]$ as

$$\mathcal{G}_{\mu, \gamma}^\nu \left[\frac{\varepsilon}{\nu} \zeta, \beta b \right] \psi = \sqrt{1 + \left| \nabla \left(\frac{\varepsilon}{\nu} \zeta \right) \right|^2} \partial_n \Phi|_{z = \frac{\varepsilon}{\nu} \zeta},$$

with Φ solving the boundary value problem

$$\begin{cases} \nu^2 \mu \partial_x^2 \Phi + \nu^2 \gamma^2 \mu \partial_y^2 \Phi + \partial_z^2 \Phi = 0, & \frac{1}{\nu}(-1 + \beta b) \leq z \leq \frac{\varepsilon}{\nu} \zeta, \\ \Phi|_{z = \frac{\varepsilon}{\nu} \zeta} = \psi, & \partial_n \Phi|_{z = \frac{1}{\nu}(-1 + \beta b)} = 0, \end{cases}$$

(as always in this paper, $\partial_n \Phi$ stands for the upwards conormal derivative associated to the elliptic equation). As remarked in [52, 17, 16], the equations (A.3) are equivalent to a set of two equations on the free surface parameterization ζ and the trace of the velocity potential at the surface $\psi = \Phi|_{z = \varepsilon/\nu \zeta}$ involving the Dirichlet-Neumann operator.

Namely,

$$\begin{cases} \partial_t \zeta - \frac{1}{\mu \nu^2} \mathcal{G}_{\mu, \gamma}^{\nu}[\frac{\varepsilon}{\nu} \zeta, \beta b] \psi = 0, \\ \partial_t \psi + \zeta + \frac{\varepsilon}{2\nu} |\nabla^{\gamma} \psi|^2 - \frac{\varepsilon \mu}{\nu^3} \frac{(\frac{1}{\mu} \mathcal{G}_{\mu, \gamma}^{\nu}[\frac{\varepsilon}{\nu} \zeta, \beta b] \psi + \nu \nabla^{\gamma}(\varepsilon \zeta) \cdot \nabla^{\gamma} \psi)^2}{2(1 + \varepsilon^2 \mu |\nabla^{\gamma} \zeta|^2)} = 0. \end{cases} \quad (\text{A.4})$$

In order to derive the system (1.4), let $\mathcal{G}_{\mu, \gamma}[\varepsilon \zeta, \beta b] \cdot$ be the Dirichlet-Neumann operator $\mathcal{G}_{\mu, \gamma}^{\nu}[\frac{\varepsilon}{\nu} \zeta, \beta b] \cdot$ corresponding to the case $\nu = 1$. One will easily check that

$$\forall \nu > 0, \quad \mathcal{G}_{\mu, \gamma}[\varepsilon \zeta, \beta b] = \frac{1}{\nu} \mathcal{G}_{\mu, \gamma}^{\nu}[\frac{\varepsilon}{\nu} \zeta, \beta b],$$

so that plugging this relation into (A.4) yields

$$\begin{cases} \partial_t \zeta - \frac{1}{\mu \nu} \mathcal{G}_{\mu, \gamma}[\varepsilon \zeta, \beta b] \psi = 0, \\ \partial_t \psi + \zeta + \frac{\varepsilon}{2\nu} |\nabla^{\gamma} \psi|^2 - \frac{\varepsilon \mu}{\nu} \frac{(\mathcal{G}_{\mu, \gamma}[\varepsilon \zeta, \beta b] \psi + \nabla^{\gamma}(\varepsilon \zeta) \cdot \nabla^{\gamma} \psi)^2}{2(1 + \varepsilon^2 \mu |\nabla^{\gamma} \zeta|^2)} = 0. \end{cases}$$

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