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Classification of the virtually cyclic subgroups of the pure braid groups of the projective plane

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Abstract

We classify the (finite and infinite) virtually cyclic subgroups of the pure braid groups $P_n(\mathbb{R}P^2)$ of the projective plane. The maximal finite subgroups of $P_n(\mathbb{R}P^2)$ are isomorphic to the quaternion group of order 8 if $n = 3$, and to $\mathbb{Z}_4$ if $n \geqslant 4$. Further, for all $n \geqslant 3$, up to isomorphism, the following groups are the infinite virtually cyclic subgroups of $P_n(\mathbb{R}P^2)$: $\mathbb{Z}$, $\mathbb{Z}_2 \times \mathbb{Z}$ and the amalgamated product $\mathbb{Z}_4 *_{\mathbb{Z}_2} \mathbb{Z}_4$.

1 Introduction

The braid groups B_n of the plane were introduced by E. Artin in 1925 [A1, A2]. Braid groups of surfaces were studied by Zariski [Z]. They were later generalised by Fox to braid groups of arbitrary topological spaces via the following definition [FoN]. Let M be a compact, connected surface, and let $n \in \mathbb{N}$. We denote the set of all ordered n-tuples of distinct points of M, known as the n^{th} *configuration space of* M, by:

$$F_n(M) = \left\{ (p_1, \ldots, p_n) \mid p_i \in M \text{ and } p_i \neq p_j \text{ if } i \neq j \right\}.$$

Configuration spaces play an important rôle in several branches of mathematics and have been extensively studied, see [CG, FH] for example.

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The symmetric group S_n on n letters acts freely on $F_n(M)$ by permuting coordinates. The corresponding quotient will be denoted by $D_n(M)$. The n^{th} *pure braid group* $P_n(M)$ (respectively the n^{th} *braid group* $B_n(M)$) is defined to be the fundamental group of $F_n(M)$ (respectively of $D_n(M)$).

Together with the 2-sphere $\mathbb{S}^2$, the braid groups of the real projective plane $\mathbb{R}P^2$ are of particular interest, notably because they have non-trivial centre [VB, GG1], and torsion elements [VB, Mu]. Indeed, Van Buskirk showed that among the braid groups of compact, connected surfaces, $B_n(\mathbb{S}^2)$ and $B_n(\mathbb{R}P^2)$ are the only ones to have torsion [VB]. Let us recall briefly some of the properties of $B_n(\mathbb{R}P^2)$ [GG1, Mu, VB].

If $\mathbb{D}^2 \subseteq \mathbb{R}P^2$ is a topological disc, there is a group homomorphism $\iota\colon\ B_n \longrightarrow B_n(\mathbb{R}P^2)$ induced by the inclusion. If $\beta \in B_n$ then we shall denote its image $\iota(\beta)$ simply by β. A presentation of $B_n(\mathbb{R}P^2)$ was given in [VB], and of $P_n(\mathbb{R}P^2)$ in [GG3]. The first two braid groups of $\mathbb{R}P^2$ are finite: $B_1(\mathbb{R}P^2) = P_1(\mathbb{R}P^2) \cong \mathbb{Z}_2$, $P_2(\mathbb{R}P^2)$ is isomorphic to the quaternion group $\mathcal{Q}_8$ of order 8 and $B_2(\mathbb{R}P^2)$ is isomorphic to the generalised quaternion group of order 16. For $n \geqslant 3$, $B_n(\mathbb{R}P^2)$ is infinite. The pure braid group $P_3(\mathbb{R}P^2)$ is isomorphic to a semi-direct product of a free group of rank 2 by $\mathcal{Q}_8$ [VB]; an explicit action was given in [GG1] (see also the proof of Proposition 18).

The so-called 'full twist' braid of $B_n(\mathbb{R}P^2)$ is defined by $\Delta_n = (\sigma_1 \cdots \sigma_{n-1})^n$. For $n \geqslant 2$, Δ_n is the unique element of $B_n(\mathbb{R}P^2)$ of order 2 [GG1], and it generates the centre of $B_n(\mathbb{R}P^2)$ [Mu]. The finite order elements of $B_n(\mathbb{R}P^2)$ were characterised by Murasugi [Mu] (see Theorem 6, Section 3), however their orders are not clear, even for elements of $P_n(\mathbb{R}P^2)$. In [GG1], we proved that for $n \geqslant 2$, the torsion of $P_n(\mathbb{R}P^2)$ is 2 and 4, and that of $B_n(\mathbb{R}P^2)$ is equal to the divisors of $4n$ and $4(n-1)$.

The classification of the finite subgroups of $B_n(\mathbb{S}^2)$ and $B_n(\mathbb{R}P^2)$ is an interesting problem, and helps us to better understand their group structure. In the case of $\mathbb{S}^2$, this was undertaken in [GG2, GG4]. It is natural to ask which finite groups are realised as subgroups of $B_n(\mathbb{R}P^2)$. As for $B_n(\mathbb{S}^2)$, one common property of such subgroups is that they are finite periodic groups of cohomological period 2 or 4. Indeed, by Proposition 6 of [GG1], the universal covering X of $F_n(\mathbb{R}P^2)$ is a finite-dimensional complex which has the homotopy type of $\mathbb{S}^3$. Thus any finite subgroup of $B_n(\mathbb{R}P^2)$ acts freely on X, and so has period 2 or 4 by Proposition 10.2, Section 10, Chapter VII of [Br]. Since Δ_n is the unique element of order 2 of $B_n(\mathbb{R}P^2)$, and it generates the centre $Z(B_n(\mathbb{R}P^2))$, the Milnor property must be satisfied for any finite subgroup of $B_n(\mathbb{R}P^2)$.

In Section 2, we start by determining the maximal finite subgroups of $P_n(\mathbb{R}P^2)$:

PROPOSITION 1. *Up to isomorphism, the maximal finite subgroups of $P_n(\mathbb{R}P^2)$ are:*

(a) $\mathbb{Z}_2$ *if* $n = 1$.

(b) $\mathcal{Q}_8$ *if* $n = 2, 3$.

(c) $\mathbb{Z}_4$ *if* $n \geqslant 4$.

In Section 3, we simplify Murasugi's characterisation of the torsion elements of $B_n(\mathbb{R}P^2)$ (see Proposition 7), which enables us to show that within $B_n(\mathbb{R}P^2)$, there are two conjugacy classes of subgroups isomorphic to $\mathbb{Z}_4$ lying in $P_n(\mathbb{R}P^2)$ (see Proposition 9).

The rest of the paper is devoted to determining the infinite virtually cyclic subgroups of $P_n(\mathbb{R}P^2)$ (recall that a group is said to be *virtually cyclic* if it has a cyclic subgroup of finite index). This was initially motivated by a question of Stratos Prassidis concerning the determination of the algebraic K-theory of the braid groups of $\mathbb{S}^2$ and

$\mathbb{R}P^2$. It has been shown recently that the full and pure braid groups of these surfaces satisfy the Fibered Isomorphism Conjecture of T. Farrell and L. Jones [BJPL, JPML1, JPML2]. This implies that the algebraic K-theory groups of their group rings may be computed by means of the algebraic K-theory groups of their virtually cyclic subgroups via the assembly maps. More information on these topics may be found in [BLR, FJ, JP]. As well as helping us to better understand these braid groups, this provides us with additional reasons to find their virtually cyclic subgroups.

In Section 4, we recall the criterion due to Wall of an infinite virtually cyclic group G as one which has a finite normal subgroup F such that G/F is isomorphic to $\mathbb{Z}$ or to the free product $\mathbb{Z}_2 * \mathbb{Z}_2$ (this being the case, we shall say that G is of *Type I* or *Type II* respectively). Wall's result enables us to establish a list of the possible infinite virtually cyclic subgroups of a given group G, if one knows its finite subgroups (which is the case for $P_n(\mathbb{R}P^2)$). The real difficulty lies in deciding whether the groups belonging to this list are effectively realised as subgroups of G. In Lemma 16 we give a useful criterion to decide whether a group is of Type II.

In the case of $P_n(\mathbb{S}^2)$, since there are only two finite subgroups for $n \geqslant 3$, the trivial group and that generated by the full twist (which are the elements of the centre of $P_n(\mathbb{S}^2)$), it is then easy to see that its infinite virtually cyclic subgroups are isomorphic to $\mathbb{Z}$ or to $\mathbb{Z}_2 \times \mathbb{Z}$.

Since the structures of the finite subgroups of $P_3(\mathbb{R}P^2)$ and $P_n(\mathbb{R}P^2)$ differ, we separate the discussion of their virtually cyclic subgroups. However, it turns out that up to isomorphism, they have the same infinite virtually cyclic subgroups:

THEOREM 2. *Let $n \geqslant 3$. Up to isomorphism, the infinite virtually cyclic subgroups of $P_n(\mathbb{R}P^2)$ are $\mathbb{Z}$, $\mathbb{Z}_2 \times \mathbb{Z}$ and $\mathbb{Z}_4 *_{\mathbb{Z}_2} \mathbb{Z}_4$.*

For $n = 3$, the result will be proved in Theorem 17 (see Section 5.1), while for $n \geqslant 4$, it will be proved in Theorem 19 (see Section 5.2).

As a immediate corollary of Proposition 1 and Theorem 2, we obtain the classification of the virtually cyclic subgroups of $P_n(\mathbb{R}P^2)$:

COROLLARY 3. *Let $n \in \mathbb{N}$. Up to isomorphism, the virtually cyclic subgroups of $P_n(\mathbb{R}P^2)$ are:*

(a) The trivial group $\{e\}$ and $\mathbb{Z}_2$ if $n = 1$.
(b) $\{e\}$, $\mathbb{Z}_2$, $\mathbb{Z}_4$ and $\mathcal{Q}_8$ if $n = 2$.
(c) $\{e\}$, $\mathbb{Z}_2$, $\mathbb{Z}_4$, $\mathcal{Q}_8$, $\mathbb{Z}$, $\mathbb{Z}_2 \times \mathbb{Z}$ and $\mathbb{Z}_4 *_{\mathbb{Z}_2} \mathbb{Z}_4$ if $n = 3$.
(d) $\{e\}$, $\mathbb{Z}_2$, $\mathbb{Z}_4$, $\mathbb{Z}$, $\mathbb{Z}_2 \times \mathbb{Z}$ and $\mathbb{Z}_4 *_{\mathbb{Z}_2} \mathbb{Z}_4$ if $n \geqslant 4$.

One of the key results needed in the proof of Theorem 2 is that $P_n(\mathbb{R}P^2)$ has no subgroup isomorphic to $\mathbb{Z}_4 \times \mathbb{Z}$. This fact allows us to eliminate several possible Type I and Type II subgroups, and has the following interesting corollary which we prove at the end of Section 5.2:

COROLLARY 4. *Let $n \geqslant 2$, and let $x \in P_n(\mathbb{R}P^2)$ be an element of order 4. Then its centraliser $Z_{P_n(\mathbb{R}P^2)}(x)$ in $P_n(\mathbb{R}P^2)$ is equal to $\langle x \rangle$.*

The study of the finite subgroups of $B_n(\mathbb{R}P^2)$ and of the infinite virtually cyclic subgroups of $B_n(\mathbb{S}^2)$ and $B_n(\mathbb{R}P^2)$ is the subject of work in progress.

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2 Finite subgroups of $P_n(\mathbb{R}P^2)$

In this section, we characterise the finite subgroups of $P_n(\mathbb{R}P^2)$ by proving Proposition 1. In Theorem 4, Corollary 19 and Proposition 23 respectively of [GG1], we obtained the following results:

THEOREM 5 ([GG1]). *Let* $n \geqslant 2$. *Then:*

(a) $B_n(\mathbb{R}P^2)$ *has an element of order* ℓ *if and only if* ℓ *divides either* $4n$ *or* $4(n-1)$.
(b) the (non-trivial) torsion of $P_n(\mathbb{R}P^2)$ *is precisely* 2 *and* 4.
(c) the full twist Δ_n *is the unique element of* $B_n(\mathbb{R}P^2)$ *of order* 2.

It was shown in [Mu] that Δ_n generates the centre of $B_n(\mathbb{R}P^2)$. Using the projection $P_{n+1}(\mathbb{R}P^2) \longrightarrow P_n(\mathbb{R}P^2)$ given by forgetting the last string, and induction, one may check that Δ_n also generates the centre of $P_n(\mathbb{R}P^2)$.

It follows from Theorem 5 that the maximal (finite) cyclic subgroups of $B_n(\mathbb{R}P^2)$ are isomorphic to $\mathbb{Z}_{4n}$ or $\mathbb{Z}_{4(n-1)}$, and that those of $P_n(\mathbb{R}P^2)$ are isomorphic to $\mathbb{Z}_4$.

Proof of Proposition 1. Since $P_1(\mathbb{R}P^2) \cong \mathbb{Z}_2$ and $P_2(\mathbb{R}P^2) \cong \mathcal{Q}_8$, the result follows easily for $n = 1, 2$. So suppose that $n \geqslant 3$. Consider the Fadell-Neuwirth short exact sequence:

$$1 \longrightarrow P_{n-2}(\mathbb{R}P^2 \setminus \{x_1, x_2\}) \longrightarrow P_n(\mathbb{R}P^2) \xrightarrow{p_*} P_2(\mathbb{R}P^2) \longrightarrow 1,$$

where p_* corresponds geometrically to forgetting the last $n-2$ strings. Let H be a finite subgroup of $P_n(\mathbb{R}P^2)$. Then $p_*|_H$ is injective. To see this, suppose that $x, y \in H$ are such that $p_*(x) = p_*(y)$. Then $xy^{-1} \in H \cap \operatorname{Ker}(p_*)$. But $\operatorname{Ker}(p_*)$ is torsion free, so $x = y$, which proves the claim. In particular, $|H| \leqslant 8$, and since the torsion of $P_n(\mathbb{R}P^2)$ is 2 and 4, and $P_n(\mathbb{R}P^2)$ has a unique element of order 2, it follows that H is isomorphic to one of $\{e\}$, $\mathbb{Z}_2$ (so is $\langle \Delta_n \rangle$), $\mathbb{Z}_4$ or $\mathcal{Q}_8$. If $n = 3$ then the above short exact sequence splits [VB], and so $P_3(\mathbb{R}P^2) \cong \mathbb{F}_2 \rtimes \mathcal{Q}_8$. Hence the finite subgroups of $P_3(\mathbb{R}P^2)$ are those of $\mathcal{Q}_8$. Now let $n \geqslant 4$. Then $\mathcal{Q}_8$ is not realised as a subgroup of $P_n(\mathbb{R}P^2)$. For suppose that $H < P_n(\mathbb{R}P^2)$, where $H \cong \mathcal{Q}_8$. From above, it follows that $p_*|_H$ is an isomorphism, so p_* admits a section. But this would contradict Theorem 3 of [GG1]. The result follows from Theorem 5(b). □

3 Murasugi's characterisation of the finite order elements of $B_n(\mathbb{R}P^2)$

In this section, our aim is to reorganise the classification of Murasugi [Mu] of the finite order elements of $B_n(\mathbb{R}P^2)$ in a form that shall be more convenient for our purposes. As well as being useful in its own right, we shall make use of this reorganisation to prove that there are precisely two conjugacy classes in $B_n(\mathbb{R}P^2)$ of subgroups of pure braids of order 4. This shall be applied in the classification of the virtually cyclic subgroups of $P_n(\mathbb{R}P^2)$, notably to show that $P_n(\mathbb{R}P^2)$ has no subgroup isomorphic to $\mathbb{Z}_4 \times \mathbb{Z}$ (see Propositions 18 and 22).

Up to conjugacy, the finite order elements of $B_n(\mathbb{R}P^2)$ may be characterised as follows:

THEOREM 6 ([MU]). *Every element of $B_n(\mathbb{R}P^2)$ of finite order is conjugate to one of the following elements:*

$$\begin{aligned} A_1(n,r,s,q) &= (\rho_r\sigma_{r-1}\cdots\sigma_1)^s(\sigma_{r+1}\cdots\sigma_{n-1})^q \\ A_2(n,r,s,q) &= (\rho_r\sigma_{r-1}\cdots\sigma_1)^s(\sigma_{r+1}\cdots\sigma_{n-1}\sigma_{r+1})^q, \end{aligned}$$

where $(n-r,q) \approx (2r,s)$ in the first case, and $(n-r-1,q) \approx (2r,s)$ in the second. The relation $\approx$ is defined by:

$$(a,b) \approx (c,d) \text{ if there exists } (m,k) \neq (0,0) \text{ such that } m(a,b) = k(c,d).$$

This characterisation may be simplified as follows:

PROPOSITION 7. *Let $n \geqslant 2$. Then every finite order element of $B_n(\mathbb{R}P^2)$ is conjugate to a power of some $A_i(n,r,2r/l,p/l)$, $i=1,2$, where*

$$p = \begin{cases} n-r & \text{if } i=1 \\ n-r-1 & \text{if } i=2, \end{cases}$$

and $l = \gcd(p,2r)$.

REMARK 8. The characterisation of the finite order elements of $B_n(\mathbb{R}P^2)$ thus appears more transparent than that given in Theorem 6, in the sense that these elements may be determined directly from n and r (without involving the integers s,q, nor the relation $\approx$).

Proof of Proposition 7. From Theorem 6, every finite order element of $B_n(\mathbb{R}P^2)$ is conjugate to some $A_i(n,r,s,q)$, where $m(p,q) = k(2r,s)$. Thus every finite order element of $B_n(\mathbb{R}P^2)$ is conjugate to a power of some $A_i(n,r,s/\lambda,q/\lambda)$, where $\lambda = \gcd(s,q)$. We consider the following cases separately:

(a) If $r=0$ then $m=s=0$ and $k \neq 0$. Then $A_i(n,r,s/\lambda,q/\lambda) = A_i(n,0,0,1) = A_i(n,r,2r/l,p/l)$ as required.

(b) If either $i=2$ and $r=n-1$, or $i=1$ and $r=n$ then $p=0$. Thus $k=0$ and $m \neq 0$, so $q=0$. Hence $A_i(n,r,s/\lambda,q/\lambda) = A_i(n,r,1,0) = A_i(n,r,2r/l,p/l)$ as required.

(c) Suppose that either $i \in \{1,2\}$ and $1 \leqslant r \leqslant n-2$, or $i=1$ and $r=n-1$, so $p \neq 0$. Since $mp = 2rk$, $mq = ks$ and $(m,k) \neq (0,0)$, it follows that $m,k \neq 0$, $m/k = s/q = 2r/p$ (if $s=0$ then $q=0$, but then $A_i(n,r,s,q)$ would be trivial), and so $s'p = 2rq'$, where $s' = s/\lambda$ and $q' = q/\lambda$ are coprime. We thus obtain $2r = ls'$ and $p = lq'$, so $\frac{(2r,p)}{l} = \frac{(s,q)}{\lambda}$. Hence every finite order element of $B_n(\mathbb{R}P^2)$ is conjugate to a power of some $A_i(n,r,2r/l,p/l)$.

□

In Proposition 26 of [GG1], we proved that the following elements of $B_n(\mathbb{R}P^2)$:

$$a = \sigma_{n-1}^{-1} \cdots \sigma_1^{-1} \rho_1$$
$$b = \sigma_{n-2}^{-1} \cdots \sigma_1^{-1} \rho_1$$

are of order $4n$ and $4(n-1)$ respectively. By Remark 27 of [GG1], we have

$$\begin{cases} \alpha = a^n = \rho_n \cdots \rho_1 \\ \beta = b^{n-1} = \rho_{n-1} \cdots \rho_1. \end{cases} \tag{1}$$

It is clear that α and β are pure braids of order 4. Using Proposition 7, we now show that any element in $P_n(\mathbb{R}P^2)$ of order 4 is conjugate in $B_n(\mathbb{R}P^2)$ to one of these two elements (or their inverses):

PROPOSITION 9. *Let $n \geqslant 2$. Then every element of order 4 of $P_n(\mathbb{R}P^2)$ is conjugate in $B_n(\mathbb{R}P^2)$ to one of α and β or their inverses.*

REMARK 10. Since the Abelianisation of $B_n(\mathbb{R}P^2)$ is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$ (cf. the presentation of $B_n(\mathbb{R}P^2)$ given in [VB]), and the generator σ_i, $1 \leqslant i \leqslant n-1$ (resp. ρ_j, $1 \leqslant j \leqslant n$) is sent to the generator of the first (resp. second) $\mathbb{Z}_2$-factor, α and β are not conjugate in $B_n(\mathbb{R}P^2)$.

Proof of Proposition 9. For given n and r, define

$$\omega = \begin{cases} \sigma_{r+1} \cdots \sigma_{n-1} & \text{if } i=1 \\ \sigma_{r+1} \cdots \sigma_{n-1} \sigma_{r+1} & \text{if } i=2. \end{cases}$$

Up to conjugacy and powers, the finite order elements of $B_n(\mathbb{R}P^2)$ appear in one of the three cases given in the proof of Proposition 7. Among these elements, we search for pure braids of order 4.

(a) We have that $A_i(n,0,0,1) = \omega$. If $i=1$ then $\omega = \sigma_1 \cdots \sigma_{n-1}$ is of order $2n$, and its permutation is $(1,n,\cdots,2)$. The smallest $j \geqslant 1$ for which ω^j becomes pure is thus $j=n$, but then $\omega^n = \Delta_n$. If $i=2$ then $\omega = \sigma_1 \cdots \sigma_{n-1}\sigma_1$ is of order $2(n-1)$, its permutation is $(1,n,\cdots,3)$, and $\omega^{n-1} = \Delta_n$. The smallest $j \geqslant 1$ for which ω^j becomes pure is indeed $j = n-1$. In either case, the powers of ω yield no pure braids of order 4.

(b) Suppose that either $i=2$ and $r=n-1$, or $i=1$ and $r=n$. Using the relation $\rho_{i+1} = \sigma_i^{-1}\rho_i\sigma_i^{-1}$, $1 \leqslant i \leqslant n-1$ in $B_n(\mathbb{R}P^2)$ (see [VB, Mu, GG1]), we have that

$$A_i(n,r,1,0) = \rho_r \sigma_{r-1} \cdots \sigma_1 = \begin{cases} b & \text{if } r = n-1 \\ a & \text{if } r = n. \end{cases}$$

Studying the permutations of a and b, we observe that the smallest power of each of these elements which is a pure braid is $b^{n-1} = \beta$ and $a^n = \alpha$ (which are of order 4).

(c) If $i = 1$ and $r = n-1$ then we have $A_1(n,r,2r/l,p/l) = A_1(n,n-1,2(n-1),1) = (\rho_{n-1}\sigma_{n-2}\cdots\sigma_1)^{2(n-1)} = b^{2(n-1)} = \Delta_n$ which is of order 2. So we may suppose that $1 \leqslant r \leqslant n-2$. Let $\xi = \rho_r\sigma_{r-1}\cdots\sigma_1$ and $y_i = A_i(n,r,2r/l,p/l) = \xi^{2r/l}\omega^{p/l}$. By equation (5.1) at the bottom of page 79 of [Mu], $y_i^l = \xi^{2r}\omega^p = \Delta_n$, so the order of y_i divides $2l$. Now the permutation of ω is a p-cycle, and since l divides p, the permutation of $\omega^{p/l}$ is a product of p/l disjoint l-cycles. Thus the smallest $j \geqslant 1$ for which $(\omega^{p/l})^j$ becomes a pure braid is $j = l$. But y_i^l is the full twist, and so the only pure braids among the powers of y_i are Δ_n and the identity. In particular, no power of y_i can yield a pure braid of order 4.

We conclude that up to conjugacy and inverses, the only elements of $P_n(\mathbb{R}P^2)$ of order 4 are α and β, and that they only occur in the second case ($i = 2$ and $r = n-1$, or $i = 1$ and $r = n$). This completes the proof of the proposition. ☐

As a corollary of the proof of Proposition 9, we are able to determine explicitly the orders of the torsion elements given by Proposition 7 as follows.

COROLLARY 11. *Let $n \geqslant 2$. Then for all $i = 1,2$, and all $r = 0,1,\ldots,n$, the element $y_i = A_i(n,r,2r/l,p/l) = \xi^{2r/l}\omega^{p/l}$ of $B_n(\mathbb{R}P^2)$ is of order $2l = 2\gcd(p,2r)$.*

Proof. If $1 \leqslant r \leqslant n-2$, we deduce from the above proof that y_i is of order $2l$. If $r = 0$ then $l = p$, and $y_i = \omega$ which is of order $2l$. If $i = 1$ and $r = n-1$ then $l = 1$, and $y_i = \Delta_n$ is indeed of order 2. Finally, if $i = 2$ and $r = n-1$ or $i = 1$ and $r = n$ then $p = 0$ and $l = 2r$. Hence $y_1 = \xi = a$ which is of order $4n$, and $y_2 = \xi = b$ which is of order $4(n-1)$. The result follows. ☐

Set $l_1(n,r) = \gcd(2r, n-r)$ and $l_2(n,r) = \gcd(2r, n-r-1)$. For $0 \leqslant r \leqslant n-1$, clearly $l_2(n,r) = l_1(n-1,r)$, so it suffices to know the values of $l_1(n,r)$. An alternative manner to express the order $2l_1(n,r)$ of y_1 is given by the following:

COROLLARY 12. *Let $n \geqslant 2$. Let*

$$k = \begin{cases} \frac{n-r}{2} & \text{if } n \text{ and } r \text{ are even} \\ n-r & \text{otherwise.} \end{cases}$$

Then

$$2l_1(n,r) = \begin{cases} 2\gcd(n,r) & \text{if } k \text{ is odd} \\ 4\gcd(n,k) & \text{if } k \text{ is even.} \end{cases}$$

Proof. Suppose first that n and r are even, so $k = \frac{n-r}{2}$. If k is odd then $l_1(n,r) = \gcd(2r, n-r) = 2\gcd(r,k) = 2\gcd\left(\frac{r}{2}, \frac{n-r}{2}\right) = \gcd(r, n-r) = \gcd(n,r)$. If k is even then $l_1(n,r) = \gcd(2r, n-r) = \gcd(2n, n-r) = 2\gcd\left(n, \frac{n-r}{2}\right) = 2\gcd(n,k)$, which yields the result in this case.

Now suppose that at least one of n and r is odd, so $k = n-r$. If k is odd then $l_1(n,r) = \gcd(2r, n-r) = \gcd(r, n-r) = \gcd(n,r)$. If k is even then both n and r are odd, and $l_1(n,r) = \gcd(2r, n-r) = 2\gcd(r, n-r) = 2\gcd(n, n-r) = 2\gcd(n,k)$. ☐

4 Virtually cyclic groups

We start this section by recalling the definition of a virtually cyclic group.

DEFINITION 13. A group is *virtually cyclic* if it contains a cyclic subgroup of finite index.

REMARKS 14.
(a) Every finite group is virtually cyclic.
(b) Every infinite virtually cyclic group contains a normal subgroup of finite index.

The following criterion is due to Wall:

THEOREM 15 ([P, W]). *Let G be a group. Then the following are equivalent.*
(a) G is a group with two ends.
(b) G is an infinite virtually cyclic group.
(c) G has a finite normal subgroup F such that G/F is $\mathbb{Z}$ or $\mathbb{Z}_2 * \mathbb{Z}_2$.
Equivalently, G is of the form:
(i) $F \rtimes \mathbb{Z}$, or
(ii) $G_1 *_F G_2$, where $[G_i : F] = 2$ for $i = 1, 2$.

If a virtually cyclic group G satisifies (i), we shall say that it is of *Type I*, while if it satisifies (ii), we shall say that it is of *Type II*.

We have already obtained the finite subgroups of $P_n(\mathbb{R}P^2)$ in Proposition 1. Theorem 15 allows us to establish a list of their *possible* infinite virtually cyclic subgroups. However, the difficulty is to decide whether the groups belonging to this list are effectively realised as subgroups of $P_n(\mathbb{R}P^2)$.

The following lemma gives a practical criterion for deciding whether a given infinite group is virtually cyclic of Type II, and shall be applied frequently in what follows.

LEMMA 16. *Let $G = G_1 *_F G_2$ be a virtually cyclic group of Type II, and let $\varphi\colon G_1 *_F G_2 \longrightarrow H$ be a homomorphism such that the restriction of φ to each G_i is injective. Then φ is injective if and only if $\varphi(G)$ is infinite.*

Proof. Clearly the given condition is necessary. So suppose that $\varphi(G)$ is infinite. By Theorem 15, we have the short exact sequence

$$1 \longrightarrow F \longrightarrow G_1 *_F G_2 \longrightarrow \mathbb{Z}_2 * \mathbb{Z}_2 \longrightarrow 1.$$

Let x (resp. y) denote the generator of the first (resp. second) copy of $\mathbb{Z}_2$. Then $\left\langle xy^{-1} \right\rangle \cong \mathbb{Z}$ is a normal subgroup of $\mathbb{Z}_2 * \mathbb{Z}_2$, and $\mathbb{Z}_2 * \mathbb{Z}_2 \cong \mathbb{Z} \rtimes \mathbb{Z}_2$, where $\mathbb{Z}_2$ may be taken as being generated by the $\left\langle xy^{-1} \right\rangle$-coset of x.

Let K be the preimage of $\left\langle xy^{-1} \right\rangle$ under the projection $G \longrightarrow \mathbb{Z}_2 * \mathbb{Z}_2$. Then we have a short exact sequence

$$1 \longrightarrow F \longrightarrow K \longrightarrow \mathbb{Z} \longrightarrow 1,$$

and thus $K \cong F \rtimes \mathbb{Z}$ is virtually cyclic of Type I, and is of index 2 in G. This gives rise to the following commutative diagram of short exact sequences:

$$\begin{array}{ccccccccc}
 & & & & 1 & & 1 & & \\
 & & & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & F & \hookrightarrow & K & \longrightarrow & \mathbb{Z} & \longrightarrow & 1\\
 & & \| & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & F & \hookrightarrow & G & \longrightarrow & \mathbb{Z}_2 * \mathbb{Z}_2 & \longrightarrow & 1\\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \mathbb{Z}_2 & = & \mathbb{Z}_2 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 1 & & 1 & &
\end{array}$$

Let $w \in G_1 \backslash F$, and consider the image of w in $G_1 *_F G_2$, which by abuse of notation we also denote by w. Since w is of finite order it cannot belong to K (for then it would have to be mapped onto 0 in $\mathbb{Z}$, which is impossible), and hence $G = K \coprod wK$. So by hypothesis, $\varphi(K)$ is infinite.

Notice that the result that we are aiming to prove is true for K i.e. if we consider $\varphi|_K$, and suppose that the restriction of φ to F is injective then the fact that $\varphi(K)$ is infinite implies that $\varphi|_K$ is injective. Indeed, identifying K with $F \rtimes \mathbb{Z}$, let $k = (x, m) \in K$ belong to the kernel of $\varphi|_K$, where $x \in F$ and $m \in \mathbb{Z}$. Since $\varphi(K)$ is infinite, $\varphi|_{\mathbb{Z}}$ is injective. If $m \neq 0$ then $e_H = \varphi(k) = \varphi((x, 0)).\,\varphi((e_G, m))$. But $\varphi((e_G, m))$ is of infinite order, while $\varphi((x, 0))$ is of finite order, a contradiction. So $m = 0$, $x = e_F$ by injectivity of $\varphi|_F$, and so $\varphi|_K$ is injective.

Now $F \subset G_1$, so $\varphi|_F$ is injective. It follows then from the previous paragraph that $\varphi|_K$ is injective. Furthermore, $\varphi|_{wK}$ is injective since any two elements differ by an element of K.

Finally, to prove the result, it suffices to show that for all $k, k' \in K$, $\varphi(k) \neq \varphi(wk')$, or equivalently that for all $k'' \in K$, $\varphi(w) \neq \varphi(k'')$. Suppose on the contrary that there exists $k'' \in K$ such that $\varphi(w) = \varphi(k'')$. Then $\varphi(w^2) = \varphi(k''^2)$. Now $w^2 \in K$ since K is of index 2 in G, and so $w^2 = k''^2$ by injectivity of $\varphi|_K$. Thus $k'' \in K$ is of finite order, and so belongs to F. Hence $w, k'' \in G_1$, $w \neq k''$ (as $w \in G_1 \backslash F$), and so $\varphi(w) \neq \varphi(k'')$ by injectivity of $\varphi|_{G_1}$. This yields a contradiction. Hence φ is injective. ☐

5 Virtually cyclic subgroups of $P_n(\mathbb{R}P^2)$

We now turn to the study of the virtually cyclic subgroups of $P_n(\mathbb{R}P^2)$. As $P_1(\mathbb{R}P^2) \cong \mathbb{Z}_2$ and $P_2(\mathbb{R}P^2) \cong \mathcal{Q}_8$, it is trivial to determine their virtually cyclic subgroups. We thus suppose from now on that $n \geqslant 3$. Since the structure of the finite subgroups differ for $n = 3$ and $n \geqslant 4$, we treat these two cases separately. Further, by Proposition 1, up to isomorphism, we already know their finite subgroups. So in what follows, we shall seek their infinite virtually cyclic subgroups.

5.1 Virtually cyclic subgroups of $P_3(\mathbb{R}P^2)$

In this section, we prove Theorem 17 which is the case $n = 3$ of Theorem 2.

THEOREM 17. *Up to isomorphism, the infinite virtually cyclic subgroups of $P_3(\mathbb{R}P^2)$ are $\mathbb{Z}$, $\mathbb{Z}_2 \times \mathbb{Z}$ and $\mathbb{Z}_4 *_{\mathbb{Z}_2} \mathbb{Z}_4$.*

The key result needed to determine the Type I subgroups of $P_3(\mathbb{R}P^2)$ is the following:

PROPOSITION 18. *The pure braid group $P_3(\mathbb{R}P^2)$ has no subgroup isomorphic to $\mathbb{Z}_4 \times \mathbb{Z}$.*

Proof of Proposition 18. Suppose that $P_3(\mathbb{R}P^2)$ possesses a subgroup G which is isomorphic to $\mathbb{Z}_4 \times \mathbb{Z}$. By [VB], the Fadell-Neuwirth short exact sequence

$$1 \longrightarrow P_1(\mathbb{R}P^2 \setminus \{x_1, x_2\}) \longrightarrow P_3(\mathbb{R}P^2) \xrightarrow{p_*} P_2(\mathbb{R}P^2) \longrightarrow 1$$

splits. Using the notation of page 765 of [GG1], the kernel is a free group which we write as $\mathbb{F}_2 = \mathbb{F}_2(x, y)$, where $x = \rho_3$ and $y = \rho_3^{-1} B_{2,3}$. The quotient $P_2(\mathbb{R}P^2)$ is isomorphic to $\mathcal{Q}_8$, and is generated by ρ_1 and ρ_2 which are of order 4. In Corollary 11 of [GG1], we exhibited an explicit section s_* for p_* given by $s_*(\rho_1) = \tau_1$, $s_*(\rho_2) = \tau_2$, and $s_*(\rho_1\rho_2) = \tau_3 = \tau_1\tau_2$, where the action on the kernel is as follows:

$$\begin{aligned} \tau_1 x \tau_1^{-1} &= y & \tau_2 x \tau_2^{-1} &= y^{-1} & \tau_3 x \tau_3^{-1} &= x^{-1} \\ \tau_1 y \tau_1^{-1} &= x & \tau_2 y \tau_2^{-1} &= x^{-1} & \tau_3 x \tau_3^{-1} &= y^{-1}. \end{aligned}$$

Further, by Proposition 21 of [GG1], there are precisely three conjugacy classes in $P_3(\mathbb{R}P^2)$ of elements of order 4 whose representatives may be taken to be τ_1, τ_2 and τ_3. So conjugating G by an element of $P_3(\mathbb{R}P^2)$ if necessary, and identifying $P_3(\mathbb{R}P^2)$ with $\mathbb{F}_2 \rtimes \mathcal{Q}_8$, we may suppose that its $\mathbb{Z}_4$ factor is generated by $(1, \tau_i)$ for some $i = 1, 2, 3$. Let (z, ξ) generate the $\mathbb{Z}$-factor of G for some $z \in \mathbb{F}_2$ and $\xi \in \mathcal{Q}_8$. Since G is the direct product of these two factors, we have:

$$\begin{aligned} (1, \tau_i).(z, \xi) &= (z, \xi).(1, \tau_i) \\ (\tau_i z \tau_i^{-1}, \tau_i \xi) &= (z, \xi \tau_i). \end{aligned}$$

Thus $z = \tau_i z \tau_i^{-1}$. But from the form of the action of the τ_i on x and y given above, this is not possible unless $z = 1$, in which case (z, ξ) is of finite order, a contradiction. ☐

Proof of Theorem 17. Let G be an infinite virtually cyclic subgroup of $P_3(\mathbb{R}P^2)$. We first suppose that it is of Type I. By Theorem 15, G must be isomorphic to one of the following groups: $\mathbb{Z}$, $\mathbb{Z}_2 \rtimes \mathbb{Z}$, $\mathbb{Z}_4 \rtimes \mathbb{Z}$ or $\mathcal{Q}_8 \rtimes \mathbb{Z}$. Clearly the two groups $\mathbb{Z}$ and $\mathbb{Z}_2 \times \mathbb{Z}$ are realised as subgroups of $P_3(\mathbb{R}P^2)$, and there is no non-trivial semi-direct product $\mathbb{Z}_2 \rtimes \mathbb{Z}$. We saw in Proposition 18 that G cannot be isomorphic to $\mathbb{Z}_4 \times \mathbb{Z}$. Since $\mathbb{Z}_4$ is a subgroup of $\mathcal{Q}_8$, it follows that $\mathcal{Q}_8 \times \mathbb{Z}$ cannot be realised as a subgroup of $P_3(\mathbb{R}P^2)$ either.

The possible (non-trivial) semi-direct products are not realised either as subgroups of $P_3(\mathbb{R}P^2)$. Indeed, since $\mathrm{Aut}\,(\mathbb{Z}_4) \cong \mathbb{Z}_2$, it follows that the subgroup $\mathbb{Z}_4 \rtimes 2\mathbb{Z}$ would in fact be a direct product, abstractly isomorphic to $\mathbb{Z}_4 \times \mathbb{Z}$. Similarly, since $\mathrm{Aut}\,(\mathcal{Q}_8) \cong S_4$, the subgroup $\mathcal{Q}_8 \rtimes 12\mathbb{Z}$ would be a direct product, abstractly isomorphic to $\mathcal{Q}_8 \times \mathbb{Z}$. Thus $\mathbb{Z}$ and $\mathbb{Z}_2 \times \mathbb{Z}$ are the only groups realised as Type I subgroups of $P_3(\mathbb{R}P^2)$.

Now suppose that G is of Type II. By Theorem 15, G is isomorphic to one of $\mathbb{Z}_2 * \mathbb{Z}_2$, $\mathbb{Z}_4 *_{\mathbb{Z}_2} \mathbb{Z}_4$ or $\mathcal{Q}_8 *_{\mathbb{Z}_4} \mathcal{Q}_8$. The first case is ruled out since $P_3(\mathbb{R}P^2)$ has just one element of order 2. If $P_3(\mathbb{R}P^2)$ had a subgroup isomorphic to $\mathcal{Q}_8 *_{\mathbb{Z}_4} \mathcal{Q}_8$ then by the proof of Lemma 16, it would also have a subgroup isomorphic to $\mathbb{Z}_4 \rtimes \mathbb{Z}$ which as we just saw cannot happen. The realisation of $\mathbb{Z}_4 *_{\mathbb{Z}_2} \mathbb{Z}_4$ as a subgroup of $P_n(\mathbb{R}P^2)$ occurs for all $n \geqslant 3$, and will be proved below in Proposition 20. This completes the proof of Theorem 17. □

5.2 Virtually cyclic subgroups of $P_n(\mathbb{R}P^2)$, $n \geqslant 4$

In this section, we classify the virtually cyclic subgroups of $P_n(\mathbb{R}P^2)$ in the general case. In fact we obtain exactly the same result as in the case $n = 3$.

THEOREM 19. *Let $n \geqslant 4$. Up to isomorphism, the infinite virtually cyclic subgroups of $P_n(\mathbb{R}P^2)$ are $\mathbb{Z}$, $\mathbb{Z}_2 \times \mathbb{Z}$ and $\mathbb{Z}_4 *_{\mathbb{Z}_2} \mathbb{Z}_4$.*

In order to prove Theorem 19, we first state and prove the following two propositions concerning the realisation of $\mathbb{Z}_4 *_{\mathbb{Z}_2} \mathbb{Z}_4$ and the non-realisation of $\mathbb{Z}_4 \times \mathbb{Z}$.

PROPOSITION 20. *Let $n \geqslant 3$. Then $P_n(\mathbb{R}P^2)$ possesses a subgroup isomorphic to $\mathbb{Z}_4 *_{\mathbb{Z}_2} \mathbb{Z}_4$.*

Proof of Proposition 20. Let $H = \langle \alpha, \beta \rangle < P_n(\mathbb{R}P^2)$, where α, β are as defined in equation (1). Since α, β are of order 4 and $\alpha^2 = \beta^2$, there is a natural surjective homomorphism $\mathbb{Z}_4 *_{\mathbb{Z}_2} \mathbb{Z}_4 \longrightarrow H$ that is injective on the $\mathbb{Z}_4$-factors. Further $\rho_n = \alpha\beta^{-1} \in H$ is of infinite order, so $\langle \alpha \rangle \cap \langle \beta \rangle = \langle \Delta_n \rangle$ and H is of infinite order. We conclude from Lemma 16 that $H \cong \mathbb{Z}_4 *_{\mathbb{Z}_2} \mathbb{Z}_4$. □

REMARK 21. A similar argument may be applied in order to show that if $n \geqslant 4$ then any two distinct subgroups of $P_n(\mathbb{R}P^2)$ of order 4 generate a subgroup isomorphic to $\mathbb{Z}_4 *_{\mathbb{Z}_2} \mathbb{Z}_4$. The fact that this subgroup is infinite follows directly from Proposition 1.

PROPOSITION 22. *For all $n \geqslant 1$, $P_n(\mathbb{R}P^2)$ has no subgroup isomorphic to $\mathbb{Z}_4 \times \mathbb{Z}$.*

Proof. Since $P_1(\mathbb{R}P^2)$ and $P_2(\mathbb{R}P^2)$ are finite, we may suppose that $n \geqslant 3$. We argue by induction on n. If $n = 3$ then the result was proved previously in Proposition 18. So suppose that the result is true for $P_n(\mathbb{R}P^2)$, and let us consider $P_{n+1}(\mathbb{R}P^2)$. Suppose that $P_{n+1}(\mathbb{R}P^2)$ has a subgroup H isomorphic to $\mathbb{Z}_4 \times \mathbb{Z}$. Let $x, g \in P_{n+1}(\mathbb{R}P^2)$ generate respectively the $\mathbb{Z}_4$- and $\mathbb{Z}$-factors. Since $P_{n+1}(\mathbb{R}P^2)$ is normal in $B_{n+1}(\mathbb{R}P^2)$, by conjugating H by an element of $B_{n+1}(\mathbb{R}P^2)$ if necessary, we may suppose further by Proposition 9 that x is equal to one of $\alpha = a^{n+1}$ or $\beta = b^n$.

Consider the usual projection $p_*\colon P_{n+1}(\mathbb{R}P^2) \longrightarrow P_n(\mathbb{R}P^2)$, consisting in forgetting the last string. Since $\operatorname{Ker}(p_*) = P_1(\mathbb{R}P^2 \setminus \{x_1, \ldots, x_n\})$ is torsion free, p_* is injective on finite subgroups of $P_{n+1}(\mathbb{R}P^2)$, and so $x' = p_*(x)$ is of order 4. Let $g' = p_*(g)$. If g' is of infinite order then $\langle x', g' \rangle$ is a subgroup of $P_n(\mathbb{R}P^2)$ isomorphic to $\mathbb{Z}_4 \times \mathbb{Z}$, which is ruled out by the induction hypothesis. So g' must be of finite order. Since g' and x' commute, they generate a finite Abelian subgroup of $P_n(\mathbb{R}P^2)$ which by Proposition 1 must be isomorphic to $\mathbb{Z}_4$. In particular, $g' \in \langle x' \rangle$, and there exists $0 \leqslant j \leqslant 3$ such that $z = gx^{-j} \in \operatorname{Ker}(p_*)$. Then z is of infinite order, H is generated by z and x, and is isomorphic to the direct product $\langle x \rangle \times \langle z \rangle \cong \mathbb{Z}_4 \times \mathbb{Z}$. So replacing g by z if necessary, we may suppose that $g \in \operatorname{Ker}(p_*)$.

If such a g were indeed to exist, it would be a non-trivial fixed point of the action of conjugation by x on $\operatorname{Ker}(p_*)$. So let us study this action, and show that there are no such fixed points, which will thus yield a contradiction.

Now $\operatorname{Ker}(p_*) = \pi_1(\mathbb{R}P^2 \setminus \{x_1, \ldots, x_n\}, x_{n+1})$ is a free group of rank n for which a basis is:

$$\{\rho_{n+1}, B_{1,n+1}, \ldots, B_{n-1,n+1}\}, \tag{2}$$

where for $i = 1, \ldots, n$,

$$B_{i,n+1} = \sigma_n \cdots \sigma_{i+1}\sigma_i^2\sigma_{i+1}^{-1} \cdots \sigma_n^{-1}.$$

By equation (7) of [GG1], we have

$$\rho_{n+1}^{-2} = \sigma_n \cdots \sigma_2\sigma_1^2\sigma_2 \cdots \sigma_n,$$

which is equal to $B_{1,n+1} \cdots B_{n-1,n+1}B_{n,n+1}$. So

$$\sigma_n^2 = B_{n,n+1} = B_{n-1,n+1}^{-1} \cdots B_{1,n+1}^{-1}\rho_{n+1}^{-2}.$$

Further, by using the Artin relations of $B_n(\mathbb{R}P^2)$ (those involving just the σ_i), for all $1 \leqslant i \leqslant n-1$, we obtain the following useful identities:

$$\begin{cases} \sigma_i B_{i,n+1}\sigma_i^{-1} = B_{i+1,n+1} \\ \sigma_i^{-1}B_{i,n+1}\sigma_i = B_{i,n+1}B_{i+1,n+1}B_{i,n+1}^{-1}. \end{cases} \tag{3}$$

We now calculate the action of α and β on each of the elements of the basis given in equation (2). It was proved on page 777 of [GG1] that conjugation by a^{-1} permutes cyclically the $2(n+1)$ elements

$$\sigma_1, \sigma_2, \ldots, \sigma_n, a^{-1}\sigma_n a, \sigma_1^{-1}, \sigma_2^{-1}, \sigma_n^{-1}, a^{-1}\sigma_n^{-1}a,$$

and permutes cyclically the $2(n+1)$ elements

$$\rho_1, \rho_2, \ldots, \rho_n, \rho_{n+1}, \rho_1^{-1}, \rho_2^{-1}, \ldots, \rho_n^{-1}, \rho_{n+1}^{-1}.$$

In particular, conjugation by $\alpha^{-1} = a^{-(n+1)}$ sends each of σ_i, $1 \leqslant i \leqslant n$, and ρ_j, $1 \leqslant j \leqslant n+1$, to its inverse. By induction, we have that for all $1 \leqslant i \leqslant n-1$,

$$\alpha^{-1}B_{i,n+1}\alpha = \rho_{n+1}^2 B_{1,n+1} \cdots B_{i-1,n+1}B_{i,n+1}^{-1}B_{i-1,n+1}^{-1} \cdots B_{1,n+1}^{-1}\rho_{n+1}^{-2}.$$

To see this, first let $i = 1$. Then:

$$\begin{aligned} \alpha^{-1}B_{1,n+1}\alpha &= \alpha^{-1}\sigma_n \cdots \sigma_2\sigma_1^2\sigma_2^{-1} \cdots \sigma_n^{-1}\alpha \\ &= \sigma_n^{-1} \cdots \sigma_2^{-1}\sigma_1^{-2}\sigma_2^{-1} \cdots \sigma_n^{-1}.\, \sigma_n \cdots \sigma_2\sigma_1^{-2}\sigma_2^{-1} \cdots \sigma_n^{-1}.\, \sigma_n \cdots \sigma_2\sigma_1^2\sigma_2 \cdots \sigma_n \\ &= \rho_{n+1}^2 B_{1,n+1}^{-1}\rho_{n+1}^{-2}, \end{aligned}$$

using equation (7) of [GG1]. Now suppose by induction that the result holds for $1 \leqslant i \leqslant n-2$. Then using equation (3), we have

$$\begin{aligned} \alpha^{-1}B_{i+1,n+1}\alpha &= \alpha^{-1}\sigma_i B_{i,n+1}\sigma_i^{-1}\alpha \\ &= \sigma_i^{-1}\left(\rho_{n+1}^2 B_{1,n+1} \cdots B_{i-1,n+1}B_{i,n+1}^{-1}B_{i-1,n+1}^{-1} \cdots B_{1,n+1}^{-1}\rho_{n+1}^{-2}\right)\sigma_i \\ &= \rho_{n+1}^2 B_{1,n+1} \cdots B_{i-1,n+1}\sigma_i^{-1}B_{i,n+1}^{-1}\sigma_i B_{i-1,n+1}^{-1} \cdots B_{1,n+1}^{-1}\rho_{n+1}^{-2} \\ &= \rho_{n+1}^2 B_{1,n+1} \cdots B_{i-1,n+1}B_{i,n+1}B_{i+1,n+1}^{-1}B_{i,n+1}^{-1}B_{i-1,n+1}^{-1} \cdots B_{1,n+1}^{-1}\rho_{n+1}^{-2}, \end{aligned}$$

as required.

In other words, the action of conjugating each of these generators by α^{-1} is equal to $\iota_{\rho_{n+1}^2} \circ \varphi$, where $\iota_{\rho_{n+1}^2}$ is conjugation by ρ_{n+1}^2, and φ is the automorphism of $\operatorname{Ker}(p_*)$ given by:

$$\begin{aligned} \varphi(\rho_{n+1}) &= \rho_{n+1}^{-1} \\ \varphi(B_{i,n+1}) &= B_{1,n+1} \cdots B_{i-1,n+1} B_{i,n+1}^{-1} B_{i-1,n+1}^{-1} \cdots B_{1,n+1}^{-1}, \end{aligned}$$

for $i = 1, \ldots, n-1$. Notice further that φ induces an automorphism of the subgroup $K = \langle B_{1,n+1}, \ldots, B_{n-1,n+1} \rangle$ of $\operatorname{Ker}(p_*)$.

Since $\alpha = \rho_{n+1}\beta$, for all $y \in \operatorname{Ker}(p_*)$,

$$\beta^{-1} y \beta = \alpha^{-1} \rho_{n+1} y \rho_{n+1}^{-1} \alpha = \rho_{n+1}^{-1} \alpha^{-1} y \alpha \rho_{n+1},$$

and so conjugation by β^{-1} on each of the given basis elements is equal to $\iota_{\rho_{n+1}} \circ \varphi$.

Suppose first that $x = \alpha$, and let $z = \rho_{n+1}^{\varepsilon_0} w_1 \rho_{n+1}^{\varepsilon_1} \cdots w_{k-1} \rho_{n+1}^{\varepsilon_k}$ be a fixed point of the action of conjugation by x, where $k \geqslant 0$, for $i = 1, \ldots, k-1$, $w_i \in K$ is a non-trivial word, $\varepsilon_j \in \mathbb{Z}$ for $j = 0, 1, \ldots, k$, and $\varepsilon_j \neq 0$ for $j = 1, \ldots, k-1$. Now $\alpha^{-1} \rho_{n+1} \alpha = \rho_{n+1}^{-1}$, so we must have $k \geqslant 1$, and then

$$\alpha^{-1} z \alpha = \rho_{n+1}^{2-\varepsilon_0} \varphi(w_1) \rho_{n+1}^{-\varepsilon_1} \cdots \varphi(w_{k-1}) \rho_{n+1}^{-\varepsilon_k - 2}.$$

Since φ induces an automorphism of K, there is no cancellation between successive terms of this expression. It thus follows that $k = 1$, $\varepsilon_0 = 1$ and $\varepsilon_1 = -1$ and $\varphi(w_1) = w_1$. Now $\varphi|_K = \iota_{B_{1,n+1}} \circ \varphi'$, where φ' is the automorphism of K defined by:

$$\begin{aligned} \varphi'(B_{1,n+1}) &= B_{1,n+1}^{-1} \\ \varphi'(B_{i,n+1}) &= B_{2,n+1} \cdots B_{i-1,n+1} B_{i,n+1}^{-1} B_{i-1,n+1}^{-1} \cdots B_{2,n+1}^{-1}, \end{aligned}$$

for $i = 2, \ldots, n-1$. Similarly, φ' restricts to an automorphism of the subgroup $K' = \langle B_{2,n+1}, \ldots, B_{n-1,n+1} \rangle$.

Set $w_1 = B_{1,n+1}^{\delta_0} v_1 B_{1,n+1}^{\delta_1} \cdots v_{t-1} B_{1,n+1}^{\delta_t}$, where $t \geqslant 0$, for $i = 1, \ldots, t-1$, $v_i \in K'$ is a non-trivial word, $\delta_j \in \mathbb{Z}$ for $j = 0, 1, \ldots, t$, and $\delta_j \neq 0$ for $j = 1, \ldots, t-1$. Now $\varphi(B_{1,n+1}) = B_{1,n+1}^{-1}$, so we must have $t \geqslant 1$, and then

$$\varphi(w_1) = B_{1,n+1}^{1-\delta_0} \varphi'(v_1) B_{1,n+1}^{-\delta_1} \cdots \varphi'(v_{t-1}) B_{1,n+1}^{-\delta_t - 1}.$$

Again, since K' is invariant under φ', there is no cancellation between successive terms in this expression. But w_1 is a (non-trivial) fixed point of φ, so we must have $2\delta_0 = 1$ which yields a contradiction. Hence there exists no $g \in P_{n+1}(\mathbb{R}P^2)$ of infinite order which commutes with α. We now repeat the same analysis for β, from which we obtain:

$$\beta^{-1} z \beta = \rho_{n+1}^{1-\varepsilon_0} \varphi(w_1) \rho_{n+1}^{-\varepsilon_1} \cdots \varphi(w_{k-1}) \rho_{n+1}^{-\varepsilon_k - 1}.$$

Thus $\varepsilon_0 \notin \mathbb{Z}$, a contradiction. Hence there exists no $g \in P_{n+1}(\mathbb{R}P^2)$ of infinite order that commutes with β. $\square$

Proof of Theorem 19. Let H be an infinite virtually cyclic subgroup of $P_n(\mathbb{R}P^2)$. If H is of Type I then by Proposition 1, H is isomorphic to either $\mathbb{Z}$, $\mathbb{Z}_2 \times \mathbb{Z}$ (both of which are realised), $\mathbb{Z}_4 \times \mathbb{Z}$ or $\mathbb{Z}_4 \rtimes \mathbb{Z}$. But $\mathbb{Z}_4 \times \mathbb{Z}$ and $\mathbb{Z}_4 \rtimes \mathbb{Z}$ are ruled out by Proposition 22. If H is of Type II then either $H \cong \mathbb{Z}_2 * \mathbb{Z}_2$, which is not possible since $P_n(\mathbb{R}P^2)$ has only one subgroup isomorphic to $\mathbb{Z}_2$, or $H \cong \mathbb{Z}_4 *_{\mathbb{Z}_2} \mathbb{Z}_4$, which as we saw in Proposition 20, is realised. This completes the proof of Theorem 19, as well as that of Theorem 2. □

REMARK 23. We saw that in $B_n(\mathbb{R}P^2)$, there are two conjugacy classes of subgroups of order 4 lying in $P_n(\mathbb{R}P^2)$. It would be interesting to know how many such conjugacy classes exist in $P_n(\mathbb{R}P^2)$.

As a corollary of Theorem 2, we obtain the following result:

COROLLARY 24. *Let $n \geqslant 4$. Then the virtually cyclic subgroups of $P_n(\mathbb{R}P^2)$ are $\{e\}$, $\mathbb{Z}_2$, $\mathbb{Z}_4$, $\mathbb{Z}$, $\mathbb{Z}_2 \times \mathbb{Z}$ and $\mathbb{Z}_4 *_{\mathbb{Z}_2} \mathbb{Z}_4$.* □

We also obtain Corollary 4:

Proof of Corollary 4. Clearly $\langle x \rangle \subseteq Z_{P_n(\mathbb{R}P^2)}(x)$. So suppose that $z \in Z_{P_n(\mathbb{R}P^2)}(x)$. It follows from Proposition 22 that z is of finite order. Then x and z generate a finite Abelian subgroup of $P_n(\mathbb{R}P^2)$ which contains $\langle x \rangle$. So $\langle x, z \rangle$ is isomorphic to $\mathbb{Z}_4$, and thus $z \in \langle x \rangle$. □

REMARK 25. For $x \in P_n(\mathbb{R}P^2)$ of order 4, we believe that the centraliser of $Z_{B_n(\mathbb{R}P^2)}(x)$ in $B_n(\mathbb{R}P^2)$ is finite cyclic of order $4n$ or $4(n-1)$ depending on whether x is conjugate in $B_n(\mathbb{R}P^2)$ to α or β. In order to justify this, notice first that $Z_{B_n(\mathbb{R}P^2)}(x)$ cannot contain any element z of infinite order, since there would exist some $1 \leqslant j \leqslant n!$ such that $z^j \in P_n(\mathbb{R}P^2)$, and so $z^j \in Z_{P_n(\mathbb{R}P^2)}(x)$, a contradiction. Then $Z_{B_n(\mathbb{R}P^2)}(x)$ contains only elements of finite order, and so is itself finite. To prove this, suppose that $Z_{B_n(\mathbb{R}P^2)}(x)$ is infinite. Then $K = Z_{B_n(\mathbb{R}P^2)}(x) \cap P_n(\mathbb{R}P^2)$ is infinite.
– If $n \geqslant 4$ then K contains two distinct copies of $\mathbb{Z}_4$, and by Remark 21 they together generate a copy of $\mathbb{Z}_4 *_{\mathbb{Z}_2} \mathbb{Z}_4$ which contains elements of infinite order.
– If $n = 3$ then K contains an infinite number of elements of order 4. Identifying $P_3(\mathbb{R}P^2)$ with $\mathbb{F}_2 \rtimes \mathcal{Q}_8$ as in the proof of Proposition 18, there exist $i \in \{1, 2, 3\}$ and $z_1, z_2 \in \mathbb{F}_2(x, y)$, $z_1 \neq z_2$, such that $(z_1, \tau_i), (z_2, \tau_i) \in K$. But then

$$(z_1, \tau_i).\,(z_2, \tau_i)^{-1} = (z_1, \tau_i).\,(\tau_i^{-1} z_2^{-1} \tau_i, \tau_i^{-1}) = (z_1 z_2^{-1}, 1) \in K.$$

In both cases, we deduce a contradiction, and thus $Z_{B_n(\mathbb{R}P^2)}(x)$ is finite. Further, by conjugating x by an element of $B_n(\mathbb{R}P^2)$ if necessary, we may suppose that $x = \alpha = a^n$ or $x = \beta = b^{n-1}$, in which case $\langle a \rangle \subseteq Z_{B_n(\mathbb{R}P^2)}(x)$ or $\langle b \rangle \subseteq Z_{B_n(\mathbb{R}P^2)}(x)$ respectively. Investigations into the structure of the finite subgroups of $B_n(\mathbb{R}P^2)$ suggest that $\langle a \rangle$ and $\langle b \rangle$ are maximal finite subgroups of $B_n(\mathbb{R}P^2)$. If this is indeed the case then $Z_{B_n(\mathbb{R}P^2)}(x) = \langle a \rangle$ or $Z_{B_n(\mathbb{R}P^2)}(x) = \langle b \rangle$ as claimed. A similar argument would show that the normaliser of $\langle x \rangle$ in $B_n(\mathbb{R}P^2)$ is also finite cyclic of order $4n$ or $4(n-1)$.

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