Sphere Rigidity in the Euclidean Space

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Abstract

In this article, we prove new stability results for almost-Einstein hypersurfaces of the Euclidean space, based on previous eigenvalue pinching results. Then, we deduce some comparable results for almost-umbilic hypersurfaces and new characterizations of geodesic spheres.

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Mathematical Subject Clasification: 53A07, 53A10, 53C20, 53C24.

1 Introduction

The well-konwn Alexandrov theorem [1] says that embedded hypersurfaces in $\mathbb{R}^{n+1}$ with constant mean curvature are geodesic spheres. This result is not true for only immersed hypersurfaces. For instance, the so-called Wente's tori (see [14]) are examples of compact surfaces with constant mean curvature in $\mathbb{R}^3$, which are not geodesic spheres. Other examples of higher genus are known (see [8] for instance).

For immersed hypersurfaces of constant mean curvature, an additional assumption is needed. One condition is given by the Hopf theorem [7], which

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says that constant mean curvature spheres immersed in $\mathbb{R}^{n+1}$ are geodesic spheres.

In this paper, we give a new rigidity theorem for spheres, where we replace the topological assumption by a metric assumption. Precisely, it is easy to see that hypersurfaces of $\mathbb{R}^{n+1}$ with constant mean curvature and constant scalar curvature are geodesic spheres. This result comes from the fact that a hypersurface of constant mean curvature and constant scalar curvature is totally umbilic. Here, we give a new rigidity result with a weaker assumption on the scalar curvature. Namely, we show

Theorem 1. *Let (M^n, g) be a compact, connected and oriented Riemannian manifold without boundary isometrically immersed into $\mathbb{R}^{n+1}$. Let h be a positive constant. Then, there exists $\varepsilon > 0$ such that if*

(1) $H = h$ and

(2) $|\mathrm{Scal} - s| \leqslant \varepsilon$,

for some constant s, then M is the sphere $\mathbb{S}^n\left(\frac{1}{h}\right)$ with its standard metric.

We derive this theorem from results about almost umbilic hypersurfaces that we prove in Sect. 4, and based on a previous eignevalue pinching result given in [12].

2 Preliminaries

Let (M^n, g) be a n-dimensional compact, connected, oriented Riemannian manifold without boundary, isometrically immersed into the $(n+1)$-dimensional Euclidean space $(\mathbb{R}^{n+1}, can)$. The second fundamental form B of the immersion is the bilinear symmetric form defined by

$$B(Y, Z) = -g\left(\overline{\nabla}_Y \nu, Z\right),$$

where $\overline{\nabla}$ is the Riemannian connection on $\mathbb{R}^{n+1}$ and ν the outward normal unit vector field on M.

From B, we can define the mean curvature,

$$H = \frac{1}{n}\mathrm{tr}\,(B),$$

and, more generally, the higher order mean curvatures,

$$H_r = \frac{1}{\begin{pmatrix} n \\ r \end{pmatrix}}\sigma_r(\kappa_1, \cdots, \kappa_n),$$

where σ_r is the r-th symmetric polynomial and $\kappa_1, \cdots, \kappa_n$ are the principal curvatures of the immersion. By convention, we set $H_0 = 1$.

Note that $H_1 = H$ and from the Gauss equation, H_2 is, up to a mutliplicative constant, the scalar curvature. Namely, we have $H_2 = \frac{1}{n(n-1)}\mathrm{Scal}$.

These extrinsic curvatures satisfy the well-known Hsiung-Minkowski formula, for $1 \leqslant r \leqslant n$,

$$\int_M \big(H_{r-1} - H_r \langle X, \nu\rangle\big) dv_g = 0, \tag{1}$$

where X is the position vector of the immersion. They also satisfy the following inequalities if H_r is a positive function:

$$H_r^{\frac{1}{r}} \leq H_{r-1}^{\frac{1}{r-1}} \leq \cdots \leq H_2^{\frac{1}{2}} \leq H. \tag{2}$$

Moreover, Reilly [10] proved some upper bounds for the first eigenvalue of the Laplacian for hypersurfaces of $\mathbb{R}^{n+1}$ in terms of higher order mean curvatures. Precisely, he shows

$$\lambda_1(M) \left(\int_M H_{r-1} dv_g\right)^2 \leqslant \frac{n}{\mathrm{Vol}(M)} \int_M H_r^2 dv_g, \tag{3}$$

with equality only for the geodesic hyperspheres of $\mathbb{R}^{n+1}$.

By the Hölder inequality, we obtain for $p \geqslant 2$,

$$\lambda_1(M) \leqslant n \frac{||H_r||_{2p}^2 \mathrm{Vol}(M)^{2-\frac{1}{p}}}{\left(\int_M H_{r-1} dv_g\right)^2}.$$

Now, for $p \geqslant 2$ and $1 \leqslant r \leqslant n$, we define $k_{p,r} = \dfrac{||H_r||_{2p}^2 \mathrm{Vol}(M)^{2-\frac{1}{p}}}{\left(\int_M H_{r-1} dv_g\right)^2}$, which are the constants involved in Theorem 2.

The main tool in the proof of Theorem 1 is the following pinching result, associated with these inequalities, that we proved in [12].

Theorem A (Roth [12]). *Let (M^n, g) be a compact, connected, oriented Riemannian manifold without boundary isometrically immersed in $\mathbb{R}^{n+1}$. Assume that* $\mathrm{Vol}(M) = 1$ *and let $r \in \{1, \cdots, n\}$ such that $H_r > 0$. Then for any $p \geqslant 2$ and any $\theta \in]0, 1[$, there exists a constant K_θ depending only on n, $||H||_\infty$, $||H_r||_{2p}$ and θ such that if the pinching condition*

$$0 \geqslant \lambda_1(M) \left(\int_M H_{r-1} dv_g\right)^2 - n\mathrm{Vol}(M)^{2-1/p} ||H_r||_{2p}^2 > -K_\theta \tag{P_{K_θ}}$$

is satisfied, then M is diffeomorphic and θ-quasi-isometric to $\mathbb{S}^n\left(\sqrt{\frac{n}{\lambda_1}}\right)$.

Remark 1. *Note that if $p \geqslant \frac{n}{2r}$; then K_θ does not depend on $||H_r||_{2p}$.*

3 Almost Einstein hypersurfaces

It is a well-known fact that an Einstein (Scal >0) hypersurface of the Euclidean space $\mathbb{R}^{n+1}$ is a round sphere. This was proved by Thomas [13] and independently by Fialkow [4] in the 30's. Recently, Grosjean [6] gave a new proof based on an upper bound of the first eigenvalue of the Laplacian involving the scalar curvature.

From this approach, we showed in [12] that almost-Einstein hypersurfaces of $\mathbb{R}^{n+1}$ are close to round spheres. Namely,

Theorem B (Roth [12]). *Let (M^n,g) be a compact, connected, oriented Riemannian manifold without boundary isometrically immersed in $\mathbb{R}^{n+1}$, $n\geqslant 3$. Let $\theta\in]0,1[$. If (M^n,g) is almost-Einstein, that is, $||\mathrm{Ric}-(n-1)kg||_\infty\leqslant\varepsilon$ for a positive constant k, with ε small enough depending on n, k, $||H||_\infty$ and θ, then M is diffeomorphic and θ-quasi-isometric to $\mathbb{S}^n\left(\sqrt{\frac{1}{k}}\right)$*

By θ-quasi-isometric, we understand that there exists a diffeomorphism F from M into $\mathbb{S}^n\left(\sqrt{\frac{1}{k}}\right)$ such that, for any $x\in M$ and for any unitary vector $u\in T_xM$, we have

$$\left||d_xF(u)|^2-1\right|\leqslant\theta.$$

This theorem is a corollary of our pinching result for the first eigenvalue of the Laplacian (Theorem A).

In this article, we consider almost-Einstein hypersurfaces of $\mathbb{R}^{n+1}$ in a weaker sense, namely for the L^q-norm, that is, $||\mathrm{Ric}-(n-1)kg||_q\leqslant\varepsilon$ for some positive constant k and a sufficiently small ε. We prove that for some suitable constants k, such manifolds are close to round spheres. Precisely, we prove the following

Theorem 2. *Let (M^n,g) be a compact, connected, oriented Riemannian manifold without boundary isometrically immersed in $\mathbb{R}^{n+1}$. Let $\theta\in]0,1[$, if (M^n,g) satisfies $||\mathrm{Ric}-(n-1)k_{p,r}g||_q\leqslant\varepsilon$ for some sufficiently small ε depending on n, q, $||H||_\infty$ and θ, then M is diffeomorphic and θ-quasi-isometric to $\mathbb{S}^n\left(\sqrt{\frac{1}{k_{p,r}}}\right)$.*

The constants $k_{p,r}$ in the theorem are defined from the higher order mean curvature (see Sect. 2).

After giving the proof of this theorem, we will give some applications to almost-umbilic hypersurfaces. Finally, we derive some applications to almost constant mean curvature and almost constant scalar curvature, and then conclude with the proof of theorem 1

The proof of Theorem 2 is based on the above pinching result combined with a lower bound for the first eigenvalue of the Laplacian due to Aubry [2]. Assume that $||\mathrm{Ric} - (n-1)kg||_q \leqslant \varepsilon(n,q,k)$ for a positive constant k, $q > \frac{n}{2}$ and ε small enough, then from Theorem 1.1 of [2], we deduce that $\lambda_1(\Delta)$ satisfies

$$\lambda_1(\Delta) \geqslant nk(1 - C_\varepsilon), \tag{4}$$

where C_ε is an explicit constant such that $C_\varepsilon \longrightarrow 0$ when $\varepsilon \longrightarrow 0$.

Now, with the particular choice of $k = k_{p,r}$, we get:

$$\lambda_1(M)\left(\int_M H_{r-1} dv_g\right)^2 - n\mathrm{Vol}(M)^{2-1/p}||H_r||_{2p}^2 > -K_\varepsilon$$

for some constant K_ε such that $K_\varepsilon \longrightarrow 0$ when $\varepsilon \longrightarrow 0$.

Let $\theta \in]0,1[$, we choose $\varepsilon(n,q,k,\theta)$ small enough such that K_ε is small enough in Theorem A to obtain a diffeomorphism and θ-quasi-isometry between M and $\mathbb{S}^n\left(\sqrt{\frac{1}{k_{p,r}}}\right)$. $\square$

Now, we will deduce from Theorem 2 some Corollaries for almost-umbilic hypersurfaces of $\mathbb{R}^{n+1}$.

4 Almost umbilic hypersurfaces

First, we give the following theorem, which a direct application of Theorem B.

Theorem 3. *Let (M^n, g) be a compact, connected, oriented Riemannian manifold without boundary isometrically immersed in $\mathbb{R}^{n+1}$. Let $\theta \in]0,1[$. If (M^n, g) is almost-umbilic, that is, $||B - kg||_\infty \leqslant \varepsilon$ for a positive constant k, with ε small enough depending on n, k and θ then M is diffeomorphic and θ-quasi-isometric to $\mathbb{S}^n\left(\frac{1}{k}\right)$.*

Proof : We recall the once traced Gauss formula

$$\mathrm{Ric}(Y,Y) = nH\left\langle B(Y), Y\right\rangle - \left\langle B(Y), B(Y)\right\rangle, \tag{5}$$

for a tangent vector field Y. From (5) and $||B - kg||_\infty \leqslant \varepsilon$, we deduce

$$\begin{aligned}\mathrm{Ric}(Y,Y) &\geqslant nk^2||Y||^2(1-\varepsilon)^2 - k^2||Y||^2(1+\varepsilon)^2 \\ &\geqslant (n-1)k^2||Y||^2 - \alpha_n(\varepsilon)||Y||^2,\end{aligned}$$

where α_n is a positive function such that $\alpha_n(\varepsilon) \longrightarrow 0$ when $\varepsilon \longrightarrow 0$.
Similarly, we get

$$\mathrm{Ric}(Y,Y) \leqslant (n-1)k^2||Y||^2 + \alpha_n(\varepsilon)||Y||^2.$$

Finally, we have

$$||\mathrm{Ric} - (n-1)k^2 g||_\infty \leqslant \alpha_n(\varepsilon),$$

which implies, by Theorem B, that for ε small enough, M is diffeomorphic and θ-quasi-isometric to $\mathbb{S}^n\left(\frac{1}{k}\right)$. □

Now, from Theorem 2, it is possible to obtain, in some particular cases, comparable results for almost-umbilic hypersurfaces in an L^q-sense. We recall that the umbilicity tensor is defined by

$$\tau = B - H\mathrm{Id}\,.$$

It is a well-known fact that if M is umbilic, *i.e.*,$\tau = 0$, and if M is compact, then M is a geodesic sphere. Here, we prove the following stability result for almost umbilic hypersurfaces.

Theorem 4. *Let (M^n, g) be a compact, connected, oriented Riemannian manifold without boundary isometrically immersed in $\mathbb{R}^{n+1}$. Let $q > \frac{n}{2}$. For any $\theta \in]0,1[$, there exists two constants $\varepsilon_i(\theta, n, ||H||_\infty)$, $i = 1, 2$, such that if*

1. *$||\tau||_{2q} \leqslant \varepsilon_1$,*
2. *$||H^2 - k_{p,r}||_q \leqslant \varepsilon_2$, for $p \geqslant 4$ and $1 \leqslant r \leqslant n$,*

then M is diffeomorphic and θ-quasi-isometric to $\mathbb{S}^n\left(\frac{1}{\sqrt{k_{p,r}}}\right)$

Remark 2. *Note that for $r = 1$, the result is due to Grosjean (see [5]), using a pinching result which involves only H_1 (see [3]).*

Proof. We recall the Gauss formula for hypersurfaces of $\mathbb{R}^{n+1}$:

$$\mathrm{Ric} = nHB - B^2.$$

From this, we deduce that

$$\begin{aligned}\mathrm{Ric} - (n-1)H^2 g &= nHB - B^2 - (n-1)H^2 g\\ &= (n-2)H\tau - \tau^2,\end{aligned}$$

which implies

$$\begin{aligned}
||\mathrm{Ric}-(n-1)kg||_q &\leqslant ||\mathrm{Ric}-(n-1)H^2g||_q+(n-1)\sqrt{n}||(H^2-k)||_q\\
&\leqslant (n-2)||H||_\infty||\tau||_{2q}+||\tau||_{2q}^2+(n-1)\sqrt{n}||(H^2-k)||_q\\
&\leqslant (n-2)||H||_\infty\varepsilon_1+\varepsilon_1^2+(n-1)\sqrt{n}\varepsilon_2
\end{aligned}$$

Now, we conclude by taking ε_1 and ε_2 small enough depending on n, $||H||_\infty$ and θ in order to apply Theorem 2 and obtain the θ-quasi-isometry. □

Then, we deduce the following corollary which is to compare with Theorem 3 .

Corollary 1. *Let (M^n,g) be a compact, connected, oriented Riemannian manifold without boundary isometrically immersed in $\mathbb{R}^{n+1}$. Let $\theta\in]0,1[$. If (M^n,g) is almost-umbilic, that is, $||B-\sqrt{k_{p,r}}g||_{2q}\leqslant\varepsilon$, for $q>\frac{n}{2}$, with ε small enough depending on n, $||H||_\infty$ and θ then M is diffeomorphic and θ-quasi-isometric to $\mathbb{S}^n\left(\frac{1}{\sqrt{k_{p,r}}}\right)$.*

Proof : A simple computation shows that

$$||H^2-k_{p,r}||_{2q}\leqslant\alpha_1||B-\sqrt{k_{p,r}}g||_{2q},\ \text{and}$$

$$||\tau||_{2q}\leqslant\alpha_2||B-\sqrt{k_{p,r}}g||_{2q},$$

for two constants α_1 and α_2 depending on n and $||H||_\infty$. Since we assume that $||B-\sqrt{k_{p,r}}g||_{2q}\leqslant\varepsilon$, we get

1. $||H^2-k_{p,r}||_{2q}\leqslant\alpha_1\varepsilon$,
2. $||\tau||_{2q}\leqslant\alpha_2\varepsilon$.

For ε small enough, the assumptions of Theorem 4 are satisfied and we can conclude that M is diffeomorphic and quasi-isometric to $\mathbb{S}^n\left(\frac{1}{\sqrt{k_{p,r}}}\right)$. □

Remark 3. *We want to point out that this corollary is an improvement of Theorem 3 only in some sense. Indeed, we improve the L^∞-proximity to an L^{2q}-proximity, but this corollary is valid only for some special constants $k_{p,r}$ and not for any positive constant as in Theorem 3. Nevertheless, this result is sufficient to deduce an interesting application for $r=2$.*

5 Proof of Theorem 1

In this section, we will give the proof of Theorem 1. First, using Corollary 1, we will show that hypersurfaces of $\mathbb{R}^{n+1}$ with almost constant mean and scalar curvatures are close to a geodesic sphere. Precisely, we show the following

Theorem 5. *Let (M^n, g) be a compact, connected and oriented Riemannian manifold without boundary isometrically immersed into $\mathbb{R}^{n+1}$. Let $h > 0$ and $\theta \in]0,1[$. Then, there exists $\varepsilon(n,h,\theta) > 0$ so that if*

(1) $|H - h| \leqslant \varepsilon$, *and*

(2) $|\mathrm{Scal} - s| \leqslant \varepsilon$,

for some constant s, then $|s - n(n-1)h^2| \leqslant A(n,h)\varepsilon$ and M is diffeomorphic and θ-quasi-isometric to $\mathbb{S}^n\left(\frac{1}{h}\right)$.

Proof : First, we show that h and s are related.

Lemma 1. *The two constants h and s satisfy*

$$h = \frac{1}{n(n-1)}s + A\varepsilon,$$

where A is constant depending only on n and h.

Proof : We recall the so-called Hsiung-Minkowski formula:

$$\int_M \left(H - \frac{1}{n(n-1)} \mathrm{Scal}\, \langle X, \nu \rangle \right) dv_g = 0. \tag{6}$$

By assumption, we have $H(x) = h + f_1(x)\varepsilon$ and $\mathrm{Scal}\,(x) = h + f_2(x)\varepsilon$, with f_1 and f_2 two functions satisfying $|f_1(x)| \leqslant 1$ and $|f_2(x)| \leqslant 1$. Now, by (6), we have

$$\begin{aligned}
0 &= \int_M \left(h + \varepsilon f_1(x) - \left(\frac{1}{n(n-1)} s + \varepsilon f_2(x) \right) \langle X, \nu \rangle \right) dv_g \\
&= h\mathrm{Vol}(M) + \varepsilon \int_M f_1(x) dv_g - \frac{1}{n(n-1)} \int_M s \langle X, \nu \rangle\, dv_g \\
&\quad - \frac{\varepsilon}{n(n-1)} \int_M f_2(x) \langle X, \nu \rangle\, dv_g \\
&= A_1\varepsilon + h\mathrm{Vol}(M) - \frac{s}{n(n-1)h} \int_M h \langle X, \nu \rangle\, dv_g \\
&= A_1\varepsilon + h\mathrm{Vol}(M) - \frac{s}{n(n-1)h} \int_M (H(x) - \varepsilon f_1(x)) \langle X, \nu \rangle\, dv_g \\
&= A_2\varepsilon + h\mathrm{Vol}(M) - \frac{s}{n(n-1)h} \int_M H \langle X, \nu \rangle\, dv_g
\end{aligned}$$

But, we know that $\int_M H \langle X, \nu \rangle \, dv_g = \mathrm{Vol}(M)$, which yields

$$A_2\varepsilon + h - \frac{s}{n(n-1)h} = 0.$$

So, we deduce that $s = n(n-1)h^2 + A\varepsilon$, where A is a constant depending only on n and h. □

Now, we use Corollary 1 to conclude. For this, we need to estimate the umbilicity tensor. By the Gauss formula, we have

$$\begin{aligned} |\tau|^2 &\leqslant n(n-1)H^2 - \mathrm{Scal} \\ &\leqslant n(n-1)\left(h + \varepsilon f_1(x)\right)^2 - \left(s + \varepsilon f_2(x)\right) \\ &\leqslant n(n-1)h^2 - s + \varepsilon h(x) \\ &\leqslant A'\varepsilon. \end{aligned}$$

Then, from the definition of $k_{p,r}$ and the assumptions, we can see easily that $|H^2 - k_{p,2}| \leqslant A''\varepsilon$. So the assumptions of Corollary 1 are satisfied and we can conclude that M is diffeomorphic and quasi-isometric to $\mathbb{S}^n\left(\frac{1}{k}\right)$ if we choose ε small enough depending only on n, θ and h. □

Now, we can deduce Theorem 1 from this result.

Proof of Theorem 1. For $\varepsilon(n,h)$ small enough, we know, from Theorem 5, that M is diffeomorphic to $\mathbb{S}^n$ (we take $\theta = \frac{1}{2}$ in Theorem 5). So we deduce that the immersion of M into $\mathbb{R}^{n+1}$ is in fact an embedding. Since M has constant positive mean curvature, by the Alexandrov theorem, M is the sphere of corresponding radius, that is, $\mathbb{S}^n\left(\frac{1}{h}\right)$. □

From Theorem 5, and using the Alexandrov theorem for constant scalar curvature due to Ros [11, 9], we get the following corollary.

Corollary 2. *Let (M^n, g) be a compact, connected and oriented Riemannian manifold without boundary isometrically immersed into $\mathbb{R}^{n+1}$. Let s be a positive constant. Then, there exists $\varepsilon > 0$ such that if*

(1) $\mathrm{Scal} = s$

(2) $|H - h| \leqslant \varepsilon$,

for some constant h, then M is the sphere $\mathbb{S}^n\left(\sqrt{\frac{n(n-1)}{s}}\right)$.

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