OUTER FACTOR 2-D MA MODELS FOR PURELY INDETERMINISTIC FIELDS AND WOLD-TYPE TEXTURE DECOMPOSITIONS

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ABSTRACT

In this paper, we propose a new method to compute the parameters of finite approximations of 2-D MA infinite models associated with purely indeterministic stationary fields, by using spectral factorizations. This method is developed in the framework of model based approaches for texture analysis and synthesis. The proposed parameter estimation method is then exploited to derive a scheme to separate the indeterministic and the deterministic components of a texture in Wold decomposition based models. Unlike existing approaches, the resulting decomposition scheme does not require any support detection technique in the spectral domain.

1. INTRODUCTION

Nowadays, a great deal of interest has been paid to texture analysis and synthesis, especially in the fields of image compression, image resolution enhancement, etc. [9,10]. In that case, when looking at model based approaches, the parameter estimation issue must be addressed.

Among the approaches of image modeling that use random processes, one consists in using texture models based on the 2-D Wold decomposition [1,2]. This result extends the original 1-D result [6] which was initially used for time series. In this 2-D approach, the texture is represented as a regular stationary field that can be decomposed in orthogonal components:

- ✓ A purely indeterministic field characterized by a continuous spectral density,
- ✓ a deterministic field which is itself decomposed in two orthogonal components: an harmonic field corresponding to the 2-D harmonics components (i.e. sharp peaks in the frequency domain) and an evanescent field which represents periodic fields characterized by straight lines in the frequency domain.

In [2], Francos et *al.* present a generic model for each Wold component. It is known that the most general model for the purely indeterministic component is the moving average (MA) representation [1]. Then, under some assumptions[1], an autoregressive (AR) representation is suggested in [2].

In order to carry out the parameter estimation, the Wold components are usually separated. Thus, Francos [2] proposes an iterative procedure to detect the support of the deterministic field in the texture periodogram. It consists in searching the sharpest peaks. For this purpose, a threshold is introduced and initially set to the maximal value of the periodogram. Then, it is gradually lowered as long as only sharp peaks are detected. The purely indeterministic field is obtained by subtracting the estimated deterministic field from the original texture. However, it is difficult to define a stopping rule. In addition, this approach cannot be used for a large variety of textures.

A more robust method for Wold decomposition is presented by Liu and Picard in [7]. Their algorithm is based on the intrinsic fundamental-harmonic relationships to identify harmonic frequencies and on the Hough transform to detect spectral evanescent components. This method has the drawback of searching of the support of each deterministic component individually.

In this paper, two aspects of the Wold-based models for texture analysis are studied:

- ✓ modeling and parameter estimation of the purely indeterministic field,
- ✓ separation of the deterministic components and the purely indeterministic ones.

Our contribution is twofold. Firstly, we propose to represent the purely indeterministic field as a finite approximation of an infinite 2-D MA expansion. The MA parameter estimation is then based on the Taylor representation of a two-variable analytic function, called outer function, whose square absolute value on the bi-torus of this function equals almost everywhere the density of the spectral measure of the purely

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[1] The purely indeterministic field has an AR representation if its spectral density is positive in the unit bicircle and analytic in some neighborhood of it.

indeterministic field. Thus, the proposed MA representation differs from the limited order 2-D MA models whose parameters are obtained from the truncated correlation function by algebraic methods (i.e. as proposed in [8]).

Secondly, we present a new decomposition scheme. Unlike existing approaches, the proposed scheme does not require any support detection techniques in the spectral domain. This scheme is based on the proposed MA parameter estimation method applied to the mixed texture. The estimated parameters are then exploited in a filtering and separation procedure.

The remainder of the paper is organized as follows: section 2 deals with the theoretical background and the proposed MA parameter estimation scheme. In section 3, the implementation procedure is presented. In section 4, we derive the decomposition scheme to separate the deterministic and the purely indeterministic sub-fields. An example is provided to illustrate the algorithm.

2. THEORETICAL BACKGROUND

Let us first recall how infinite MA representations can be theoretically associated to purely indeterministic fields.

Let $\{y(m,n)\}_{(m,n)\in\mathbf{Z}^2}$ be any stationary field of zero-mean random variables with finite variance, in the Hilbert space $H = L^2(\Omega)$, for some probability space Ω, and denote by μ_y the spectral measure of the field y on the bi-torus $\mathbf{T}^2 := \{z\in\mathbf{C} : |z| = 1\}^2$.

Given the Helson-Lowdenslager extension [1] of the Wold decomposition theorem, the field $y(m,n)$ is uniquely decomposable as the orthogonal sum:

$$y(m,n) = w(m,n) + v(m,n) \tag{1}$$

where $\{w(m,n)\}_{(m,n)\in Z^2}$ and $\{v(m,n)\}_{(m,n)\in Z^2}$ are respectively purely indeterministic and deterministic fields, with their spectral measures denoted μ_w and μ_v.

It is known that, if $\{w(m,n)\}$ is not zero, then μ_w is absolutely continuous and μ_v is singular, so that the spectral measure of y decomposes as: $d\mu_y = \varphi dm_2 + \mu_v$, where φ is the spectral density of μ_w with respect to the normalized Lebesgue measure m_2 on $\mathbf{T}^2$.

The construction of the purely indeterministic $w(m,n)$ field is based on the so-called innovations of the field with respect to some fixed total group order $\leq$ on $\mathbf{Z}^2$. More precisely, consider the field $\{p(m,n)\}_{(m,n)\in\mathbf{Z}^2}$ obtained by projecting each sample $y(m,n)$ onto the closed linear span of all the past samples $y(m',n')$, i.e. $(m',n') < (m,n)$. The field $i(m,n) = y(m,n) - p(m,n)$ is called the innovation field of $\{y(m,n)\}$. The innovation field $\{i(m,n)\}$ is then a white process with variance σ^2 given by the Szegö formula:

$$\sigma^2 = \|i(0,0)\|^2_{L^2(\Omega)} = \exp\left(\int_{\mathbf{T}^2}\log(\varphi)dm_2\right). \tag{2}$$

The purely indeterministic field $\{w(m,n)\}$ is obtained by projecting the field $\{y(m,n)\}$ onto the closed linear span of all the samples of the innovation field $\{i(m,n)\}_{(m,n)\in\mathbf{Z}^2}$. Since the field $\{w(m,n)\}$ lies in the closed linear span of the innovation white field $\{i(m,n)\}$, the stationarity condition implies that each sample of $\{w(m,n)\}$ is an infinite linear combination of innovation samples. Therefore it admits an infinite 2-D MA representation:

$$w(m,n) = \sum_{(0,0)\preceq(k,l)} a(k,l)\, i(m-k,n-l), \tag{3}$$

driven by the white noise $\{i(m,n)\}$, where $a(k,l)$ are some square summable coefficients. Thus, the transfer function f associated to the model:

$$f(z_1,z_2) = \sum_{(k,l)\geq(0,0)} a(k,l) z_1^k z_2^l \tag{4}$$

belongs to $L^2(\mathbf{T}^2)$ and must satisfy the relation:

$$\sigma^2 |f(z_1,z_2)|^2 = \varphi(z_1,z_2) \text{ a.e. } \mathbf{T}^2 \tag{5}$$

Now, if one replaces in the MA representation (3) the innovation process $\{i(m,n)\}$ by some other white process of the same variance, and the coefficients $a(k,l)$ by the coefficients $a'(k,l)$ of some other function f' with the same modulus on $\mathbf{T}^2$, the resulting field $\{w'(m,n)\}$ obviously has the same spectral measure $d\mu_{w'} = \varphi dm_2$ as $\{w(m,n)\}$. Therefore, $\{w(m,n)\}$ and $\{w'(m,n)\}$ are isomorphic. To describe the field $\{w(m,n)\}$, one can make any particular choice of MA parameters provided they satisfy (5), and this choice affects only the driving white noise.

For practical purposes (see also [3], [4]), one can assume that the representation (3) has all the coefficients in the first quadrant of $\mathbf{Z}^2$. The function f may thus be considered analytic, and hence belongs to the Hardy space $H^2(\mathbf{D}^2)$. With this additional assumption, it is well known [5] that all the solutions of the equation (5) are factors of the form

$$f(z_1,z_2) = u(z_1,z_2)F(z_1,z_2) \tag{6}$$

where u is any inner function (i.e. u has modulus 1 a.e. on $\mathbf{T}^2$) and F is the *unique* outer function (see [5]) such that $|F|^2 = \varphi/\sigma^2$ almost everywhere on $\mathbf{T}^2$.

A special solution for the equation (5) is therefore the outer function F itself (i.e. $u \equiv 1$ in (6)), which has two advantages: the first is that F is bounded and has no zeroes in the unit bi-disk, meaning that the corresponding MA filter (3) and its inverse are both BIBO-stable. The second advantage is that F is uniquely determined by the density φ, through the formula (see [5]):

$$F(z_1,z_2)= \exp\left(\frac{1}{4\pi^2} \int_0^{2\pi}\int_0^{2\pi} \frac{e^{it_1}+z_1}{e^{it_1}-z_1} \frac{e^{it_2}+z_2}{e^{it_2}-z_2} \log \varphi^{1/2}(e^{it_1},e^{it_2}) dt_1 dt_2 \right) \tag{7}$$

The parameters of the MA model (3), i.e. the Taylor coefficients of F, are then easily obtained by the classical Cauchy formula.

3. PROPOSED METHOD

The proposed method uses the numerical computation of the outer factor F described above, through the following approach: consider an image representing a texture, regarded as a realization of a stationary random field. Under a standard ergodicity assumption, the density of the spectral measure of the underlying random field can be assimilated to the power spectral density (PSD) of the texture. Using the PSD, one can perform a discrete computation of a finite set of coefficients of the outer function F associated to this density, and hence obtain a finite 2-D MA approximation of the MA model (3).

The proposed estimation method is resumed in the following three steps:

1. Estimation of the PSD $\hat{\varphi}_{ww}(e^{j\omega_1},e^{j\omega_2})$ of the texture.
2. Estimation of the outer function:

$$F(r_1e^{j\theta_1},r_2e^{j\theta_2})= \exp\int_{\omega_1=0}^{2\pi}\int_{\omega_2=0}^{2\pi} \frac{1}{2(2\pi)^2}\left(\frac{e^{j\omega_1}+r_1e^{j\theta_1}}{e^{j\omega_1}-r_1e^{j\theta_1}}\right)\left(\frac{e^{j\omega_2}+r_2e^{j\theta_2}}{e^{j\omega_2}-r_2e^{j\theta_2}}\right)\log\left|\hat{\varphi}_{ww}(e^{j\omega_1},e^{j\omega_2})\right| d\omega_1 d\omega_2, \tag{8}$$

3. Estimation of the Taylor coefficients of the outer function

$$F(z_1,z_2)=\sum_{m,n}\hat{a}_{m,n} z_1^m z_2^n \ , \ (z_i = r_i e^{j\theta_i}, 0<r_i<1) \tag{9}$$

by using the Cauchy formula:

$$a_{m,n} = \frac{1}{(2\pi)^2 r_1^m r_2^n} \int_{\theta_2=0}^{2\pi}\int_{\theta_1=0}^{2\pi} \hat{F}(r_1e^{j\theta_1},r_2e^{j\theta_2}) e^{jm\theta_1}, e^{jn\theta_2} d\theta_1 d\theta_2 \tag{10}$$

In practice, the PSD is estimated via the periodogram:

$$\hat{P}_y(k,l)=\frac{1}{N^2}\left|Y(k,l)\right|^2 \tag{11}$$

where $Y(k,l)$ denotes the $N\times N$ points 2-D discrete Fourier transform (DFT) of $y(k,l)$.

The discrete version of the outer factor computation is:

$$F_r(k,l)= \exp\sum_{m=0}^{N-1}\sum_{n=0}^{N-1}\frac{1}{2N^2}\left(\frac{e^{j\frac{2\pi}{N}m}+re^{j\frac{2\pi}{N}k}}{e^{j\frac{2\pi}{N}m}-re^{j\frac{2\pi}{N}k}}\right)\left(\frac{e^{j\frac{2\pi}{N}n}+re^{j\frac{2\pi}{N}l}}{e^{j\frac{2\pi}{N}n}-re^{j\frac{2\pi}{N}l}}\right)\log(\hat{P}_y(k,l)) \tag{12}$$

The Taylor coefficients are then obtained as follows:

$$a(m,n)=\frac{1}{N^2 r^{n+m}}\sum_{k=0}^{N-1}\sum_{l=0}^{N-1}F_r(k,l)e^{jmk(\frac{2\pi}{N})}e^{jnl(\frac{2\pi}{N})} \tag{13}$$

for some chosen $0<r<1$.

In figure 1 an example of reconstruction is presented. The original texture (a) is synthesized using a 6×4 2-D MA filter driven by a white noise field. The reconstruction texture (b) is generated using a 6×4 2-D MA model as described above and an arbitrary white noise[2].

4. A NEW DECOMPOSITION SCHEME

In the following, we derive a decomposition approach to separate the deterministic and purely indeterministic components of a texture. Equation (1) can be written as follows:

$$\begin{aligned} y(m,n) &= w(m,n)+v(m,n) \\ &= \sum_{(0,0)\preceq(k,l)} a(k,l)u(m-k,n-l)+v(m,n) \end{aligned} \tag{14}$$

where u denotes a white noise with same variance as the innovation field $\{i(m,n)\}$ and $\{a(k,l)\}$ the MA parameters of the purely indeterministic field w. Taking the z transform[3] of equation (14) yields:

$$\begin{aligned} Y(z_1,z_2) &= W(z_1,z_2)+V(z_1,z_2) \\ &= F(z_1,z_2)U(z_1,z_2)+V(z_1,z_2) \end{aligned} \tag{15}$$

where $F(z_1,z_2)$ is the outer factor defined as follows:

$$F(z_1,z_2)=\sum_{m,n}a(m,n)z^m z^n \tag{16}$$

Now, by filtering the original image with $F^{-1}(z_1,z_2)$, one obtains a field $Y_{FIL}(z_1,z_2)$ whose z-transform is given by:

[2] The 2-D MA model is truncated to 6×4 given that the other coefficients are negligible.

[3] Capital letters indicate z-transforms for the corresponding fields.

$$Y_{FIL}(z_1,z_2) := F^{-1}(z_1,z_2)Y(z_1,z_2) = U(z_1,z_2) + F^{-1}(z_1,z_2)V(z_1,z_2) \quad (17)$$

i.e. the sum of the z-transform of the white noise u and of some deterministic field. Thus, to compute the deterministic field v in the original texture, one has to filter out the white noise u from (17), and then to restore v by filtering again the result with F. See figure 2.

As mentioned in the theoretical section, the fact that F is an outer function (and hence has no zeroes in the bi-disk) insures that the inverse filtering (17) is stable. This motivates the particular choice of the outer factor

Step 1: Filter the FFT of the original image with the transfer outer function

$$F^{-1}(z_1,z_2) = \exp\left(\frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi}\frac{e^{it_1}+z_1}{e^{it_1}-z_1}\frac{e^{it_2}+z_2}{e^{it_2}-z_2}(-\log\varphi^{1/2}(e^{it_1},e^{it_2}))dt_1dt_1\right)$$

(computed with the discrete equations (12), (13))

Step 2: Estimate the variance of the white noise u

$$\|u(0,0)\| = \|i(0,0)\| = \exp\left(\frac{1}{2N^2}\sum_{k=0}^{N-1}\sum_{l=0}^{N-1}\log\left(\frac{1}{N^2}|Y_{FIL}(k,l)|^2\right)\right)$$

Step 3: Filter out the white noise. To this end, a binary mask is generated, by using the threshold $\alpha N\sqrt{\|u(0,0)\|}$ in $Y_{FIL}(z_1,z_2)$. This value is issued from: $\|u(0,0)\| = |U(z_1,z_2)|^2/N^2$. The value α is an overestimation factor that ensures the removal of the white noise. This factor compensates the variance of the white noise spectrum estimate $U(z_1,z_2)$. In practice we choose α between 4 and 5.

The resulting mask is dilated by using a standard 4-connected structuring element. The dilated mask is used as a filter in the spectral domain (see figure 4).

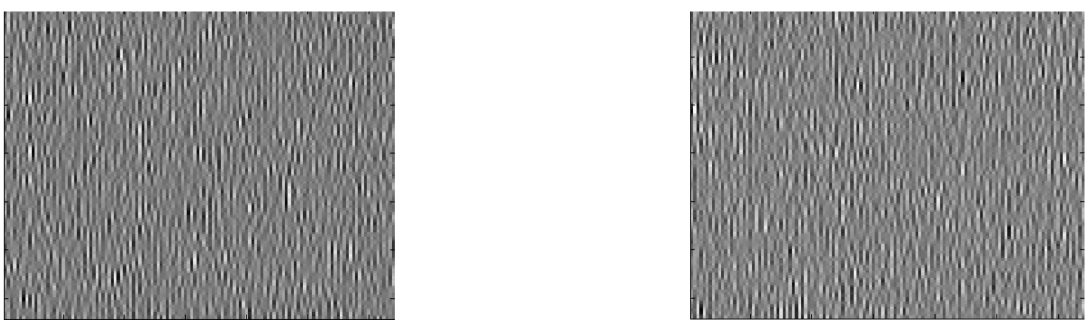
Step 4: Filter the result of Step 3 with the outer factor F, computed as indicated in the previous section, and obtain an estimation of the deterministic part of the texture.

An example is presented in figure 3, illustrating the decomposition of a texture into its deterministic and purely indeterministic parts. In figure 3, we present the original image (a), the deterministic field (b) extracted from (a) at the end of Step 4 and the purely indeterministic field (c) obtained by subtracting (b) from (a).

In figure 4, we present intermediary results related to the filtering out of noise u in Step 3: (a) the filtered image in frequency domain Y_{FIL}, (b) binary mask obtained from Y_{FIL} using the threshold $\alpha N\sqrt{\|u(0,0)\|}$, (c) dilated binary mask. The dilated binary mask is used as a filter in the frequency domain for recovering the deterministic component (i.e. an evanescent field).

6. CONCLUSIONS

In this paper, we propose to use the outer factor of the spectral power density of a texture to compute a 2-D MA model for the purely indeterministic part of the texture. The method is then used to derive a Wold decomposition scheme to separate the purely indeterministic and the deterministic parts of a texture. Unlike existing approaches, our decomposition scheme does not require any support detection techniques in the spectral domain.



(a) Original purely indeterministic field (b) 2-D MA reconstruction

Figure 1. Analysis / Synthesis Example.

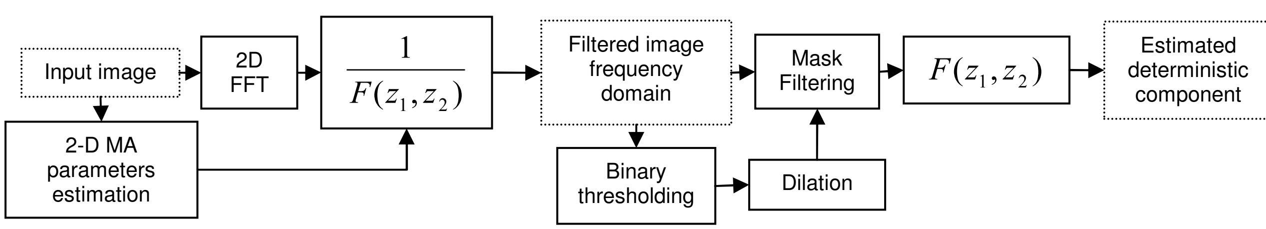
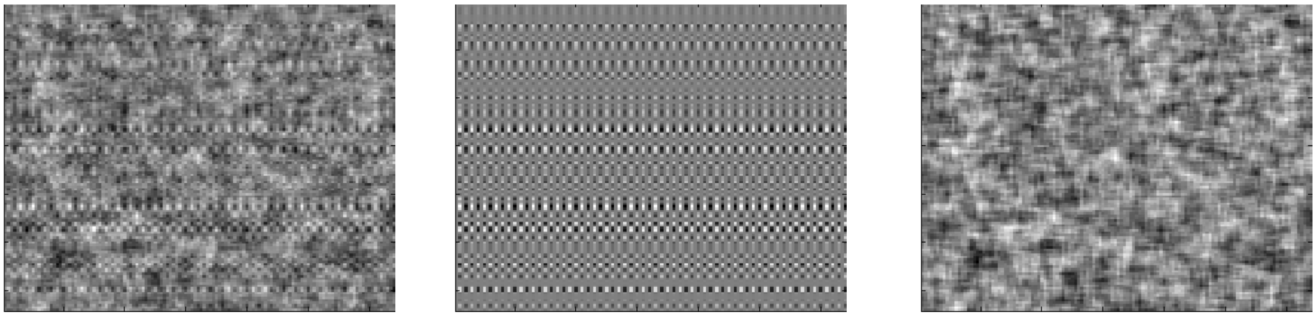


Figure 2. Purely indeterministic / deterministic decomposition scheme



(a) Original image Y (b) Estimated deterministic field V (c) Estimated purely indeterministic field Y-V

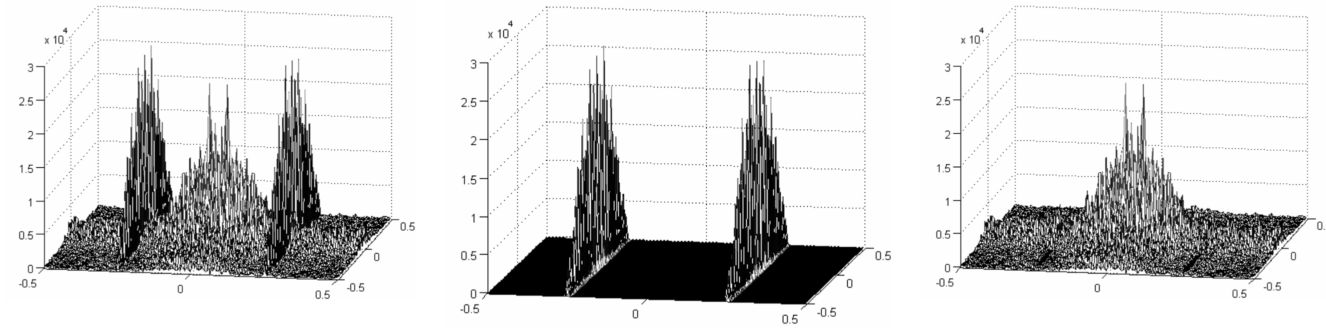
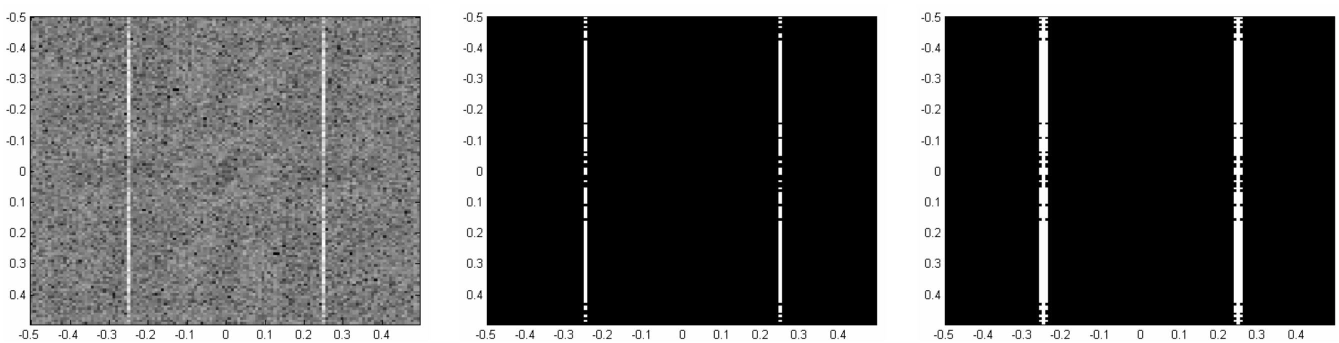


Figure 3. Wold Decomposition example. First row: spatial domain. Second row: frequency domain (linear scale).



(a) Filtered image (frequency domain) (b) Binary mask obtained by thresholding (c) Dilation of binary mask (with 4 connected structuring element)

Figure 4. Decomposition algorithm details (Step 3).

7. REFERENCES

[1] H. Helson D. Lowdenslager, “Prediction theory and Fourier series in several variables, I,II,” *Acta Math*., vol.99, pp.165-202, 1958, vol.106, pp.175-213, 1962.

[2] J. M. Francos, A. Z. Meiri and B. Porat, “A Unified Texture Model Based on a 2-D Wold Like Decomposition,” *IEEE Trans. Signal Process,* vol. 41, pp. 2665-2678, August 1993.

[3] L. Gavarecki, V. Mandrekar, P. Richard, “Proper Moving Average Representations and Outer Functions in Two Variables” *Georgian Math. J.*, vol. 8, No. 2, pp.275-281, 2001.

[4] H. Korezlioglu, and P. Loubaton, “Spectral factorization of wide Sense stationary processes on Z2,” *Journal of Multivariate Analysis*, vol. 19, no. 1, pp. 24-27, 1986.

[5] W. Rudin, *Function theory in polydiscs*, Benjamin, New York, 1969.

[6] H. Wold, “A study in the analysis of stationary time series”, Almquist & Wiksell, Stockholm, 1938

[7] F. Liu and R. W. Picard, “A Spectral 2-D Wold Decomposition Algorithm for Homogeneous Random Fields,” *ICASSP'99*, Phoenix, Arizona, March 1999, vol. 4, pp. 3501-3504.

[8] J. M. Francos and B. Friedlander, “Parameter Estimation of Two-Dimensional Moving Average Random Fields,” *IEEE Trans. Signal Process*., vol. 46, pp. 2157-2165, August 1998.

[9] L. Ghouti, A. Bouridane, M.K., “Image compression using texture modelling,” *ISCAS 2005*, vol. 3, pp.2313-2316, May 2005.

[10] N. Komodakis, “Image completion using global optimization,” *Proc. of CVPR’06*, vol.1, pp. 442-452, 2006.