Compact Perturbations and Factorizations of Closed Range Operators

Ioana Serban, Flavius Turcu
Université Bordeaux 1, LAPS
351, Cours de la Liberation
33400 Talence, France
serban@tsi.u-bordeaux1.fr
turcu@tsi.u-bordeaux1.fr

Abstract: In this paper it is shown that if A et B are closed range operators in a Hilbert space for which the equation $B = XA$ has at least a solution, then the compacteness of $A - B$ is equivalent to the existence of a solution X such that $X - I$ is compact. This result has several consequences on the description of the compact perturbations of particular classes of operators.

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Let H be an infinitely dimension separable complex Hilbert space and $B(H)$ the Banach algebra of bounded linear operators on H. Several papers deal with the study of perturbations $B = A + K$ of isometry A in $B(H)$, when K runs over certain classes of compact operators, such as rank one operators (in [5], [6]), or finite rank operators (see [1], [2], [3]). A first question is to caracterize the situations when the perturbation B is still an isometry, or at least a contraction. For K a rank one operator, it is shown in [6] that $B = A + K$ is an isometry (resp. contraction) if and only if $K = (\alpha - 1)h \otimes A^*h$, where h is an unitary vector in H and $|\alpha| = 1$ (resp. $|\alpha| \leq 1$).

This condition is equivalent to the existence of a factorization $B = XA$ where for some unitary (respectively contraction) X such that $X - I$ is of rank 1 (more precisely $X = I + (\alpha - 1)h \otimes h$). In [8] we show that this is still true for K of finite rank (resp. an arbitrary compact), with an factor X such that $X - I$ is of finite rank (resp. compact).

It is clear that factorizations of type $B = XA$ always exist for any isometry A and any contraction B (see [4]), so the fact that $A - B$ is also compact (or finite rank) is equivalent to the existence of a factor X such that $X - I$ is of the same type as $A - B$.

The aim of this article is to show that this equivalence still holds for any closed range operator A and any operator B such that $\mathrm{Im}\ B^* \subset \mathrm{Im}\ A^*$ (i.e. for which there exists at least one factorization $B = XA$).

We remind the Douglas factorization criterion [4], which we use in a dual form:

Proposition 1 *Let A and B in $B(H)$. The following affirmations are equivalent:*

1) $\mathrm{Im}\ B^* \subset \mathrm{Im}\ A^*$*;*

2) There is$\lambda > 0$ *such that* $B^*B \leq \lambda^2 A^*A$*;*

3) There is an operator X in $B(H)$ such that $B = XA$.

In this case, there is a unique operator X_0 verifying the following additional conditions: $\overline{\mathrm{Im}\ X_0} = \overline{\mathrm{Im}\ B}$ *and* $\overline{\mathrm{Im}\ X_0^*} \subset \overline{\mathrm{Im}\ A}$*.*

The unique factor X_0 is called *the reduced solution* of the equation $B = XA$, and also satisfies

$\|X_0\| = \inf_{\lambda>0}\{\lambda \in \mathbb{R}_+ : B^*B \leq \lambda^2 A^*A\}$ and $\text{Ker } X_0 \cap \text{Im } A = A(\overline{\text{Im } A^*} \cap \text{Ker } B)$.

If in addition Im A is closed then $\text{Ker } X_0 = \overline{A(\text{Im } A^* \cap \text{Ker } B)} \oplus (\text{Im } A)^{\perp}$.

For a partial isometry W in $B(H)$ we denote by $s_i(W) := W^*W$ the initial support of W (the orthogonal projection on the orthogonal of Ker (W)) and by $s_f(W) := WW^*$ the final support of W (the orthogonal projection on Im (W)). We also note by $i(W) = s_i(W)H$ the initial subspace and by $f(W) = s_f(W)H$ the final subspace of W.

The next lemma shows the behaviour of the polar decompositions of A and B and the reduced solution X_0 in Douglas criterion, when the operator A has a closed range and $A - B$ is compact.

Lemma 2 *Let A a closed range operator in $B(H)$ and B in $B(H)$ such that $A - B$ is compact and* Im $A^* \supset$ Im B^*. *If $A = V|A|$ and $B = W|B|$ are the polar decompositions of A et B, then:*

1) $s_i(V) - s_i(W)$ is a finite rank projection;

2) The operators $V - W$ and $s_f(V) - s_f(W)$ are compact in $B(H)$; 3) If X_0 is the reduced solution of the equation $B = XA$ then $X_0 - s_f(V)$ is compact (i.e. the restriction of $X_0 - I$ to Im A *is a compact operator from* Im A *to H.*

Proof. Note first that $A - B$ is compact and the operator $A|_{\text{Im } A^*} : \text{Im } A^* \to H$ is semi-Fredholm, then the operator $B|_{\text{Im } A^*} : \text{Im } A^* \to H$ is also semi-Fredholm, so in particular B has closed range.

1) Obviously $s_i(V) - s_i(W)$ is the orthogonal projection on $K_0 = \text{Im } A^* \cap \text{Ker } B$. To see that the dimension of K_0 is finite, let M be the closed unit ball of K_0. The operator A is bounded from below on Im A^* so there exists $\delta > 0$ such that $\delta M \subset AM = (A - B)M$. But this last set is compact, therefore the unit ball M is necessarily compact, so K_0 is finite dimensional.

2) Let A' and B' in $B(H)$ the inverses of $|A| : \text{Im } A^* \mapsto \text{Im } A^*$ and $|B| : \text{Im } B^* \mapsto \text{Im } B^*$, extended with 0 on Ker A and Ker B respectively. We have: $|A|A' = A'|A| = s_i(V)$ and $|B|B' = B'|B| = s_i(W)$. As $s_i(V)s_i(W) = s_i(W)$, we have:

$$A' - B' = A's_i(V) - s_i(W)B' = A'(|B| - |A|)B' + A'(s_i(V) - s_i(W))$$

so $A' - B'$ is compact. But then

$$V - W = Vs_i(V) - Ws_i(W) = V|A|A' - W|B|B' = AA' - BB' = A(A' - B') + (A - B)B'.$$

which shows that $V - W$ is compact. Finaly,

$$s_f(V) - s_f(W) = VV^* - WW^* = (V - W)V^* + W(V^* - W^*)$$

is also a compact operator.

3) If X_0 is the reduced solution of $B = XA$, then

$$X_0 - s_f(V) = X_0 s_f(V) - s_f(V) = (X_0 - 1)Vs_i(V)V^* = (B - A)A'V^*$$

so $X_0 - s_f(V)$ is compact and the proof is complete. □

The previous lemma shows that the compacity of $X_0 - I$ holds only on the range of A. We now show that X_0 can be extended on the orthogonal of Im A while preserving this compacity condition. We do this by the means of a caracterisation (shown in [8]), of couples of closed subspaces of H that are ranges of isometries with compact difference, stated here in a slightly generelized form:

Proposition 3 *Let H_1 and H_2 two closed subspaces of H of same dimension and P_1, P_2 the coresponding orthogonal projections. The following affirmations are equivalent:*

1) There exist two partial isometries V_1 and V_2 whith the same initial support, such that $H_i = V_i H$ for $i = 1, 2$ and $V_1 - V_2$ is compact;

2) $P_1 - P_2$ is compact and $\dim(H_1 \cap H_2^\perp) = \dim(H_2 \cap H_1^\perp)$.

Moreover, V_1 and V_2 can be chosen isometric iff $\dim H_1 = \dim H_2 = \aleph_0$.

Clearly, the condition 2) of the previous proposition remains the same when the two subspaces H_1 and H_2 are replaced by their orthogonals. This symmetry is essential in order to prove the main result:

Theorem 4 *Let A a closed range operator in $B(H)$ and B in $B(H)$ such that $\mathrm{Im}\ A^* \supset \mathrm{Im}\ B^*$. The following affirmations are equivalent:*

1) $A - B$ is compact;

2) There exists an operator X in $B(H)$ such that $X - I$ is compact and $B = XA$;

If that is the case, one can choose X such that:

$\mathrm{Ker}\ X = A(\mathrm{Im}\ A^* \cap \mathrm{Ker}\ B)$ *and* $\|X\| = \max\{1, \inf_\lambda\{B^*B \le \lambda^2 A^*A\}\}$.

In particulary:

3) If $\mathrm{Im}\ A^* = \mathrm{Im}\ B^*$ then X can be chosen invertible.

4) If $B^*B \le A^*A$ then X can be chosen contractive.

5) If $A^*A = B^*B$ then X can be chosen unitary.

Proof. The non-trivial part is the fact that 1) implies 2). Let $A = V|A|$ and $B = W|B|$ the polar decomposistions of A and B, and X_0 the reduced solution of the equation $B = XA$. Let's first observe that the codimensions of the ranges of A and B are either both finite or both infinite. Indeed, if for instance the range codimension of A is finite, then A and B are simultaneously right semi-Fredholm, so the range codimension of B is finite. It is enough to treat the infinite case, because if the two codimensions are finite, $X = X_0$ already satisfies the conclusion.

By the Lemma 2 the dimension $d = \dim(\mathrm{Im}\ A^* \cap \mathrm{Ker}\ B)$ is finite, so one can choose a subspace K_0 of $(\mathrm{Im}\ B)^\perp$ such that $\dim K_0 = d$. Let W_0 an arbitrary partial isometry whith $i(W_0) = \mathrm{Im}\ A^* \cap \mathrm{Ker}\ B$ and $f(W_0) = K_0$ (so W_0 is finite rank), and let $W' = W + W_0$. We have then $i(W') = i(V) = \mathrm{Im}\ A^*$, $f(W') = \mathrm{Im}\ B \oplus K_0$, and moreover $V - W' = (V - W) - W_0$ is compact (by the lemma 2). However, $\mathrm{Im}\ A$ and $\mathrm{Im}\ B \oplus K_0$ verify the condition 1) of the Proposition 3, so it is the same for their orthogonals. Thus there exist two partial isometries Y and Z such that $Y - Z$ is compact, $i(Y) = i(Z)$, $f(Y) = (\mathrm{Im}\ A)^\perp$ and $f(Z) = (\mathrm{Im}\ B)^\perp \ominus K_0 \subset (\mathrm{Im}\ B)^\perp$. Let U_0 be the reduced solution of $Z = UY$ (which is in fact an partial isometry that acts unitarily between $(\mathrm{Im}\ A)^\perp$ and $\mathrm{Im}\ B)^\perp \ominus K_0$). by the Lemma 2 we know that $U - s_f(Y)$ est compact.

Let's set $X = X_0 + U_0$. Obviously $B = XA$ and moreover $X - I = (X_0 - s_f(V)) + (U_0 - s_f(Y))$ is compact, which ends the proof of the implication. Moreover, by the construction of U_0 and the Douglas criterion, we have:

$$\|X\| = \max\{\|U_0\|, \|X_0\|\} = \max\{1, \inf_\lambda\{B^*B \le \lambda^2 A^*A\}\}.$$

For the suplementary statements:

3) If $\mathrm{Im}\ A^* = \mathrm{Im}\ B^*$ then $\mathrm{Ker}\ X_0 = \mathrm{Im}\ A^* \cap \mathrm{Ker}\ B = (0)$, so $\mathrm{Ker}\ X = \mathrm{Ker}\ X_0 = (0)$. But as X is Fredholm of zero index, this implies that X is invertible.

4)If $A^*A \leq B^*B$ then $\|X_0\| \leq 1$, so $\|X\| \leq 1$.

5) Finaly, if $A^*A = B^*B$ then X_0 acts unitarily between Im A and Im B, while U_0 acts unitarily between (Im $A)^\perp$ and (Im $A)^\perp$, so X is unitary, and the proof is complete.

□

This result is not true if the closed range condition on A is dropped. For example, if A is any dense range compact in H, X_0 an unitary such that $X_0 - I$ is not compact, and $B = X_0 A$, then $A - B$ is trivialy compact, but X_0 is the unique solution of the equation $B = XA$.

Corollary 5 *If A and B are in $B(H)$ such that A is an isometry and $A - B$ is a compact operator then B is a contraction (resp. isometry) iff there exists a contraction (resp. unitary) X in $B(H)$ such that $B = XA$ and that $X - I$ is compact.*

In [8] it is shown that in this last corolary one can replace "compact" by "finite rank". A close look at the proofs shows that the same can be done in Theorem 4, but only under the additional hypothesis that $|A| - |B|$ is also a finite rank operator.

More precisely, this additional condition does not follow in general from the fact that $A - B$ has finite rank (as it happens to its analoguous condition in the compact case) because if the difference $R_1 - R_2$ of two positive operators has finite rank, this doesn't necessarily imply that $R_1^{1/2} - R_2^{1/2}$ is finite rank (in our case $R_1 = A^*A$ and $R_2 = B^*B$).

Such an example can be found as follows (cf. H. Bercovici, private communication): consider $0 < R_1 < R_2$ two positive operators such that $R_2 - R_1$ has rank one and the image of $R_2 - R_1$ is cyclic for R_2. Then $R_1^{1/2} - R_2^{1/2}$ is injective, and thus cannot have finite rank. To see this take x in $Ker(R_1^{1/2} - R_2^{1/2})$ and write

$$R_i^{1/2} = -\frac{1}{\pi}\int_0^\infty \sqrt{t}((t + R_i)^{-1} - t^{-1})dt \qquad (i = 1, 2)$$

.

Then x lies in $Ker((t + R_1)^{-1} - (t + R_2)^{-1})$ for $t > 0$, hence in $Ker((R_2 - R_1)(t + R_2)^{-1}$ for $t > 0$, which implies that $(t + R_2)^{-1}x$ is orthogonal to the range of $R_2 - R_1$ for $t > 0$. But then $R_2^n x$ is orthogonal to the range of $R_2 - R_1$ for $n \geq 0$ which is itself cyclic for R_2, so necessarily $x = 0$, meaning that $R_1^{1/2} - R_2^{1/2}$ is not a finite rank operator.

However, in the particular case when A is an isometry, the fact that $A - B$ has finite rank implies that $A^*A - B^*B = I - B^*B$ has finite rank, so B^*B is diagonalizable and hence $B^*B - (B^*B)^{1/2}$ has finite rank, therefore $|A| - |B| = (I - B^*B) + (B^*B - (B^*B)^{1/2})$ has finite rank.

Corollary 6 *Let A in $B(H)$ bounded from below operator. All the perturbations of A by compact operators are of type XA with X in $B(H)$ such that $X - I$ is compact.*

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