TUTORIAL ON SYSTEM IDENTIFICATION USING FRACTIONAL DIFFERENTIATION MODELS

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Abstract: This paper presents a tutorial on system identification using fractional differentiation models. The tutorial starts with some general aspects on time and frequency-domain representations, time-domain simulation, and stability of fractional models. Then, an overview on system identification methods using fractional models is presented. Both equation-error and output-error-based models are detailed.

Keywords: Fractional differentiation, fractional integration, identification, simulation, output error models, equation error models, state variable filter.

1. INTRODUCTION

Although fractional (non integer) operators remained for a long time purely a mathematical concept, the rise of digital computers offered an easy way for simulating numerically non integer integro-differentiation of mathematical functions.

The last two decades have witnessed considerable development in the use of fractional differentiation in various fields. Fractional differentiation is now an important tool for the international scientific and industrial communities. In that scope, the 1st IFAC Workshop on Fractional Differentiation and its Applications (FDA'04) was hold in 2004 in France. The use of fractional differentiation models in system identification was initiated in the late nineties and the beginning of this century (Le Lay, 1998; Lin, 2001; Cois, 2002; Aoun, 2005).

Fractional models are now enough mature and are widely used in representing some diffusive phenomena (thermal diffusion, electrochemical diffusion) and in modeling viscoelastic materials.

1.1 Mathematical background

A fractional mathematical model is based on fractional differential equation:

$$y(t) + b_1\mathbf{D}^{\beta_1}y(t) + \cdots + b_{m_B}\mathbf{D}^{\beta_{m_B}}y(t) = a_0\mathbf{D}^{\alpha_0}u(t) + a_1\mathbf{D}^{\alpha_1}u(t) + \cdots + a_{m_A}\mathbf{D}^{\alpha_{m_A}}u(t) \quad (1)$$

where differentiation orders, $\beta_1 < \beta_2 < \ldots < \beta_{m_B}$, $\alpha_0 < \alpha_1 < \ldots < \alpha_{m_A}$, are allowed to be non-integer positive numbers. The concept of differentiation to an arbitrary order (non-integer),

$$\mathbf{D}^\gamma \triangleq \left(\frac{d}{dt}\right)^\gamma \qquad \forall\gamma \in \mathbb{R}_+^*$$

was defined in the 19^{th} century by Riemann and Liouville. The γ fractional derivative of $x(t)$ is defined as being an integer derivative of order $m = \lfloor\gamma\rfloor + 1$ ($\lfloor . \rfloor$ stands for the floor operator) of a non-integer integral of order $1-(m-\gamma)$ (Samko *et al.*, 1993):

$$\mathbf{D}^\gamma x(t) = \mathbf{D}^m\left(\mathbf{I}^{m-\gamma}x(t)\right) \triangleq \frac{1}{\Gamma(m-\gamma)}\left(\frac{d}{dt}\right)^m \int_0^t \frac{x(\tau)\,d\tau}{(t-\tau)^{1-(m-\gamma)}} \quad (2)$$

where $t > 0$, $\forall\gamma \in \mathbb{R}_+^*$, and the Euler's Γ function is defined as:

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt \qquad \forall x \in \mathbb{R}^* \setminus \{\mathbb{N}^-\} \tag{3}$$

A discrete-time definition of fractional derivative was proposed by Grünwald (1867), $\forall\gamma \in \mathbb{R}_+^*$:

$$\mathbf{D}^\gamma x(t) = \lim_{h\to 0} \frac{1}{h^\gamma} \sum_{k=0}^{\infty} (-1)^k \binom{\gamma}{k} x(t-kh) \tag{4}$$

where Newton's binomial $\binom{\gamma}{k}$ is generalized to non-integer orders by the use of Euler's Γ function:

$$\binom{\gamma}{k} = \frac{\Gamma(\gamma+1)}{\Gamma(k+1)\Gamma(\gamma-k+1)} \tag{5}$$

Equation (4) is generally used in time-domain simulations of fractional differentiation. As Newton's binomial $\binom{\gamma}{k}$ does not converge rapidly to zero with k when γ is non integer, the computation of $\mathbf{D}^\gamma x(t)$ depends on all values of $x(t)$ between 0 and t (supposing that $x(t)$ is relaxed at $t = 0$, *i.e.* $x(t) = 0\ \forall t < 0$). Since fractional derivatives of a function depend on whole its past, fractional operators are known to have long memory behavior.

A more concise algebraic tool can be used to represent fractional systems: the Laplace transform (Oldham and Spanier, 1974):

$$\mathscr{L}\{\mathbf{D}^\gamma x(t)\} = s^\gamma X(s) \quad \text{if } x(t) = 0\ \forall t < 0$$

This property allows to write the fractional differential equation (1), provided $u(t)$ and $y(t)$ are relaxed at $t = 0$, in a transfer function form:

$$F(s) = \frac{\sum_{i=0}^{m_A} a_i s^{\alpha_i}}{1 + \sum_{j=1}^{m_B} b_j s^{\beta_j}} \tag{6}$$

where $(a_i, b_j) \in \mathbb{R}^2$, $(\alpha_i, \beta_j) \in \mathbb{R}_+^2$, $\forall i = 0, 1, \ldots, m_A$, $\forall j = 1, 2, \ldots, m_B$.

Definition 1. A transfer function $F(s)$ is commensurate of order γ iff it can be written as $F(s) = S(s^\gamma)$, where $S = \frac{T}{R}$ is a rational function with T and R two co-prime polynomials. Moreover, the commensurate order γ is the biggest number satisfying the aforementioned condition. □

In other words, the commensurate order γ is defined as the biggest real number such that all differentiation orders are integer multiples of γ.

A modal form transfer function can then be obtained, provided (6) is strictly proper:

$$F(s) = \sum_{k=1}^{N} \sum_{q=1}^{v_k} \frac{A_{k,l}}{(s^\gamma - s_k)^q}, \tag{7}$$

where $s_k, k = 1, \cdots, N$ are known as the s^γ-poles of integer multiplicity q.

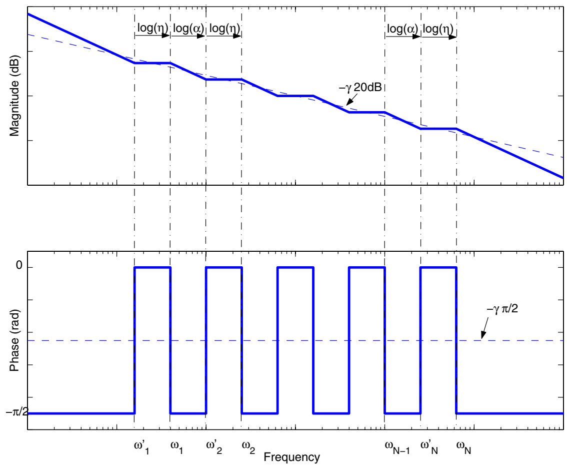

Fig. 1. Recursive approximation of a fractional differentiator with poles and zeroes

1.2 Stability theorem

(Matignon, 1998, revisited) A commensurate (of order γ) transfer function $F(s)$ is BIBO stable iff

$$0 < \gamma < 2 \tag{8}$$

and for every s^γ-pole, $s_k \in \mathbb{C}$ of $F(s)$:

$$|\arg(s_k)| > \gamma\frac{\pi}{2} \tag{9}$$

1.3 Equivalence with rational models

Due to the consideration that real physical systems generally have bandlimited fractional behavior and due to the practical limitations of input and output signals (Shannon's cut-off frequency for the upper band and the spectrum of the input signal for the lower band), fractional operators are usually approximated by high order rational models. As a result, a fractional model and its rational approximation have the same dynamics within a limited frequency band. The most commonly used approximation of s^γ in the frequency band $[\omega_A, \omega_B]$ is the recursive distribution of zeros and poles proposed by Oustaloup (1983). Trigeassou *et al.* (1999) suggested to use an integrator outside the frequency range $[\omega_A, \omega_B]$ instead of a gain:

$$s^{-\gamma} \to s^{-\gamma}_{[\omega_A,\omega_B]} = \frac{C_0}{s}\left(\frac{1+\frac{s}{\omega_A}}{1+\frac{s}{\omega_B}}\right)^{1-\gamma} \approx \frac{C_0}{s}\prod_{k=1}^{N}\frac{1+\frac{s}{\omega'_k}}{1+\frac{s}{\omega_k}} \tag{10}$$

where $\omega_i = \alpha\omega'_i$, $\quad \omega'_{i+1} = \eta\omega'_i$ and

$$\gamma = 1 - \frac{\log\alpha}{\log\alpha\eta} \tag{11}$$

α and η are real parameters which depend on the differentiation order γ. The bigger N the better the approximation of the integrator $s^{-\gamma}$.

2. SYSTEM IDENTIFICATION

Frequency-domain system identification using fractional models was initiated by the Ph. D. thesis of (Le Lay, 1998). Time-domain system identification using fractional differentiation models was initiated by the Ph.D. theses of Le Lay (1998), Lin (2001), and Cois (2002). Mainly two classes of models were developed: Equation-Error-based models and Output-Error-based models, both of which are presented in this section. Recently Malti *et al.* (2005) synthesized fractional orthogonal bases generalizing Laguerre, Kautz and BOG bases to fractional differentiation orders.

2.1 Equation-error models

Equation-error-based models are linear in coefficients. The identified system is assumed to be initially at rest, modeled by (1), and characterized by input/output coefficient's vector:

$$\theta = [a_0 \dots a_{m_A} b_1 \dots b_{m_B}]^{\mathbf{T}} \quad (12)$$

A priori knowledge is generally used to fix the differentiation orders $\alpha_0, \dots, \alpha_{m_A}, \beta_1, \dots, \beta_{m_B}$. Usually a commensurate order γ is chosen and then all its multiples fixed up to a given order, say β_{m_B}, α_{m_A} is generally set to $\beta_{m_B} - \gamma$ for strictly proper systems.

$$F(s) = \frac{\sum_{i=0}^{\frac{\alpha_{m_A}}{\gamma}} a_i s^{i\gamma}}{1 + \sum_{j=1}^{\frac{\beta_{m_B}}{\gamma}} b_j s^{j\gamma}} \quad (13)$$

Consider observed data $u(t)$ and $y^*(t) = y(t) + p(t)$, where $p(t)$ is a perturbation signal, collected at regular samples: $k_0T_s, (k_0+1)T_s, \dots, (k_0 + K - 1)T_s$. The most basic estimation method consists of computing fractional derivatives of input/output signals from sampled data by applying (4). The output can be written in a regression form:

$$y(t) = \phi^*(t)\theta \quad (14)$$

where parameters and regression vectors are respectively given by (12) and:

$$\phi^*(t) = \begin{bmatrix} \mathbf{D}^{\alpha_0} u(t) \cdots \mathbf{D}^{\alpha_{m_A}} u(t) \\ -\mathbf{D}^{\beta_1} y^*(t) \cdots - \mathbf{D}^{\beta_{m_B}} y^*(t) \end{bmatrix} \quad (15)$$

The estimated parameters vector $\hat{\theta}$ of θ is obtained by minimizing the quadratic norm of the error:

$$J\left(\hat{\theta}\right) = \mathbf{E}^{\mathrm{T}}\mathbf{E} \quad (16)$$

where:

$$\mathbf{E} = \begin{bmatrix} \varepsilon(k_0 T_{\mathrm{s}}) & \varepsilon((k_0+1)T_{\mathrm{s}}) \\ & \cdots \varepsilon((k_0+K-1)T_{\mathrm{s}}) \end{bmatrix}^{\mathrm{T}}$$

and

$$\varepsilon(t) = y^*(t) - \phi^*(t)\hat{\theta}$$

The minimum of J is given by the classical least squares:

$$\hat{\theta}_{\mathbf{opt}} = \left(\mathbf{\Phi}^{*\mathrm{T}}\mathbf{\Phi}^*\right)^{-1} \mathbf{\Phi}^{*\mathrm{T}}\mathbf{Y}^*$$

where:

$$\mathbf{\Phi}^* = \left[\phi^{*T}(k_0T_s) \quad \phi^{*T}((k_0+1)T_s) \quad \dots \right. \\ \left. \phi^{*T}((k_0+K-1)T_s)\right]^{\mathrm{T}} \quad (17)$$

As in the integer case, fractional differentiation of noisy signals amplifies the noise. Hence, a linear transformation (low-pass filter) can be applied to (14) so as to obtain a linear continuous regression of filtered input, $u_f(t)$, and output, $y_f^*(t)$, signals:

$$y_f(t) = \phi_f^*(t)\theta \quad (18)$$

where

$$\phi_f^*(t) = \begin{bmatrix} \mathbf{D}^{\alpha_0} u_f(t) \cdots \mathbf{D}^{\alpha_{m_A}} u_f(t) \\ -\mathbf{D}^{\beta_1} y_f^*(t) \cdots - \mathbf{D}^{\beta_{m_B}} y_f^*(t) \end{bmatrix} \quad (19)$$

The filter is generally chosen to be causal, stationary, and low-pass. Among the possible filters, Cois *et al.* (2001) extend the concept of State Variable Filters (SVF) to fractional differentiation systems. They propose to use the following fractional filter:

$$H(s) = \frac{A}{\alpha_0 + \alpha_1 s^\gamma + \dots + \alpha_{N_f-1} s^{\gamma(N_f-1)} + s^{\gamma N_f}}$$

γN_f is filter's order. The design must respect the following specifications:

- $N_f > \max(\beta_{m_B}, \alpha_{m_A})$
- Coefficients $\alpha_0, \alpha_1, \dots, \alpha_{N_f-1}$ must be chosen such that $H(s)$ is stable.

A particular choice of SVF, proposed by Cois *et al.* (2001), is the fractional Poisson's filter:

$$H(s) = \frac{1}{\left(\left(\frac{s}{\omega_f}\right)^{\frac{1}{n}} + 1\right)^{N_f}} =$$

$$\frac{\omega_f^{nN_f}}{s^{nN_f} + \binom{N_f}{1}\omega_f^n s^{n(N_f-1)} \cdot\cdot \binom{N_f}{N_f-1}\omega_f^{n(N_f-1)} s^n + \omega_f^{nN_f}}$$

which is simply an extension of the rational Poisson's filter to fractional differentiation orders. Frequency ω_f is fixed by the user according to the frequency characteristics of the system to be identified (close to the highest corner frequency). The state vector, composed of fractional derivatives of filtered input or output signals, is defined by:

$$x_f = \begin{bmatrix} \mathbf{D}^{(N_f-1)\gamma} z_f(t), \mathbf{D}^{(N_f-2)\gamma} z_f(t), \\ \dots, \mathbf{D}^{\gamma} z_f(t), z_f(t) \end{bmatrix}^{\mathrm{T}} \quad (20)$$

where z_f denotes either u_f or y_f. The fractional state space representation of the filter is given by:

$$\mathbf{D}^\gamma x_f(t) = A_f x_f(t) + B_f z_f(t) \quad (21)$$

where $A_f = -$

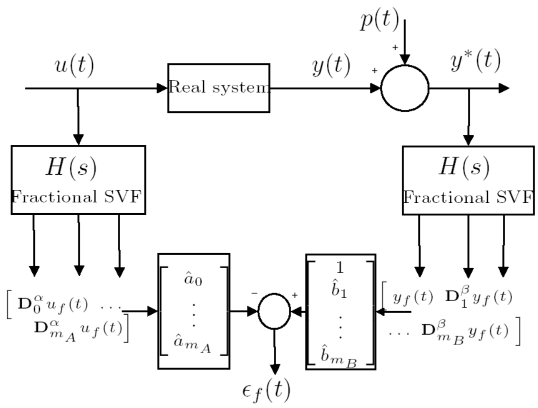

Fig. 2. Fractional state variable filters

$$\begin{bmatrix} \binom{N_f}{1}\omega_f^{\gamma} & \binom{N_f}{2}\omega_f^{2\gamma} & \cdots & \binom{N_f}{N_f-1}\omega_f^{\gamma(N_f-1)} & \omega_f^{\gamma N_f} \\ -1 & 0 & \cdots & \cdots & 0 \\ 0 & \ddots & 0 & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & -1 & 0 \end{bmatrix}$$

and

$$B_f = \left[\omega_f^{\gamma N_f} \; 0 \ldots 0\right]^T$$

Each state represents the derivative of a given order of input or output signals (see figure (2)). Fractional Poisson's filters are simulated using (4)

The estimated parameters vector $\hat{\theta}$ of θ is now obtained by minimizing the quadratic norm of the filtered equation error:

$$J\left(\hat{\theta}\right) = \mathbf{E_f}^{\mathrm{T}}\mathbf{E_f} \tag{22}$$

where:

$$\mathbf{E_f} = \begin{bmatrix} \varepsilon_f\left(k_0 T_{\mathrm{s}}\right) & \varepsilon_f\left(\left(k_0+1\right)T_{\mathrm{s}}\right) \\ & \cdots \varepsilon_f\left(\left(k_0+K-1\right)T_{\mathrm{s}}\right) \end{bmatrix}^{\mathrm{T}}$$

and

$$\varepsilon_f\left(t\right) = y_f^*\left(t\right) - \phi_f^*(t)\hat{\theta},$$

ϕ_f^* being defined by (19).

The solution is given by the classical least squares:

$$\hat{\theta} = \left(\mathbf{\Phi}_{\mathrm{f}}{}^{\mathrm{T}}\mathbf{\Phi}_f\right)^{-1}\mathbf{\Phi}_f{}^{\mathrm{T}}\mathbf{Y}_f^* \tag{23}$$

where:

$$\mathbf{\Phi}_f^* = \left[\phi_f^{*T}(k_0T_s) \quad \phi_f^{*T}\left((k_0+1)T_s\right) \quad \ldots \quad \phi_f^{*T}\left((k_0+K-1)T_s\right)\right] \tag{24}$$

As in the classical case, Cois *et al.* (2001) showed that the least squares estimator (23) is biased in presence of noisy output. To eliminate the bias, they propose to use instrumental variable method. Parameters are estimated according to:

$$\hat{\theta}_{\mathrm{opt}}^{\mathrm{IV}} = \left(\mathbf{\Phi}_f^{\mathrm{IV}\,\mathrm{T}}\mathbf{\Phi}_f^*\right)^{-1}\mathbf{\Phi}_f{}^{\mathrm{IV}\,\mathrm{T}}\mathbf{Y}_f^*$$

where $\mathbf{\Phi}_f^{IV}$ is the regression matrix formed of derivatives of filtered inputs and derivatives of instrumental variables. The authors also suggest to optimize instruments by an iterative method.

2.2 Output-error models

Output-error-based models allow simultaneous estimation of differentiation orders and model parameters. Mainly, two identification methods were proposed in the literature. They differ in the way fractional derivatives are simulated and, at the same time, the way differentiation orders are estimated. The first approach is based on discrete-time simulation of fractional models and assumes that the fractional behavior is present in the whole frequency band, whereas the second one is based on continuous-time simulation of fractional models and assumes that the fractional behavior is present in a limited frequency band.

2.2.1. Method based on discrete-time simulation of fractional models Here, the system to be identified is assumed to be initially at rest, modeled by (6). It is now characterized by input/output vector formed of coefficients and differentiation orders $\theta = [a_0, \ldots a_{m_A}, b_1, \ldots b_{m_B}, \alpha_0, \ldots \alpha_{m_A}, \beta_1, \ldots \beta_{m_B}]$.

When the number of parameters in (6) is high, optimization algorithms applied on θ are ill-conditioned due to the absence of constraints on differentiation orders. One way for introducing a constrained optimization on differentiation orders and, at the same time, limiting the number of parameters consists of optimizing the commensurate order γ instead of all differentiation orders. In this case, the fractional transfer function (6) is rewritten in a commensurate form as in (13).

Numerator and denominator orders, respectively α_{m_A} and β_{m_B} (both multiples of γ), are fixed as in classical rational models. Henceforth, the system is entirely characterized by coefficients' vector: $\theta = [a_0, \ldots, a_{m_A}, b_1, \ldots, b_{m_B}, \gamma]$. As far as identification of stable systems is concerned, the commensurate order can be constrained to $]0, 2[$ (see condition (8) of stability theorem).

Considering observed data $u(t)$ and $y^*(t) = y(t) + p(t)$, $p(t)$ being an output white noise, the quadratic norm:

$$J\left(\hat{\theta}\right) = \sum_{k=k_0}^{k_0+K-1} \varepsilon^2\left(kT_s, \hat{\theta}\right) \tag{25}$$

of output error:

$$\varepsilon\left(kT_s, \hat{\theta}\right) = y^*\left(kT_s\right) - \hat{y}\left(kT_s, \hat{\theta}\right) \tag{26}$$

is now minimized. Model's output $\hat{y}(kT_s, \hat{\theta})$ being non linear in $\hat{\theta}$, gradient-based algorithms, such as the Marquardt algorithm (Marquardt, 1963), are used to estimate $\hat{\theta}$ iteratively:

$$\hat{\theta}_{i+1} = \hat{\theta}_i - \left\{ \left[\mathbf{J}''_{\theta\theta} + \xi \mathbf{I} \right]^{-1} \mathbf{J}'_{\theta} \right\}_{\theta=\hat{\theta}_i} \tag{27}$$

$$\begin{cases} \mathbf{J}'_{\theta} = -2 \displaystyle\sum_{k=k_0}^{k_0+K-1} \varepsilon(kT_s)\, \mathbf{S}\left(kT_s, \hat{\theta}\right) \text{: gradient} \\ \mathbf{J}''_{\theta\theta} \approx 2 \displaystyle\sum_{k=k_0}^{k_0+K-1} \mathbf{S}\left(kT_s, \hat{\theta}\right) \mathbf{S}^{\mathrm{T}}\left(kT_s, \hat{\theta}\right) \text{: hessian} \\ \mathbf{S}\left(kT_s, \hat{\theta}\right) = \dfrac{\partial \hat{y}\left(kT_s, \hat{\theta}\right)}{\partial \theta} \text{: output sensitivity} \\ \xi \quad : \quad \text{Marquardt parameter} \end{cases} \tag{28}$$

Output sensitivity functions can be computed by differentiating (13) with respect to a_i, b_i, γ. In doing so, one can notice the presence of $\ln(s)$, in $\frac{\partial F(s)}{\partial \gamma}$. Consequently, $\frac{\partial F(s)}{\partial \gamma}$ is computed numerically rather than analytically.

The idea of optimizing the commensurate order instead of all differentiation orders was first introduced in (Cois *et al.*, 2000) who chose to write the transfer function (13) in a modal form as in (7). They however constrained all s^{γ}-poles to be real and of multiplicity one ($v_k = 1, \quad \forall k$). In general, s^{γ}-poles can be real or complex conjugate, and of multiplicity greater or equal to one.

2.2.2. Method based on continuous-time simulation of fractional models Trigeassou *et al.* (1999) take as a building block of fractional models a non integer integrator bounded in the frequency band as shown in (10) and described in section 1.3. The estimation of differentiation order is carried out by estimating the parameters α and η of the recursive distribution of poles and zeros (10). Once α and η known, the differentiation order γ is deduced according to (11).

For the sake of simplicity, consider the following fractional differential system:

$$\mathbf{D}^{\gamma} y(t) + a_0 y(t) = b_0 u(t) \tag{29}$$

Identification algorithm System is identified in an output error context as defined in section 2.2.1.

In the case, the fractional behavior is believed to be naturally limited in a frequency band, say $[\omega_A, \omega_B]$, then authors propose to estimate the parameter vector:

$$\theta^T = [a_0, b_0, \alpha, \eta] \tag{30}$$

In all cases the optimized criterion is defined as in (25) and (26). The coefficients are computed recursively according to (27) and (28). Sensitivity functions are now obtained by computing partial derivatives of (10) with respect to each of the parameter of (30).

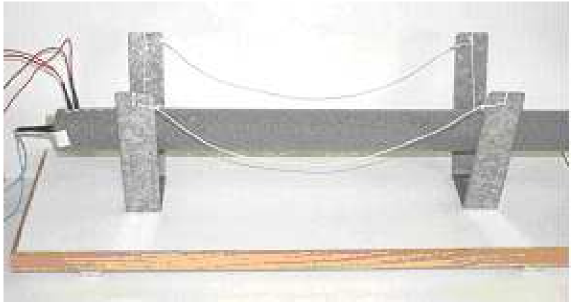

Fig. 3. Insulated long aluminium rod heated by a resistor

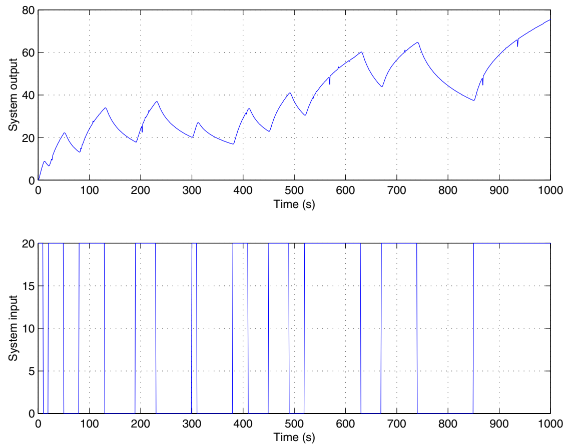

Fig. 4. Estimation data

3. EXAMPLE

To illustrate the use of fractional models in system identification, a semi-infinite dimensional thermal system is considered. It is constituted of a long aluminium rod heated by a resistor. To ensure unidirectional heat transfer, the entire surface of the rod is insulated. The temperature of the rod is measured at a distance $x = 5mm$ from the heated end (figure (3)).

The thermal system is considered as a semi-infinite plane homogenous medium initially at ambient temperature. Losses on the surface where the thermal flux is applied are neglected. Cois *et al.* (2000) have shown that the analytical model linking the flux density applied on the outgoing normal surface of the medium to the temperature measured at an abscissa x inside the medium has a commensurate order of 0.5.

First of all, the system was identified by applying equation-error model and more precisely the SVF method. Identification data are plot on figure (4). The commensurate order was set to 0.5 and the following three-parameter model was obtained:

$$H_1(s) = \frac{0.256 s^{0.5} - 0.002}{2.585 s^{1.5} + s} \tag{31}$$

Then, equation error model was applied and the commensurate order optimized. The following

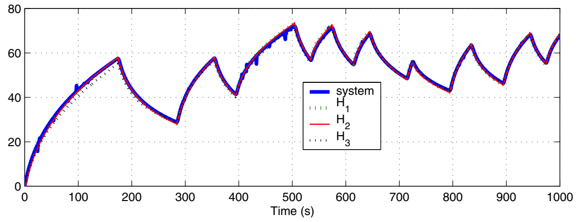

Fig. 5. Validation data

five-parameter model was obtained:

$$H_2(s) = \frac{0.319}{s^{0.483} + 0.016} - \frac{0.530}{s^{0.483} + 0.608} \quad (32)$$

The optimal commensurate order is close to 0.5 as in the analytic model Cois *et al.* (2000).

Next, for comparison purposes, a twelve parameters rational model was identified:

$$H_3(s) = \frac{-0.01s^5 + 0.45s^4 - 0.07s^3 + 1.55s^2 + 0.03s + 10^{-5}}{s^6 + 0.16s^5 + 13.04s^4 + 1.09s^3 + 31.23s^2 + 1.70s + 0.01}$$

The normalized mean squared errors computed on validation data for both fractional models are close to each other: $\text{NMSE}(\text{H}_1) \approx \text{NMSE}(\text{H}_2) \approx 2 \times 10^{-4}$; whereas the normalized mean squared error of the rational model is: $\text{NMSE}(\text{H}_1) \approx 6 \times 10^{-4}$.

As shown on validation data of figure (5), the identified models give satisfactory results.

4. CONCLUSION AND OUTLOOKS

This paper presents a tutorial on system identification using fractional differentiation models. Mainly equation-error and output-error models were detailed. In the former differentiation orders are fixed and only model's parameters are estimated. In the latter both differentiation orders and model's coefficients are estimated. One way for limiting the number of parameters consists of estimating the commensurate order and fixing all its multiples. For the time being, only white additive noise was considered. All model classes including colored noise should be extended to fractional differential orders. Moreover, system identification using stochastic signals is worth consideration.

Multiple other questions regarding fractional system identification remain unanswered. One of the most challenging is how to take into account initial conditions? This question cannot be answered as easily as in the rational case because a non-integer derivative of a signal depends on its whole past. Lorenzo and Hartley (2000) showed that the effect of the past can be considered by taking into account an initialization function instead of a limited number of points. Can such a function be identified?

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