Asymptotic Properties of the Detrended Fluctuation Analysis of Long Range Dependence Processes

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Abstract— In the past few years, a certain number of authors have proposed analysis methods of the time series built from a long range dependence noise. One of these methods is the Detrended Fluctuation Analysis (DFA), frequently used in the case of physiological data processing. The aim of this method is to highlight the long-range dependence of a time series with trend. In this study asymptotic properties of the DFA of the fractional Gaussian noise are provided. Those results are also extended to a general class of stationary long-range dependent processes. As a consequence, the convergence of the semi-parametric estimator of the Hurst parameter is established. However, several simple examples also show that this method is not at all robust in the case of trends.

Index Terms— Detrended ¤uctuation analysis, fractional Gaussian noise, stationary process, self-similar process, Hurst parameter, trend, long-range dependent processes.

I. INTRODUCTION

IN the past few years, numerous methods of analysis of a trended long range process have been proposed. One of these methods is the Detrended Fluctuation Analysis (DFA), frequently used in the case of physiological data processing in particular the heartbeat signals recorded on healthy or sick subjects [17], [20], [25]-[27]. Indeed, it can be interesting to £nd some constants among the ¤uctuations of physiological data. The Hurst parameter of the original signal, or the self-similarity parameter of the aggregated signal could be a new way of interpreting and explaining a physiological behavior.

The DFA method is a version for time series with trend of the method of aggregated variance used for a long-memory stationary process (see for instance [20]). It consists in 1. aggregating the process by windows with £xed length, 2. detrending the process from a linear regression in each window, 3. computing the standard deviation of the residual errors (the DFA function) for all data, 4. estimating the coef£cient of the power law from a log-log regression of the DFA function on the length of the chosen window. After the £rst stage, the process is supposed to behave like a self-similar process with stationary increments added to a trend. The second stage is supposed to remove the trend. Finally, the third and fourth stages are identical to those of the aggregated method (for zero-mean stationary process).

The processing of experimental data, and in particular

J.M. Bardet is with Samos-Matisse-Ces, University of Paris1 (e-mail: bardet@univ-paris1.fr).

I. Kammoun is with Samos-Matisse-Ces, University of Paris1 (e-mail: imen.kammoun@malix.univ-paris1.fr).

physiological data, exhibits a major problem which is the non-stationarity of the signal. Hu *et al.* [17] have studied different types of non-stationarities associated with examples of trends (linear, sinusoidal and power-law trends) and deduced their effect on an added noise and the kind of competition which exists between this two signals. They have also explained (see Chen *et al.* [8]) the effects of three other types of non-stationarities, which are often encountered in real data. The DFA method was applied to signals having some segments removed, with random spikes or with different local behavior. The results were compared with the case of stationary correlated signals.

In Taqqu *et al.* [33], the case of the fractional Gaussian noise (FGN) is studied. A theoretical proof to the power law followed by the expectation of the DFA function of this process is established. This is an important £rst step in order to prove the convergence of the estimator of the Hurst parameter. The study we propose here constitutes an accomplishment of this work. Indeed the convergence rate of the Hurst parameter estimator is obtained, in a semi-parametric frame.

The paper is organized as follows. In Section II, the DFA method is presented and two general properties are proved. Section III is devoted to providing asymptotic properties (illustrated beforehand by simulations) of the DFA function in the case of the FGN. Section IV contains an extension of these results for a general class of stationary long-range dependent processes. Finally, in Section V, the method is proved not to be robust in different particular cases of trended processes. Indeed the trend is dominant in the case of power law and polynomial trends, where the slope of the DFA log-log regression line for trended processes is always close to 2, or in the case of a piecewise constant trend, where the slope is estimated at $\frac{3}{2}$, which dominates the Hurst exponent. The proofs of the different results are in Appendix I.

II. DEFINITIONS AND FIRST PROPERTIES OF THE DFA METHOD

Notation and preliminaries

In the following we shall use the following notations. Let $(Y(1), \ldots, Y(N))$ be a sample of a time series $(Y(n))_{n \in \mathbb{N}}$. Let us denote the "discrete integration" of this sample

$$X(k) = \sum_{i=1}^{k} Y(i) \quad \text{for } k \in \{1, \ldots, N\}. \tag{1}$$

For $j \in \{1, \ldots, [N/n]\}$, let us define $s_{j,k} = n(j-1)+k$ with $k = \{1, \ldots, n\}$ and the vectors

$$X^{(j)} = (X(1 + n(j-1)), \ldots, X(nj))'$$
$$\text{and} \quad \tilde{X}^{(j)} = (\tilde{X}(1 + n(j-1)), \ldots, \tilde{X}(nj))'.$$

Let E_j be the vector subspace of $\mathbb{R}^n$ generated by the two vectors of $\mathbb{R}^n$, $e_1 = (1, \ldots, 1)'$ and $e_2 = ((j-1)n+1, (j-1)n+2, \ldots, nj)'$ and $E_j^\perp$ its orthogonal vector subspace. Finally, let us define P_A the matrix of the orthogonal projection on a vector subspace A of $\mathbb{R}^n$.

The Detrended Fluctuation Analysis (DFA)

The DFA method was introduced in [26]. The aim of this method is to highlight the self-similarity of a time series with trend.

1) The first step of the DFA method is the calculation of $(X(1), \ldots, X(N))$.
2) The second step is a division of $\{1, \ldots, N\}$ in $[N/n]$ windows of length n (for $x \in \mathbb{R}$, $[x]$ is the integer part of x). In each window, the least-square regression line is computed, which represents the linear trend of the process in the window. Then, we denote by $\widehat{X}_n(k)$ for $k = 1, \ldots, N$ the process formed by this piecewise linear interpolation. Then the DFA function is the standard deviation of the residuals obtained from the difference between $X(k)$ and $\widehat{X}_n(k)$, therefore,

$$F(n) = \sqrt{\frac{1}{n \cdot [N/n]} \sum_{k=1}^{n \cdot [N/n]} \left(X(k) - \widehat{X}_n(k) \right)^2}$$

3) The third step consists in a repetition of the second step with different values $(n_1, \ldots, n_m)$ of the window's length. Then the graph of $\log F(n_i)$ by $\log n_i$ is drawn. The slope of the least-square regression line of this graph provides an estimation of the self-similarity parameter of the $(X(k))_{k \in \mathbb{N}}$ process or the Hurst parameter of the $(Y(n))_{n \in \mathbb{N}}$ process (see above for the explanations).

From the construction of the DFA method, it is interesting to define the restriction of the DFA function in a window. Thus, for $n \in \{1, \ldots, N\}$, one defines the partial DFA function computed in the j-th window, *i.e.*

$$F_j^2(n) = \frac{1}{n} \sum_{i=n(j-1)+1}^{nj} (X(i) - \widehat{X}_n(i))^2 \qquad (2)$$

for $j \in \{1, \ldots, [N/n]\}$. Then, it is obvious that

$$F^2(n) = \frac{1}{[N/n]} \sum_{j=1}^{[N/n]} F_j^2(n). \qquad (3)$$

Remark: In Hu *et al.'s* and Kantelhardt *et al.'s* papers (for details see [17], [19] and [20]), the definition of the time series $(X(n))_{n \in \mathbb{N}}$ computed from $(Y(n))_{n \in \mathbb{N}}$ is different from (1), *i.e.*

$$\tilde{X}(k) = \sum_{i=1}^{k} (Y(i) - \overline{Y}_N), \quad \text{for} \ \ k \in \{1, \ldots, N\}$$

with $\overline{Y}_N = \dfrac{1}{N} \displaystyle\sum_{j=1}^{N} Y(j)$.

It is obvious that in both definitions, $(X(k) - \widehat{X}_n(k))$ is the same and therefore the value of $F(n)$ is the same.

Lemma 2.1: With the previous notations, let $\tilde{F}(n)$ be the DFA function built from $(\tilde{X}(k))$, *i.e.*

$$\tilde{F}(n) = \sqrt{\frac{1}{n \cdot [N/n]} \sum_{k=1}^{n \cdot [N/n]} \left(\tilde{X}(k) - \widehat{\tilde{X}}_n(k) \right)^2}.$$

Then for $n \in \{1, \ldots, N\}$, $F(n) = \tilde{F}(n)$.

Proof: In j-th window, we have $\tilde{X}^{(j)} = X^{(j)} - e_2 \cdot \overline{Y}_N$,

$$F_j^2(n) = \frac{1}{n} \left(P_{E_j^\perp} \cdot X^{(j)} \right)' \cdot P_{E_j^\perp} \cdot X^{(j)}$$
$$\text{and} \quad \tilde{F}_j^2(n) = \frac{1}{n} \left(P_{E_j^\perp} \cdot \tilde{X}^{(j)} \right)' \cdot P_{E_j^\perp} \cdot \tilde{X}^{(j)}.$$

As $e_2 \cdot \overline{Y}_N \in E_j$, $P_{E_j^\perp} = P_{E_j^\perp} \cdot X^{(j)}$ and thus $F_j^2(n) = \tilde{F}_j^2(n)$. This implies that $F(n) = \tilde{F}(n)$. $\square$

In order to simplify the following proofs, we prove that the application of the DFA to a stationary process yields a stationary process again.

Lemma 2.2: Let $\{Y(t), t \geq 0\}$ be a stationary process. Then, the time series $(F_j^2(n))_{1 \leq j \leq [N/n]}$ is a stationary process.

Proof: In each window j,

$$X^{(j)} - X(s_{j,1}) \cdot e_1 \ \overset{\mathcal{L}}{=} \ X^{(1)} - X(1) \cdot e_1. \qquad (4)$$

Indeed

$$X^{(j)} - X(s_{j,1}) \cdot e_1 = \left(0, Y(s_{j,2}), \ldots, \sum_{k=2}^{n-1} Y(s_{j,k}), \sum_{k=2}^{n} Y(s_{j,k}) \right)$$

$$\text{and} \quad X^{(1)} - X(1) \cdot e_1 = \left(0, Y(2), \ldots, \sum_{k=2}^{n-1} Y(k), \sum_{k=2}^{n} Y(k) \right).$$

As $(Y(2), \ldots, Y(n)) \overset{\mathcal{L}}{=} (Y(s_{j,2}), \ldots, Y(s_{j,n}))$, then with $g : \mathbb{R}^{n-1} \to \mathbb{R}^{n-1}$ a Borelian function defined by $g(y_2, \ldots, y_n) = (y_2, \ldots, \sum_{k=2}^{n-1} y_k, \sum_{k=2}^{n} y_k)$, it is clear that $g(Y(2), \ldots, Y(n)) \overset{\mathcal{L}}{=} g(Y(s_{j,2}), \ldots, Y(s_{j,n}))$ and therefore (4) is true.

$$F_j^2(n) = \frac{1}{n} \left(P_{E_j^\perp} \cdot X^{(j)} \right)' \cdot P_{E_j^\perp} \cdot X^{(j)}$$
$$= \frac{1}{n} \left(X^{(j)} - X(s_{j,1}) \cdot e_1 \right)' \cdot P_{E_j^\perp} \cdot \left(X^{(j)} - X(s_{j,1}) \cdot e_1 \right),$$

with $P_{E_j^\perp} \cdot e_1 = 0_{\mathbb{R}^n}$. But $E_1 = E_j$ and thus $E_j^\perp = E_1^\perp$. Therefore, with (4), we obtain $F_j^2(n) \overset{\mathcal{L}}{=} F_1^2(n)$ for all $j \in \{1, \ldots, [N/n]\}$.

Moreover, for all $m \in \mathbb{N}^*$, $(j_1, \ldots, j_m) \in \{1, \ldots, [N/n]\}^m$ and $t \in \mathbb{N}^*$, the same reasoning can be used again

for the case of vectors $(F_{j_1}^2(n), \ldots, F_{j_m}^2(n))$ and $(F_{j_1+t}^2(n), \ldots, F_{j_m+t}^2(n))$. Indeed,

$$\left(X^{(j_1)'} - X(s_{j_1,1}) \cdot e_1', \ldots, X^{(j_m)'} - X(s_{j_m,1}) \cdot e_1' \right)' \stackrel{\mathcal{L}}{=}$$
$$\left(X^{(j_1+t)'} - X(s_{j_1+t,1}) \cdot e_1', \ldots, X^{(j_m+t)'} - X(s_{j_m+t,1}) \cdot e_1' \right)'$$

and $P_{E_{j_1}} = \cdots = P_{E_{j_m}} = P_{E_{j_1+t}} = \cdots = P_{E_{j_m+t}}$. This achieves the proof. $\square$

III. ASYMPTOTIC PROPERTIES OF THE DFA FUNCTION FOR A FGN

In this section, we study the asymptotic behavior (both the sample size N and the length of window n increase to ∞) of the DFA when $(Y(k))_{k \in \mathbb{N}}$ is a stationary Gaussian process called a fractional Gaussian noise (FGN), *i.e.* $(X(1), \ldots, X(N))$ is a Gaussian process having stationary increments and called a fractional Brownian motion (FBM). First, let us remind some definitions and properties of both these processes.

Definition and first properties of the FBM and the FGN

Let $\{X^H(t), t \in \mathbb{R}\}$ be a FBM with parameters $H \in]0, 1[$ and $\sigma^2 > 0$, *i.e.* a real zero mean Gaussian process satisfying

1) $X^H(0) = 0$ a.s.
2) $E[(X^H(t) - X^H(s))^2] = \sigma^2 |t - s|^{2H} \quad \forall (t, s) \in \mathbb{R}^2$.

Here are some properties of a FBM $\{X^H(t), t \in \mathbb{R}\}$ (see more details in Samorodnitsky and Taqqu [30])

- The process $\{X^H(t), t \in \mathbb{R}\}$ has stationary increments. As a consequence, if we denote $\{Y^H(t), t \in \mathbb{R}\}$ the process defined by $Y^H(t) = X^H(t+1) - X^H(t)$ for $t \in \mathbb{R}$, then $\{Y^H(t), t \in \mathbb{R}\}$ is a zero-mean stationary Gaussian process also called a FGN.
- $\{X^H(t), t \in \mathbb{R}\}$ is a self-similar process satisfying $\forall c > 0$, $X^H(ct) \stackrel{\mathcal{L}}{=} c^H X^H(t)$ and H is also called the exponent (or parameter) of self-similarity.
- The covariance function of a FBM $\{X^H(t), t \in \mathbb{R}\}$, for all $(s, t) \in \mathbb{R}^2$ is

$$\text{Cov}(X^H(t), X^H(s))$$
$$= \frac{\sigma^2}{2}(|s|^{2H} + |t|^{2H} - |t - s|^{2H}). \tag{5}$$

- The covariance function of a FGN $\{Y^H(t), t \in \mathbb{R}\}$, for all $(s, t) \in \mathbb{R}^2$ is

$$\text{Cov}(Y^H(t), Y^H(s))$$
$$= \frac{\sigma^2}{2}(|t - s + 1|^{2H} + |t - s - 1|^{2H} - 2|t - s|^{2H}). \tag{6}$$

Therefore, $\text{Cov}(Y^H(t), Y^H(s)) \sim H(2H-1)|t-s|^{2H-2}$ when $|t - s| \to \infty$: when $1/2 < H < 1$, Y^H is a long memory process (see also (11) below) and H is the Hurst (or long range dependent) parameter of Y^H.

Some numerical results of the DFA applied to the FGN

The following Figures 1 and 2 show an example of the DFA method applied to a FGN with different values of H ($H = 0.6$ in the first figure and $H = 0.2, 0.4, 0.5, 0.7, 0.8$ in the second one, with $N = 10000$ in both cases). Such a sample path is generated with a circulant matrix algorithm (see for instance Bardet *et al.*, [7]).
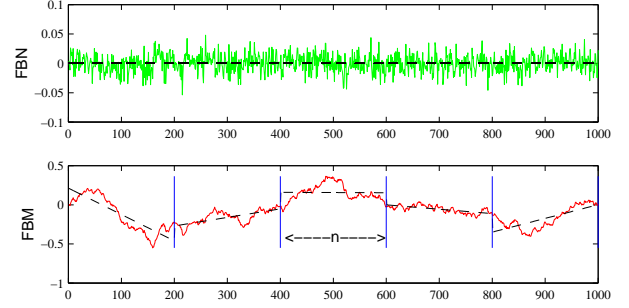

Fig. 1. The two first steps of the DFA method applied to a path of a discretized FGN (with $H = 0.6$ and $N = 10000$)

On the right side of Figure 2 appear the different estimations of H computed from the DFA method. Those values have to be compared with theoretical ones. The results seem to be quite good and it seems that, under certain conditions, the asymptotic behavior of the DFA function $F(n)$ can be written as

$$F(n) \simeq c(\sigma, H) \cdot n^H \tag{7}$$

where c is a positive function depending only on σ and H (see its expression above).
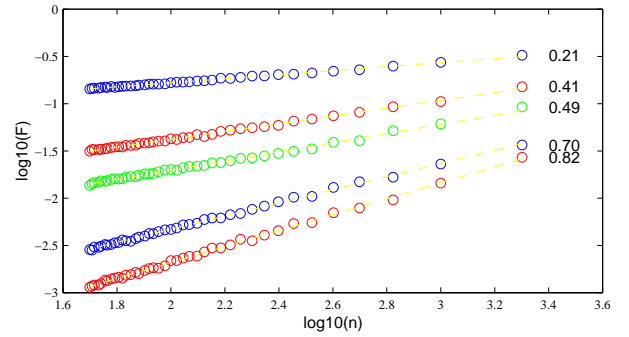

Fig. 2. Results of the DFA method applied to a path of a discretized FGN for different values of $H = (0.2, 0.4, 0.5, 0.7, 0.8)$ (also with $N = 10000$)

The approximation (7) explains that the slope of the least-square regression line of $(\log F(n_i))$ by $\log(n_i)$ for different values of n_i provides an estimation of H. We now provide a mathematical proof of this result.

Let $\{X^H(t), t \geq 0\}$ be a FBM, built as a cumulated sum of a FGN $\{Y^H(t), t \geq 0\}$. We first give some asymptotic properties of $F_1^2(n)$.

Property 3.1: Let $\{X^H(t),\, t \geq 0\}$ be a FBM with parameters $0 < H < 1$ and $\sigma^2 > 0$. Then, for n and j large enough,

1. $(F_1^2(n)) \qquad\qquad = \sigma^2 f(H) \cdot n^{2H}\left(1 + O(\frac{1}{n})\right),$
2. $\mathrm{Var}\left(F_1^2(n)\right) \qquad = \sigma^4 g(H) \cdot n^{4H}\left(1 + O(\frac{1}{n})\right),$
3. $\mathrm{Cov}(F_1^2(n), F_j^2(n)) = \sigma^4 h(H) \cdot n^{4H} \cdot j^{2H-3}\cdot$
$$\left(1 + O(\tfrac{1}{n}) + O(\tfrac{1}{j})\right),$$

with $f(H) = \dfrac{(1-H)}{(2H+1)(H+1)(H+2)}$, g depending only on H, see (20), and $h(H) = \dfrac{H^2(H-1)(2H-1)^2}{48(H+1)(2H+1)(2H+3)}$.

The proofs of these results (and of the others) are provided in Appendix I.
In order to obtain a central limit theorem for the logarithm of the DFA function, we consider normalized DFA functions

$$\tilde{S}_j(n) = \frac{F_j^2(n)}{n^{2H}\sigma^2 f(H)} \quad \text{and} \quad \tilde{S}(n) = \frac{F^2(n)}{n^{2H}\sigma^2 f(H)} \qquad (8)$$

for $n \in \{1, \ldots, N\}$ and $j \in \{1, \ldots, [N/n]\}$.
As a consequence, for $n \in \{1, \ldots, N\}$, the stationary time series $(\tilde{S}_j(n))_{1 \leq j \leq [N/n]}$ satisfy

$$\begin{cases} (\tilde{S}_j(n)) & = 1 + O(\frac{1}{n}) \\[2mm] \mathrm{Var}(\tilde{S}_j(n)) & = \frac{g(H)}{f(H)^2}\left(1 + O(\frac{1}{n})\right) \\[2mm] \mathrm{Cov}(\tilde{S}_1(n), \tilde{S}_j(n)) & = \frac{h(H)}{f(H)^2} \cdot \frac{1}{j^{3-2H}} \cdot \\[2mm] & \qquad \left(1 + O(\frac{1}{n}) + O(\frac{1}{j})\right) \end{cases} \qquad (9)$$

Under certain conditions on the asymptotic length n of the windows, one proves a central limit theorem satis£ed by the logarithm of the empirical mean $\tilde{S}(n)$ of the random variables $(\tilde{S}_j(n))_{1 \leq j \leq [N/n]}$.

Property 3.2: Under the previous assumptions and notations, let $n \in \{1, \ldots, N\}$ be such that $N/n \to \infty$ and $N/n^3 \to 0$ when $N \to \infty$. Then

$$\sqrt{\left[\frac{N}{n}\right]} \cdot \log(\tilde{S}(n)) \xrightarrow[\substack{n \to \infty \\ N \to \infty}]{\mathcal{L}} \mathcal{N}(0, \gamma^2(H))),$$

where $\gamma^2(H) > 0$ depends only on H.

This result can be obtained for different lengths of windows satisfying the conditions $N/n \to \infty$ and $N/n^3 \to 0$. Let $(n_1, \ldots, n_m)$ be such different window lengths. Then, one can write for N and n_i large enough

$$\log(\tilde{S}(n_i)) \simeq \frac{1}{\sqrt{[N/n_i]}} \cdot \varepsilon_i \quad \Longrightarrow$$

$$\log(F(n_i)) \simeq H \cdot \log(n_i) + \frac{1}{2}\log(\sigma^2 f(H)) + \frac{1}{\sqrt{[N/n_i]}} \cdot \varepsilon_i,$$

with $\varepsilon_i \sim \mathcal{N}(0, \gamma^2(H))$. As a consequence, a linear regression of $\log(F(n_i))$ on $\log(n_i)$ provides an estimation of H. More

precisely,

Proposition 3.3: Under the previous assumptions and notations, let $n \in \{1, \ldots, N\}$, $m \in {}^*\backslash\{1\}$, $r_i \in \{1, \ldots, [N/n]\}$ for each i with $r_1 < \cdots < r_m$ and $n_i = r_i n$ be such that $N/n \to \infty$ and $N/n^3 \to 0$ when $N \to \infty$. Let $\widehat{H}$ be the estimator of H from the linear regression of $\log(F(r_i \cdot n))$ on $\log(r_i \cdot n)$, *i.e.*

$$\widehat{H} = \frac{\sum_{i=1}^m (\log(F(r_i \cdot n)) - \overline{\log(F)})(\log(r_i \cdot n) - \overline{\log(n)})}{\sum_{i=1}^m (\log(r_i \cdot n) - \overline{\log(n)})^2}.$$

Then $\widehat{H}$ is a consistant estimator of H such that

$$[(\widehat{H} - H)^2] \leq C(H, m, r_1, \ldots, r_m)\frac{1}{[N/n]} \qquad (10)$$

with $C(H, m) > 0$.

IV. EXTENSION OF THE RESULTS FOR A GENERAL CLASS OF A LONG-RANGE DEPENDENT PROCESS

Let $\{Y(k),\, k \in \ \}$ be a stationary zero mean long-range dependent process with a Hurst parameter $H \in]\frac{1}{2}, 1[$. More precisely, let $r_Y(k)$ be the autocorrelation function of this process and let us assume that there exists a slowly varying function $L(k)$ such that:

$$r_Y(k) \sim k^{2H-2}L(k) \ , \text{ as } k \to \infty. \qquad (11)$$

Under different additional assumptions on Y, Davydov [9], Taqqu [32], Dobrushin and Major [11], Giraitis and Surgailis [15] and other authors have studied the asymptotic behavior of the Donsker line and obtained the following convergence,

$$\left(L(n)^{-\frac{1}{2}} n^{-H} \sum_{i=1}^{[nt]} Y(i)\right)_{t>0} \xrightarrow[n \to \infty]{\mathcal{D}} \left(\sigma \cdot B_H(t)\right)_{t>0}, \qquad (12)$$

with $\sigma > 0$ and B_H a fractional Brownian motion. Remind that $Z = \{Z(k),\, k \in \ \}$ is a linear process when

$$Z(k) = \sum_{i=-\infty}^{\infty} a_i \,\xi_{k-i} \ \text{ for } k \in \ ,$$

with (a_k) a sequence of real numbers and (ξ_n) a sequence of zero mean i.i.d.r.v. Then,

Theorem 4.1: (Davydov, Taqqu) Let $Y = \{Y(k),\, k \in \ \}$ be a stationary zero mean long-range dependent process satisfying assumption (11). Then, if:
- Y is a linear process,
- or Y is a function of a Gaussian process with Hermite rank $r = 1$,

then (12) holds, and the convergence takes place in the Skorohod space.

Limit theorems are also obtained by Dobrushin and Major [11], Giraitis and Surgailis [15] and Ho and Hsing [16] for sums of polynomials of linear (or moving average) process with slowly decreasing coef£cients a_i. It is obtained under the hypothesis that (ξ_n) are i.i.d standard normal random variable and that the Polynomial Hermit rank satis£es $2 \cdot r < (1-H)^{-1}$.

So, in this case of general class of LRD process, the aggregated process $(X(k))$ has roughly the same behavior as a fractional Brownian motion and the previous asymptotic results of the DFA method can be applied. But Property 3.1 and Proposition 3.3 cannot be proved under so general assumptions. Indeed, the proofs of such results use a very precise expression of the covariance and a stronger version of assumption (11) is necessary. Hence, the covariance r_Y of the stationary process Y is now supposed to satisfy $r_Y \in \mathcal{H}(H, \beta, C)$ with

$$\mathcal{H}(H, \beta, C) = \Big\{ r, \; r(k) = C \cdot k^{2H-2}\big(1 + O(1/k^\beta)\big)$$
$$\text{when } k \to \infty \Big\}, \quad (13)$$

with $1/2 < H < 1$, $C > 0$ and $\beta > 0$. In such a semi-parametric frame, the previous proofs are still valid and :

Theorem 4.2: Let $Y = \{Y(k), k \in \quad\}$ be a Gaussian stationary zero mean long-range dependent process with covariance $r_Y \in \mathcal{H}(H, \beta, C)$. Then, Property 3.1 holds with the addition of $O(1/n^\beta)$ in each expansion. Moreover, if $N = o\big(n^{\max(2\beta+1,3)}\big)$, Property 3.2 and Proposition 3.3 hold.

As a consequence of this theorem, if $0 < \beta \leq 1$, the DFA method provides a semi-parametric estimator of H with the well-known minimax rate of convergence for the estimation of Hurst parameter in this semi-parametric setting (see for instance Giraitis *et al.* [13]), *i.e.*

$$\limsup_{N \to \infty} \; \sup_{r_Y \in \mathcal{H}(H, \beta, C)} N^{2\beta/(1+2\beta)} \; [(\widehat{H} - H)^2] < +\infty.$$

However, if $\beta \geq 1$, this result is replaced with $\limsup_{N \to \infty} \sup_{r_Y \in \mathcal{H}(H, \beta, C)} N^{2/3} [(\widehat{H} - H)^2] < +\infty$ ($\beta = 2$ for instance for FGN or Gaussian FARIMA(p, d, q)). Thus, the DFA estimator is not rate optimal for all $\beta > 0$ like local Whittle, local log-periodogram or wavelet based estimators are (see respectively [29], [14] and [24]).

V. CASES OF PARTICULAR TRENDED LONG-RANGE DEPENDENT PROCESSES

In this Section, two general examples of trended long-range dependent processes are considered and it is proved that the DFA method in such cases yields a biased and unusable estimation of the Hurst parameter.
In order to consider trended processes, the following lemma for two independent processes could be considered :

Lemma 5.1: Let $Y = \{Y(k), k \in \quad\}$ and $Y' = \{Y'(k), k \in \quad\}$ be two independent processes, with $(Y(k)) = 0$ for all $k \in \quad$, and let us denote respectively F_Y^2, $F_{Y'}^2$ and $F_{Y+Y'}^2$ the DFA functions associated to Y, Y' and $Y + Y'$. Then, for $n \in \{1, \cdots, N\}$,

$$(F_{Y+Y'}^2(n)) = (F_Y^2(n)) + (F_{Y'}^2(n)).$$

Proof: With X and X' the aggregated processes associated to Y and Y', it is obvious that

$$(F_{Y+Y'}^2(n))$$
$$= \frac{1}{n \cdot [N/n]} \sum_{k=1}^{n \cdot [N/n]} \left(\Big(X(k) + X'(k) - \widehat{X}_n(k) - \widehat{X}'_n(k) \Big)^2 \right)$$
$$= (F_Y^2(n)) + (F_{Y'}^2(n)) + \frac{2}{n \cdot [N/n]} \cdot$$
$$\sum_{k=1}^{n \cdot [N/n]} \left(\big(X(k) - \widehat{X}_n(k) \big)\big(X'(k) - \widehat{X}'_n(k) \big) \right).$$

From the independence of X and X' and thanks to the assumption $(Y(k)) = 0$ for all $k \in \quad$ which implies $(X(k)) = 0$ and $(\widehat{X}(k)) = 0$ for all $k \in \quad$, we deduce $\left(\big(X(k) - \widehat{X}_n(k) \big)\big(X'(k) - \widehat{X}'_n(k) \big) \right) = 0$. $\square$

Let $Y = \{Y(k), k \in \quad\}$ be a Gaussian stationary zero mean long-range dependent process satisfying assumption (13) (for instance, Y is a FGN) and let $f : \quad \mapsto \quad$ be a deterministic function. From Lemma 5.1, it is obvious that for $n \in \{1, \cdots, N\}$,

$$(F_{Y+f}^2(n)) = (F_Y^2(n)) + (F_f^2(n)). \quad (14)$$

Moreover, let us denote respectively $F_{Y,j}^2$ and $F_{f,j}^2$ the DFA function of Y and f relating to window $j \in \{1, \ldots, [\frac{N}{n}]\}$. Then, with few changes in the proof of Lemma 5.1,

$$(F_{Y+f,j}^2(n)) = (F_{Y,j}^2(n)) + (F_{f,j}^2(n)). \quad (15)$$

Case of power law and polynomial trends

First, let us assume that there exists $\lambda > 0$ and $a \in \quad$ such that

$$f(t) = a(t^{\lambda+1} - (t-1)^{\lambda+1}), \quad \text{for } t \geq 1.$$

Then, the associated integrated function is

$$g(k) = \sum_{i=1}^{k} f(i) = ak^{\lambda+1}.$$

For this kind of trend,

Property 5.1: For $f(t) = a(t^{\lambda+1} - (t-1)^{\lambda+1})$, with $\gamma(a, N, \lambda)$ a real number depending only on a, N and λ, $\log F_f(n) \simeq 2 \log n + \gamma(a, N, \lambda)$ for $n \to \infty$.

Thus, it appears that a linear regression of $\log F_f(n_i)$ and $\log(n_i)$ for different values of n_i will provide a slope 2 for any $\lambda > 0$. This result is confirmed by several simulations made for various values of $\lambda > 0$, a and $(n_1, \ldots, n_m)$.
This result can also be used to deduce similar results for polynomial trends.

Property 5.2: Let us assume that there exists $p \in$ * and a family $(a_j)_{0 \leq j \leq p}$ with $a_p \neq 0$ such that for $k \in \quad$, $f(k) = a_p k^p + \cdots + a_0$. Then, $\implies \log F_{a_p k^p + \cdots + a_0}(n) \simeq 2 \log n + \gamma(a_p, N, p)$ for $n \to \infty$.

Using relations (14) and (15), the previous results for trends can be used for deducing the behavior of the DFA function of trended long range dependent processes. Hence, in both previous cases of trends, there exists $C > 0$ such that

$$
\begin{aligned}
(F_{Y+f}^2(n)) &= C \cdot n^4 N^{2\lambda-2}\Big(1 + O(\tfrac{1}{n}) + O(\tfrac{1}{[\frac{N}{n}]})\Big) \\
&\quad + \sigma'^2 f(H) \cdot n^{2H}\Big(1 + O(\tfrac{1}{n^{\min(1,\beta)}})\Big) \\
&\simeq C \cdot n^4 N^{2\lambda-2}.
\end{aligned}
$$
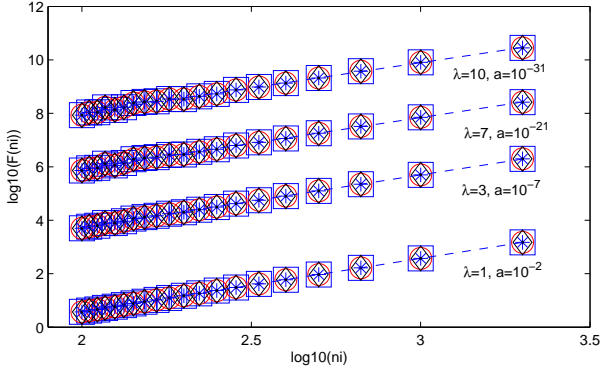

Fig. 3. Relation between $\log F_{Y+f}(n_i)$ and $\log n_i$ in the case of a power law trend ($N = 10000$, $H = 0.2$ ($\square$), $H = 0.4$ ($\bigcirc$), $H = 0.5$ ($\Diamond$), $H = 0.7$ ($*$) and $H = 0.8$ ($\cdot$))

Thus, it is clear that the trend is dominant for a large n and the graph exhibiting the relation between $\log F_{Y+f}(n_i)$ and $\log n_i$ for different power law trends and different coefficients H confirms this (the estimated slope is always close to 2).

Case of a piecewise constant trend

Let us assume now that f is a step function of the form
$$
f(t) = \sum_{i=0}^{m-1} a_i \ _{]t_i,t_{i+1}]} \text{ with } t_0 = 0, \ t_m = N \text{ and } m \in \ ^*.
$$
The associated integrated series is
$$
g(k) = \sum_{i=0}^{m-1}\Big(\sum_{s=0}^{i}(a_{s-1}-a_s)t_s + a_i k\Big) \ _{]t_i,t_{i+1}]} \text{ with } a_{-1} = 0.
$$

For $j \in \{1,\ldots,[N/n]\}$, the partial DFA function $F_{fj}^2(n)$ is null unless there exist i_p with $p \in \{1,\ldots,r\}$ and $(r,i_r) \in \{1,\ldots,m-1\}^2$ such that $t_{i_p} \in [(j_p-1)n + \tau n, j_p n - \tau n]$ with $\tau \in]0,\frac{1}{2}[$. In such a case, we calculate the partial DFA function:

$$
\begin{aligned}
F_{f,j_p}^2(n) &= \frac{1}{n}\sum_{k=1}^{n}(g(k+(j_p-1)n) - \widehat{g}_n(k+(j_p-1)n))^2 \\
&= \frac{1}{n}\Big(G^{(j_p)'} \cdot P_{E_{j_p}^\perp} \cdot G^{(j_p)}\Big).
\end{aligned}
$$

If we consider the first window, the partial DFA function can be estimated from below :

$$
F_{f,1}^2(n) \geq \frac{1}{n}\Big(\sum_{k=1}^{\tau n}(g(k) - \widehat{g}_n(k))^2 + \sum_{k=n-\tau n}^{n}(g(k) - \widehat{g}_n(k))^2\Big)
$$

where the $n \times 1$ vector $(g(k) - \widehat{g}_n(k))_{1 \leq k \leq n} = P_{E_1^\perp} \cdot G^{(1)}$ with :
$$
\begin{aligned}
G^{(1)} = \big(&a_0 \cdot 1, \ldots, a_0 \cdot t_1, (a_0 - a_1)t_1 + a_1 \cdot (t_1 + 1), \ldots, \\
&(a_0 - a_1)t_1 + a_1 \cdot n\big)'.
\end{aligned}
$$
Then,
$$
\sum_{k=1}^{\tau n}(g(k) - \widehat{g}_n(k))^2 = \Big(J_{\tau n} \cdot P_{E_1^\perp} \cdot G^{(1)}\Big)' \cdot \Big(J_{\tau n} \cdot P_{E_1^\perp} \cdot G^{(1)}\Big)
$$

where $J_{\tau n}$ is a square matrix of order n with ones in the τn first terms of the diagonal and zeros elsewhere. When we approximate sums by integrals, this expression can be written as follows :

$$
\begin{aligned}
\sum_{k=1}^{\tau n}(g(k) - \widehat{g}_n(k))^2 = n^3\Big(&\int_0^\tau\Big(\int_0^1 a_0 y - \big(a_0 x \cdot \ _{x\leq\frac{t_1}{n}} + \\
&(a_1 x + (a_0 - a_1)\tfrac{t_1}{n}) \ _{x>\frac{t_1}{n}}\big)(4 - 6(x+y) + 12xy)dx\Big)^2 dy\Big)\cdot \\
&\Big(1 + O(\tfrac{1}{n})\Big).
\end{aligned}
$$

For $\tau \in]0,\frac{1}{2}[$, the second term can be developed in the same way by replacing $J_{\tau n}$ by $J_{n-\tau n}$ which is $Id - J_{\tau n}$. Then, this term can be approximated by :

$$
\begin{aligned}
\sum_{k=n-\tau n}^{n}(g(k) - \widehat{g}_n(k))^2 = n^3\Big(&\int_{1-\tau}^1\Big(\int_0^1 (a_0 - a_1)\tfrac{t_1}{n} + a_1 y \\
&- \big(a_0 x \cdot \ _{x\leq\frac{t_1}{n}} + (a_1 x + (a_0 - a_1)\tfrac{t_1}{n}) \ _{x>\frac{t_1}{n}}\big)(4 - 6(x+y) \\
&+ 12xy)dx\Big)^2 dy\Big)\Big(1 + O(\tfrac{1}{n})\Big).
\end{aligned}
$$

Then after developing the two terms, we deduce that there exists a positive number $c(a_0,\ldots,a_{i_p},t_{i_p},\tau)$ such that the partial DFA function in the j_p-th window, where $t_{i_p} \in [(j_p-1)n + \tau n, j_p n - \tau n]$ for $p \in \{1,\ldots,r\}$ and n large enough, can be written as :

$$
F_{f,j_p}^2(n) \geq c(a_0,\ldots,a_{i_p},t_{i_p},\tau)n^2. \tag{16}
$$

Then if we suppose that there exists only one change point or a definite number of windows $j_1,\ldots,j_r$, there exists $c'(a_0,\ldots,a_{i_r},t_{i_1},\ldots,t_{i_r},\tau) > 0$ such that the DFA function relating to f is :

$$
\begin{aligned}
F_f^2(n) &= \frac{1}{[\frac{N}{n}]}\sum_{j=j_1}^{j_r} F^2_{f,j}(n) \geq \\
&c'(a_0,\ldots,a_{i_r},t_{i_1},\ldots,t_{i_r},\tau)n^3 N^{-1}\Big(1 + O(\tfrac{1}{n})\Big).
\end{aligned}
$$

If we consider the signal formed by the superposition between the trend and a long range dependent process, we point out that $(F_Y^2(n)) = \sigma'^2 f(H) \cdot n^{2H}\Big(1 + O(\tfrac{1}{n^{\min(1,\beta)}})\Big)$. We can deduce, from the previous conditions on n and N ($N/n \to \infty$ and $N = o(n^{\min(3,2\beta+1)})$), that the trend is dominant for large n.

Then, for different values $(n_1,\ldots,n_m)$, the graph tracing the relation between $\log F_{f+Y}(n_i)$ and $\log(n_i)$ (Figure 4), shows a slope estimated at $\frac{3}{2}$.
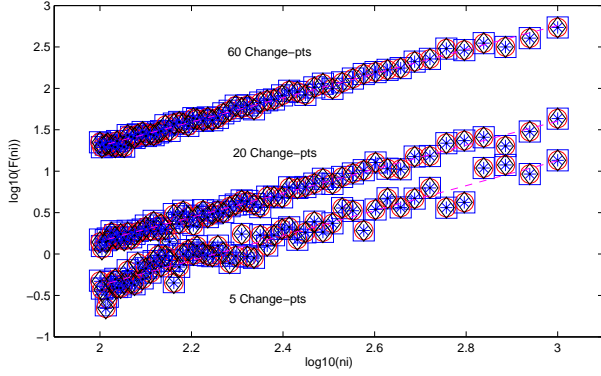
Fig. 4. Relation between $\log F_{f+Y}(n_i)$ and $\log n_i$ in the case of a trend with change points ($N = 20000$, $H = 0.2$ ($\square$), $H = 0.4$ ($\bigcirc$), $H = 0.5$ ($\Diamond$), $H = 0.7$ ($*$) and $H = 0.8$ ($\cdot$))

VI. CONCLUSION

In the semi-parametric frame of a long-memory stationary process, we have shown, using the DFA method, that the estimator of the long range dependence parameter is convergent with a reasonable (but not always optimal) convergence rate. However, in numerous cases of trended long-range dependent process (with perhaps the only exception of a constant trend), this estimator does not converge. Indeed the effect of long-memory property is dominated by the effect of the trend in the case of power law or polynomial trends. The slope for the trended processes of the DFA log-log regression is always close to 2 and in the case of a piecewise constant trend, the slope is estimated at $\frac{3}{2}$. The DFA method is therefore not robust at all and should not be applied for trended processes.

The wavelet-based method provides a more efficient and robust estimator of the Hurst parameter especially when a polynomial trended LRD (or self-similar) process is considered. Indeed, Abry *et al.* [1] remarked that all polynomial trend of degree M is without effects on the estimator of the Hurst parameter as soon as the mother wavelet has its M first vanishing moments. Therefore, the larger M, the more robust the estimator is. Moreover, in the semi-parametric frame of general class of stationary Gaussian LRD processes, it was established by Moulines *et al.* [24] that the estimator of the Hurst parameter converges with an optimal convergence rate (following the minimax criteria) when an optimal length of windows is known. Bardet *et al.* [6] proposed an adaptive estimator and obtained an optimal convergence rate up to logarithmic factor. Finally, the wavelet based estimator can be computed by Mallat's fast cascade algorithm which is a very fast algorithm (the equivalent for wavelet transform of FFT for Fourier transform) for computing wavelet coefficients. Thus, computing time of wavelet based estimator is smaller than DFA estimator one.

APPENDIX I

Proof of Property 3.1: 1. From the proof of Lemma 2.2 and with its notations, we obtain

$$F_1^2(n) = \frac{1}{n} \left(X^{(1)} - P_{E_1} \cdot X^{(1)}\right)' \cdot \left(X^{(1)} - P_{E_1} \cdot X^{(1)}\right)$$
$$= \frac{1}{n} \left(X^{(1)'} \cdot X^{(1)} - X^{(1)'} \cdot P_{E_1} \cdot X^{(1)}\right).$$

As a consequence,

$$(F_1^2(n)) = \frac{1}{n} \left(\text{trace}(\Sigma_n) - \text{trace}(P_{E_1} \cdot \Sigma_n)\right),$$

where Σ_n is the covariance matrix of $X^{(1)}$ and is such that

$$\Sigma_n = \text{Cov}(X(i), X(j))_{1 \leq i,j \leq n}$$
$$= \frac{\sigma^2}{2} \left(|i|^{2H} + |j|^{2H} - |i-j|^{2H}\right)_{1 \leq i,j \leq n}.$$

But, $\quad \text{trace}(\Sigma_n) = \sigma^2 \sum_{i=1}^{n} |i|^{2H} = \sigma^2 n^{2H+1} \left(\frac{1}{n} \sum_{i=1}^{n} |\frac{i}{n}|^{2H}\right)$

$= \sigma^2 n^{2H+1} \left(\int_0^1 x^{2H} dx + O(\frac{1}{n})\right)$. Therefore, on the one hand,

$$\text{trace}(\Sigma_n) = \frac{\sigma^2}{2H+1} n^{2H+1} \cdot \left(1 + O(\frac{1}{n})\right) \quad (17)$$

and on the other hand, it is well known that P_{E_1} is a $(n \times n)$ square matrix such that

$$P_{E_1} = \frac{2}{n(n-1)} \left((2n+1) - 3(i+j) + 6\frac{i \cdot j}{1+n}\right)_{1 \leq i,j \leq n}.$$

Then, after some straightforward computations, we obtain the formula

$$\text{trace}(P_{E_1} \cdot \Sigma_n) = \frac{\sigma^2 n^{2H+1} n^2}{n(n-1)} \sum_{p=1}^{n} \sum_{q=1}^{n} \left[\frac{1}{n^2} \left((2 + \frac{1}{n}) - 3 \cdot \right. \right.$$
$$\left. \left. \frac{p+q}{n} + \frac{6p \cdot q}{n(1+n)}\right) \left(|\frac{q}{n}|^{2H} + |\frac{p}{n}|^{2H} - |\frac{q-p}{n}|^{2H}\right)\right].$$

In order to clarify the formula, we approximate these sums by integrals

$$\text{trace}(P_{E_1} \cdot \Sigma_n) = \sigma^2 n^{2H+1} \cdot \left(1 + O(\frac{1}{n})\right) \cdot$$
$$\int_0^1 \int_0^1 \left[\left(2 - 3(x+y) + 6xy\right)\left(x^{2H} + y^{2H} - |x-y|^{2H}\right)\right] dx dy.$$

After the calculation of this integral and a simplification with formula (17), we get the result

$$\text{trace}(\Sigma_n) - \text{trace}(P_{E_1} \cdot \Sigma_n) = \sigma^2 f(H) \cdot n^{2H+1} \cdot \left(1 + O(\frac{1}{n})\right)$$

and therefore the formula of $(F_1^2(n))$.

2. From the previous notations and the property of the trace of a product of matrices,

$$\text{Var}(F_1^2(n)) = \frac{1}{n^2} \left[(X^{(1)'} \cdot P_{E_1^\perp} \cdot X^{(1)} \cdot X^{(1)'} \cdot P_{E_1^\perp} \cdot X^{(1)}) \right.$$
$$\left. - \left((X^{(1)'} \cdot P_{E_1^\perp} \cdot X^{(1)})\right)^2 \right]$$

$$= \frac{1}{n^2} \left[\text{trace}(\Sigma_n \cdot \Sigma_n) - \text{trace}(P_{E_1} \cdot \Sigma_n \cdot \Sigma_n)\right] \quad (18)$$

The development of the £rst term provides the following asymptotic expansion

$$\text{trace}(\Sigma_n \cdot \Sigma_n) = \frac{\sigma^4}{4} \sum_{i=1}^{n} \sum_{p=1}^{n} \left(|i|^{2H} + |p|^{2H} - |i-p|^{2H} \right)^2 =$$

$$\frac{\sigma^4}{4} n^{4H+2} \left(1 + O\left(\frac{1}{n}\right)\right) \int_0^1 \int_0^1 \left(|x|^{2H} + |y|^{2H} - |x-y|^{2H} \right)^2 dx\, dy.$$

The calculation of this integral provides the following simpli-£ed expression

$$\text{trace}(\Sigma_n \cdot \Sigma_n) = \frac{\sigma^4}{4} n^{4H+2} \left(1 + O\left(\frac{1}{n}\right)\right) \cdot$$
$$\left[\frac{1}{4H+1} + \frac{1}{(4H+1)(4H+2)} - 2\frac{(\Gamma(2H+1))^2}{\Gamma(4H+3)} \right]. \quad (19)$$

The same development can be made for the second term

$$\text{trace}(P_{E_1} \cdot \Sigma_n \cdot \Sigma_n) = \frac{\sigma^4}{2} n^{4H+2} \left(1 + O\left(\frac{1}{n}\right)\right) \cdot$$

$$\int_0^1 \int_0^1 \int_0^1 \left(|x|^{2H} + |y|^{2H} - |y-x|^{2H} \right) \left(|x|^{2H} + |z|^{2H} - |x-z|^{2H} \right)$$
$$\cdot \left(2 - 3(y+z) + 6yz \right) dx\, dy\, dz.$$

After the computation of this last integral, and using relations (18) and (19)

$$\left[\text{trace}(\Sigma_n \cdot \Sigma_n) - \text{trace}(P_{E_1} \cdot \Sigma_n \cdot \Sigma_n) \right]$$
$$= \sigma^4 \cdot g(H)\, n^{4H+2} \left(1 + O\left(\frac{1}{n}\right)\right),$$

with
$$g(H) = \frac{1}{2} \left(-\frac{(16H^2 + 24H + 17)(\Gamma(2H+1))^2}{(4H+5)\Gamma(4H+4)} \right. \quad (20)$$
$$+ \frac{H+1}{(2H+1)(4H+1)} + \frac{7H+3}{2(2H+1)^2(H+1)} - \frac{3}{2(H+1)^2}$$
$$\left. + \frac{3(4H+3)}{2(2H+1)^2(H+1)^2(4H+5)} - \frac{4}{(2H+1)^2(4H+3)} \right).$$

Then, using the relations (18), we obtain
$$\text{Var}\left(F_1^2(n) \right) = \sigma^4 \cdot g(H) \cdot n^{4H} \left(1 + O\left(\frac{1}{n}\right)\right).$$

3. An asymptotic expansion of the covariance between two DFA functions in two suf£ciently far windows can be provided. Indeed,

$$\text{Cov}(F_1^2(n), F_j^2(n)) = \frac{1}{n^2} \cdot$$
$$\text{Cov}\left(\left(X^{(1)} - \widehat{X}^{(1)}\right)' \left(X^{(1)} - \widehat{X}^{(1)}\right) \left(X^{(j)} - \widehat{X}^{(j)}\right)' \left(X^{(j)} - \widehat{X}^{(j)}\right) \right)$$
$$= \frac{1}{n^2} \left(\text{trace}\left(\Sigma^{(1,j)} \cdot \Sigma^{(1,j)} \right) - \text{trace}\left(P_{E_1} \cdot \Sigma^{(1,j)} \cdot \Sigma^{(1,j)} \right) \right),$$

because $P_{E_1^\perp} = P_{E_j^\perp}$ and with $\Sigma^{(1,j)}$ the covariance matrix $(X^{(1)} \cdot X^{(j),}) = (\sigma_{k,k'}^{(1,j)})_{1 \le k,k' \le n}$. As usual, this formula can be developed by

$$\text{Cov}(F_1^2(n), F_j^2(n)) =$$
$$\frac{1}{n^2} \left(\sum_{k=1}^{n} \sum_{k'=1}^{n} \sigma_{k,k'}^{(1,j)} \cdot \sigma_{k',k}^{(1,j)} - \sum_{i=1}^{n} \sum_{k'=1}^{n} \sum_{k=1}^{n} p_{i,k} \cdot \sigma_{k,k'}^{(1,j)} \cdot \sigma_{k',i}^{(1,j)} \right),$$

with
$$\sigma_{k,k'}^{(1,j)} = \frac{\sigma^2}{2} \left(|k+nj|^{2H} + |k'|^{2H} - |k-k'+nj|^{2H} \right)_{1 \le k,k' \le n}$$

and with $P_{E_1} = (p_{i,j})_{1 \le i,j \le n}$ such that
$$p_{i,j} = \frac{2}{n(n-1)} \left((2n+1) - 3(i+j) + 6\frac{i \cdot j}{1+n} \right).$$

Now, we consider the asymptotic expansion of this formula when n is large enough

$$\text{Cov}(F_1^2(n), F_j^2(n)) = \frac{\sigma^4}{4} n^{4H} \left(1 + O\left(\frac{1}{n}\right)\right) \left(\int_0^1 \int_0^1 \left(|x+j|^{2H} \right.\right.$$
$$\left. + y^{2H} - |x-y+j|^{2H} \right) \left(|y+j|^{2H} + x^{2H} - |y-x+j|^{2H} \right) dx\, dy$$
$$- \int_0^1 \int_0^1 \int_0^1 (4 - 6(x+z) + 12xz) \left(|x+j|^{2H} + y^{2H} - |x-y+j|^{2H} \right)$$
$$\left. \left(|y+j|^{2H} + z^{2H} - |y-z+j|^{2H} \right) dx\, dy\, dz \right).$$

In order to obtain an asymptotic expansion of this formula when j is large enough (*i.e.* both windows are taken away from one another), a Taylor expansion in j up to order 3 is necessary. After calculating and simplifying the integrals, we get the result. □

Proof of Property 3.2: We divide the proof into 3 steps:

• Step 1: one proves that $[N/n] \cdot \text{Var}(\tilde{S}(n)) \to \gamma^2(H)$, where $\gamma^2(H)$ depends only on H, when $[N/n] \to \infty$. Indeed,

$$\text{Var}(\tilde{S}(n)) = \frac{1}{[\frac{N}{n}]^2} \sum_{j=1}^{[\frac{N}{n}]} \sum_{j'=1}^{[\frac{N}{n}]} \text{Cov}(\tilde{S}_j(n), \tilde{S}_{j'}(n))$$

$$= \frac{1}{[\frac{N}{n}]} \text{Var}(\tilde{S}_j(n)) + \frac{2}{[\frac{N}{n}]^2} \sum_{j=1}^{[\frac{N}{n}]} \left([\frac{N}{n}] - j \right) \text{Cov}(\tilde{S}_1(n), \tilde{S}_j(n))$$

due to the stationarity.

However, with properties (9), one deduces that when $[N/n] \to \infty$, $\sum_{j=1}^{[\frac{N}{n}]} \text{Cov}(\tilde{S}_1(n), \tilde{S}_j(n))$ and $\sum_{j=1}^{[\frac{N}{n}]} j \cdot \text{Cov}(\tilde{S}_1(n), \tilde{S}_j(n))$ converge, because there exists $C \ge 0$ such that $\left| \text{Cov}(\tilde{S}_1(n), \tilde{S}_j(n)) \right| \le C \cdot j^{2H-3}$ and $0 < H < 1$.

Therefore, there exists $\gamma^2(H)$ depending only on H such that
$$\lim_{[N/n] \to \infty} [N/n] \cdot \text{Var}(\tilde{S}(n)) = \gamma^2(H). \quad (21)$$

• Step 2: the proof of a central limit theorem for $\tilde{S}(n)$ when $[N/n] \to \infty$ can be obtained from the same method as in the proof of Proposition 2.1 in Bardet [4] (Theorem 3 in Soulier [31] leads to the same result).

Indeed, $\tilde{S}(n) = \frac{1}{n^{2H+1} \sigma^2 f(H) \cdot [N/n]} \sum_{i=1}^{n \cdot [N/n]} Z_i^2$, where the zero-mean Gaussian vector $Z = (Z_1, \ldots, Z_{n \cdot [N/n]})$ has the covariance matrix $P \cdot \Sigma \cdot P$, where P is a diagonal block matrix with each block consisting of (n,n) matrix $P_{E_1^\perp}$ and Σ is the covariance matrix of an FBM time series (each (n,n)

block is $\Sigma^{(i,j)}$ with the previous notations). Using a Lindeberg condition, $\tilde{S}(n)$ satisfies the following central limit theorem :

$$\sqrt{[N/n]} \cdot \Big(\tilde{S}(n) - \ (S(n))\Big) \xrightarrow[{[N/n]\to\infty}]{\mathcal{L}} \mathcal{N}(0, \gamma^2(H))), \quad (22)$$

if $\lambda = \|P \cdot \Sigma \cdot P\|$, the supremum of the eigenvalues of the symmetrical matrix $P \cdot \Sigma \cdot P$, is such that

$$\lambda = o\Big(\frac{1}{\sqrt{[N/n]}}\Big). \quad (23)$$

But, using Lemma 4.1 and following the proof of Proposition 2.1 in Bardet [4],

$$\lambda \leq \frac{1}{n^{2H+1}\sigma^2 f(H) \cdot [N/n]} \cdot$$
$$\max_{i\in\{1,\ldots,n\cdot[N/n]\}} \Big(\sum_{j=1}^{n\cdot[N/n]} |\mathrm{Cov}(Z_i, Z_j)|\Big)$$

$$\lambda \leq \frac{1}{\sqrt{2}[N/n]} \max_{i\in\{1,\ldots,n\cdot[N/n]\}} \Big(\sum_{j=1}^{n\cdot[N/n]} \frac{\sqrt{|\mathrm{Cov}(Z_i^2, Z_j^2)|}}{n^{2H+1}\sigma^2 f(H)}\Big)$$

$$\leq \frac{1}{\sqrt{2}[N/n]} \max_{i\in\{1,\ldots,[N/n]\}} \Big(\sum_{j=1}^{[N/n]} \sqrt{\mathrm{Cov}(\tilde{S}_i(n), \tilde{S}_j(n))}\Big)$$

$$\leq \frac{\sqrt{2}}{[N/n]} \Big(\sum_{j=1}^{[N/n]} \sqrt{\mathrm{Cov}(\tilde{S}_1(n), \tilde{S}_j(n))}\Big).$$

So, there exists $C(H) > 0$ depending only on H such that

$$\lambda \leq C(H) \cdot \frac{1}{[N/n]} \sum_{j=1}^{[N/n]} \Big(\sqrt{j^{2H-3}} + \frac{c}{n}\Big) \text{ third line of (9)}$$
$$\leq C'(H) \cdot \Big([N/n]^{H-3/2} + \frac{1}{n}\Big). \quad (24)$$

Therefore if $\frac{1}{n} = o\Big(\frac{1}{\sqrt{[N/n]}}\Big)$ (*i.e.* $N = o(n^3)$), (23) and (22) are proved.

• Step 3: Now, $(\tilde{S}(n)) = 1 + O(\frac{1}{n})$ for n large enough. Then, if $\sqrt{[N/n]} \cdot \frac{1}{n} \to 0$, that is $N/n^3 \to 0$,

$$\sqrt{[N/n]} \cdot \Big(\tilde{S}(n) - 1\Big) \xrightarrow[{[N/n]\to\infty}]{\mathcal{L}} \mathcal{N}(0, \gamma^2(H))).$$

The classical Delta method allows the passage between a central limit theorem for $\tilde{S}(n)$ and a central limit theorem for $\log(\tilde{S}(n))$ (thanks to the regularity properties of the function logarithm). □

Proof of Proposition 3.3: It is possible to write $\widehat{H} = (1,0) \cdot (Z' \cdot Z)^{-1} \cdot Z' \cdot F$, where Z is the $(m,2)$ matrix such that $Z = \begin{pmatrix} \log(r_1 \cdot n) & 1 \\ \vdots & \vdots \\ \log(r_m \cdot n) & 1 \end{pmatrix}$ and

$F = \begin{pmatrix} \log(F(r_1 \cdot n)) \\ \vdots \\ \log(F(r_m \cdot n)) \end{pmatrix}$. Then

$\mathrm{Var}(\widehat{H})$
$$= (1,0) \cdot (Z' \cdot Z)^{-1} \cdot Z' \cdot \mathrm{Cov}(F, F) \cdot Z \cdot (Z' \cdot Z)^{-1} \cdot (1,0)'$$
$$\leq \|(1,0) \cdot (Z' \cdot Z)^{-1} \cdot Z'\|^2 \cdot \|\mathrm{Cov}(F, F)\|$$
$$\leq \|(1,0) \cdot (Z' \cdot Z)^{-1} \cdot Z'\|^2 \cdot 2m \cdot \prod_{i=1}^{m} r_i \cdot \gamma^2(H) \cdot \frac{1}{[N/n]}.$$

Since $\|(1,0) \cdot (Z' \cdot Z)^{-1} \cdot Z'\|$ only depends on $r_1, \ldots, r_m$, the proof of Proposition 3.3 is completed. □

Proof of Theorem 4.2: From the assumptions on Y and r_Y, if $i \geq j \geq 1$,

$$\mathrm{Cov}(X(i), X(j)) = \sum_{k=1}^{i}\sum_{\ell=1}^{j} \mathrm{Cov}(Y(k), Y(\ell)) =$$

$$\sum_{k=1}^{i}(i-k)r_Y(k) + \sum_{k=1}^{j}(j-k)r_Y(k) - \sum_{k=1}^{i-j}(i-j-k)r_Y(k).$$

As a consequence, for all $(i,j) \in \{1, \ldots, n\}^2$,

$$\mathrm{Cov}(X(i), X(j)) = C \cdot \Big(\int_0^1 (1-u)u^{2H-2}du\Big) \cdot$$
$$\Big(i^{2H}\big(1 + O(\frac{1}{i^{\min(\beta,1)}})\big) + j^{2H}\big(1 + O(\frac{1}{j^{\min(\beta,1)}})\big) -$$
$$|i-j|^{2H}\big(1 + O(\frac{1}{(1+|i-j|)^{\min(\beta,1)}})\big)\Big).$$

Now, this covariance can be used for the proofs replacing the previous ones. This implies

1. $(F_1^2(n)) = \sigma'^2 f(H) \cdot n^{2H}\big(1 + O(\frac{1}{n^{\min(\beta,1)}})\big),$
2. $\mathrm{Var}(F_1^2(n)) = \sigma'^4 g(H) \cdot n^{4H}\big(1 + O(\frac{1}{n^{\min(\beta,1)}})\big),$
3. $\mathrm{Cov}(F_1^2(n), F_j^2(n)) = \sigma'^4 h(H) \cdot n^{4H} \cdot j^{2H-3} \cdot$
$$\big(1 + O(\frac{1}{n^{\min(\beta,1)}}) + O(\frac{1}{j})\big),$$

with $\sigma'^2 = 2C \cdot \Big(\int_0^1 (1-u)u^{2H-2}du\Big)$. The proofs of property 3.2 are the same as in the case of the FGN except that in (24),

$$\lambda \leq C(H) \cdot \frac{1}{[N/n]} \sum_{j=1}^{[N/n]} \Big(\sqrt{j^{2H-3}} + \frac{c}{n} + \frac{c}{n^\beta}\Big)$$
$$\leq C'(H) \cdot \Big([N/n]^{H-3/2} + \frac{1}{n} + \frac{1}{n^\beta}\Big).$$

So, if $\frac{1}{n} + \frac{1}{n^\beta} = o\Big(\frac{1}{\sqrt{[N/n]}}\Big)$ therefore $N = o(n^{\max(2\beta+1,3)})$, the central limit theorem as well as Proposition 3.3 are proved following the same proof as in the case of the FGN. □

Proof of Property 5.1: In the j-th window, with $j \in \{1, \ldots, [N/n]\}$, let us consider E_j the vector

subspace de£ned above and de£ne the vector $G^{(j)} = a((1 + n(j-1))^{\lambda+1}, \ldots, (nj)^{\lambda+1})'$. We have

$$F_{f,j}^2(n) = \frac{1}{n}\left(G^{(j)'} \cdot G^{(j)} - G^{(j)'} \cdot P_{E_j} \cdot G^{(j)}\right)$$

An explicit asymptotic expansion (in n and N) of this partial DFA function can be obtained by approximating sums by integrals. Then,

$$F_{f,j}^2(n) = a^2 n^{2\lambda+2}\left(1 + O\left(\frac{1}{n}\right)\right)\left(\int_0^1 \int_0^1 (x + j - 1)^{2\lambda+2} - \right.$$
$$\left. (4 - 6(x + y) + 12xy)(x + j - 1)^{\lambda+1}(y + j - 1)^{\lambda+1} dx dy\right)$$

Moreover, using Taylor expansion in j up to order 3, one obtains

$$F_{f,j}^2(n) = \alpha(a, \lambda) \cdot n^{2\lambda+2} j^{2\lambda-2}\left(1 + O\left(\frac{1}{n}\right) + O\left(\frac{1}{j}\right)\right), \quad (25)$$

and it implies that the DFA function relating to f can be written as

$$F_f^2(n) = \frac{1}{[N/n]}\sum_{j=1}^{[N/n]} F_{f,j}^2(n)$$
$$= \beta(a, \lambda) \cdot n^4 N^{2\lambda-2}\left(1 + O\left(\frac{1}{n}\right) + O\left(\frac{1}{\left[\frac{N}{n}\right]}\right)\right),$$

with $\alpha(a, \lambda)$, $\beta(a, \lambda)$ two positive numbers depending only on a and λ. $\square$

Proof of Property 5.2: Let $f(k) = a_p k^p + \cdots + a_0 \implies g(k) = \sum_{i=1}^k f(i) = b_{p+1} k^{p+1} + \cdots + b_0$, with $b_{p+1} \neq 0$, *i.e.* the associated integrated function is also a polynomial function. From the expression of the partial DFA function and with the asymptotic expansion (25) depending on the degree λ, for n and N large enough,

$$F_{a_p k^p + \cdots + a_0, j}^2(n) = F_{a_p k^p, j}^2(n)\left(1 + O\left(\frac{1}{n}\right) + O\left(\frac{1}{j}\right)\right)$$

(the power of n in the partial DFA function relating to $a_p k^p$ is greater than the ones in the partial DFA function relating to the other monomes). This approximation leads to the following expression of the DFA function of a polynomial function,

$$F_{a_p k^p + \cdots + a_0}^2(n) =$$
$$\beta(b_{p+1}) \cdot n^4 N^{2\lambda-2}\left(1 + O\left(\frac{1}{n}\right) + O\left(\frac{1}{\left[\frac{N}{n}\right]}\right)\right). \quad \square$$

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REFERENCES

[1] Abry P., Veitch D., Flandrin P., *Long-range dependence: revisiting aggregation with wavelets.*, J. Time Ser. Anal. 19, no. 3, 253-266, 1998.
[2] Absil P.A., Sepulchre R., Bilge A., Gérard P., *Nonlinear analysis of cardiac rhythm ¤uctuations using DFA method*, J. Physica A : Statistical mechanics and its applications, 235-244, 1999.
[3] Arcones M., *Limit theorems for nonlinear functionals of a stationary Gaussian sequence of vectors.*, Ann. Probab. 22, 2242-2274, 1994.
[4] Bardet J.M., *Testing for the presence of self-similarity of Gaussian time series having stationary increments*, J. of Time Series Anal. 21, 497-516, 2000.
[5] Bardet J.M., *Statistical study of the wavelet analysis of fractional Brownian motion*, IEEE Trans. Inform. Theory. 48, 991-999, 2002.
[6] Bardet J.M., Bibi H., Jouini A. *Adaptive wavelet based estimator of the memory parameter for stationary Gaussian processes*, Preprint of Samos, University Paris 1, 2006.
[7] Bardet J.M., Lang G., Oppenheim G., Philippe A., Taqqu M.S. *Generators of long-range dependent processes : A survey*. In *Long-range Dependence : Theory and Applications*, Birkhauser (2002).
[8] Chen Z., Ivanov P.C., Hu K., Stanley H.E, *Effect of nonstationarities on detrended ¤uctuation analysis*, Physical Review E, Vol. 65, 041107, 2002.
[9] Davydov Y.A., *The invariance principal for stationry processes*, Theory Probab. Appl. 15, 487-498, 1970.
[10] Delignières D., *L'analyse des processus stochastiques*, "Sport, Performance, Santé", EA 2991, Université Montpellier I, janvier 2001.
[11] Dobrushin R.L., Major P., *Non-central limit theorems for nonlinear functionals of Gaussian £elds*, Z. Wahrsch. Verw. Gebiete 50, 27-52, 1979.
[12] Gao J.B., Cao Y., Lee J.M, *Principal component analysis of $1/f^\alpha$ noise*, Physics Letters, A 314, 392-400, 2003.
[13] Giraitis L., Robinson, P. and Samarov, A., *Rate optimal semi-parametric estimation of the memory parameter of the Gaussian time series with long range dependence*, J. Time Ser. Anal., 18, 49-61, 1997.
[14] Giraitis L., Robinson, P. and Samarov, A., *Adaptive Semiparametric Estimation of the Memory Parameter*, Journal of Multivariate Analysis, Vol. 72, Issue 2, 183-207, 2000.
[15] Giraitis L., Surgailis D., *CLT and other limit theorems for functionals of Gaussian processes*, Z. Wahrsch. Verw. Gebiete 70, 191-212, 1985.
[16] Ho H.C., Hsing T., *Limit theorems for functionals of moving averages*, The Annals of Probability, Vol. 25, No. 4, 1636-1669, 1997.
[17] Hu K., Ivanov P.C., Chen Z., Carpena P., Stanley H.E, *Effect of trends on detrended ¤uctuation analysis*, Physical Review E, Vol. 64, 011114, 2001.
[18] Hwa R.C., Ferree T.C.,*Fluctuation analysis of human electroencephalogram*, Nonlinear phenomena in complex systems, 302-307, 2002.
[19] Kantelhardt J.W., Ashkenazy Y., Ivanov P.C., Bunde A., Havlin S., Penzel T., Peter J.H., Stanley H.E., *Characterization of sleep stages by correlations in the magnitude and sign of heartbeat increments*, Physical Review E, Vol. 65, 051908, 2002.
[20] Kantelhardt J.W., Koscielny-Bunde E., Rego H.A.H., Havlin S., Bunde A., *Detecting Long- range Correlations with Detrended Fluctuation Analysis*, Physica A, 295, 441-454, 2001.
[21] Karasik R., Sapir N., Ashkenazy Y., Ivanov P.C., Dvir I., Lavie P., Havlin S., *Correlation differences in heartbeat ¤uctuations during rest and exercise*, Physical Review E 66, 062902, 2002.
[22] Martinis M., Knezevic A., Krstacic G., Vargovic E., *Changes in the Hurst exponent of heartbeat intervals during physical activities*, Physics 0212029, 2002.
[23] Masugi M., *Detrended ¤uctuation analysis of IP-network traf£c using a two-dimensional topology map*, Phys. A 337, no. 3-4, 664–678, 2004.
[24] Moulines E., Roueff F., Taqqu M.S., *On the Spectral Density of the Wavelet Coef£cients of Long-Memory Time Series with Application to the Log-Regression Estimation of the Memory Parameter*, JTSA, Vol. 28, issue 2, 155-187, 2007.
[25] Nagarajan R., Kavasseri R.G, *Minimizing the effect of sinusoidal trends in detrended ¤uctuation analysis*, (in press) International Journal of Bifurcations and Chaos, Vol. 15, No. 2, 2005.
[26] Peng C.K., Buldyrev S.V., Havlin S., Simons M., Stanley H.E., Goldberger A.L., *Mosaic organization of DNA nucleotides*, Physical Review E, Vol. 49, 1685-1689, 1994.
[27] Peng C.K., Hausdorff J.M., Goldberger A.L., *Fractal mechanisms in neural control: Human heartbeat and gait dynamics in health and disease*. In: Walleczek J, ed. Nonlinear Dynamics, Self-Organization, and Biomedicine. Cambridge University Press, 1999.

[28] Peng C.K., Havlin S., Stanley H.E., Goldberger A.L., *Quanti£cation of scaling exponents and crossover phenomena in nonstationary heartbeat time series*, Chaos 5, 82, 1995.

[29] Robinson, P.M.*Gaussian semiparametric estimation of long range dependence*, The Annals of statistics, 23:1630-1661, 1995.

[30] Samorodnitsky G., Taqqu M.S., *Stable non-Gaussian Random Processes*, Chapman and Hall, 1994.

[31] Soulier P., *Moment bounds and central limit theorem for functions of Gaussian vectors*, Statist. Probab. Lett. 54, 193-203, 2001.

[32] Taqqu M.S., *Weak Convergence to Fractional Brownian Motion and to Rosenblatt Process*, Zeit. Wahr. verw. Geb., 31, 287-302, 1975.

[33] Taqqu M.S., Teverovsky V., Willinger W., *Estimators for long-range dependence: an empirical study*, Fractals, Vol. 3, No. 4, 785-788, 1995.

[34] Veitch D., Abry P., *A wavelet-based joint estimator of the parameters of long-range dependence.* IEEE Trans. Inform. Theory, Vol. 45, No. 3, 878–897, 1999.