

# About the frequencies of some patterns in digital planes. Application to area estimators

extended version

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**Abstract.** In this paper we prove that the function giving the frequency of a class of patterns of digital planes with respect to the slopes of the plane is continuous and piecewise affine, moreover the regions of affinity are precised. It allows to prove some combinatorial properties of a class of patterns called  $(m, n)$ -cubes. This study has also some consequences on local estimators of area: they never converge to the exact area when the resolution tends to zero for almost all region of plane. Actually we can prove with the same technics that this result is true for the regions of hyperplanes for any dimension  $d \geq 3$ .

**Keywords:** Digital Plane, Pattern,  $(m, n)$ -cube, Area Estimator, Local Estimator, Multigrid Convergence.

## 1 Introduction

Digital Planes are very classical objects of Discrete Geometry. Their combinatorics have been studied in a lot of papers (for example [1,2,3,4,5,6,7,8,9]), for a recent review on the subject, see [10]. In this paper we are interested in a class of patterns called  $(m, n)$ -cubes which are intuitively the pieces of digital planes of size  $m \times n$ . These objects have been studied for quite a long time, for example it is well-known that the number of  $(m, n)$ -cubes appearing in a digital plane is always less than  $mn$  ([4,5,6,8,9]). These  $(m, n)$ -cubes can be used in different domains of image analysis for example for normal vector estimation, area estimation ([11], see also second section of this paper), form reconstruction. The originality of this paper is the study of not only the presence or not of a  $(m, n)$ -cube in a digital plane, but also the frequency of this  $(m, n)$ -cube in all the digital plane.

The main result of this paper is that the function giving the frequency of  $(m, n)$ -cubes of digital planes with respect to the slopes of the plane is continuous and piecewise affine. Moreover we will see that the study of the frequency allows to prove some combinatorial properties on the  $(m, n)$ -cubes.

In a second part of the paper, we use our study about frequencies to prove some results about local estimators of area. A local estimator of area simply consists to decompose a surface into little pieces (in fact similar to  $(m, n)$ -cubes)

and to sum some weights which correspond to the pieces. The study of the frequency of the  $(m, n)$ -cubes allows to prove that even for planar regions these estimators are not correct in the sense that, if the discrete regions are obtained from a continuous plane, then the estimated area does not converge to the exact area for almost all slope of plane when the resolution of the discretization tends to zero. It is in fact a generalization to 3D of [12]. Actually we can prove with the same technics that all the results of this paper are true for the regions of hyperplanes for any dimension  $d \geq 3$ .

## 2 Preliminaries

Let  $a, b \in \mathbb{N}$  and  $a \leq b$ . The discrete interval  $\{a, a + 1, \dots, b - 1, b\}$  is denoted  $\llbracket a, b \rrbracket$ . For  $x \in \mathbb{R}$ ,  $\lfloor x \rfloor$  (resp.  $\langle x \rangle$ ) denotes the integral part (resp. the fractional part) of  $x$ . So,  $x = \lfloor x \rfloor + \langle x \rangle$  with  $\lfloor x \rfloor \in \mathbb{Z}$ ,  $\lfloor x \rfloor \leq x < \lfloor x \rfloor + 1$  and  $0 \leq \langle x \rangle < 1$ . If  $m \in \mathbb{N}^*$  and  $n \in \mathbb{Z}$  then  $n \pmod{m}$  denotes the congruence class of  $n$  modulo  $m$  and corresponds to the set  $\{n + km \mid k \in \mathbb{Z}\}$ . The set of congruence classes modulo  $n$  is denoted  $\mathbb{Z}/n\mathbb{Z}$ . For any set  $E$ ,  $\text{card}(E)$  denotes the cardinality of  $E$ . We refer in all the following to a subset of  $\mathbb{R}^3$  of the form  $R = \{(x, y, \alpha x + \beta y + \gamma) \mid a \leq x \leq b \text{ and } c \leq y \leq d\}$  such that  $0 \leq \alpha, \beta \leq 1$  and  $a, b, c, d \in \mathbb{R}$  as a rectangular planar region. It corresponds to a subset of plane whose projection on the  $XY$ -plane is a rectangle with sides parallel to  $X, Y$ -axes.

In this paper all the topological notions are considered relatively to the euclidean usual topology. The measure notions are considered relatively to the Lebesgue measure on the euclidean space, for example ‘negligible set’ (set with zero measure) and ‘almost everywhere’ are considered relatively to the Lebesgue measure on the euclidean space.

## 3 Frequencies of the $(m, n)$ -cubes

In this paper we consider naive digital planes  $P_{\alpha, \beta, \gamma} = \{(x, y, \lfloor \alpha x + \beta y + \gamma \rfloor) \mid (x, y) \in \mathbb{Z}^2\}$  with  $0 \leq \alpha, \beta \leq 1$  and  $\gamma \in \mathbb{R}$ . So a naive plane is functional in its  $x, y$  coordinates:  $z = p_{\alpha, \beta, \gamma}(x, y) = \lfloor \alpha x + \beta y + \gamma \rfloor$  for all  $(x, y, z) \in P_{\alpha, \beta, \gamma}$ . Moreover we fix two positive integers  $m$  and  $n$  and we define  $\mathcal{F}_{m, n} = \llbracket 0, m - 1 \rrbracket \times \llbracket 0, n - 1 \rrbracket$ .

**Definition 1.** A  $(m, n)$ -pattern is a function  $w : \mathcal{F}_{m, n} \rightarrow \mathbb{Z}$ . We note  $\mathcal{M}_{m, n}$  the set of all  $(m, n)$ -patterns and the size of a  $(m, n)$ -pattern is  $m \times n$ .

We can also see a  $(m, n)$ -pattern as a set of voxels which projection in the  $XY$ -plane is  $\mathcal{F}_{m, n}$  and which has at most one point in each line parallel the third coordinate direction.

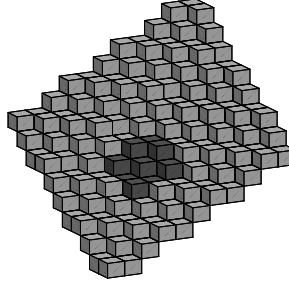
In all the following, a pattern of size less than  $m \times n$  corresponds to a  $(m', n')$ -pattern where  $m' \leq m$ ,  $n' \leq n$  and  $(m, n) \neq (m', n')$ .

A  $(m, n)$ -cube is a  $(m, n)$ -pattern which can be extracted from a naive digital plane, more precisely:

**Definition 2.** The  $(m, n)$ -cube at position  $(i, j)$  of the digital plane  $P_{\alpha, \beta, \gamma}$  is the  $(m, n)$ -pattern  $w$  defined by  $w(i', j') = p_{\alpha, \beta, \gamma}(i + i', j + j') - p_{\alpha, \beta, \gamma}(i, j)$  for any  $(i', j') \in \mathcal{F}_{m, n}$ . It is denoted  $w_{i, j}(\alpha, \beta, \gamma)$ .

So a  $(m, n)$ -cube is simply a piece of a digital plane which projection in the  $XY$ -plane is a translated of  $\mathcal{F}_{m, n}$ . Fig.1 corresponds to a  $(3, 3)$ -cube in a digital plane.

Note that for all  $i, j \in \mathbb{Z}$  and  $\alpha, \beta, \gamma \in \mathbb{R}$ ,  $w_{i, j}(\alpha, \beta, \gamma) = w_{0, 0}(\alpha, \beta, \alpha i + \beta j + \gamma)$ .



**Fig. 1.** A  $(3, 3)$ -cube in a digital plane

Let  $C_{i, j}^{\alpha, \beta} = 1 - \langle \alpha i + \beta j \rangle$  for  $(i, j) \in \mathcal{F}_{m, n}$ , and  $\sigma^{\alpha, \beta}$  be a bijection from  $\llbracket 1, mn \rrbracket$  to  $\mathcal{F}_{m, n}$  such that the sequence  $(B_i^{\alpha, \beta})_{0 \leq i \leq mn}$  defined by  $B_i^{\alpha, \beta} = C_{\sigma^{\alpha, \beta}(i)}^{\alpha, \beta}$  for  $1 \leq i \leq mn$  and  $B_0^{\alpha, \beta} = 0$ , is increasing.

We recall some known results (see for example [5]).

**Proposition 1.** For all  $\alpha, \beta, \gamma \in \mathbb{R}$  we have:

1. The  $(k, l)$ -th point of the  $(m, n)$ -cube at position  $(i, j)$  of the digital plane  $P_{\alpha, \beta, \gamma}$  can be computed by the formula:

$$w_{i, j}(\alpha, \beta, \gamma)(k, l) = \begin{cases} \lfloor \alpha k + \beta l \rfloor & \text{if } \langle \alpha i + \beta j + \gamma \rangle < C_{k, l}^{\alpha, \beta} \\ \lfloor \alpha k + \beta l \rfloor + 1 & \text{otherwise} \end{cases}$$

2. The  $(m, n)$ -cube  $w_{i, j}(\alpha, \beta, \gamma)$  only depends on the interval  $[B_h^{\alpha, \beta}, B_{h+1}^{\alpha, \beta}[$  containing  $\langle \alpha i + \beta j + \gamma \rangle$ .
3. For all  $h \in \llbracket 1, mn - 1 \rrbracket$ , if  $[B_h^{\alpha, \beta}, B_{h+1}^{\alpha, \beta}[$  is not empty ( $B_h^{\alpha, \beta} < B_{h+1}^{\alpha, \beta}$ ), then there exist  $i, j$  such that  $\langle \alpha i + \beta j + \gamma \rangle \in [B_h^{\alpha, \beta}, B_{h+1}^{\alpha, \beta}[$  and thus the number of  $(m, n)$ -cubes in the digital plane  $P_{\alpha, \beta, \gamma}$  is equal to  $\text{card}(\{C_{k, l}^{\alpha, \beta} \mid (k, l) \in \mathcal{F}_{m, n}\})$ . We have, in particular,  $\text{card}(\{C_{k, l}^{\alpha, \beta} \mid (k, l) \in \mathcal{F}_{m, n}\}) \leq mn$ .

So, we have  $w_{0,0}(\alpha, \beta, \gamma) = w_{0,0}(\alpha, \beta, \langle \gamma \rangle)$  and thus  $w_{i,j}(\alpha, \beta, \gamma) = w_{0,0}(\alpha, \beta, \langle \alpha i + \beta j + \gamma \rangle)$  for all  $\alpha, \beta, \gamma \in \mathbb{R}$  and  $(i, j) \in \mathbb{Z}^2$ .

By Proposition 1, the set of  $(m, n)$ -cubes of the digital plane  $P_{\alpha, \beta, \gamma}$  depends only on  $\alpha, \beta$  and it is denoted  $\mathcal{C}_{m, n, \alpha, \beta}$ . In all the following,  $\mathcal{U}_{m, n}$  denotes the set of all the  $(m, n)$ -cubes. So,  $\mathcal{U}_{m, n} = \bigcup_{(\alpha, \beta) \in [0, 1]^2} \mathcal{C}_{m, n, \alpha, \beta}$ .

**Definition 3 ([13]).** Let  $w$  be a  $(m, n)$ -cube, then the pre-image  $\text{PI}(w)$  of  $w$  is the set of the triple  $(\alpha, \beta, \gamma) \in [0, 1]^3$  such that  $w$  is the  $(m, n)$ -cube at position  $(0, 0)$  of the digital plane  $P_{\alpha, \beta, \gamma}$ .

**Remark.** It is easy to see that  $\text{PI}(w)$  is a convex polyhedron defined by the inequalities  $w(k, l) \leq k\alpha + l\beta + \gamma < w(k, l) + 1$  for  $(k, l) \in \mathcal{F}_{m, n}$ . Moreover the set of the  $\gamma' \in [0, 1]$  such that  $(\alpha, \beta, \gamma') \in \text{PI}(w_{i,j}(\alpha, \beta, \gamma))$  is exactly the interval  $[B_h^{\alpha, \beta}, B_{h+1}^{\alpha, \beta}[$  containing  $\langle \alpha i + \beta j + \gamma \rangle$ .

The last remark leads to the following definition:

**Definition 4.** The  $\gamma$ -frequency of a  $(m, n)$ -cube  $w$  for the slopes  $(\alpha, \beta)$  (denoted  $\text{freq}_{\alpha, \beta}(w)$ ) is the length of the interval  $I^{\alpha, \beta}(w) = \{\gamma \in [0, 1] \mid (\alpha, \beta, \gamma) \in \text{PI}(w)\}$ . (so the function  $TP : \text{PI}(w) \rightarrow \mathbb{R}$  such that  $TP(\alpha, \beta) = \text{freq}_{\alpha, \beta}(w)$  is the tomographic projection of  $\text{PI}(w)$  w.r.t. the third coordinate direction).

**Definition 5.** The overlapping frequency of a  $(m, n)$ -cube in the digital plane  $P_{\alpha, \beta, \gamma}$  is

$$\lim_{N \rightarrow +\infty} \frac{\text{card}(\{(i, j) \in \llbracket -N, N \rrbracket^2 \mid w_{i,j}(\alpha, \beta, \gamma) = w\})}{(2N + 1)^2}$$

if the limit exists. It is denoted  $\text{overfreq}_{\alpha, \beta, \gamma}(w)$ .

$$\text{So, } \text{overfreq}_{\alpha, \beta, \gamma}(w) = \lim_{N \rightarrow +\infty} \frac{\text{card}(\{(i, j) \in \llbracket -N, N \rrbracket^2 \mid \langle \alpha i + \beta j + \gamma \rangle \in I^{\alpha, \beta}(w)\})}{(2N + 1)^2}$$

We have the following properties:

**Proposition 2.** For any  $\alpha, \beta \in [0, 1]$  and  $\gamma \in \mathbb{R}$  we have:

1.  $w \in \mathcal{C}_{m, n, \alpha, \beta}$  if and only if  $\text{freq}_{\alpha, \beta}(w) > 0$ .
2.  $\text{overfreq}_{\alpha, \beta, \gamma}(w) = \text{freq}_{\alpha, \beta}(w)$

*Proof.* 1. If  $w \in \mathcal{C}_{m, n, \alpha, \beta}$ , then there exists  $(i, j) \in \mathbb{Z}^2$  and  $\gamma \in \mathbb{R}$  such that  $w = w_{i,j}(\alpha, \beta, \gamma) = w_{0,0}(\alpha, \beta, \langle \alpha i + \beta j + \gamma \rangle)$ . So,  $\langle \alpha i + \beta j + \gamma \rangle \in I^{\alpha, \beta}(w)$ . Then  $\text{freq}_{\alpha, \beta}(w) = \mu(I^{\alpha, \beta}(w)) > 0$  because  $I^{\alpha, \beta}(w)$  is a non-empty interval of the form  $[A, A'[,$

Conversely if  $\text{freq}_{\alpha, \beta}(w) > 0$  then  $I^{\alpha, \beta}(w) \neq \emptyset$ . So, by Proposition 1, for all  $\gamma \in \mathbb{R}$  there exists  $(i, j) \in \mathbb{Z}^2$  such that  $\langle \alpha i + \beta j + \gamma \rangle \in I^{\alpha, \beta}(w)$  which implies that  $w \in \mathcal{C}_{m, n, \alpha, \beta}$ .

2. We prove now that  $\text{overfreq}_{\alpha, \beta, \gamma}(w) = \text{freq}_{\alpha, \beta}(w)$  for any  $\alpha, \beta \in [0, 1]$  and  $\gamma \in \mathbb{R}$ .

- (a) Suppose first that  $\alpha$  and  $\beta$  are both rational and let  $\gamma \in \mathbb{R}$ , then  $\alpha = \frac{p_1}{q_1}$  and  $\beta = \frac{p_2}{q_2}$  with  $p_i$  and  $q_i$  co-prime.  
Put  $p = \gcd(p_1, p_2)$ ,  $p'_1 = \frac{p_1}{p}$ ,  $p'_2 = \frac{p_2}{p}$ ,  $q = \gcd(q_1, q_2)$ ,  $q'_1 = \frac{q_1}{q}$ ,  $q'_2 = \frac{q_2}{q}$ ,  $q_3 = \frac{q_1 q_2}{q}$  and  $r = \frac{pq}{q_1 q_2} = \frac{p}{q_3}$ .  
Put  $k_0 = \lfloor \gamma q_3 \rfloor$  and  $\gamma' = \gamma - \frac{\lfloor \gamma q_3 \rfloor}{q_3}$ . Then  $0 \leq \gamma' < \frac{1}{q_3}$ .  
As  $p'_1 q'_2$  and  $p'_2 q'_1$  are co-prime, then  $\{p'_1 q'_2 x + p'_2 q'_1 y \mid (x, y) \in \mathbb{Z}^2\} = \mathbb{Z}$  and thus  $\{\alpha x + \beta y \mid (x, y) \in \mathbb{Z}^2\} = r\{p'_1 q'_2 x + p'_2 q'_1 y \mid (x, y) \in \mathbb{Z}^2\} = r\mathbb{Z}$ , which implies that  $\{\langle \alpha x + \beta y + \gamma \rangle \mid (x, y) \in \mathbb{Z}^2\} = \{\gamma' + \frac{i}{q_3} \mid i \in \llbracket 0, q_3 - 1 \rrbracket\}$  because  $p$  and  $q_3$  are co-prime.

Consider the function  $f : \mathbb{Z}^2 \rightarrow \mathbb{Z}/q_3\mathbb{Z}$  such that for all  $(x, y) \in \mathbb{Z}^2$ ,  $f(x, y) = p'_1 q'_2 x + p'_2 q'_1 y \pmod{q_3}$ . Then  $f$  is surjective and for all  $(x, y) \in \mathbb{Z}^2$ ,  $f(x + q_1, y + q_2) = f(x, y)$ . So,  $f$  can be viewed as a function from  $\mathbb{Z}/q_1\mathbb{Z} \times \mathbb{Z}/q_2\mathbb{Z}$  to  $\mathbb{Z}/q_3\mathbb{Z}$  and in this case we use the notation  $\hat{f}$  instead  $f$ .

Then  $\hat{f}$  is a surjective morphism of groups and thus for all  $y \in \mathbb{Z}/q_3\mathbb{Z}$ ,  $\text{card}(\{x \in \mathbb{Z}/q_1\mathbb{Z} \times \mathbb{Z}/q_2\mathbb{Z} \mid \hat{f}(x) = y\}) = \text{card}(\text{Ker}(\hat{f})) = \frac{\text{card}(\mathbb{Z}/q_1\mathbb{Z} \times \mathbb{Z}/q_2\mathbb{Z})}{\text{card}(\mathbb{Z}/q_3\mathbb{Z})} = q$  where  $\text{Ker}(\hat{f})$  is the Kernel of the morphism of groups  $\hat{f}$ .

Let  $s, t \in \mathbb{Z}$  and consider the rectangle  $H(s, t) = \llbracket s, s + q_1 - 1 \rrbracket \times \llbracket t, t + q_2 - 1 \rrbracket$  ( $H(s, t)$  can be viewed as a representation of the group  $\mathbb{Z}/q_1\mathbb{Z} \times \mathbb{Z}/q_2\mathbb{Z}$ ). Then for any  $s, t \in \mathbb{Z}$  and any  $y \in \mathbb{Z}/q_3\mathbb{Z}$ ,  $\text{card}(\{x \in H(s, t) \mid \hat{f}(x) = y\}) = \text{card}(\{x \in \mathbb{Z}/q_1\mathbb{Z} \times \mathbb{Z}/q_2\mathbb{Z} \mid \hat{f}(x) = y\}) = q$ .

Let  $i \in \llbracket 0, q_3 - 1 \rrbracket$  and consider  $(x_0(i), y_0(i)) \in \mathbb{Z}^2$  such that  $\langle \alpha x_0(i) + \beta y_0(i) + \gamma \rangle = \gamma' + \frac{i}{q_3}$ .

Put  $F(i, \gamma) = \{(x, y) \in \mathbb{Z}^2 \mid \langle \alpha x + \beta y + \gamma \rangle = \gamma' + \frac{i}{q_3}\}$  and for any set  $E$ , put  $F(i, \gamma, E) = F(i, \gamma) \cap E$ .

Then  $(x, y) \in F(i, \gamma)$  if and only if  $f((x, y) - (x_0(i), y_0(i))) = 0$  because  $p, q_3$  are co-prime.

So,  $F(i, \gamma, H(s, t)) = \{(x_0(i), y_0(i)) + (x, y) \in \mathbb{Z}^2 \mid f(x, y) = 0\} \cap H(s, t) = \{(x, y) \in H(s - x_0(i), t - y_0(i)) \mid f(x, y) = 0\}$ .

Then  $\text{card}(F(i, \gamma, H(s, t))) = q$  which is independent of  $s, t \in \mathbb{Z}$  and  $i \in \llbracket 0, q_3 - 1 \rrbracket$ . Let  $N \in \mathbb{N}$  and put  $E_N = \llbracket -N, N \rrbracket^2$ .

Then  $E_N = (\bigcup_{(i,j) \in \llbracket 0, \lfloor \frac{2N+1}{q_1} \rfloor - 1 \rrbracket \times \llbracket 0, \lfloor \frac{2N+1}{q_2} \rfloor - 1 \rrbracket} H(-N + q_1 i, -N + q_2 j)) \cup E'_N$  where  $E'_N = \llbracket -N + q_1(\lfloor \frac{2N+1}{q_1} \rfloor - 1), N \rrbracket \times \llbracket -N, N \rrbracket \cup \llbracket -N, N \rrbracket \times \llbracket -N + q_2(\lfloor \frac{2N+1}{q_2} \rfloor - 1), N \rrbracket$ . So  $E_N$  is partitioned on  $\lfloor \frac{2N+1}{q_1} \rfloor \lfloor \frac{2N+1}{q_2} \rfloor$  rectangles of the form  $H(s, t)$  and  $E'_N$ .

Then,  $\text{card}(F(i, \gamma, E_N)) = \lfloor \frac{2N+1}{q_1} \rfloor \lfloor \frac{2N+1}{q_2} \rfloor q + \text{card}(F(i, \gamma, E'_N))$ .

So,

$$\lim_{N \rightarrow +\infty} \frac{\text{card}(F(i, \gamma, E_N))}{(2N+1)^2} = \lim_{N \rightarrow +\infty} \frac{\lfloor \frac{2N+1}{q_1} \rfloor \lfloor \frac{2N+1}{q_2} \rfloor q}{(2N+1)^2} +$$

$$\lim_{N \rightarrow +\infty} \frac{\text{card}(F(i, \gamma, E'_N))}{(2N+1)^2} = \frac{q}{q_1 q_2} = \frac{1}{q_3}$$

because  $\text{card}(F(i, \gamma, E'_N)) < (q_1 + q_2)(2N + 1)$ .

So, for all  $l \in \llbracket 0, q_3 - 1 \rrbracket$ ,

$$\lim_{N \rightarrow +\infty} \frac{\text{card}(\{(i, j) \in E_N \mid \langle \alpha i + \beta j + \gamma \rangle = \gamma' + \frac{l}{q_3}\})}{(2N + 1)^2} = \frac{1}{q_3}.$$

Put  $L = \{\langle \alpha i + \beta j \rangle \mid (i, j) \in \mathcal{F}_{m,n}\} = \frac{1}{q_3}\{l_1, l_2, \dots, l_h\}$  where  $0 = l_1 < l_2 < \dots < l_h$ . Then  $B_0^{\alpha, \beta} = 0$  and  $B_i^{\alpha, \beta} = 1 - \frac{l_{h-i+1}}{q_3} = \frac{q_3 - l_{h-i+1}}{q_3}$  for  $i = 1, \dots, h$ .

Let  $k \in \llbracket 0, h - 1 \rrbracket$  and  $w$  such that  $I^{\alpha, \beta}(w) = [B_k^{\alpha, \beta}, B_{k+1}^{\alpha, \beta}[$ . Then

$$\begin{aligned} \text{overfreq}_{\alpha, \beta, \gamma}(w) &= \lim_{N \rightarrow +\infty} \frac{\text{card}(\{(i, j) \in E_N \mid \langle \alpha i + \beta j + \gamma \rangle \in [B_k^{\alpha, \beta}, B_{k+1}^{\alpha, \beta}[ \})}{(2N + 1)^2} \\ &= M \frac{1}{q_3} \quad \text{where } M = \text{card}(\{l \mid \gamma' + \frac{l}{q_3} \in [B_k^{\alpha, \beta}, B_{k+1}^{\alpha, \beta}[ \}) \\ &= B_{k+1}^{\alpha, \beta} - B_k^{\alpha, \beta} \quad \text{because } 0 \leq \gamma' < \frac{1}{q_3} \\ &= \text{freq}_{\alpha, \beta}(w) \end{aligned}$$

(b) Suppose now that  $\alpha$  or  $\beta$  is irrational. By Corollary 6 of Appendix A we have

$$\begin{aligned} \text{overfreq}_{\alpha, \beta, \gamma}(w) &= \lim_{N \rightarrow +\infty} \frac{\text{card}(\{(i, j) \in E_N \mid \langle \alpha i + \beta j + \gamma \rangle \in I^{\alpha, \beta}(w)\})}{(2N + 1)^2} \\ &= \mu(I^{\alpha, \beta}(w)) \end{aligned}$$

So, for all  $\alpha, \beta \in [0, 1]$ ,  $\text{overfreq}_{\alpha, \beta, \gamma}(w) = \text{freq}_{\alpha, \beta}(w) \quad \square$

**Definition 6.** A function  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$  is called a piecewise affine function if there exists a finite collection  $(C_i)_{i \in I}$  of open convex subsets of  $\mathbb{R}^3$  and affine functions  $f_i : \mathbb{R}^3 \rightarrow \mathbb{R}$  for  $i \in I$ , such that :

- $C_i \cap C_{i'} = \emptyset$  for  $i, i' \in I$  and  $i \neq i'$ ,
- $\bigcup_{i \in I} \overline{C_i} = \mathbb{R}^2$  and
- the restriction of  $f$  to  $C_i$  is  $f_i$  for all  $i \in I$  (for all  $i \in I$ ,  $f(x) = f_i(x)$  for all  $x \in C_i$ ).

*Property 1.* Let  $f, g : \mathbb{R}^2 \rightarrow \mathbb{R}$  be two piecewise affine functions. Then  $-f, f + g, f - g, \max(f, g)$  and  $\min(f, g)$  are also piecewise affine functions.

*Proof.* Let  $(C_i)_{i \in I}$  and  $(f_i)_{i \in I}$  (respectively  $(D_j)_{j \in J}$  and  $(g_j)_{j \in J}$ ) collections of open convex sets and affine functions corresponding to  $f$  (respectively to  $g$ ) by the last definition.

Then  $(C_i)_{i \in I}$  and  $(-f_i)_{i \in I}$  (respectively  $(C_i \cap D_j)_{(i, j) \in I \times J}$  and  $(f_i + g_j)_{(i, j) \in I \times J}$ ) correspond to  $-f$  (respectively to  $f + g$ ). Then  $-f$  and  $f + g$  are piecewise affine functions and thus  $f - g$  is a piecewise affine function.

Let  $(i, j) \in I \times J$  and let  $H_{i,j}^1 = \{x \in \mathbb{R}^3 \mid f_i(x) > g_j(x)\} \cap C_i \cap D_j$  and  $H_{i,j}^2 = \{x \in \mathbb{R}^3 \mid f_i(x) < g_j(x)\} \cap C_i \cap D_j$ . Then collections  $(H_{i,j}^k)_{(k, i, j) \in \{1, 2\} \times I \times J}$

and  $(h_{i,j}^k)_{(k,i,j) \in \{1,2\} \times I \times J}$  such that for all  $(i,j) \in I \times J$ ,  $h_{i,j}^1(x) = f_i(x)$  for all  $x \in H_{i,j}^1$  and  $h_{i,j}^2(x) = g_i(x)$  for all  $x \in H_{i,j}^2$  correspond to the function  $\max(f, g)$ . As  $\min(f, g) = -\max(-f, -g)$  then  $\min(f, g)$  is a piecewise affine function  $\square$

**Theorem 1.** For any  $(m, n)$ -cube  $w$ , the function  $(\alpha, \beta) \mapsto \text{freq}_{\alpha, \beta}(w)$  is a continuous function which is piecewise affine.

*Proof.*  $PI(w) = \{(\alpha, \beta, \gamma) \in \mathbb{R}^3 \mid w(k, l) \leq \alpha k + \beta l + \gamma < w(k, l) + 1 \text{ for all } (k, l) \in \mathcal{F}_{m,n}\}$ .

Then  $I^{\alpha, \beta}(w) = [\max_{(l,k) \in \mathcal{F}_{m,n}} (w(k, l) - \alpha k - \beta l), \min_{(l,k) \in \mathcal{F}_{m,n}} (w(k, l) + 1 - \alpha k - \beta l)]$ .

So,  $\text{freq}_{\alpha, \beta}(w) = \max(0, \min_{(l,k) \in \mathcal{F}_{m,n}} (w(k, l) + 1 - \alpha k - \beta l) - \max_{(l,k) \in \mathcal{F}_{m,n}} (w(k, l) - \alpha k - \beta l))$ . Affine functions, max and min are continuous functions. Then  $(\alpha, \beta) \mapsto \text{freq}_{\alpha, \beta}(w)$  is a continuous function which is piecewise affine because it is composition of continuous functions and by Property 1 it is piecewise affine function  $\square$

**Proposition 3.** Let  $(\alpha_1, \beta_1), (\alpha_2, \beta_2), (\alpha_3, \beta_3)$  be points of  $[0, 1]^2$  and  $T$  be the convex hull of these three points. Let  $(\alpha_0, \beta_0) \in T$  and consider  $\lambda_1, \lambda_2, \lambda_3 \geq 0$  such that  $(\alpha_0, \beta_0) = \sum_{i=1}^3 \lambda_i (\alpha_i, \beta_i)$  and  $\sum_{i=1}^3 \lambda_i = 1$  ( $\lambda_1, \lambda_2, \lambda_3$  are barycentric coordinates of  $(\alpha_0, \beta_0)$  relatively to  $(\alpha_1, \beta_1), \dots, (\alpha_3, \beta_3)$ ). Suppose moreover that the function  $(\alpha, \beta) \mapsto \text{freq}_{\alpha, \beta}(w)$  is affine on  $T$  for any  $(m, n)$ -cube  $w$ , then

$$\mathcal{C}_{m,n,\alpha_0,\beta_0} = \bigcup_{1 \leq i \leq 3 \text{ and } \lambda_i \neq 0} \mathcal{C}_{m,n,\alpha_i,\beta_i}$$

*Proof.* By affinity of  $(\alpha, \beta) \mapsto \text{freq}_{\alpha, \beta}(w)$  on  $T$  we have:

$$\text{freq}_{\alpha_0,\beta_0}(w) = \sum_{i=1}^3 \lambda_i \text{freq}_{\alpha_i,\beta_i}(w)$$

If  $w \notin \mathcal{C}_{m,n,\alpha_0,\beta_0}$  then by Proposition 2,  $\text{freq}_{\alpha_0,\beta_0}(w) = 0$  and so for any  $i$ ,  $\text{freq}_{\alpha_i,\beta_i}(w) = 0$  or  $\lambda_i = 0$  because  $\lambda_1, \lambda_2, \lambda_3 \geq 0$  which implies that for any  $i$ , if  $\lambda_i \neq 0$  then  $w \notin \mathcal{C}_{m,n,\alpha_i,\beta_i}$ . Conversely as  $\lambda_1, \lambda_2, \lambda_3 \geq 0$  and  $\sum_{i=1}^3 \lambda_i = 1$ , if  $w \in \mathcal{C}_{m,n,\alpha_0,\beta_0}$ , then by Proposition 2,  $\text{freq}_{\alpha_0,\beta_0}(w) > 0$  and thus, there must exist a  $i \in \{1, 2, 3\}$  such that  $\lambda_i \neq 0$  and  $\text{freq}_{\alpha_i,\beta_i}(w) > 0$   $\square$

**Remarks:**

By Proposition 3, we have:

- If the points  $(\alpha_1, \beta_1), (\alpha_2, \beta_2), (\alpha_3, \beta_3)$  are non-aligned and  $(\alpha_0, \beta_0)$  is in the interior of  $T$ . So  $\lambda_1, \lambda_2, \lambda_3 > 0$  and then

$$\mathcal{C}_{m,n,\alpha_0,\beta_0} = \mathcal{C}_{m,n,\alpha_1,\beta_1} \cup \mathcal{C}_{m,n,\alpha_2,\beta_2} \cup \mathcal{C}_{m,n,\alpha_3,\beta_3}$$

- If  $(\alpha_0, \beta_0)$  is in the interior of the segment defined by  $(\alpha_1, \beta_1), (\alpha_2, \beta_2)$  so,  $\lambda_1, \lambda_2 > 0$  and  $\lambda_3 = 0$  and then

$$\mathcal{C}_{m,n,\alpha_0,\beta_0} = \mathcal{C}_{m,n,\alpha_1,\beta_1} \cup \mathcal{C}_{m,n,\alpha_2,\beta_2}$$

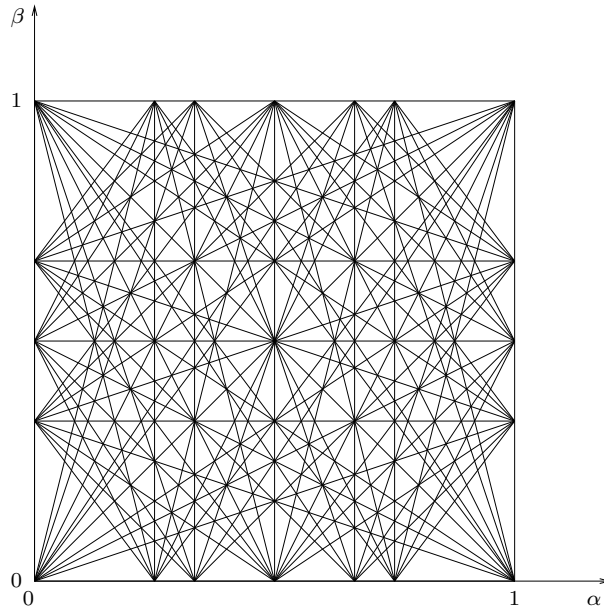
We will now precise the domains where the function  $(\alpha, \beta) \mapsto \text{freq}_{\alpha, \beta}(w)$  is affine: Let  $D_{u,v,w}$  be the line  $\{(\alpha, \beta) \in \mathbb{R}^2 \mid \alpha u + \beta v + w = 0\}$  and

$$E_{m,n} = \bigcup_{(u,v,w) \in \llbracket -m+1, m-1 \rrbracket \times \llbracket -n+1, n-1 \rrbracket \times \mathbb{Z}} D_{u,v,w} \cap [0, 1]^2.$$

$E_{m,n}$  involves only straight lines  $D_{u,v,w}$  such that  $D_{u,v,w} \cap [0, 1]^2 \neq \emptyset$  and so we must only consider the straight lines  $D_{u,v,w}$  such that  $|w| \leq |u| + |v|$  and thus  $E_{m,n}$  involves only a finite number of straight lines.

$E_{m,n}$  is called Hyper Farey fan in [1] and Farey's diagram in [8].

Fig.2 corresponds to Farey's diagram for  $m = 4$  and  $n = 3$ .



**Fig. 2.** Farey's diagram for  $m = 4$  and  $n = 3$

**Lemma 1.** Let  $\mathcal{O}$  be a connected component of  $[0, 1]^2 \setminus E_{m,n}$  and  $(k, l) \in \mathcal{F}_{m,n}$  then the function  $(\alpha, \beta) \mapsto \lfloor \alpha k + \beta l \rfloor$  is constant on  $\mathcal{O}$  and so the function  $(\alpha, \beta) \mapsto C_{k,l}^{\alpha, \beta}$  is affine on  $\mathcal{O}$ .

*Proof.* Suppose that  $(\alpha, \beta) \mapsto \lfloor \alpha k + \beta l \rfloor$  is not constant on  $\mathcal{O}$  and let  $(\alpha, \beta), (\alpha', \beta') \in \mathcal{O} \subset [0, 1]^2 \setminus E_{m,n}$  such that  $h = \lfloor \alpha k + \beta l \rfloor < h' = \lfloor \alpha' k + \beta' l \rfloor$  ( $h' \geq h + 1$ ) and consider the straight line  $D_{k,l,-(h+1)}$ . Then  $\alpha k + \beta l - (h + 1) < 0$  and  $\alpha' k + \beta' l - (h + 1) > 0$  and so  $(\alpha, \beta), (\alpha', \beta')$  are not in the same connected component of  $[0, 1]^2 \setminus E_{m,n}$  because they are separated by the straight line  $D_{k,l,-(h+1)}$ .

Thus for any  $(k, l) \in \mathcal{F}_{m,n}$  and for all connected component  $\mathcal{O}$  of  $[0, 1]^2 \setminus E_{m,n}$  the function  $(\alpha, \beta) \mapsto \lfloor \alpha k + \beta l \rfloor$  is constant in  $\mathcal{O}$ .

As  $(\alpha, \beta) \mapsto \lfloor \alpha k + \beta l \rfloor$  is constant in  $\mathcal{O}$  and  $C_{k,l}^{\alpha,\beta} = 1 - \langle \alpha k + \beta l \rangle = -\alpha k - \beta l + (1 + \lfloor \alpha k + \beta l \rfloor)$ , then the function  $(\alpha, \beta) \mapsto C_{k,l}^{\alpha,\beta}$  is affine on  $\mathcal{O}$   $\square$

**Lemma 2.** *Let  $\mathcal{O}$  be a connected component of  $[0, 1]^2 \setminus E_{m,n}$  and  $(\alpha, \beta) \in \mathcal{O}$  then all the  $C_{k,l}^{\alpha,\beta}$  for  $(k, l) \in \mathcal{F}_{m,n}$  are distinct.*

*Proof.* Let  $(\alpha, \beta) \in \mathcal{O} \subset [0, 1]^2 \setminus E_{m,n}$  and suppose that there exist  $(k, l), (k', l') \in \mathcal{F}_{m,n}$  and  $(k, l) \neq (k', l')$  such that  $C_{k,l}^{\alpha,\beta} = C_{k',l'}^{\alpha,\beta}$  which implies that  $\alpha k + \beta l - h = \alpha k' + \beta l' - h'$  where by Lemma 1,  $h = \lfloor \alpha k + \beta l \rfloor$  (respectively  $h' = \lfloor \alpha k' + \beta l' \rfloor$ ) is constant on  $\mathcal{O}$ . Then  $(\alpha, \beta) \in D_{k-k', l-l', h-h'}$ . This is absurd because as  $k - k' \in \llbracket -m+1, m-1 \rrbracket$  and  $l - l' \in \llbracket -n+1, n-1 \rrbracket$  then  $(\alpha, \beta) \in E_{m,n}$  but by hypothesis  $(\alpha, \beta) \in \mathcal{O} \subset [0, 1]^2 \setminus E_{m,n}$   $\square$

**Theorem 2.** *The function  $(\alpha, \beta) \mapsto \text{freq}_{\alpha,\beta}(w)$  is affine on the closure of any connected component of  $[0, 1]^2 \setminus E_{m,n}$  for all  $w \in \mathcal{U}_{m,n}$ . Moreover for any  $(\alpha, \beta), (\alpha', \beta') \in [0, 1]^2 \setminus E_{m,n}$ :  $\mathcal{C}_{m,n,\alpha,\beta} = \mathcal{C}_{m,n,\alpha',\beta'}$  if and only if  $(\alpha, \beta)$  and  $(\alpha', \beta')$  are in the same connected component of  $[0, 1]^2 \setminus E_{m,n}$ .*

*Proof.* We consider a connected component  $\mathcal{O}$  of  $[0, 1]^2 \setminus E_{m,n}$ . Suppose that there exist  $(\alpha_1, \beta_1), (\alpha_2, \beta_2) \in \mathcal{O}$  and  $(k_1, l_1), (k_2, l_2)$  two different couples of  $\mathcal{F}_{m,n}$  such that  $C_{k_1,l_1}^{\alpha_1,\beta_1} < C_{k_2,l_2}^{\alpha_1,\beta_1}$  and  $C_{k_1,l_1}^{\alpha_2,\beta_2} > C_{k_2,l_2}^{\alpha_2,\beta_2}$  then by continuity of the functions  $(\alpha, \beta) \mapsto C_{k,l}^{\alpha,\beta}$  and connectivity of  $\mathcal{O}$  there exists  $(\alpha, \beta) \in \mathcal{O}$  such that  $C_{k_1,l_1}^{\alpha,\beta} = C_{k_2,l_2}^{\alpha,\beta}$  which is impossible by Lemma 2. So,  $(\alpha, \beta) \mapsto \sigma^{\alpha,\beta}$  is constant on  $\mathcal{O}$ . But by Lemma 2, for  $(\alpha, \beta) \in \mathcal{O}$  we know that all the  $C_{k,l}^{\alpha,\beta}$  for  $(k, l) \in \mathcal{F}_{m,n}$  are distinct. Then  $(\alpha, \beta) \mapsto \mathcal{C}_{m,n,\alpha,\beta}$  is constant on  $\mathcal{O}$ .

Let  $(\alpha_0, \beta_0) \in \mathcal{O}$  and put  $\sigma_0 = \sigma^{\alpha_0,\beta_0}$  and  $\mathcal{C}_0 = \mathcal{C}_{m,n,\alpha_0,\beta_0}$ .

So, for all  $(\alpha, \beta) \in \mathcal{O}$ ,  $B_h^{\alpha,\beta} = C_{\sigma_0(h)}^{\alpha,\beta}$  for all  $h \in \llbracket 1, mn \rrbracket$ . Thus, by Lemma 1  $(\alpha, \beta) \mapsto B_h^{\alpha,\beta}$  is affine on  $\mathcal{O}$  for all  $h \in \llbracket 1, mn \rrbracket$ .

As for  $w \in \mathcal{C}_0$ ,  $\text{freq}_{\alpha,\beta}(w) = B_{h+1}^{\alpha,\beta} - B_h^{\alpha,\beta}$  for some  $h \in \llbracket 1, mn-1 \rrbracket$ , then  $(\alpha, \beta) \mapsto \text{freq}_{\alpha,\beta}(w)$  is affine in  $\mathcal{O}$  for all  $w \in \mathcal{C}_0$ .

Now let  $w \in (\mathcal{U}_{m,n} \setminus \mathcal{C}_0)$  and  $(\alpha, \beta) \in \mathcal{O}$ , then  $I^{\alpha,\beta}(w) = \emptyset$  which implies that  $\text{freq}_{\alpha,\beta}(w) = 0$ .

Then for all  $w \in \mathcal{U}_{m,n}$ ,  $(\alpha, \beta) \mapsto \text{freq}_{\alpha,\beta}(w)$  is affine on any connected component of  $[0, 1]^2 \setminus E_{m,n}$ .

But by Theorem 1, for all  $w \in \mathcal{U}_{m,n}$ , the function  $(\alpha, \beta) \mapsto \text{freq}_{\alpha,\beta}(w)$  is a continuous function which is piecewise affine. Then for  $w \in \mathcal{U}_{m,n}$  and for any connected component  $\mathcal{O}$ , the function  $(\alpha, \beta) \mapsto \text{freq}_{\alpha,\beta}(w)$  is affine on  $\overline{\mathcal{O}}$  where  $\overline{\mathcal{O}}$  is the closure of  $\mathcal{O}$ .

Now it remains to prove that if  $(\alpha, \beta)$  and  $(\alpha', \beta')$  are not in the same connected component of  $[0, 1]^2 \setminus E_{m,n}$  then  $\mathcal{C}_{m,n,\alpha,\beta} \neq \mathcal{C}_{m,n,\alpha',\beta'}$ .

Let  $(\alpha, \beta) \in [0, 1]^2$  and  $(k, l), (k', l') \in \mathcal{F}_{m,n}$ , put:

$$- PZ(\alpha, \beta, k, l) = \{w(k, l) \mid w \in \mathcal{C}_{m,n,\alpha,\beta}\} \text{ and}$$

$$- PZ_2(\alpha, \beta, k, l, k', l') = \{(w(k, l), w(k', l')) \mid w \in \mathcal{C}_{m,n,\alpha,\beta}\}.$$

$PZ(\alpha, \beta, k, l)$  (respectively  $PZ_2(\alpha, \beta, k, l, k', l')$ ) is the set of the restrictions of the  $(m, n)$ -cubes of  $\mathcal{C}_{m,n,\alpha,\beta}$  to the sub-window  $\{(k, l)\}$  (respectively  $\{(k, l), (k', l')\}$ ). Let  $(\alpha, \beta)$  and  $(\alpha', \beta')$  be two points in two distinct connected components of  $[0, 1]^2 \setminus E_{m,n}$ . Then there exists a line  $D_{u,v,w}$  of  $E_{m,n}$  such that  $u\alpha + v\beta + w$  and  $u\alpha' + v\beta' + w$  have not the same sign. Suppose for example that  $u\alpha + v\beta + w < 0$  and  $u\alpha' + v\beta' + w > 0$ . By definition of  $E_{m,n}$  there exist  $(k, l), (k', l') \in \mathcal{F}_{m,n}$  such that  $u = k' - k$  and  $v = l' - l$ . We have  $k'\alpha + l'\beta < k\alpha + l\beta - w$  and  $k'\alpha' + l'\beta' > k\alpha' + l\beta' - w$ . Let  $q_1 = \lfloor \alpha k + \beta l \rfloor$ ,  $q_2 = \lfloor \alpha k' + \beta l' \rfloor$ ,  $q'_1 = \lfloor \alpha' k + \beta' l \rfloor$ ,  $q'_2 = \lfloor \alpha' k' + \beta' l' \rfloor$ .

Then  $PZ(\alpha, \beta, k, l) = \{q_1\}$  or  $\{q_1, q_1 + 1\}$  and  $PZ(\alpha', \beta', k, l) = \{q'_1\}$  or  $\{q'_1, q'_1 + 1\}$ . So, if  $q_1 \neq q'_1$  then  $PZ(\alpha, \beta, k, l) \neq PZ(\alpha', \beta', k, l)$  and thus  $\mathcal{C}_{m,n,\alpha,\beta} \neq \mathcal{C}_{m,n,\alpha',\beta'}$ . We have similarly  $\mathcal{C}_{m,n,\alpha,\beta} \neq \mathcal{C}_{m,n,\alpha',\beta'}$  if  $q_2 \neq q'_2$ .

So we can suppose now  $q_1 = q'_1$  and  $q_2 = q'_2$ . We have:

$$\begin{aligned} \langle k'\alpha + l'\beta \rangle &< \langle k\alpha + l\beta \rangle + q_1 - q_2 - w \\ \langle k'\alpha' + l'\beta' \rangle &> \langle k\alpha' + l\beta' \rangle + q_1 - q_2 - w \end{aligned}$$

as  $\langle x \rangle$  is always in  $[0, 1]$ , then we have  $q_1 - q_2 - w = 0$ , and so  $C_{k',l'}^{\alpha,\beta} > C_{k,l}^{\alpha,\beta}$  and  $C_{k',l'}^{\alpha',\beta'} < C_{k,l}^{\alpha',\beta'}$ .

We deduce that  $PZ_2(\alpha, \beta, k, l, k', l') = \{(q_1, q_2), (q_1 + 1, q_2)\}$  or  $\{(q_1, q_2), (q_1 + 1, q_2), (q_1 + 1, q_2 + 1)\}$  and  $PZ_2(\alpha', \beta', k, l, k', l') = \{(q_1, q_2), (q_1, q_2 + 1)\}$  or  $\{(q_1, q_2), (q_1, q_2 + 1), (q_1 + 1, q_2 + 1)\}$ . Thus,  $\mathcal{C}_{m,n,\alpha,\beta} \neq \mathcal{C}_{m,n,\alpha',\beta'}$  which proves the claim  $\square$

**Corollary 1.** *Let  $\mathcal{O}$  be a connected component of  $[0, 1]^2 \setminus E_{m,n}$ . Then  $\mathcal{O}$  is a convex polygon and if  $p_1, p_2, p_3$  are distinct vertexes of the polygon  $\mathcal{O}$  then*

1. for any point  $p \in \mathcal{O}$ ,  $\mathcal{C}_{m,n,p} = \mathcal{C}_{m,n,p_1} \cup \mathcal{C}_{m,n,p_2} \cup \mathcal{C}_{m,n,p_3}$  and
2. for any point  $p$  in the interior of the segment of vertexes  $p_1, p_2$ ,  $\mathcal{C}_{m,n,p} = \mathcal{C}_{m,n,p_1} \cup \mathcal{C}_{m,n,p_2}$ .

*Proof.* The function  $(\alpha, \beta) \mapsto \mathcal{C}_{m,n,\alpha,\beta}$  is constant on  $\mathcal{O}$ . By Theorem 2, for all  $w \in \mathcal{U}_{m,n}$ , the function  $(\alpha, \beta) \mapsto \text{freq}_{\alpha,\beta}(w)$  is affine on  $\overline{\mathcal{O}}$  and we conclude by using Proposition 3  $\square$

**Corollary 2.** *The number of  $(m, n)$ -cubes is in  $O((m+n)^2 m^3 n^3)$ .*

*Proof.* Any line of equation  $ux + vy + w = 0$  with  $|w| > |u| + |v|$  does not intersect the square  $[0, 1]^2$ , so  $E_{m,n}$  is composed of at most  $(2m+1)(2n+1)(2(m+n+1)+1) = f(m, n)$  lines. Thanks to Theorem 2 and Corollary 3 all the  $(m, n)$ -cubes appear in the vertices of the connected components of  $[0, 1]^2 \cap E_{m,n}$ . Each vertex is the intersection of two lines of  $E_{m,n}$  so there are at most  $f(m, n)^2$  such vertices. Each vertex corresponds to at most  $mn$   $(m, n)$ -cubes, so in total there are at most  $((2m+1)(2n+1)(2(m+n+1)+1))^2 mn$   $(m, n)$ -cubes, which proves the claim  $\square$

Corollary 2 gives an upper bound for the number of  $(m, n)$ -cubes. In the following, we will give a lower bound for this number.

**Definition 7.** Let  $m \in \mathbb{N}^*$

1. Let  $\alpha, \gamma \in [0, 1]$ . The set  $S(m, \alpha, \gamma) = \{(x, \lfloor \alpha x + \gamma \rfloor \mid x \in \llbracket 0, m-1 \rrbracket\}$  is called a digital segment of size  $m$ .
2.  $\mathcal{S}_m = \{S(m, \alpha, \gamma) \mid \alpha, \gamma \in [0, 1]\}$  is the set of all digital segments.

*Property 2.* [14,15]

1.  $\text{card}(\mathcal{S}_m) = 1 + \sum_{i=1}^m (m-i+1)\varphi(i)$  where  $\varphi$  is the Euler's totient function ( $\varphi(i) = \text{card}(\{j \mid 1 \leq j < i \text{ and } i \text{ and } j \text{ are co-prime}\})$ ).
2.  $\text{card}(\mathcal{S}_m) = \frac{m^3}{\pi^2} + O(m^2 \log(m))$ .

*Property 3.* Let  $m, n \in \mathbb{N}^*$  and  $\alpha, \beta, \gamma_1, \gamma_2 \in [0, 1]$  and consider the two digital segments  $S(m, \alpha, \gamma_1), S(n, \beta, \gamma_2)$ . Then, there exists  $w \in \mathcal{U}_{m,n}$  such that  $S(m, \alpha, \gamma_1) = \{(i, w(i, 0)) \mid i \in \llbracket 0, m-1 \rrbracket\}$  and  $S(n, \beta, \gamma_2) = \{(j, w(0, j)) \mid j \in \llbracket 0, n-1 \rrbracket\}$ .

*Proof.* By [15,5], there exist  $i, j \in \mathbb{Z}$  such that  $S(m, \alpha, \gamma_1) = \{(x-i, \lfloor \alpha x \rfloor - \lfloor \alpha i \rfloor) \mid x \in \llbracket i, m+i-1 \rrbracket\}$  and  $S(n, \beta, \gamma_2) = \{(x-j, \lfloor \beta y \rfloor - \lfloor \beta j \rfloor) \mid y \in \llbracket j, n+j-1 \rrbracket\}$ . Then  $w = w_{i,j}(\alpha, \beta, 0)$  verifies the conditions of the property  $\square$

**Corollary 3.**  $\text{card}(\mathcal{U}_{m,n}) \geq \text{card}(\mathcal{S}_m)\text{card}(\mathcal{S}_n)$ . So,  $\text{card}(\mathcal{U}_{m,n}) \geq \frac{1}{\pi^4} m^3 n^3 + O(m^2 n^2 \log(m) \log(n))$ .

*Proof.* By Property 3, we have  $\text{card}(\mathcal{U}_{m,n}) \geq \text{card}(\mathcal{S}_m)\text{card}(\mathcal{S}_n)$  and the second assertion is a direct consequence of Property 2  $\square$

Corollaries 2 and 3 imply that there exist two constant numbers  $k_1, k_2$  such that  $k_1 m^3 n^3 < \text{card}(\mathcal{U}_{m,n}) < k_2 (m+n)^2 m^3 n^3$ .

## 4 Application to local estimators

A digital surface is the discretization of a surface of  $\mathbb{R}^3$ . We investigate in this section the local estimators of the area of digital surface in the digital space  $r\mathbb{Z}^3$  of resolution  $r$ .

The local estimator of area is obtained by associating a weight  $p(w)$  to each pattern  $w \in \mathcal{M}(m, n)$  of size  $m \times n$  where  $\mathcal{M}(m, n)$  is the set of all  $m \times n$ -patterns, then any digital surface  $\mathcal{S}_r$ , can be obtained by concatenation of elements in  $\mathcal{M}(m, n)$  with perhaps a pattern  $\varepsilon_i, \varepsilon'_j$  of size less than  $m \times n$ . In other words  $\mathcal{S}_r$  can be viewed as a bi-dimensional word on the alphabet of patterns of size less or equal to  $m \times n$ . If

$$\mathcal{S}_r = \begin{array}{cccccc} w_{1,1} & w_{2,1} & \cdots & w_{M,1} & \varepsilon_1 \\ w_{1,2} & w_{2,2} & \cdots & w_{M,2} & \varepsilon_2 \\ \vdots & \vdots & \dots & \vdots & \vdots \\ w_{1,N} & w_{2,N} & \cdots & w_{M,N} & \varepsilon_N \\ \varepsilon'_1 & \varepsilon'_2 & \cdots & \varepsilon'_M & \varepsilon'_{M+1} \end{array}$$

where  $w_{i,j} \in \mathcal{M}(m,n)$  for all  $i,j$ , then we define the area of  $\mathcal{S}_r$  by  $\mathfrak{S}_{r,m,n,p}(\mathcal{S}_r) = r^2 \sum_{i,j} p(w_{i,j})$  (i.e. we neglect the contribution of the digital surfaces  $\varepsilon_i$  and  $\varepsilon'_j$ ).

Actually, we investigate the following problem:

Does there exist  $m,n$  and  $p(\cdot)$  such that for any surface  $\mathcal{S} \in \mathbb{R}^3$  the areas  $\mathfrak{S}_{r,m,n,p}(\mathcal{S}_r)$  converge to the area of  $\mathcal{S}$  where  $r$  tends to 0? (i.e.  $\mathcal{S}_r$  is a discretization of  $\mathcal{S}$ ).

In this section, we study this problem for a particular class of surfaces: the set of rectangular planar regions. Moreover we suppose that the discretization operator  $\delta_r$  restricted to these regions is the ‘‘Bresenham’’ discretization.

We consider the rectangular planar region  $R = \{(x,y, \alpha x + \beta y + \gamma) \mid a \leq x \leq b \text{ and } c \leq y \leq d\}$  such that  $0 \leq \alpha, \beta \leq 1$ , the other cases could be deduced by symmetry.

Its ‘‘Bresenham’’ discretization in  $r\mathbb{Z}^3$  is

$$R_r = r\{(x,y, \lfloor \alpha x + \beta y + \frac{\gamma}{r} \rfloor) \mid (x,y) \in \llbracket \lfloor \frac{a}{r} \rfloor, \lfloor \frac{b}{r} \rfloor \rrbracket \times \llbracket \lfloor \frac{c}{r} \rfloor, \lfloor \frac{d}{r} \rfloor \rrbracket\}$$

We fix  $m,n$  as a positive integers. As it has been explained for surfaces, the discrete region  $R_r$  can be seen as the bi-dimensional word

$$\begin{array}{cccccc} w_{1,1} & w_{2,1} & \cdots & w_{M,1} & \varepsilon_1 & \\ w_{1,2} & w_{2,2} & \cdots & w_{M,2} & \varepsilon_2 & \\ \vdots & \vdots & \cdots & \vdots & \vdots & \\ w_{1,N} & w_{2,N} & \cdots & w_{M,N} & \varepsilon_N & \\ \varepsilon'_1 & \varepsilon'_2 & \cdots & \varepsilon'_M & \varepsilon'_{M+1} & \end{array}$$

where  $M = \lfloor \frac{\lfloor \frac{b}{r} \rfloor - \lfloor \frac{a}{r} \rfloor + 1}{m} \rfloor$  and  $N = \lfloor \frac{\lfloor \frac{d}{r} \rfloor - \lfloor \frac{c}{r} \rfloor + 1}{n} \rfloor$  and for all  $i,j$ ,  $w_{i,j}$  is a pattern of size  $m \times n$  and  $\varepsilon_i, \varepsilon'_j$  are patterns of size less than  $m \times n$ .

We construct  $\mathfrak{S}_{r,m,n,p}$  as the local estimator of measure by using a weight function  $p : \mathcal{U}_{m,n} \rightarrow \mathbb{R}$ . Then  $\mathfrak{S}_{r,m,n,p}$  is defined by:

$$\begin{aligned} \mathfrak{S}_{r,m,n,p}(R_r) &= r^2 \sum_{i,j} p(w_{i,j}) \\ &= r^2 \sum_{w \in \mathcal{U}_{m,n}} n(w, R_r, r) p(w) \end{aligned}$$

Where  $n(w, R_r, r)$  is the number of  $(i,j) \in \llbracket 1, M \rrbracket \times \llbracket 1, N \rrbracket$  such that  $w_{i,j} = w$ .

The central question of this section can be formulated as the following, does there exist  $m,n$  and  $p(\cdot)$  such that, for any rectangular planar region  $R$ , the estimation  $\mathfrak{S}_{r,m,n,p}(R_r)$  converges to the area of  $R$  when the resolution  $r$  tends to 0?

We will prove in this section that the response is almost everywhere no.

Actually, we will prove that for almost all rectangular planar regions  $R$ , the estimation  $\mathfrak{S}_{r,m,n,p}(R_r)$  does not converge to the area of  $R$  when the resolution  $r$  tends to 0.

The result of this section is an extension of the results of [12] for estimating area of rectangular planar regions of  $\mathbb{R}^3$ .

Let  $a, b, c, d \in \mathbb{R}$  such that  $a < b$  and  $c < d$  and  $0 \leq \alpha, \beta \leq 1$ . Let  $r > 0$  be the resolution of the discrete space  $r\mathbb{Z}^3$ . Let  $R = \{(x, y, \alpha x + \beta y + \gamma) \mid a \leq x \leq b \text{ and } c \leq y \leq d\}$ . So the ‘‘Bresenham’’ discretization of  $R$  in  $r\mathbb{Z}^3$  is  $R_r = r\{(x, y, \lfloor \alpha x + \beta y + \frac{\gamma}{r} \rfloor) \mid (x, y) \in [\lfloor \frac{a}{r} \rfloor, \lfloor \frac{b}{r} \rfloor] \times [\lfloor \frac{c}{r} \rfloor, \lfloor \frac{d}{r} \rfloor]\}$ . Put  $M_r = \lfloor \frac{\lfloor \frac{b}{r} \rfloor - \lfloor \frac{a}{r} \rfloor + 1}{m} \rfloor$  and  $N_r = \lfloor \frac{\lfloor \frac{d}{r} \rfloor - \lfloor \frac{c}{r} \rfloor + 1}{n} \rfloor$ .

Then

$$R_r = \begin{matrix} w_{1,1} & w_{2,1} & \cdots & w_{M_r,1} & \varepsilon_{1,r} \\ w_{1,2} & w_{2,2} & \cdots & w_{M_r,2} & \varepsilon_{2,r} \\ \vdots & \vdots & \dots & \vdots & \vdots \\ w_{1,N_r} & w_{2,N_r} & \cdots & w_{M_r,N_r} & \varepsilon_{N_r,r} \\ \varepsilon'_{1,r} & \varepsilon'_{2,r} & \cdots & \varepsilon'_{M_r,r} & \varepsilon'_{M_r+1,r} \end{matrix}$$

where  $w_{i,j} \in \mathcal{U}_{m,n}$  for  $(i, j) \in \llbracket 1, N_r \rrbracket \times \llbracket 1, M_r \rrbracket$  and  $\varepsilon_{i,r}, \varepsilon'_{j,r}$  are of size less than  $m \times n$  for  $i \in \llbracket 1, N_r \rrbracket$  and  $j \in \llbracket 1, M_r + 1 \rrbracket$ .

Consider  $\mathfrak{S}_{r,m,n,p}(R_r) = r^2 \sum_{1 \leq i \leq N_r} \sum_{1 \leq j \leq M_r} p(w_{i,j})$  as an approximation of the area of the region  $R$  (i.e. we neglect the contributions of the  $\varepsilon_{i,r}$  and  $\varepsilon'_{j,r}$ ). Put  $DA_{m,n,p}(R) = \lim_{r \rightarrow 0} \mathfrak{S}_{r,m,n,p}(R_r)$ .

In all the following, put  $E_r = (\llbracket \lfloor \frac{a}{r} \rfloor, \lfloor \frac{b}{r} \rfloor \rrbracket \times \llbracket \lfloor \frac{c}{r} \rfloor, \lfloor \frac{d}{r} \rfloor \rrbracket) \cap ((m\mathbb{Z} + \lfloor \frac{a}{r} \rfloor) \times (n\mathbb{Z} + \lfloor \frac{c}{r} \rfloor))$  and  $S_r = (\lfloor \frac{b}{r} \rfloor - \lfloor \frac{a}{r} \rfloor + 1)(\lfloor \frac{d}{r} \rfloor - \lfloor \frac{c}{r} \rfloor + 1)$

**Definition 8.** The non-overlapping frequency  $F_r^{\alpha,\beta,\gamma,a,b,c,d}$  of a pattern  $w$  of size  $m \times n$  in  $R_r$  is defined by:

$$F_r^{\alpha,\beta,\gamma,a,b,c,d} = \frac{\text{card}(\{(x, y) \in E_r \mid w_{x,y}(\alpha, \beta, \frac{\gamma}{r}) = w\})}{S_r}$$

**Lemma 3.** Let  $\alpha, \beta \in [0, 1]$  such that  $\alpha$  or  $\beta$  is irrational,  $\gamma, a, b, c, d \in \mathbb{R}$ ,  $w \in \mathcal{C}_{m,n,\alpha,\beta}$ . Then

$$F^{\alpha,\beta,\gamma,a,b,c,d} = \lim_{r \rightarrow 0} F_r^{\alpha,\beta,\gamma,a,b,c,d} = \frac{1}{mn} \text{freq}_{\alpha,\beta}(w)$$

In particular  $F^{\alpha,\beta,\gamma,a,b,c,d}$  does not depend on  $\gamma, a, b, c$ , and  $d$ .

*Proof.*

$$\begin{aligned} F^{\alpha,\beta,\gamma,a,b,c,d} &= \lim_{r \rightarrow 0} \frac{\text{card}(\{(x, y) \in E_r \mid w_{x,y}(\alpha, \beta, \frac{\gamma}{r}) = w\})}{S_r} \\ &= \lim_{r \rightarrow 0} \frac{\text{card}(\{(x, y) \in E_r \mid (\alpha x + \beta y + \frac{\gamma}{r}) \in I^{\alpha,\beta}(w)\})}{S_r} \end{aligned}$$

So, if we take  $p = m$ ,  $q = n$ ,  $\gamma_r = \frac{1}{r}\gamma$  and  $I = I^{\alpha,\beta}(w)$  in Theorem 4 of the Appendix A, then we have  $F^{\alpha,\beta,\gamma,a,b,c,d} = \frac{1}{mn}\mu(I^{\alpha,\beta}(w)) = \frac{1}{mn}\text{freq}_{\alpha,\beta}(w)$  because by Proposition 2  $\text{overfreq}_{\alpha,\beta,\gamma}(w) = \mu(I^{\alpha,\beta}(w))$   $\square$

**Theorem 3.** *Let  $\mathcal{O}$  be a connected component of  $[0, 1]^2 \setminus E_{m,n}$ . Then there exist  $u, v, t \in \mathbb{R}$  such that  $DA_{m,n,p}(R) = (b-a)(d-c)(u\alpha + v\beta + t)$  for all rectangular planar regions  $R = \{(x, y, \alpha x + \beta y + \gamma) \mid a \leq x \leq b \text{ and } c \leq y \leq d\}$  such that  $\alpha, \beta \in \mathcal{O}$  and  $\alpha$  or  $\beta$  is irrational.*

*In other words,  $DA_{m,n,p}(\cdot)$  is an affine function in  $(\alpha, \beta)$  for  $(\alpha, \beta) \in (\mathcal{O} \setminus \mathbb{Q}^2)$ .*

*Proof.* We suppose that the weight function  $p : \mathcal{U}_{m,n} \rightarrow \mathbb{R}$  associates to each pattern  $w$  of size  $m \times n$  a weight  $p(w)$ . The digital region  $R_r$  can be seen as the bi-dimensional word:

$$R_r = \begin{array}{cccccc} w_{1,1} & w_{2,1} & \cdots & w_{M_r,1} & \varepsilon_{1,r} & \\ w_{1,2} & w_{2,2} & \cdots & w_{M_r,2} & \varepsilon_{2,r} & \\ \vdots & \vdots & \dots & \vdots & \vdots & \\ w_{1,N_r} & w_{2,N_r} & \cdots & w_{M_r,N_r} & \varepsilon_{N_r,r} & \\ \varepsilon'_{1,r} & \varepsilon'_{2,r} & \cdots & \varepsilon'_{M_r,r} & \varepsilon'_{M_r+1,r} & \end{array}$$

where  $M_r = \lfloor \frac{\lfloor \frac{b}{r} \rfloor - \lceil \frac{a}{r} \rceil + 1}{m} \rfloor$  and  $N_r = \lfloor \frac{\lfloor \frac{d}{r} \rfloor - \lceil \frac{c}{r} \rceil + 1}{n} \rfloor$ ,  $w_{i,j} \in \mathcal{U}_{m,n}$  for  $(i, j) \in \llbracket 1, N_r \rrbracket \times \llbracket 1, M_r \rrbracket$  and  $\varepsilon_{i,r}, \varepsilon'_{j,r}$  are of size less than  $m \times n$  for  $i \in \llbracket 1, N_r \rrbracket$  and  $j \in \llbracket 1, M_r + 1 \rrbracket$ .

So, the approximated area of the digital region  $R_r$  is :

$$\begin{aligned} \mathfrak{S}_{r,m,n,p}(R_r) &= r^2 \sum_{1 \leq i \leq M_r} \sum_{1 \leq j \leq N_r} p(w_{i,j}) \\ &= r^2 \sum_{w \in \mathcal{U}_{m,n}} n(w, R_r, r) p(w) \end{aligned}$$

where  $n(w, R_r, r) = \text{card}(\{(x, y) \in ((m\mathbb{Z} + \lceil \frac{a}{r} \rceil) \times (n\mathbb{Z} + \lceil \frac{c}{r} \rceil)) \cap (\llbracket \frac{a}{r} \rceil, \llbracket \frac{b}{r} \rceil] \times \llbracket \frac{c}{r} \rceil, \llbracket \frac{d}{r} \rceil]) \mid w_{x,y}(\alpha, \beta, \frac{\gamma}{r}) = w\}$  which is the number of occurrences of the pattern  $w$  in the bi-dimensional word  $R_r$ . So,

$$\begin{aligned} DA_{m,n,p}(R) &= \lim_{r \rightarrow 0} r^2 \sum_{w \in \mathcal{U}_{m,n}} n(w, R_r, r) p(w) \\ &= \lim_{r \rightarrow 0} r^2 S_r \sum_{w \in \mathcal{U}_{m,n}} \frac{n(w, R_r, r)}{S_r} p(w) \\ &= (b-a)(d-c) \sum_{w \in \mathcal{U}_{m,n}} \frac{1}{mn} \text{freq}_{\alpha,\beta}(w) p(w) \quad (\text{By Lemma 3}) \end{aligned}$$

So, according to Theorem 2,  $DA_{m,n,p}(\cdot)$  is an affine function in  $(\alpha, \beta)$  for  $(\alpha, \beta) \in (\mathcal{O} \setminus \mathbb{Q}^2)$   $\square$

**Corollary 4.** *The set of  $(\alpha, \beta) \in ([0, 1]^2 \setminus E_{m,n})$  such that  $\alpha$  or  $\beta$  is irrational and  $DA_{m,n,p}(R) = \text{area}(R)$  is a negligible (relatively to the Lebesgue measure on the euclidean space) where for  $a, b, c, d \in \mathbb{R}$ ,  $R = \{(x, y, \alpha x + \beta y + \gamma) \mid a \leq x \leq b \text{ and } c \leq y \leq d\}$ .*

*Proof.* We consider a connected component  $\mathcal{O}$  of  $[0, 1]^2 \setminus E_{m,n}$ . By Theorem 3, there exist  $u, v, t \in \mathbb{R}$  such that the estimated area of the rectangular planar region  $R$  is  $DA_{m,n,p}(R) = (b-a)(d-c)(u\alpha + v\beta + t)$  for  $\alpha$  or  $\beta$  is irrational. The exact area of  $R$  is  $\text{area}(R) = (b-a)(d-c)\sqrt{1 + \alpha^2 + \beta^2}$ . So we have:

$$DA_{m,n,p}(R) = \text{area}(R) \iff (u\alpha + v\beta + t)^2 = 1 + \alpha^2 + \beta^2$$

Which is equivalent to  $(u^2 - 1)\alpha^2 + (v^2 - 1)\beta^2 + 2(uv\alpha\beta + ut\alpha + vt\beta) + t^2 - 1 = 0$ . But, the last equation corresponds to an object of Lebesgue measure greater than 0 only when  $u^2 - 1 = 0$ ,  $v^2 - 1 = 0$ ,  $t^2 - 1 = 0$ ,  $uv = 0$ ,  $ut = 0$  and  $vt = 0$  which never happens. So, the last equation corresponds to a curve in  $\mathbb{R}^2$  (which is the intersection of conic and the region  $\mathcal{O}$ ) and thus, for  $(\alpha, \beta) \in \mathcal{O}$ , the estimated area can be equal to the exact area for only  $(\alpha, \beta)$  in a set included in the intersection of a conic and the region  $\mathcal{O}$  which corresponds to a negligible set.

But,  $[0, 1]^2 \setminus E_{m,n}$  contains only a finite number of connected components. Thus, the set of  $(\alpha, \beta) \in ([0, 1]^2 \setminus (E_{m,n} \cup \mathbb{Q}^2))$  such that the estimated area is equal to the exact area is a negligible set because it is a finite union of negligible sets  $\square$

**Corollary 5.** *For any  $m, n \in \mathbb{N}^*$  and any weight function  $p(\cdot)$  the set of  $(\alpha, \beta) \in [0, 1]^2$  such that the rectangular planar region  $R = \{(x, y, \alpha x + \beta y + \gamma) \mid a \leq x \leq b \text{ and } c \leq y \leq d\}$  (where  $\gamma, a, b, c, d \in \mathbb{R}$ ) satisfies  $\text{area}(R) = DA_{m,n,p}(R)$  is a negligible set. So, for any  $m, n \in \mathbb{N}^*$  and any weight function  $p(\cdot)$ , for all rectangular planar regions  $R$  with the parameters  $\alpha, \beta \in [0, 1]$ , we have  $\text{area}(R) \neq DA_{m,n,p}(R)$  almost everywhere.*

*Proof.* By Corollary 5, we have, for almost all rectangular planar regions  $R$  with parameters  $(\alpha, \beta) \in ([0, 1]^2 \setminus E_{m,n})$   $\text{area}(R) \neq DA_{m,n,p}(R)$ . But  $\mathbb{Q}^2$  is infinite countable set and  $E_{m,n}$  is a finite set of straight lines. So  $E_{m,n} \cup \mathbb{Q}^2$  is a negligible set. So, for all rectangular planar regions  $R$  with the parameters  $\alpha, \beta \in [0, 1]$ ,  $\text{area}(R) \neq DA_{m,n,p}(R)$  almost everywhere  $\square$

## 5 Conclusion

In this paper we have seen that the frequencies of the  $(m, n)$ -cubes of digital planes satisfy nice properties: they constitute a continuous piecewise affine function in the slopes of the digital planes. This has consequences on the combinatorics of  $(m, n)$ -cubes, in particular on the asymptotic behavior of the number of  $(m, n)$ -cubes when  $m$  and  $n$  tend to infinity.

Moreover it has also consequences on local estimators of area as it permits to prove rigorously that any local estimator of area is never multigrid-convergent: for almost all region of plane it does not converge to the true area. This result

is a generalization of a result in dimension two proved in [12]. Actually we can prove with the same technics that this result is true for the equivalent notions for any finite dimension.

## A Appendix: Frequencies for irrational parameters

In all the following  $\alpha, \beta \in \mathbb{R}$  such that  $\alpha$  or  $\beta \in (\mathbb{R} \setminus \mathbb{Q})$ ,  $\gamma : \mathbb{R}^* \rightarrow \mathbb{R}^*$  (i.e. In the following we note  $\gamma_r$  for  $\gamma(r)$ ),  $a, b, c, d \in \mathbb{R}$  such that  $a < b$  and  $c < d$  and  $i, j \in \mathbb{N}$ .

Put  $E_r = (\llbracket \lceil \frac{a}{r} \rceil, \lfloor \frac{b}{r} \rfloor \rrbracket \times \llbracket \lceil \frac{c}{r} \rceil, \lfloor \frac{d}{r} \rfloor \rrbracket) \cap ((p\mathbb{Z} + \lceil \frac{a}{r} \rceil) \times (q\mathbb{Z} + \lceil \frac{c}{r} \rceil))$  and  $S_r = (\lfloor \frac{b}{r} \rfloor - \lceil \frac{a}{r} \rceil + 1)(\lfloor \frac{d}{r} \rfloor - \lceil \frac{c}{r} \rceil + 1)$  and consider the linear form :

$l_{r;p,q,a,b,c,d} : L^1([0, 1]) \rightarrow \mathbb{R}$  such that for all  $f \in L^1([0, 1])$

$$l_{r;p,q,a,b,c,d}(f) = \frac{1}{S_r} \sum_{(x,y) \in E_r} f((\alpha x + \beta y + \gamma_r)).$$

**Lemma 4.** Let  $h \in \mathbb{Z}^*$  and put  $e_h : t \mapsto e^{2\pi hit}$ . Then

$$\lim_{r \rightarrow 0} l_{r;p,q,a,b,c,d}(e_h) = 0$$

*Proof.*

$$\begin{aligned} l_{r;p,q,a,b,c,d}(e_h) &= \frac{1}{S_r} \sum_{\substack{x \in (p\mathbb{Z} + \lceil \frac{a}{r} \rceil), x \in \llbracket \lceil \frac{a}{r} \rceil, \lfloor \frac{b}{r} \rfloor \rrbracket \text{ and} \\ y \in (q\mathbb{Z} + \lceil \frac{c}{r} \rceil), y \in \llbracket \lceil \frac{c}{r} \rceil, \lfloor \frac{d}{r} \rfloor \rrbracket}} e_h(\alpha x + \beta y + \gamma_r). \\ &= \frac{1}{S_r} e_h(\gamma_r) \left( \sum_{\substack{x \in (p\mathbb{Z} + \lceil \frac{a}{r} \rceil) \text{ and} \\ x \in \llbracket \lceil \frac{a}{r} \rceil, \lfloor \frac{b}{r} \rfloor \rrbracket}} e_h(\alpha x) \right) \left( \sum_{\substack{y \in (q\mathbb{Z} + \lceil \frac{c}{r} \rceil) \text{ and} \\ y \in \llbracket \lceil \frac{c}{r} \rceil, \lfloor \frac{d}{r} \rfloor \rrbracket}} e_h(\beta y) \right). \end{aligned}$$

Assume that  $\alpha \in (\mathbb{R} \setminus \mathbb{Q})$ . Then

$$|l_{r;p,q,a,b,c,d}(e_h)| \leq \frac{1}{(\lfloor \frac{b}{r} \rfloor - \lceil \frac{a}{r} \rceil + 1)(\lfloor \frac{d}{r} \rfloor - \lceil \frac{c}{r} \rceil + 1)} F^{|1 - e_h((E+1)h\alpha)|} \text{ where } E = \lfloor \frac{\lfloor \frac{b}{r} \rfloor - \lceil \frac{a}{r} \rceil}{p} \rfloor + 1 \text{ and } F = \lfloor \frac{\lfloor \frac{d}{r} \rfloor - \lceil \frac{c}{r} \rceil}{q} \rfloor + 1$$

$$\text{So } |l_{r;p,q,a,b,c,d}(e_h)| \leq \frac{1}{q} \left( 1 + \frac{q-1}{\lfloor \frac{d}{r} \rfloor - \lceil \frac{c}{r} \rceil + 1} \right) \frac{1}{\lfloor \frac{b}{r} \rfloor - \lceil \frac{a}{r} \rceil + 1} \frac{2}{|1 - e_h(p\alpha)|}.$$

$$\text{Thus } \lim_{r \rightarrow 0} l_{r;p,q,a,b,c,d}(e_h) = 0 \text{ because } \lim_{r \rightarrow 0} \frac{q-1}{\lfloor \frac{d}{r} \rfloor - \lceil \frac{c}{r} \rceil + 1} = 0,$$

$$\lim_{r \rightarrow 0} \frac{1}{\lfloor \frac{b}{r} \rfloor - \lceil \frac{a}{r} \rceil + 1} = 0 \text{ and } |1 - e_h(p\alpha)| \neq 0 \text{ because } \alpha \text{ is irrational.}$$

We have the same conclusion if  $\beta$  is irrational  $\square$

**Lemma 5.**

$$\lim_{r \rightarrow 0} l_{r;p,q,a,b,c,d}(e_0) = \frac{1}{pq}$$

*Proof.*  $l_{r;p,q,a,b,c,d}(e_0) = \frac{EF}{(\lfloor \frac{b}{r} \rfloor - \lceil \frac{a}{r} \rceil + 1)(\lfloor \frac{d}{r} \rfloor - \lceil \frac{c}{r} \rceil + 1)}$   
 $= \frac{1}{pq} (1 + \frac{p+\varepsilon_1}{\lfloor \frac{b}{r} \rfloor - \lceil \frac{a}{r} \rceil + 1}) (1 + \frac{q+\varepsilon_2}{\lfloor \frac{d}{r} \rfloor - \lceil \frac{c}{r} \rceil + 1})$  where  $|\varepsilon_1|, |\varepsilon_2| \leq 1$  (i.e. Look to the proof of Lemma 4 for the definition of  $E$  and  $F$ ).  
Thus,  $\lim_{r \rightarrow 0} l_{r;p,q,a,b,c,d}(e_0) = \frac{1}{pq}$   $\square$

**Proposition 4 ([12]).** Let  $f$  be a continuous function from  $[0, 1]$  to  $\mathbb{C}$ . Then

$$\lim_{r \rightarrow 0} l_{r;p,q,a,b,c,d}(f) = \frac{1}{pq} \int_0^1 f(t) dt$$

**Lemma 6 ([12]).** Let  $\mathcal{M}_r$  ( $r \in \mathbb{R}^*$ ) and  $\mathcal{M}$  be linear positive functionals on some space  $\mathcal{F}$  of real-valued function  $f : X \rightarrow \mathbb{R}$  ( $X \neq \emptyset$ ) and let  $\mathcal{L} \subseteq \mathcal{F}$  be the subspace of these functions satisfying the property

$$\lim_{r \rightarrow 0} \mathcal{M}_r(f) = \mathcal{M}(f).$$

Suppose that  $f \in \mathcal{F}$  has the property that every  $\varepsilon > 0$  there exist functions  $g_1, g_2 \in \mathcal{L}$  with  $g_1 \leq f \leq g_2$  and  $\mathcal{M}(g_2) - \mathcal{M}(g_1) < \varepsilon$ . Thus  $f \in \mathcal{L}$ .

**Theorem 4.** Let  $I \subseteq [0, 1]$  be an interval. Then

$$\lim_{r \rightarrow 0} \frac{\text{card}(\{(x,y) \in E_r \mid \langle \alpha x + \beta y + \gamma_r \rangle \in I\})}{S_r} = \frac{1}{pq} \mu(I)$$

*Proof.*  $\frac{\text{card}(\{(x,y) \in E_r \mid \langle \alpha x + \beta y + \gamma_r \rangle \in I\})}{S_r} = \frac{1}{S_r} \sum_{(x,y) \in E_r} \mathcal{X}_I(\langle \alpha x + \beta y + \gamma_r \rangle)$   
 $= l_{r;p,q,a,b,c,d}(\mathcal{X}_I)$  where  $\mathcal{X}_I$  is the characteristic function of  $I$  (i.e.  $\mathcal{X}_I(t) = 1$  if  $t \in I$  and 0 otherwise).

But  $I$  is an interval thus for every  $\varepsilon > 0$  there exist two continuous functions  $g_1, g_2$  on  $[0, 1]$  such that  $g_1 \leq \mathcal{X}_I \leq g_2$  and  $\frac{1}{pq} \int_0^1 (g_2(t) - g_1(t)) dt < \varepsilon$ . Thus

$$\lim_{r \rightarrow 0} \frac{\text{card}(\{(x,y) \in E_r \mid \langle \alpha x + \beta y + \gamma_r \rangle \in I\})}{S_r} = \frac{1}{pq} \mu(I) \quad \square$$

**Corollary 6.** Let  $I \subseteq [0, 1]$  be an interval. Then

$$\lim_{N \rightarrow \infty} \frac{\text{card}(\{(x,y) \in \llbracket -N, N \rrbracket^2 \mid \langle \alpha x + \beta y + \gamma \rangle \in I\})}{(2N+1)^2} = \mu(I)$$

*Proof.* Let  $N \in \mathbb{N}^*$ ,  $\alpha, \beta \in \mathbb{R}$  such that  $\alpha$  or  $\beta \in (\mathbb{R} \setminus \mathbb{Q})$ ,  $\gamma \in \mathbb{R}$  (i.e.  $\gamma_r = \gamma$  for all  $r \in \mathbb{R}^*$ ) and let  $I \subseteq [0, 1]$  be an interval. Then

$$l_{\frac{1}{N};1,1,-1,1,-1,1}(\mathcal{X}_I) = \frac{\text{card}(\{(x,y) \in \llbracket -N, N \rrbracket^2 \mid \langle \alpha x + \beta y + \gamma \rangle \in I\})}{(2N+1)^2}$$

So

$$\lim_{N \rightarrow \infty} \frac{\text{card}(\{(x,y) \in \llbracket -N, N \rrbracket^2 \mid \langle \alpha x + \beta y + \gamma \rangle \in I\})}{(2N+1)^2} = \mu(I) \quad \square$$

**Corollary 7.** *Let  $I \subseteq [0, 1]$  be an interval and  $p, q \in \mathbb{N}^*$ . Then*

$$\lim_{N \rightarrow \infty} \frac{\text{card}(\{(x, y) \in \llbracket -N, N \rrbracket^2 \cap (p\mathbb{Z} \times q\mathbb{Z}) \mid \langle \alpha x + \beta y + \gamma \rangle \in I\})}{(2N + 1)^2} = \frac{1}{pq} \mu(I)$$

*Proof.* Let  $N \in \mathbb{N}^*$ ,  $\alpha, \beta \in \mathbb{R}$  such that  $\alpha$  or  $\beta \in (\mathbb{R} \setminus \mathbb{Q})$ ,  $\gamma \in \mathbb{R}$  (i.e.  $\gamma_r = \gamma$  for all  $r \in \mathbb{R}^*$ ),  $p, q \in \mathbb{N}^*$  and let  $I \subseteq [0, 1]$  be an interval. Then

$$l_{\frac{1}{N}; p, q, -1, 1, -1, 1}(\mathcal{X}_I) = \frac{\text{card}(\{(x, y) \in \llbracket -N, N \rrbracket^2 \cap (p\mathbb{Z} \times q\mathbb{Z}) \mid \langle \alpha x + \beta y + \gamma \rangle \in I\})}{(2N + 1)^2}$$

So

$$\lim_{N \rightarrow \infty} \frac{\text{card}(\{(x, y) \in \llbracket -N, N \rrbracket^2 \cap (p\mathbb{Z} \times q\mathbb{Z}) \mid \langle \alpha x + \beta y + \gamma \rangle \in I\})}{(2N + 1)^2} = \frac{1}{pq} \mu(I) \quad \square$$

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