Strong convergence of the gradient in non-linear parabolic equations

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Abstract

We consider the Cauchy-Dirichlet Problem for a non-linear parabolic equation with L^1 data. We show how the concept of kinetic formulation for conservation laws [LPT94] can be be used to give a new proof of the existence of renormalized solution. To illustrate this approach, we also extend the result to the case where the equation involves a term with natural growth.

We consider the question of existence of solution to the non-linear parabolic problem

$$\begin{cases} u_t - \operatorname{div}(a(\nabla u)) = f & (x,t) \in \Omega \times (0,T), \\ u(x,0) = u_0(x) & x \in \Omega, \\ u(0,t) = 0 & (x,t) \in \Sigma \end{cases} \tag{1}$$

where Ω is a bounded subset of $\mathbb{R}^N$, $N \geq 1$, T is positive and $\Sigma = \partial\Omega \times (0,T)$. Let $p > 1$ be given. In (1), the operator $-\operatorname{div}(a(\nabla u))$ is a Leray-Lions operator of the type $-\Delta_p u$:

Assumption 1 *The function $a \in \mathcal{C}(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^N)$ satisfies: there exists $\alpha > 0$, $\beta \in C(\mathbb{R}_+, \mathbb{R}_+)$ such that*

$$a(X) \cdot X \geq \alpha |X|^p, \tag{2}$$

$$|a(X)| \leq \beta(|\xi|)|X|^{p-1}, \tag{3}$$

$$(a(X) - a(Y)) \cdot (X - Y) > 0 \tag{4}$$

for all distinct $X, Y \in \mathbb{R}^N$, where $X \cdot Y$ is the canonical scalar product of two vectors of $\mathbb{R}^N$ and $|X|$ the associated euclidean norm of X.

The framework is L^1:

Assumption 2 *The data u_0, f are L^1 functions on Ω and $\Omega \times (0,T)$ respectively.*

1 Introduction

The existence of solution (precisely, of renormalized solution, see Definition 1 below) to Problem (1) (actually even to more general problems than Problem (1)) has already been proved, we refer in particular to the paper by Blanchard, Murat, Redwane [BMR01]. Our purpose here is to give a new proof of this fact. Actually, the cornerstone in the proof of existence of solution (by means of a process of approximation) of such a non-linear parabolic Problem as (1) is the proof of the strong convergence of the gradient. We give a new method (inspired from the kinetic formulation of conservation laws developed by Perthame and coauthors [LPT94, Per02, CP03]) to prove this result.

Let us briefly summarize how and in which context the question of strong convergence of the gradient occurs. First (historically), as soon as the problem under consideration involves a non-linear function of the gradient, for example (under the hypotheses above with $p = 2$), the non-linear elliptic Problem

$$-\operatorname{div}(a(\nabla u)) = g \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega,$$

for $g \in L^2(\Omega)$. To prove existence of a solution (in $H^1(\Omega)$), it is usual to prove existence by approximation (e.g. by Galerkin approximation), i.e. for a set of data g_n converging to g. Then weak convergence in H^1 of (a subsequence of) u_n, the solution with datum g_n, although easily obtained by uniform estimate on $\|u_n\|_{H^1(\Omega)}$, is not enough to pass to the limit in the equation since a is non-linear: one has to prove the *strong convergence* of the gradient ∇u_n. This is done by use of monotonicity methods. We refer to [Min63, Bro63, LL65], and [Eva98] for a very brief and clear explaination of the technique.

Non-linear expression of the gradient also occurs after renormalization of an elliptic or parabolic equation. Actually, it occurs even if the original equation is linear. Nevertheless, renormalization for elliptic or parabolic equation has been introduced to deal with non-linear equations with data of low regularity, so that the renormalized equation involves (at least) two non-linear expressions of the gradient (see, e.g. Eq. (6) below). At any rate, it will be necessary to prove the strong convergence of the (truncates of) the gradient in order to get existence of a solution by approximation.

We give a new proof of the strong convergence of the gradient by use of an equation on the characteristic function on the level sets of the unknown, similar to the kinetic formulation for conservation laws introduced in [LPT94] (see also [Per02] and [CP03] concerning the kinetic formulation of second-order conservation laws). We intend to use it to study certain systems of reaction-diffusion equations (a forthcoming paper).

Let us conclude this introduction by few words about the concept of renormalized solutions. Introduced by DiPerna and Lions for the study of ordinary differential equations and Boltzmann Equation [DL89b, DL89a], it has been extended to non-linear elliptic equations in [BGDM93] (in parallel with the (equivalent) notion of entropy solution [BBG+95]) and has been extended to non-linear parabolic equations in [Bla93, BMR01, Lio96] (in parallel with the (equivalent) notion of entropy solution [Pri97]). It has also been extended to first-order conservation laws [BCW00, PV03].
The problem of strong convergence of the gradient (hence the question of existence of solution) has initially be solved by the method of Minty-Browder and Leray-Lions [Min63, Bro63, LL65], then extended to the case of non-linear elliptic (then parabolic) equations with less and less regular data by several methods, see, e.g. [BG92a, BM92, BGM93, BGDM93, DMMOP97, BDGO99, DMMOP99, BMR01, BP05]. Notice that our list of references to works in the fields of renormalized solutions for elliptic and parabolic equations is far from being complete.

The paper is organized as follows : in Section 2.1, we introduce the notion of renormalized solution and state the equivalent formulation by the (so-called) level-set P.D.E. In Section 2.2, we analyze this formulation and explain how it can be relaxed (although still characterizing renormalized solutions), see Theorem 2 and Theorem 3. In Section 2.3, we apply our tools to prove the convergence of an approximation to Problem (1) and thus existence of a renormalized solution to (1) (of course, we focus on the strong convergence of the gradient). In Section 3, we give the proofs of various results, which were reported at the end to let the main arguments of Section 2 stand out. Eventually, in Section 4, we extend the method to prove the existence of a renormalized solution to the Cauchy-Dirichlet Problem for a non-linear parabolic equation with a term with natural growth.

Notations : We set $Q_T := \Omega\times(-1,T)$ and $U_T := Q_T\times\mathbb{R}$. Any measurable function $v\colon \Omega\times(0,T)\to\mathbb{R}^m$ is implicitly extended to a measurable function $Q_T\to\mathbb{R}^m$ still denoted by v, with $v\equiv 0$ on $\Omega\times(-1,0)$. If ν is a Radon measure on U_T, we denote by $\widetilde{\nu}$ the measure on the whole space $\mathbb{R}^{N+2}$

$$\widetilde{\nu}(E) = \nu(U_T\cap E),\quad E\in\mathcal{B}(\mathbb{R}^{N+2})$$

and we denote by $\widetilde{\nu}_*$ be the projection of $\widetilde{\nu}$ on $\mathbb{R}_\xi$:

$$\widetilde{\nu}_*(E) = \widetilde{\nu}(\mathbb{R}^{N+1}\times E),\quad \forall E\in\mathcal{B}(\mathbb{R}).$$

2 Existence of a renormalized solution - strong convergence of the gradient

2.1 Renormalized solutions

2.1.1 Renormalized solutions

For $k>0$, we let $T_k(u)$ be the truncate of a function u at level k: $T_k(u):=\min(u,k)$ if $u\geq 0$, T_k odd.

Definition 1 *A function $u\in L^\infty(0,T;L^1(\Omega))$ is said to be a renormalized solution of the problem (1) if*

1. *(Regularity of the truncates)*

$$T_k(u)\in L^p(0,T;W_0^{1,p}(\Omega)),\quad \forall k>0, \tag{5}$$

2. *(Renormalized equation) For every function $S\in W^{2,\infty}(\mathbb{R})$ such that S' has compact support, the following equation holds in the sense of distributions in Q_T:*

$$S(u)_t-\operatorname{div}(S'(u)a(\nabla u))=S(u_0)\otimes\delta_{t=0}+S'(u)f-S''(u)a(\nabla u)\cdot\nabla u, \tag{6}$$

3. *(Recovering at infinity)*

$$\lim_{k\to+\infty}\int_{Q_T\cap\{k<|u|<k+1\}}a(\nabla u)\cdot\nabla u dxdt=0. \tag{7}$$

2.1.2 Level-set P.D.E.

For $\alpha\in\mathbb{R}$, $\xi\in\mathbb{R}$, we set $\chi_\alpha(\xi)=\mathbf{1}_{0<\xi<\alpha}-\mathbf{1}_{\alpha<\xi<0}$. This is the "equilibrium function" in the kinetic formulation of conservation laws.

Theorem 1 *A function $u\in L^\infty(0,T;L^1(\Omega))$ is a renormalized solution of the problem (1) if, and only if, it has the regularity of the truncates (5) and it satisfies*

1. *(Level-set P.D.E.) The function $(x,t,\xi)\mapsto\chi_{u(x,t)}(\xi)$, denoted by χ_u, is solution in $\mathcal{D}'(U_T)$ of the equation*

$$\partial_t\chi_u-\operatorname{div}(a(\nabla u)\delta_{u=\xi})=\chi_{u_0}\otimes\delta_{t=0}+f\delta_{u=\xi}+\partial_\xi\mu, \tag{8}$$

where μ is defined by

$$\mu:=a(\nabla u)\cdot\nabla u\delta_{u=\xi}, \tag{9}$$

2. *(Recovering at infinity)*

$$\lim_{k\to+\infty}\int_{Q_T\cap\{k<|u|<k+1\}}a(\nabla u)\cdot\nabla u dxdt=0.$$

Proof of Theorem 1 : see Section 3.1.

2.1.3 Definition of the distributions $a(\nabla u)\delta_{u=\xi}$ and μ

The (vector-valued) distribution

$$a(\nabla u)\delta_{u=\xi}$$

is defined by its restriction to each space $\mathcal{D}_K(U_T)^N$ (the set of smooth vector-valued functions with support in the compact subset K of U_T) as

$$\langle a(\nabla u)\delta_{u=\xi},\alpha\rangle=\int_Q a(\nabla T_k(u))\cdot\alpha(x,t,T_k(u))dxdt \tag{10}$$

where $\alpha \in \mathcal{D}_K(U_T)^N$, $K \subset Q_T \times [-k,k]$. Similarly, Eq. (9), which defines the distribution μ, means that

$$\langle \mu, \alpha \rangle = \int_Q a(\nabla T_k(u)) \cdot \nabla T_k(u) \alpha(x,t,T_k(u)) dxdt \tag{11}$$

for all $\alpha \in \mathcal{D}_K(U_T)$.
By (5) and assumption (3), we have

$$\begin{aligned} |\langle a(\nabla u)\delta_{u=\xi}, \alpha\rangle| &\leq \beta(k)\|T_k(u)\|_{L^{p-1}(0,T;W_0^{1,p-1}(\Omega))}\|\alpha\|_{L^\infty(K)} \\ &\leq \beta(k)\|T_k(u)\|_{L^p(0,T;W_0^{1,p}(\Omega))}\|\alpha\|_{L^\infty(K)} \end{aligned}$$

and

$$|\langle \mu, \alpha\rangle| \leq \beta(k)\|T_k(u)\|_{L^p(0,T;W_0^{1,p}(\Omega))}\|\alpha\|_{L^\infty(K)}.$$

This shows that the right-hand sides of (10) and (11) are distributions on U_T of order 0. To prove that (10) and (11) makes sense, we must also show that their respective right-hand sides do not depend on the choice of k: suppose $k < k'$ for example, with $K \subset Q_T \times [-k,k]$, then $\alpha(x,t,T_{k'}(u)) \neq 0$ for $|u| \leq k$ only, in which case $T_k(u) = T_{k'}(u)$.

2.2 Relaxation of the definition of renormalized solution - analysis of μ

2.2.1 Analysis of μ

Since $\mu \geq 0$, μ is represented by a non-negative Radon measure on U_T (equality on $C_c(U_T)$, thus in $\mathcal{D}'(U_T)$). We study the properties of the Radon measure $\widetilde{\mu}_*$ (defined at the end of the introduction).

Fact 1. For every $h \in C_c(\mathbb{R})$,

$$\int_{\mathbb{R}} h(\xi) d\widetilde{\mu}_*(\xi) = \int_{\mathbb{R}^{N+2}} h(\xi) d\widetilde{\mu}(x,t,\xi) \tag{12}$$

Proof: by definition of $\widetilde{\mu}_*$, (12) is satisfied if $h = \mathbf{1}_E$ is the characteristic function of a Borel set $E \subset \mathbb{R}$, and therefore if h is a simple function. There exists a point-wise converging sequence of simple functions with limit h with the same compact support than h: the Lebesgue dominated convergence Theorem gives the result. ∎

Fact 2. For every $h \in C_c(\mathbb{R})$ with, say, $\text{supp}(h) \subset [-k,k]$,

$$\int_{\mathbb{R}} h(\xi) d\widetilde{\mu}_*(\xi) = \int_{Q_T} a(\nabla T_k(u)) \cdot \nabla T_k(u) h(u) dxdt. \tag{13}$$

Proof: let (φ_n) be a non-negative sequence of $C_c(Q_T)$ such that $\varphi_n \uparrow 1$ everywhere on Q_T. By definition of μ, we have

$$\int_{\mathbb{R}^{N+2}} \varphi_n(x,t) h(\xi) d\widetilde{\mu}(x,t,\xi) = \int_{Q_T} a(\nabla T_k(u)) \cdot \nabla T_k(u) \varphi_n(x,t) h(u) dxdt.$$

The Lebesgue dominated convergence Theorem then gives, at the limit $[n \to +\infty]$,

$$\int_{U_T} h(\xi) d\widetilde{\mu}(x,t,\xi) = \int_{Q_T} a(\nabla T_k(u)) \cdot \nabla T_k(u) h(u) dxdt.$$

Since, by definition, $\widetilde{\mu}$ is supported in U_T, we have

$$\int_{\mathbb{R}^{N+2}} h(\xi) d\widetilde{\mu}(x,t,\xi) = \int_{Q_T} a(\nabla T_k(u)) \cdot \nabla T_k(u) h(u) dxdt$$

and we conclude by (12). ■

Fact 3. The measure $\widetilde{\mu}_*$ has no atom.

Proof: Given $k>0$, set $v=T_k(u)$. For $\xi_*\in(-k,k)$, let (h_n) be a sequence of $C_c(-k,k)$ converging monotonically to $\mathbf{1}_{\{\xi_*\}}$ (take the h_n to be tent functions for example). For every n, we have, by (13),

$$\int_{\mathbb{R}} h_n(\xi)d\widetilde{\mu}_*(\xi)=\int_{Q_T} a(\nabla v)\cdot\nabla v h_n(v)dxdt.$$

At the limit $[n\to+\infty]$, we obtain, by the Lebesgue dominated convergence Theorem,

$$\widetilde{\mu}_*(\{\xi_*\})=\int_{Q_T} a(\nabla v)\cdot\nabla v\mathbf{1}_{\{\xi_*\}}(v)dxdt,$$

and this last quantity is 0 by the Stampacchia's Theorem since, for a.e. t, $v(t)$ is in the Sobolev space $W^{1,p}$. ■

Fact 4. For every $l>k$,

$$\int_{\mathbb{R}} \mathbf{1}_{(k,l)}(\xi)d\widetilde{\mu}_*(\xi)=\int_{Q_T\cap\{k<u<l\}} a(\nabla u)\cdot\nabla u dxdt. \tag{14}$$

Proof: In the right hand-side of (14), u stands for $T_l(u)$. Let (h_n) be a sequence of $C_c(k,l)$ such that $h_n\uparrow 1$ on (k,l). For each n, we have by (13),

$$\int_{\mathbb{R}} h_n(\xi)d\widetilde{\mu}_*(\xi)=\int_{Q_T} a(\nabla u)\cdot\nabla u h_n(u)dxdt$$

where u stands for $T_l(u)$. At the limit $[n\to+\infty]$, the monotone convergence Theorem gives the result. ■

Fact 5. For $\varphi\in C_c(Q_T)$, $\varphi\geq 0$, define

$$\mu_\varphi(A):=\int_A \varphi(x,t)d\mu(x,t,\xi).$$

The measure μ_φ has the same properties than μ and its analysis follows the same lines. In particular, $\widetilde{\mu}_{\varphi,*}$ has no atoms and, for every $k>0$,

$$\widetilde{\mu}_{\varphi,*}([-k,k])=\widetilde{\mu}_{\varphi,*}((-k,k))=\int_{Q_T} a(\nabla T_k(u))\cdot\nabla T_k(u)\varphi(x,t)dxdt. \tag{15}$$

2.2.2 Relaxation of the definition of renormalized solution, Theorem 2

From Theorem 1 and the analysis of μ follows the following characterization of renormalized solutions.

Proposition 1 *Let u be a function of $L^\infty(0,T;L^1(\Omega))$ which has the regularity of the truncates (5). Define*

$$\mu:=a(\nabla u)\cdot\nabla u\delta_{u=\xi}.$$

Then u is a renormalized solution of the problem (1) if, and only if, it satisfies the equation

$$\partial_t\chi_u-\operatorname{div}(a(\nabla u)\delta_{u=\xi})=\chi_{u_0}\otimes\delta_{t=0}+f\delta_{u=\xi}+\partial_\xi\mu,$$

and if

$$\lim_{k\to\pm\infty}\widetilde{\mu}_*((k,k+1))=0. \tag{16}$$

However, a weaker form of definition can be given, as stated in the following theorem.

Theorem 2 *Let u be a function of $L^\infty(0,T;L^1(\Omega))$ which has the regularity of the truncates (5) and satisfies the condition at infinity (7). Then u is a renormalized solution of the problem (1) if, and only if, there exists a non-negative Radon measure μ on U_T such that*

$$\lim_{k\to\pm\infty}\widetilde{\mu}_*((k,k+1))=0$$

and such that the following equation is satisfied in $\mathcal{D}'(U_T)$

$$\partial_t\chi_u-\operatorname{div}(a(\nabla u)\delta_{u=\xi})=\chi_{u_0}\otimes\delta_{t=0}+f\delta_{u=\xi}+\partial_\xi\mu. \tag{17}$$

Proof of Theorem 2 : see Section 3.2.

The proof of this Theorem consists in showing that $\mu=a(\nabla u)\cdot\nabla u\delta_{u=\xi}$. It is therefore a result of structure of μ: under the hypotheses of Theorem 2, μ *has to* be the measure $a(\nabla u)\cdot\nabla u\delta_{u=\xi}$. We generalize this result in the following paragraph.

2.2.3 Structure of μ, Theorem 3

Theorem 3 *Let u be a function of $L^\infty(0,T;L^1(\Omega))$ which has the regularity of the truncates (5). Let σ be a measurable function $\Omega\times(0,T)\to\mathbb{R}^N$ such that*

$$\forall k>0,\sigma\mathbf{1}_{|u|<k}\in L^{p'}(Q_T)^N$$

and suppose that σ and u satisfy the condition at infinity

$$\lim_{k\to\pm\infty}\int_{Q_T\cap[k<|u|<k+1]}\sigma\cdot\nabla u dxdt=0.$$

Suppose that u is a quasi-renormalized solution of the problem (1) in the sense that there exists a non-negative Radon measure μ on U_T such that

$$\lim_{k\to\pm\infty}\widetilde{\mu}_*((k,k+1))=0$$

and such that the following equation is satisfied in $\mathcal{D}'(U_T)$

$$\partial_t\chi_u-\operatorname{div}(\sigma\delta_{u=\xi})=\chi_{u_0}\otimes\delta_{t=0}+f\delta_{u=\xi}+\partial_\xi\mu. \tag{18}$$

Then we have $\mu=\sigma\cdot\nabla u\delta_{u=\xi}$.

The proof of Theorem 3 is exactly the same as the proof of Theorem 2, it has just to be generalized in the case where the vector-valued function $a(\nabla u)$ is replaced by the vector-valued function σ. This parallel analysis applies in priority to the definition of the distribution $\sigma\cdot\nabla u\delta_{u=\xi}$ which is comparable to (11):

$$\langle\sigma\cdot\nabla u\delta_{u=\xi},\alpha\rangle=\int_Q(\sigma\mathbf{1}_{|u|<k})\cdot\nabla T_k(u)\alpha(x,t,T_k(u))dxdt$$

for all $\alpha\in\mathcal{D}_K(U_T)$, K compact subset of $Q_T\times[-k,k]$.
In the situation described by Theorem 3, and once the equality $\mu=\sigma\cdot\nabla u\delta_{u=\xi}$ has been proved, we can mimic the analysis of Section 2.2.1, to deduce the following result.

Corollary 1 *Under the hypotheses of Theorem 3, and given $\varphi\in C_c(Q_T)$, $\varphi\geq 0$, the measure $\widetilde{\mu}_{\varphi,*}$ has no atom and, for all $k>0$,*

$$\widetilde{\mu}_{\varphi,*}([-k,k])=\widetilde{\mu}_{\varphi,*}((-k,k))=\int_{Q_T}\sigma\cdot\nabla T_k(u)\varphi(x,t)dxdt. \tag{19}$$

2.3 Existence of a renormalized solution - Strong convergence of the gradient

2.3.1 Approximation

Let (u_0^n) and (f^n) be some approximating sequences of, respectively, u_0 and f in, respectively, $L^1(\Omega)$ and $L^1(\Omega\times(0,T))$ such that $u_0^n\in L^p(\Omega)$, $f^n\in L^{p'}(\Omega\times(0,T))$. For each n, the problem

$$\left\{\begin{array}{ll} u_t^n-\operatorname{div}(a(\nabla u^n))=f^n & (x,t)\in\Omega\times(0,T),\\ u^n(x,0)=u_0^n(x) & x\in\Omega,\\ u^n(0,t)=0 & (x,t)\in\Sigma,\end{array}\right. \tag{20}$$

has a unique solution u^n in the space of functions $v\in L^p(0,T;W_0^{1,p}(\Omega))$ with $v_t\in L^{p'}(0,T;W^{-1,p}(\Omega))$. The function u^n is a weak solution to (20), hence a renormalized solution and therefore satisfies the equation

$$\partial_t\chi_{u^n}-\operatorname{div}(a(\nabla u^n)\delta_{u^n=\xi})=\chi_{u_0^n}\otimes\delta_{t=0}+f^n\delta_{u^n=\xi}+\partial_\xi\mu^n, \tag{21}$$

where μ^n is defined by

$$\mu^n:=a(\nabla u^n)\cdot\nabla u^n\delta_{u^n=\xi}. \tag{22}$$

2.3.2 Estimates and limit equation

There are bounds independent on n on u^n in $L^\infty(0,T;L^1(\Omega))$, on $a(\nabla T_k(u^n))\cdot\nabla T_k(u^n)$ in $L^1(Q_T)$, on $\nabla T_k(u^n)$ in $L^p(Q_T)$, on $T_k(u^n)_t$ in $L^{p'}(0,T;W^{-1,p}(\Omega))+L^1(\Omega\times(0,T))$. They are obtained by multiplying the equation by $T_k(u^n)$ (see, e.g., [BMR01]).
Aubin-Simon's compactness Theorem shows that there exists a function $u\in L^\infty(0,T;L^1(\Omega))$ such that, up to a subsequence, $u^n\to u$ a.e. and in $L^1(Q_T)$. By weak compactness of L^p and $L^{p'}$, we can suppose that $T_k(u^n)\to T_k(u)$ in $L^p(0,T;W_0^{1,p}(\Omega))$-weak and $a(\nabla T_k(u^n))\to\sigma_k$ in $L^{p'}(Q_T)$-weak. Actually, by use of a diagonal process, there exists a supplementary subsequence still denoted (u^n) and a measurable function $\sigma\colon\Omega\times(0,T)\to\mathbb{R}^n$ such that

$$\forall k>0, \sigma\mathbf{1}_{|u|<k}\in L^{p'}(Q_T),\quad a(\nabla T_k(u^n))\to\sigma\mathbf{1}_{|u|<k}\text{ in }L^{p'}(Q_T)\text{ weak.}$$

The bound on $a(\nabla T_k(u^n))\cdot\nabla T_k(u^n)$ in $L^1(Q_T)$ gives a uniform bound on $\widetilde{\mu}^n(K)$ for each compact subset K of $\mathbb{R}^{N+2}$. We can therefore suppose that $(\widetilde{\mu}^n)$ converges weakly to a Radon measure $\overline{\mu}$ on $\mathbb{R}^{N+2}$. In particular, μ^n converges in $\mathcal{D}'(U_T)$ to the measure μ defined as the restriction of $\overline{\mu}$ to U_T.
With these results of convergence at hand, we pass to the limit $[n\to+\infty]$ in (21) to obtain the equation

$$\partial_t\chi_u-\operatorname{div}(\sigma\delta_{u=\xi})=\chi_{u_0}\otimes\delta_{t=0}+f\delta_{u^n=\xi}+\partial_\xi\mu, \tag{23}$$

By multiplying the equation of (20) by the function $\min(1,(u^n-(k+1))^-,(u^n+k+1)^+)$ $(k>0)$ we obtain the estimate

$$\int_{Q_T\cap\{k<|u^n|<k+1\}}a(\nabla u^n)\cdot\nabla u^n dxdt\le\int_{\Omega\cap\{|u_0^n|>k\}}|u_0^n|dx+\int_{Q_T\cap\{|u^n|>k\}}|f^n|dxdt.$$

This is

$$\widetilde{\mu}_*^n((k,k+1))+\widetilde{\mu}_*^n((-k-1,-k))\le\int_{\Omega\cap\{|u_0^n|>k\}}|u_0^n|dx+\int_{Q_T\cap\{|u^n|>k\}}|f^n|dxdt.$$

Up to a subsequence (and as a consequence of the strong convergence in L^1), there exists some functions U_0,U,F in $L^1(\Omega)$ and $L^1(U_T)$ respectively such that $|u_0^n|\le U_0$, $|u^n|\le U$, $|f^n|\le F$ a.e. This implies the uniform estimates

$$\widetilde{\mu}_*^n((k,k+1))+\widetilde{\mu}_*^n((-k-1,-k))\le\int_{\Omega\cap\{U_0>k\}}U_0dx+\int_{Q_T\cap\{U>k\}}Fdxdt$$

and

$$\left|\int_{Q_T\cap[k<|u|<k+1]}\sigma\cdot\nabla u dxdt\right|\leq\int_{\Omega\cap\{U_0>k\}}U_0dx+\int_{Q_T\cap\{U>k\}}Fdxdt$$

from which we deduce (since $\mathbb{R}^{N+1}\times(k,k+1)$ is open and thus $\overline{\mu}_*((k,k+1))\leq\liminf_{n\to+\infty}\widetilde{\mu}^n_*((k,k+1))$):

$$\overline{\mu}_*((k,k+1))+\widetilde{\mu}^n_*((-k-1,-k))\leq\int_{\Omega\cap\{U_0>k\}}U_0dx+\int_{Q_T\cap\{U>k\}}Fdxdt \tag{24}$$

and

$$\lim_{k\to\pm\infty}\int_{Q_T\cap[k<|u|<k+1]}\sigma\cdot\nabla u dxdt=0.$$

2.3.3 Strong convergence of the gradient

Since μ is defined to be the restriction to $\overline{\mu}$ to U_T, we have $0\leq\widetilde{\mu}\leq\overline{\mu}$ and, in particular, $0\leq\widetilde{\mu}_*\leq\overline{\mu}_*$. Therefore, from the estimate (24), we deduce that $\widetilde{\mu}_*(k,k+1)$ tends to 0 when $k\to\pm\infty$. Taking into account the equation (23), we are now in position to apply Theorem 3, which gives

$$\mu:=\sigma\cdot\nabla u\delta_{u=\xi}$$

in $\mathcal{D}'(U_T)$.
Now, we fix a test-function $\varphi\in C_c(Q_T)$, $\varphi\geq 0$. We use the notations of Section 2.2.1: if $\psi\in C_c(\mathbb{R}^{N+2})$, then $\psi\varphi\in C_c(\mathbb{R}^{N+2})$, therefore

$$\int_{\mathbb{R}^{N+2}}\psi(x,t,\xi)\varphi(x,t)d\widetilde{\mu}^n(x,t,\xi)\to\int_{\mathbb{R}^{N+2}}\psi(x,t,\xi)\varphi(x,t)d\overline{\mu}(x,t,\xi),$$

that is to say $(\widetilde{\mu}^n_\varphi)$ weakly converges to $\overline{\mu}_\varphi\colon A\mapsto\int_A\varphi(t,x)d\overline{\mu}(x,t,\xi)$. We deduce that $(\widetilde{\mu}^n_{\varphi,*})$ weakly converges to $\overline{\mu}_{\varphi,*}$. Indeed, if E is an open subset of $\mathbb{R}$, then $\mathbb{R}^{N+1}\times E$ is a open subset of $\mathbb{R}^{N+2}$, therefore

$$\overline{\mu}_{\varphi,*}(E)=\overline{\mu}_\varphi(\mathbb{R}^{N+1}\times E)\leq\liminf_{n\to+\infty}\widetilde{\mu}^n_\varphi(\mathbb{R}^{N+1}\times E)=\liminf_{n\to+\infty}\widetilde{\mu}^n_{\varphi,*}(E),$$

which proves the result. Besides, since φ is supported in Q_T, we have $\widetilde{\mu}_{\varphi,*}=\overline{\mu}_{\varphi,*}$ as easily checked, and thus the weak convergence of $(\widetilde{\mu}^n_{\varphi,*})$ to $\widetilde{\mu}_{\varphi,*}$. Given $k>0$, the identity (19) show that $\widetilde{\mu}_{\varphi,*}$ does not charge the boundary of $[-k,k]$. The weak convergence thus gives

$$\widetilde{\mu}^n_{\varphi,*}([-k,k])\to\widetilde{\mu}_{\varphi,*}([-k,k]).$$

By (15) and (19), this is

$$\int_{Q_T}a(\nabla T_k(u^n))\cdot\nabla T_k(u^n)\varphi dxdt\to\int_{Q_T}\sigma\cdot\nabla T_k(u)\varphi dxdt. \tag{25}$$

This proves the strong convergence of the gradient (see Section 3.4) and in particular the fact that $\sigma=a(\nabla u)$ a.e. on Q_T. In particular u is solution to the level-set p.d.e. associated to the problem (1):

$$\partial_t\chi_u-\operatorname{div}(a(\nabla u)\delta_{u=\xi})=\chi_{u_0}\otimes\delta_{t=0}+f\delta_{u^n=\xi}+\partial_\xi\mu.$$

Notice that we have also established the properties of the measure μ : it is a Radon measure on U_T which satisfies $\lim_{k\to\pm\infty}\widetilde{\mu}_*(k,k+1)=0$. By Theorem 2, u is a renormalized solution of Problem (1).

Remark 1 (Neuman Boundary conditions) *The existence of a renormalized solution to the Problem 1 with homogeneous Neumann boundary conditions instead of homogeneous Dirichlet boundary conditions can also be proved by use of the level-set PDE. This time extend u by 0 not only for negative times, but also to $\mathbb{R}^N\setminus\Omega$ and work with $Q_T:=\mathbb{R}^N\times(-1,T)$, $U_T:=Q_T\times\mathbb{R}$.*

3 Missing proofs

3.1 Proof of Theorem 1

By density of the set $\{\varphi\otimes\theta;\varphi\in\mathcal{D}(Q_T),\theta\in\mathcal{D}(\mathbb{R})\}$ in $\mathcal{D}(U_T)$, (8) is equivalent to:

$$\langle\partial_t\chi_u-\operatorname{div}(a(u,\nabla u)\delta_{u=\xi})\theta\rangle_{\mathcal{D}'(\mathbb{R}_\xi),\mathcal{D}(\mathbb{R}_\xi)}=\langle\chi_{u_0}\otimes\delta_{t=0}+f\delta_{u=\xi}+\partial_\xi\mu,\theta\rangle_{\mathcal{D}'(\mathbb{R}_\xi),\mathcal{D}(\mathbb{R}_\xi)}$$

in $\mathcal{D}'(Q_T)$ for all $\theta\in\mathcal{D}'(\mathbb{R})$. By definition of μ, this is the condition

$$\partial_t\int_{\mathbb{R}}\chi_u\theta d\xi-\operatorname{div}(\theta(u)a(u,\nabla u))=\left(\int_{\mathbb{R}}\chi_{u_0}\theta d\xi\right)\otimes\delta_{t=0}+\theta(u)f-\theta'(u)a(u,\nabla u)\cdot\nabla u \tag{26}$$

in $\mathcal{D}'(Q_T)$ for all $\theta\in\mathcal{D}'(\mathbb{R})$. By an argument of density, it appears that the set of conditions (8) is equivalent to the set of conditions (26) for θ belonging to $W^{1,\infty}(\mathbb{R})$ with compact support. We then conclude the proof of the equivalence between the renormalized equations and the level-set P.D.E. by using $S(u)=\displaystyle\int_0^u\theta(s)ds$, or conversely, $\theta=S'$, and by use of the identity

$$\int_{\mathbb{R}}\chi_u(\xi)S'(\xi)d\xi=S(u)-S(0),$$

which is satisfied for all $S\in W^{2,\infty}(\mathbb{R})$ such that S' has a compact support. ■

3.2 Proof of Theorem 2

Set $\nu:=a(\nabla u)\cdot\nabla u\delta_{u=\xi}$. We will show that $\mu=\nu$ in $\mathcal{D}'(U_T)$, which is the desired result, in view of Proposition 1. By density of the set $\{\varphi\otimes\psi\}$ with φ running in $\mathcal{D}(Q_T)$, ψ running in $\mathcal{D}(\mathbb{R})$ in the set $\mathcal{D}(U_T)$, we have to check that $\langle\mu,\varphi\otimes\psi\rangle=\langle\nu,\varphi\otimes\psi\rangle$. We first suppose that $\psi=\partial_\xi\theta$ with $\theta\in\mathcal{D}(\mathbb{R})$, so that $\langle\mu,\varphi\otimes\psi\rangle=-\langle\partial_\xi\mu,\varphi\otimes\theta\rangle$. Explicit computations on the basis of (17) then show that $\langle\mu,\varphi\otimes\psi\rangle=\langle\nu,\varphi\otimes\psi\rangle$ is equivalent to the following identity

$$-\int_0^T\int_\Omega\left(\int_{u_0}^u\theta(\xi)d\xi\right)\varphi_t+\int_0^T\int_\Omega(a(\nabla u)\cdot\nabla\varphi)\theta(u)-\int_0^T\int_\Omega f\varphi\theta(u)d\xi$$
$$=-\int_0^T\int_\Omega(a(\nabla u)\cdot\nabla u)\varphi\theta'(u).$$

By use of the rule of derivation of a product of functions in $W^{1,p}\cap L^\infty$, we obtain the more compact form

$$-\int_0^T\int_\Omega\left(\int_{u_0}^u\theta(\xi)d\xi\right)\varphi_t+\int_0^T\int_\Omega a(\nabla u)\cdot\nabla(\varphi\theta(u))-\int_0^T\int_\Omega f\varphi\theta(u)d\xi=0. \tag{27}$$

Eq. (27) can be *formally* deduced from the chain-rule formula and from the equation

$$0=\partial_t u-\operatorname{div}(a(\nabla u))-u_0\otimes\delta_{t=0}-f. \tag{28}$$

To explain for the following steps of the proof, we also remark that the equation (28) is formally deduced from Eq. (17) by integrating it from $\xi=-\infty$ to $+\infty$; indeed, that $\mu(\xi)\to0$ when $\xi\to\pm\infty$ is, still at the formal level, a consequence of the condition $\widetilde{\mu}_*((k,k+1))\to0$ when $k\to\pm\infty$. Therefore, we first derive an approximation of the equation (28): fix $k>0$, let $(\rho_n)_n$ be an approximation of the unit on $\mathbb{R}$ (ρ_n having compact support in $[-1/n,1/n]$), set $\alpha_k:=\rho_k*(\mathbf{1}_{[k,k+1]}+\mathbf{1}_{[-k-1,-k]})$, and define

$$r^k=r^k(u)=\int_{|u|}^\infty\alpha_k,\quad v^k:=\int_{\mathbb{R}}\chi_u(\xi)r_k(\xi)d\xi,\quad v_0^k:=\int_{\mathbb{R}}\chi_{u_0}(\xi)r_k(\xi)d\xi.$$

We have $v^k \in L^p(-1,T;W_0^{1,p}(\Omega)) \cap L^\infty(Q_T)$, $v_0^k \in L^\infty(\Omega)$ and $v^k \to u$, $v_0^k \to u_0$, $r^k \to 1$ when k tends to $+\infty$. Test Eq. (17) against $\varphi(t,x)r_k(\xi)$ to obtain

$$-\int_0^T\int_\Omega (v^k - v_0^k)\varphi_t + \int_0^T\int_\Omega a(\nabla u)\cdot\nabla\varphi r^k - \int_0^T\int_\Omega f\varphi r^k = \int_0^T\int_\Omega\int_{\mathbb{R}} \varphi\alpha_k d\mu. \tag{29}$$

This is the approximate form of (28). Now we want to use a kind of chain-rule formula to obtain an approximation of (27). To this purpose, we first infer from (29) the inequality

$$\left|\int_{Q_T}\varphi_t(v^k - v_0^k) - \int_0^T \langle G^k,\varphi\rangle dt\right| \le \|\varphi\|_{L^\infty}\varepsilon_k, \tag{30}$$

where $G^k := -(\mathrm{div}(a(\nabla u)r_k(u)) + fr_k(u)) \in L^{p'}(0,T;W^{-1,p'}(\Omega)) + L^1(Q)$ and $\varepsilon_k := \widetilde{\mu}_*((k-1,k+2)) + \widetilde{\mu}_*((-k-2,-k+1)) \to 0$ when $k \to +\infty$. We then consider the following lemma.

Lemma 1 *Let $\varepsilon > 0$, $v \in L^p(0,T;W_0^{1,p}(\Omega)) \cap L^1(Q_T)$, $v_0 \in L^\infty(\Omega)$ and*

$$G \in L^{p'}(0,T;W^{-1,p'}(\Omega)) + L^1(Q_T)$$

satisfy

$$\left|\int_0^T\int_\Omega \varphi_t(v - v_0) - \int_0^T\langle G,\varphi\rangle dt\right| \le \|\varphi\|_{L^\infty}\varepsilon, \tag{31}$$

for all $\varphi \in \mathcal{D}(Q_T)$. Then, for all $h \in W^{1,\infty}(\mathbb{R})$ we have

$$\left|\int_0^T\int_\Omega \varphi_t\int_{v_0}^v h(\xi)d\xi - \int_0^T\langle G,h(v)\varphi\rangle dt\right| \le \|\varphi\|_{L^\infty}\|h\|_{L^\infty}\varepsilon. \tag{32}$$

The proof of Lemma 1 is given in the following section. We apply the Lemma to (30), with $h(v) = \theta(v)$ to deduce

$$\left|\int_0^T\int_\Omega\left(\int_{v_0^k}^{v^k}\theta(\xi)d\xi\right)\varphi_t - \int_0^T\int_\Omega a(\nabla u)r_k(u)\cdot\nabla\left(\varphi\theta(v^k)\right) + \int_0^T\int_\Omega fr_k(u)\varphi\theta(v^k)\right| \le \|\varphi\otimes\theta\|_{L^\infty}\varepsilon_k.$$

By use of the Lebesgue dominated convergence Theorem, we obtain (27) at the limit $k \to +\infty$. Recall that $\psi = \partial_\xi\theta$, so that we actually proved $\partial_\xi(\mu - \nu) = 0$. By a classical Lemma in the theory of distributions, this shows $\mu - \nu$ is constant with respect to ξ (or, more precisely, that for every $\varphi \in \mathcal{D}(Q_T)$ the distribution on $\mathbb{R}$ defined by $\psi \mapsto \langle\mu - \nu, \varphi\otimes\psi\rangle$ is represented by a constant c_φ). The conditions at infinity on μ and ν then show that $c_\varphi = 0$. This being true for every φ, we have $\mu = \nu$. ■

3.3 Proof of Lemma 1

It is a variation on the proof of Lemma 4.3 in [CW99] (Lemma 4.3 of [CW99] corresponds to the case $\varepsilon = 0$).

Step 1. Suppose that v_0 additionally satisfies $v_0 \in W_0^{1,p}(\Omega)$. For $t < 0$, set $v(t) = v_0$.

Also first suppose h is non-increasing and φ non-negative or h is non-decreasing and φ non-positive. We have

$$-\|\varphi\|_{L^\infty}\varepsilon \le \int_0^T\int_\Omega\varphi_t(v - v_0) - \int_0^T\langle G,\varphi\rangle dt \le \|\varphi\|_{L^\infty}\varepsilon \tag{33}$$

for all $\varphi \in \mathcal{D}(Q_T)$ and thus, by regularity of v, G, for all $\varphi \in L^p(-1,T;W_0^{1,p}(\Omega)) \cap L^\infty(Q_T)$ with $\varphi_t \in L^{p'}(Q_T)$. To use the function $h(v)$ as a test-function in (33), we have first to regularize its dependence on t: for fixed $\varphi \in \mathcal{D}^+(Q_T)$ and for $\eta > 0$ small enough (such that $\text{supp}(\varphi) \subset \Omega \times (-1, T-2\eta]$, we set $\zeta := \varphi h(v)$,

$$\varphi_\eta \colon (x,t) \to \frac{1}{\eta}\int_{t-\eta}^{t} \zeta(x,s)ds.$$

In (33), this gives

$$\begin{aligned}
\int_0^T \langle G, \varphi_\eta\rangle dt &\leq \|\varphi\|_{L^\infty}\|h\|_{L^\infty}\varepsilon + \int_0^T\int_\Omega (\varphi_\eta)_t(v-v_0)\\
&= \|\varphi\|_{L^\infty}\|h\|_{L^\infty}\varepsilon + \int_0^T\int_\Omega \frac{1}{\eta}(\zeta(x,t)-\zeta(x,t-\eta))(v-v_0)(x,t)dxdt\\
&= \|\varphi\|_{L^\infty}\|h\|_{L^\infty}\varepsilon + \int_{\mathbb{R}}\int_\Omega \frac{1}{\eta}(\zeta(x,t)-\zeta(x,t-\eta))(v-v_0)(x,t)dxdt\\
&= \|\varphi\|_{L^\infty}\|h\|_{L^\infty}\varepsilon + \int_{\mathbb{R}}\int_\Omega \frac{1}{\eta}(v(x,t)-v(x,t+\eta))\zeta(x,t)dxdt\\
&= \|\varphi\|_{L^\infty}\|h\|_{L^\infty}\varepsilon + \int_{\mathbb{R}}\int_\Omega \frac{1}{\eta}(v(t)-v(t+\eta))h(v(t))\varphi(t)dxdt.
\end{aligned}$$

Since h is non-increasing and φ non-negative or h is non-decreasing and φ non-positive, we have the inequality

$$(v(t)-v(t+\eta))h(v(t))\varphi(t) \leq \int_{v(t)}^{v(t+\eta)} h(r)dr\varphi(t), \quad t < T$$

and deduce

$$\begin{aligned}
\int_0^T \langle G, \varphi_\eta\rangle dt &\leq \|\varphi\|_{L^\infty}\|h\|_{L^\infty}\varepsilon + \int_{\mathbb{R}}\int_\Omega \varphi(t)\frac{1}{\eta}\int_{v(t)}^{v(t+\eta)} h(r)dr\\
&= \|\varphi\|_{L^\infty}\|h\|_{L^\infty}\varepsilon + \int_{\mathbb{R}}\int_\Omega \frac{1}{\eta}(\varphi(t)-\varphi(t-\eta))\int_{v_0}^{v(t)} h(r)dr.
\end{aligned}$$

At the limit $\eta \to 0$, a first inequality is obtained

$$\int_0^T \langle G, h(v)\varphi\rangle dt \leq \|\varphi\|_{L^\infty}\|h\|_{L^\infty}\varepsilon + \int_0^T\int_\Omega \varphi_t \int_{v_0}^{v(t)} h(r)dr.$$

By use of $\varphi_\eta \colon (x,t) \to \frac{1}{\eta}\int_t^{t+\eta} \zeta(x,s)ds$ as a test-function, we derive in a similar way the second inequality

$$\int_0^T \langle G, h(v)\varphi\rangle dt \geq -\|\varphi\|_{L^\infty}\|h\|_{L^\infty}\varepsilon + \int_0^T\int_\Omega \varphi_t \int_{v_0}^{v(t)} h(r)dr$$

which gives (32). In case h is non-decreasing and φ non-negative or h is non-increasing and φ non-positive, proceed similarly (just exchanging the order of the different time-regularizations) to prove (32), then decompose h as the sum of two monotone functions and φ as the sum of two signed functions to deduce the result in the general case.

Step 2. In the general case where $v_0 \in L^\infty(Q)$, regularize v_0 by v_0^n, $v_0^n \in W_0^{1,p}(\Omega)$, $\|v_0 - v_0^n\|_{L^1(\Omega)} \leq 1/n$. Observe that, from (31), we deduce

$$\left|\int_0^T\int_\Omega \varphi_t(v-v_0^n) - \int_0^T \langle G,\varphi\rangle dt\right| \leq \|\varphi\|_{L^\infty}(\varepsilon + 1/n). \tag{34}$$

Apply Step 1. to get

$$\left|\int_0^T \int_\Omega \varphi_t \int_{v_0^n}^{v} h(\xi)d\xi - \int_0^T \langle G, h(v)\varphi\rangle dt\right| \leq \|\varphi\|_{L^\infty}\|h\|_{L^\infty}(\varepsilon + 1/n),$$

then pass to the limit $[n \to +\infty]$ to achieve the proof of Lemma 1.

3.4 Proof of the strong convergence of the gradient

We start from (25) and prove the strong convergence of the gradient by the arguments of Minty, Browder and Leray, Lions [Bro63, Min63, LL65]. Let $\varphi \in C_c(\Omega \times (0,T))$, $\varphi \geq 0$ be given. Consider the sum

$$\int_{Q_T} (a(\nabla T_k(u^n)) - a(\nabla T_k(u))) \cdot (\nabla T_k(u^n) - \nabla T_k(u))\varphi dxdt.$$

We develop the product in this last term. The result (25) yields precisely the convergence of the term with quadratic dependence on $\nabla T_k(u^n)$. The other terms have at most a linear dependence on $\nabla T_k(u^n)$ and therefore have a limit when $n \to +\infty$ by weak convergence of the gradient or of $a(\nabla T_k(u^n))$. We obtain

$$\lim_{n\to+\infty} \int_{Q_T} (a(\nabla T_k(u^n)) - a(\nabla T_k(u))) \cdot (\nabla T_k(u^n) - \nabla T_k(u))\varphi dxdt = 0.$$

Since $F_n := (a(\nabla T_k(u^n)) - a(\nabla T_k(u))) \cdot (\nabla T_k(u^n) - \nabla T_k(u))\varphi$ is non-negative (by monotony of a), this shows that $F_n \to 0$ in $L^1(Q_T)$. A subsequence of (F_n) (still denoted (F_n)) therefore converges to 0 on a set A of full measure in Q_T. Let $(x,t) \in A$ and let q be an adherence value of $(\nabla T_k(u^n))_n$ in $\overline{\mathbb{R}}^N$. Suppose that $\varphi(x,t) > 0$. The vector q has finite-valued components as a consequence of the growth of $a(\nabla T_k(u^n)) \cdot \nabla T_k(u^n)$, which gives

$$(\alpha|\nabla T_k(u^n)(x,t)|^p - C|\nabla T_k(u^n)(x,t)|)\varphi(x,t) \leq F_n(x,t) \to 0.$$

At the limit $[n \to +\infty]$ in $F_n(x,t) \to 0$, we thus obtain

$$(a(q) - a(\nabla T_k(u)(x,t))) \cdot (q - \nabla T_k(u)(x,t))\varphi(x,t) = 0$$

Since $\varphi(x,t)$ is supposed to be positive, and by strict monotony of a, we deduce that $q = \nabla T_k(u)(x,t)$. The sequence $(\nabla T_k(u^n))_n$ has only one possible adherence value and therefore converges to it. Since φ is arbitrary, we have $\nabla T_k(u^n) \to \nabla T_k(u)$ a.e. on Q_T. Together with the uniform bound on $\nabla T_k(u^n)$ in $L^p(Q_T)$, it gives the strong convergence of $\nabla T_k(u^n)$ to $\nabla T_k(u)$ in any $L^r(Q_T)$, $r < p$. Similarly, $a(\nabla T_k(u^n))$ converges to $a(\nabla T_k(u))$ a.e. and in $L^r(Q_T)$, $r < p'$. In particular, $\sigma = a(\nabla u)$ a.e.

To conclude, notice that we can recover the strong convergence $\nabla T_k(u^n) \to \nabla T_k(u)$ in $L^p_{\mathrm{loc}}(Q_T)$. Indeed, we have shown that $(a(\nabla T_k(u^n)))$ and $\nabla T_k(u^n)$ converge in $L^1(Q_T)$ to $a(\nabla T_k(u))$ and $\nabla T_k(u)$ respectively. Besides, for every $\varphi \in C_c(\Omega \times (0,T))$, we have the convergence

$$(a(\nabla T_k(u^n)) - a(\nabla T_k(u))) \cdot (\nabla T_k(u^n) - \nabla T_k(u))\varphi \to 0$$

in $L^1(Q_T)$, and these three results of convergence combine to show that $(a(\nabla T_k(u^n)) \cdot \nabla T_k(u^n)\varphi)$ converges (to $a(\nabla T_k(u)) \cdot \nabla T_k(u)\varphi$) in $L^1(Q_T)$. In particular, the family $\{a(\nabla T_k(u^n)) \cdot \nabla T_k(u^n)\varphi\}$ is equi-integrable on Q_T. By hypothesis (2), $|\nabla T_k(u^n)|^p$ is dominated by $\alpha^{-1}a(\nabla T_k(u^n))\cdot\nabla T_k(u^n)$ and, therefore, $\{|\nabla T_k(u^n)|^p\varphi\}$ is equi-integrable on Q_T. Given K a compact subset of Q_T, $\{|\nabla T_k(u^n)|^p\}$ is therefore equi-integrable on K. Together with the convergence $\nabla T_k(u^n) \to \nabla T_k(u)$ a.e., and by Vitali's Theorem, this implies $|\nabla T_k(u^n)|^p \to |\nabla T_k(u)|^p$ in $L^1(K)$. Besides, the weak convergence $\nabla T_k(u^n) \to \nabla T_k(u)$ in $L^p(K)$-weak together with the convergence $\nabla T_k(u^n) \to \nabla T_k(u)$ a.e. implies that

$$\lim_{n\to+\infty}(\|\nabla T_k(u^n)\|_{L^p(K)} - \|\nabla T_k(u^n) - \nabla T_k(u)\|_{L^p(K)}) = \|\nabla T_k(u)\|_{L^p(K)}$$

(this is a refinement of Fatou's Lemma by Brezis and Lieb [BL83]). We conclude that

$$\lim_{n\to+\infty}\|\nabla T_k(u^n) - \nabla T_k(u)\|_{L^p(K)} = 0.$$

4 Parabolic equation with a term with natural growth

In this Section, we briefly indicate how to adapt the arguments and proofs given above to solve the question of the strong convergence of the gradient (and therefore prove the existence of a renormalized solution) in the approximation by regularization and truncation of the following problem:

$$\left\{\begin{array}{ll} u_t - \mathrm{div}(a(\nabla u)) + \gamma(u)|\nabla u|^p = f & (x,t) \in \Omega \times (0,T), \\ u(x,0) = u_0(x) & x \in \Omega, \\ u(0,t) = 0 & (x,t) \in \Sigma. \end{array}\right. \tag{35}$$

We keep the same assumptions on a and on the data: assumptions (1) and (2). The function $\gamma \in C(\mathbb{R})$ is supposed to satisfies the sign condition

$$u\gamma(u) \geq 0, \quad \forall u \in \mathbb{R}. \tag{36}$$

This sign condition ensures good *a priori* estimates for the additional term $\gamma(u)|\nabla u|^p$. Actually, since the power p is the same as the power of the operator $-\mathrm{div}(a(\nabla))$, there will be an *a priori* estimates on $\gamma(u)|\nabla T_k(u)|^p$ in $L^1(U_T)$. In fact, we may as well consider a term as $\gamma(u)|\nabla u|^r$ with $1 \leq r \leq p$.

Numerous works have been devoted to the study of Problem (35) (or to its elliptic version). Let us cite in particular [BMP83, BMP89, BG92b, BGM93, Por00, SdL03] and references therein.

In case $p = 2$, $a = \mathrm{Id}$, there is a change of variables that transforms the equation in a classical Heat Equation:

$$v_t - \Delta v = g, \quad v = \int_0^u e^{-\int_0^\xi \gamma} d\xi, \quad g = f e^{-\int_0^u \gamma}.$$

It is this change of variables that we will adapt to the non-linear case by use of the kinetic formulation (or level-set PDE).
A renormalized solution to (35) is defined as a function $u \in L^\infty(0,T;L^1(\Omega))$ having the regularity of the truncates $T_k(u) \in L^p(0,T;W_0^{1,p}(\Omega))$, $\forall k > 0$, which satisfies the renormalized equation: for every function $S \in W^{2,\infty}(\mathbb{R})$ such that S' has compact support,

$$S(u)_t - \mathrm{div}(S'(u)a(\nabla u)) + S'(u)\gamma(u)|\nabla u|^p = S(u_0) \otimes \delta_{t=0} + S'(u)f - S''(u)a(\nabla u)\cdot\nabla u,$$

and satisfies the condition at infinity

$$\lim_{k\to+\infty} \int_{Q_T \cap \{k<u<k+1\}} a(\nabla u)\cdot\nabla u dx dt = 0.$$

We can also use directly the level-set PDE and define a renormalized solution to (35) as a function $u \in L^\infty(0,T;L^1(\Omega))$ having the regularity of the truncates $T_k(u) \in L^p(0,T;W_0^{1,p}(\Omega))$, $\forall k > 0$, which satisfies the equation:

$$\partial_t \chi_u - \mathrm{div}(a(\nabla u)\delta_{u=\xi}) + \gamma(\xi)|\nabla u|^p \delta_{u=\xi} = \chi_{u_0} \otimes \delta_{t=0} + f\delta_{u=\xi} + \partial_\xi \mu,$$

where $\mu := a(\nabla u)\cdot\nabla u \delta_{u=\xi}$, and satisfies the condition at infinity

$$\lim_{k\to\pm\infty} \mu_*(k,k+1) \to 0.$$

We now explain how to prove the existence of a renormalized solution to Problem (35).

Step 1. Approximation. Let (u_0^n) and (f^n) be some approximating sequences of, respectively, u_0 and f in, respectively, $L^1(\Omega)$ and $L^1(\Omega \times (0,T))$ such that $u_0^n \in L^p(\Omega)$, $f^n \in L^{p'}(\Omega \times (0,T))$. For each n, the problem

$$\left\{\begin{array}{ll} u_t^n - \mathrm{div}(a(\nabla u^n)) + \gamma(u^n)|\nabla u^n|^p = f^n & (x,t) \in \Omega \times (0,T), \\ u^n(x,0) = u_0^n(x) & x \in \Omega, \\ u^n(0,t) = 0 & (x,t) \in \Sigma, \end{array}\right. \tag{37}$$

has a unique solution u^n in the space of functions $v \in L^p(0,T;W_0^{1,p}(\Omega))$ with $v_t \in L^{p'}(0,T;W^{-1,p}(\Omega))$. The function u^n is a weak solution to (20), hence a renormalized solution and therefore satisfies the equation

$$\partial_t \chi_{u^n} - \operatorname{div}(a(\nabla u^n)\delta_{u^n=\xi}) + \gamma(\xi)|\nabla u^n|^p \delta_{u^n=\xi} = \chi_{u_0^n} \otimes \delta_{t=0} + f^n \delta_{u^n=\xi} + \partial_\xi \mu^n, \tag{38}$$

where μ^n is defined by

$$\mu^n := a(\nabla u^n) \cdot \nabla u^n \delta_{u^n=\xi}.$$

Step 2. Estimates. As in Section 2.3.2, we show that, up to a subsequence, $u_n \to u \in L^\infty(0,T;L^1(\Omega))$ in $L^1(Q_T)$, $a(\nabla T_k(u^n)) \to \sigma \mathbf{1}_{|u|<k}$ and $\widetilde{\mu}^n \to \overline{\mu}$ weakly. We also prove, by the same technique as in Section 2.3.2, the conditions at infinity

$$\lim_{k\to\pm\infty} \overline{\mu}_*((k,k+1)) = \lim_{k\to\pm\infty} \int_{Q_T\cap[k<|u|<k+1]} \sigma \cdot \nabla u dx dt = 0. \tag{39}$$

Step 3. Limit of the equation. The difficulty to understand the limit of Eq. (38) is that the term $\gamma(\xi)|\nabla u^n|^p \delta_{u^n=\xi}$ is uniformly bounded in L^1 and that no stronger *a priori* bound is available. We define

$$\Gamma_+(\xi) = \begin{cases} \dfrac{1}{\alpha}\displaystyle\int_0^\xi \gamma & \text{if } \xi > 0, \\ -\dfrac{1}{\beta}\displaystyle\int_\xi^0 \gamma & \text{if } \xi < 0. \end{cases}$$

The function Γ_+ is continuous, not C^1, on $\mathbb{R}$, but a step of regularization shows that we have

$$\partial_t e^{-\Gamma_+(\xi)} \chi_{u^n} - \operatorname{div}(e^{-\Gamma_+(\xi)} a(\nabla u^n)\delta_{u^n=\xi}) = e^{-\Gamma_+(\xi)}(\chi_{u_0^n} \otimes \delta_{t=0} + f^n \delta_{u^n=\xi}) + \partial_\xi(e^{-\Gamma_+(\xi)}\mu^n) + R$$

where

$$R := e^{-\Gamma_+(\xi)}(\gamma(\xi)(\alpha^{-1}\mathbf{1}_{\xi>0} + \beta^{-1}\mathbf{1}_{\xi<0})\mu_n - \gamma(\xi)|\nabla u^n|^p \delta_{u^n=\xi})$$

(observe that the function $\xi \mapsto \gamma(\xi)(\alpha^{-1}\mathbf{1}_{\xi>0} + \beta^{-1}\mathbf{1}_{\xi<0})$ is continuous since $\gamma(0) = 0$). Since $\mu^n := a(\nabla u^n) \cdot \nabla u^n \delta_{u^n=\xi}$, the hypotheses of coercivity and boundedness (2) and (3) on a ensure that $R \geq 0$ and, therefore, that

$$\partial_t e^{-\Gamma_+(\xi)} \chi_{u^n} - \operatorname{div}(e^{-\Gamma_+(\xi)} a(\nabla u^n)\delta_{u^n=\xi}) \geq e^{-\Gamma_+(\xi)}(\chi_{u_0^n} \otimes \delta_{t=0} + f^n \delta_{u^n=\xi}) + \partial_\xi(e^{-\Gamma_+(\xi)}\mu^n). \tag{40}$$

Similarly, we define

$$\Gamma_-(\xi) = \begin{cases} \dfrac{1}{\beta}\displaystyle\int_0^\xi \gamma & \text{if } \xi > 0, \\ -\dfrac{1}{\alpha}\displaystyle\int_\xi^0 \gamma & \text{if } \xi < 0 \end{cases}$$

and show the inequality

$$\partial_t e^{-\Gamma_-(\xi)} \chi_{u^n} - \operatorname{div}(e^{-\Gamma_-(\xi)} a(\nabla u^n)\delta_{u^n=\xi}) \leq e^{-\Gamma_-(\xi)}(\chi_{u_0^n} \otimes \delta_{t=0} + f^n \delta_{u^n=\xi}) + \partial_\xi(e^{-\Gamma_-(\xi)}\mu^n). \tag{41}$$

It is then possible to pass to the limit $[n \to +\infty]$ in (40) and (41) to obtain

$$\partial_t e^{-\Gamma_+(\xi)} \chi_u - \operatorname{div}(e^{-\Gamma_+(\xi)} \sigma \delta_{u=\xi}) \geq e^{-\Gamma_+(\xi)}(\chi_{u_0} \otimes \delta_{t=0} + f \delta_{u=\xi}) + \partial_\xi(e^{-\Gamma_+(\xi)}\mu), \tag{42}$$

$$\partial_t e^{-\Gamma_-(\xi)} \chi_u - \operatorname{div}(e^{-\Gamma_-(\xi)} a(\nabla u)\delta_{u=\xi}) \leq e^{-\Gamma_-(\xi)}(\chi_{u_0} \otimes \delta_{t=0} + f \delta_{u=\xi}) + \partial_\xi(e^{-\Gamma_-(\xi)}\mu). \tag{43}$$

What information do we extract from (42) and (43)? At a *formal* level, we can do the following computations: sum each inequality with respect to $\xi \in \mathbb{R}$ and use the condition at infinity (39) to obtain the (formal) weak equations

$$\partial_t \int_{\mathbb{R}} e^{-\Gamma_+(\xi)} \chi_u d\xi - \operatorname{div}(e^{-\Gamma_+(u)}\sigma) \geq \int_{\mathbb{R}} e^{-\Gamma_+(\xi)} \chi_{u_0} d\xi \otimes \delta_{t=0} + e^{-\Gamma_+(u)} f, \tag{44}$$

$$\partial_t \int_{\mathbb{R}} e^{-\Gamma_-(\xi)} \chi_u d\xi - \operatorname{div}(e^{-\Gamma_-(u)}\sigma) \leq \int_{\mathbb{R}} e^{-\Gamma_l(\xi)} \chi_{u_0} d\xi \otimes \delta_{t=0} + e^{-\Gamma_-(u)} f. \tag{45}$$

Multiply the first inequality by $e^{\Gamma_+(\xi)-\Gamma_-(\xi)}\delta_{u=\xi}$ and the second inequality by $e^{-\Gamma_+(\xi)+\Gamma_-(\xi)}\delta_{u=\xi}$ to obtain (still after formal computations)

$$\partial_t e^{-\Gamma_+(\xi)}\chi_u - \operatorname{div}(e^{-\Gamma_+(\xi)}\sigma\delta_{u=\xi}) \leq e^{-\Gamma_+(\xi)}(\chi_{u_0}\otimes\delta_{t=0} + f\delta_{u=\xi}) - e^{-\Gamma_+(\xi)}\sigma\cdot\nabla\delta_{u=\xi},$$
$$\partial_t e^{-\Gamma_-(\xi)}\chi_u - \operatorname{div}(e^{-\Gamma_-(\xi)}a(\nabla u)\delta_{u=\xi}) \geq e^{-\Gamma_-(\xi)}(\chi_{u_0}\otimes\delta_{t=0} + f\delta_{u=\xi}) - e^{-\Gamma_-(\xi)}\sigma\cdot\nabla\delta_{u=\xi}.$$

At last, use the identity $e^{-\Gamma_\pm(\xi)}\sigma\cdot\nabla\delta_{u=\xi} = -\partial_\xi(e^{-\Gamma_\pm(\xi)}\nu)$, where

$$\nu := \sigma\cdot\nabla u\delta_{u=\xi}$$

(this is also a very *formal* identity) to obtain

$$\partial_t e^{-\Gamma_+(\xi)}\chi_u - \operatorname{div}(e^{-\Gamma_+(\xi)}\sigma\delta_{u=\xi}) \leq e^{-\Gamma_+(\xi)}(\chi_{u_0}\otimes\delta_{t=0} + f\delta_{u=\xi}) + \partial_\xi(e^{-\Gamma_+(\xi)}\nu), \tag{46}$$
$$\partial_t e^{-\Gamma_-(\xi)}\chi_u - \operatorname{div}(e^{-\Gamma_-(\xi)}a(\nabla u)\delta_{u=\xi}) \geq e^{-\Gamma_-(\xi)}(\chi_{u_0}\otimes\delta_{t=0} + f\delta_{u=\xi}) + \partial_\xi(e^{-\Gamma_-(\xi)}\nu). \tag{47}$$

Come back to the starting point (42)-(43) to deduce the inequalities

$$\partial_\xi(e^{-\Gamma_+(\xi)}\mu) \leq \partial_\xi(e^{-\Gamma_+(\xi)}\nu),\quad \partial_\xi(e^{-\Gamma_-(\xi)}\nu) \leq \partial_\xi(e^{-\Gamma_-(\xi)}\mu). \tag{48}$$

Assume for the moment that (48) is satisfied in $\mathcal{D}'(U_T)$. A test-function $\varphi\in\mathcal{D}^+(Q_T)$ being fixed, we consider the distributions on $\mathbb{R}$ defined by

$$\mu_\varphi\colon\psi\mapsto\langle\mu,\varphi\otimes\psi\rangle,\quad \nu_\varphi\colon\psi\mapsto\langle\nu,\varphi\otimes\psi\rangle.$$

They satisfy the inequalities

$$\partial_\xi(e^{-\Gamma_+(\xi)}\mu_\varphi) \leq \partial_\xi(e^{-\Gamma_+(\xi)}\nu_\varphi),\quad \partial_\xi(e^{-\Gamma_-(\xi)}\nu_\varphi) \leq \partial_\xi(e^{-\Gamma_-(\xi)}\mu_\varphi)$$

in $\mathcal{D}'(\mathbb{R})$. Consider the first of these inequalities. Using the condition at infinity (39) for $k\to-\infty$ and classical tools of the theory of distributions, it is easy to see that we have $e^{-\Gamma_+(\xi)}\mu_\varphi \leq e^{-\Gamma_+(\xi)}\nu_\varphi$ in $\mathcal{D}'(\mathbb{R})$. But similarly, using the condition at infinity (39) for $k\to+\infty$, we obtain the converse inequality $e^{-\Gamma_+(\xi)}\mu_\varphi \geq e^{-\Gamma_+(\xi)}\nu_\varphi$. We deduce $\mu_\varphi=\nu_\varphi$ and, this being true for every $\varphi\in\mathcal{D}^+(Q_T)$, we conclude to the identity

$$\mu=\nu.$$

Step 4. Strong convergence of the gradient. The identity $\mu=\nu$ is the key point in the proof of the strong convergence of the gradient. Once this has been proved, we proceed as in Section 2.3.3. We prove in particular that $\nabla T_k(u^n)\to\nabla T_k(u)$ in $L^p_{\rm loc}(Q_T)$, and this allows to pass to the limit in Eq. (38) to obtain

$$\partial_t\chi_u - \operatorname{div}(a(\nabla u)\delta_{u=\xi}) + \gamma(\xi)|\nabla u|^p\delta_{u=\xi} = \chi_{u_0}\otimes\delta_{t=0} + f\delta_{u=\xi} + \partial_\xi(a(\nabla u)\cdot\nabla u\delta_{u=\xi}),$$

i.e. the fact that u is a renormalized solution.

Step 5. Rigorous proof of (48). This is a variation on the proof of Theorem 2 given in Section 3.2. Let us explain the main arguments. Introduce $\alpha_k := \rho_k * (\mathbf{1}_{[k,k+1]} + \mathbf{1}_{[-k-1,-k]})$, and define

$$r^k = r^k(u) = \int_{|u|}^\infty \alpha_k,\quad v^k := \int_{\mathbb{R}} e^{-\Gamma_+(\xi)}\chi_u(\xi)r_k(\xi)d\xi,\quad v_0^k := \int_{\mathbb{R}} e^{-\Gamma_+(\xi)}\chi_{u_0}(\xi)r_k(\xi)d\xi.$$

Set also

$$v^k := \int_{\mathbb{R}} e^{-\Gamma_+(\xi)}\chi_u(\xi)d\xi,\quad v_0^k := \int_{\mathbb{R}} e^{-\Gamma_+(\xi)}\chi_{u_0}(\xi)d\xi,\quad \tilde r^k = e^{-\Gamma_+(u)}r^k.$$

We have $v^k\in L^p(-1,T;W_0^{1,p}(\Omega))\cap L^\infty(Q_T)$, $v_0^k\in L^\infty(\Omega)$ and $v^k\to v$, $v_0^k\to v_0$, $r^k\to 1$ when k tends to $+\infty$. Test Eq. (42) against $\varphi(t,x)r_k(\xi)$ (with $\varphi\in\mathcal{D}^+(Q_T)$), to obtain

$$-\int_0^T\int_\Omega(v^k-v_0^k)\varphi_t + \int_0^T\int_\Omega\sigma\cdot\nabla\varphi\tilde r^k - \int_0^T\int_\Omega f\varphi\tilde r^k \leq \int_0^T\int_\Omega\int_{\mathbb{R}}\varphi e^{-\Gamma_+(\xi)}\alpha_k d\mu.$$

We deduce the inequality

$$\int_{Q_T} \varphi_t (v^k - v_0^k) - \int_0^T \langle G^k, \varphi \rangle dt \leq \|\varphi\|_{L^\infty} \varepsilon_k,$$

where $G^k := -(\mathrm{div}(\sigma \tilde{r}_k(u)) + f \tilde{r}_k(u)) \in L^{p'}(0,T; W^{-1,p'}(\Omega)) + L^1(Q)$ and $\varepsilon_k := \widetilde{\mu}_*((k-1, k+2)) + \widetilde{\mu}_*((-k-2, -k+1)) \to 0$ when $k \to +\infty$. The analogue of Lemma 1 then shows that, for every $h \in W^{1,\infty}(\mathbb{R})$, v^k satisfies the following inequality:

$$\int_{Q_T} \varphi_t \int_{v_0^k}^{v^k} h(\zeta) d\zeta - \int_0^T \langle G^k, \varphi h(v^k) \rangle dt \leq \|\varphi\|_{L^\infty} \|h\|_{L^\infty} \varepsilon_k.$$

Taking h with compact support, we obtain at the limit $k \to +\infty$ the inequality

$$\int_{Q_T} \varphi_t \int_{v_0}^{v} h(\zeta) d\zeta - \int_0^T \langle G, \varphi h(v) \rangle dt \leq 0,$$

i.e.

$$\int_{Q_T} \varphi_t \int_{v_0}^{v} h(\zeta) d\zeta - \int_{Q_T} e^{-\Gamma_+(u)} \sigma \cdot \nabla(\varphi h(v)) dt + \int_{Q_T} e^{-\Gamma_+(u)} f \varphi h(v) \leq 0. \tag{49}$$

We then fix $\theta \in \mathcal{D}(\mathbb{R})$ and apply (49) with

$$h(\zeta) := e^{-(\Gamma_- - \Gamma_+)(\phi^{-1}(\zeta))} \theta(\phi^{-1}(\zeta)), \quad \phi(\xi) := \int_0^\xi e^{-\Gamma_+},$$

in such a way that

$$\int_{v_0}^{v} h(\zeta) d\zeta = \int_{u_0}^{u} e^{-\Gamma_-(\xi)} \theta(\xi) d\xi, \quad h(v) = e^{-(\Gamma_- - \Gamma_+)(u)} \theta(u),$$

to obtain the weak form of (46). Similarly, we prove (47). As explained in Step 3., these two inequalities combined with (42) and (43) imply (48).

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