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# Absolute continuity of the representing measures of the Dunkl intertwining operator and of its dual and applications

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## Abstract

In this paper we prove the absolute continuity of the representing measures of the Dunkl intertwining operator and of its dual. Next we present some applications of this result.

**Key word :** Dunkl intertwining operator and its dual. Absolute continuity of the representing measures.

**MSC (2000) :** 33C80, 51F15, 44A15.

## Introduction

We consider the differential-difference operators on  $\mathbb{R}^d$  introduced by C.F.Dunkl in [4] and called Dunkl operators in the literature . These operators are very important in pure Mathematics and in Physics.They provide a useful tool in the study of special functions with root systems (see [3] [8]), and they are closely related to certain representations of degenerate affine Hecke algebras [2][16], moreover the commutative algebra generated by these operators has been used in the study of certain exactly solvable models of quantum mechanics, namely the Calogero-Sutherland-Moser models, which deal with systems of identical particles in a one dimensional spaces (see [10] [13] [14]).

C.F.Dunkl has proved in [6] that there exists a unique isomorphism  $V_k$  from the space of homogeneous polynomial  $\mathcal{P}_n$  on  $\mathbb{R}^d$  of degree  $n$  onto itself satisfying the transmutation relations

$$T_j V_k = V_k \frac{\partial}{\partial x_j}, \quad j = 1, \dots, d, \quad (1)$$

$$V_k(1) = 1. \quad (2)$$

This operator is called Dunkl intertwining operator. Next K.Trimèche has extended this operator to an isomorphism from  $\mathcal{E}(\mathbb{R}^d)$  (the space of  $C^\infty$ -functions on  $\mathbb{R}^d$ ) onto itself satisfying the relations (1) and (2) (see [23]).

The operator  $V_k$  possesses the integral representation

$$V_k(f)(x) = \int_{\mathbb{R}^d} f(y) d\mu_x(y), \quad f \in \mathcal{E}(\mathbb{R}^d), \quad (3)$$

where  $\mu_x$  is a probability measure on  $\mathbb{R}^d$  with support in the closed ball  $B(o, \|x\|)$  of center  $o$  and radius  $\|x\|$ . (See [17][23]).

We have studied in [23] the transposed operator  ${}^tV_k$  of the operator  $V_k$ . It has the integral representation

$${}^tV_k(f)(y) = \int_{\mathbb{R}^d} f(x) d\nu_y(x), \quad (4)$$

where  $\nu_y$  is a positive measure on  $\mathbb{R}^d$  with support in the set  $\{x \in \mathbb{R}^d / \|x\| \geq \|y\|\}$  and  $f$  in  $D(\mathbb{R}^d)$  (the space of  $C^\infty$ -functions on  $\mathbb{R}^d$  with compact support). This operator is called the dual Dunkl intertwining operator.

We have proved in [23] that the operator  ${}^tV_k$  is an isomorphism from  $D(\mathbb{R}^d)$  onto itself, satisfying the transmutation relations

$$\forall y \in \mathbb{R}^d, \quad {}^tV_k(T_j f)(y) = \frac{\partial}{\partial y_j} {}^tV_k(f)(y), \quad j = 1, \dots, d, \quad (5)$$

In this paper we prove that the measure  $\mu_x$  given by (3), is absolutely continuous with respect to the Lebesgue measure on  $\mathbb{R}^d$ . More precisely for all continuous function  $g$  on  $\mathbb{R}^d$ , we have

$$\forall x \in \mathbb{R}^d, \quad \omega_k(x) V_k(g)(x) = \int_{\mathbb{R}^d} \mathcal{K}^o(x, y) g(y) dy, \quad (6)$$

and

$$\forall x \in \mathbb{R}_{\text{reg}}^d, \quad V_k(g)(x) = \int_{\mathbb{R}^d} \mathcal{K}(x, y)g(y)dy, \quad (7)$$

where  $\mathcal{K}^o(x, \cdot)$  is a positive integrable function on  $\mathbb{R}^d$  with respect to the Lebesgue measure and with support in  $\{y \in \mathbb{R}^d / \|y\| \leq \|x\|\}$ , and  $\mathcal{K}(x, y)$  the function given by

$$\forall x \in \mathbb{R}_{\text{reg}}^d, \forall y \in \mathbb{R}^d, \mathcal{K}(x, y) = \omega_k^{-1}(x)\mathcal{K}^o(x, y). \quad (8)$$

Next we establish that for all  $y \in \mathbb{R}^d$  the measure  $\nu_y$  given by (4), is absolutely continuous with respect to the measure  $\omega_k(x)dx$  on  $\mathbb{R}^d$ , with  $\omega_k$  a positive weight function on  $\mathbb{R}^d$  which will be given in the following section. More precisely for all continuous function  $f$  on  $\mathbb{R}^d$  with compact support, we have

$$\forall y \in \mathbb{R}^d, \quad {}^tV_k(f)(y) = \int_{\mathbb{R}^d} \mathcal{K}(x, y)f(x)\omega_k(x)dx, \quad (9)$$

where  $\mathcal{K}(\cdot, y)$  is the function given by the relation (8). It is locally integrable on  $\mathbb{R}^d$  with support in  $\{x \in \mathbb{R}^d / \|x\| \geq \|y\|\}$ .

We present some applications of the relations (6),(7), in particular we prove that the Dunkl kernel  $K(-ix, z)$  satisfies

$$\forall x \in \mathbb{R}^d, \quad \lim_{\|z\| \rightarrow +\infty} \{\omega_k(x)K(-ix, z)\} = 0, \quad (10)$$

and

$$\forall x \in \mathbb{R}_{\text{reg}}^d, \quad \lim_{\|z\| \rightarrow +\infty} K(-ix, z) = 0. \quad (11)$$

Also we give a simple proof of the main result of [25].

Finally we remark that in personal communications sent to C.F.Dunkl, M.F.E.de Jeu and M.Rösler after the summer of Year 2000, we have conjectured that the measures  $\mu_x$  and  $\nu_y$  are absolutely continuous, and we have tried to solve this conjecture. Next M.F.E.de Jeu and M.Rösler have also conjectured in [12] that the measure  $\mu_x$  is absolutely continuous.

## 1 The eigenfunction of the Dunkl operators

In this section we collect some notations and results on Dunkl operators and the Dunkl kernel (see [5],[6],[9],[11]).

## 1.1 Reflection groups, root systems and multiplicity functions

We consider  $\mathbb{R}^d$  with the euclidean scalar product  $\langle \cdot, \cdot \rangle$  and  $\|x\| = \sqrt{\langle x, x \rangle}$ . On  $\mathbb{C}^d$ ,  $\|\cdot\|$  denotes also the standard Hermitian norm, while  $\langle z, w \rangle = \sum_{j=1}^d z_j \overline{w_j}$ .

For  $\alpha \in \mathbb{R}^d \setminus \{0\}$ , let  $\sigma_\alpha$  be the reflection in the hyperplan  $H_\alpha \subset \mathbb{R}^d$  orthogonal to  $\alpha$ , i.e.

$$\sigma_\alpha(x) = x - 2 \frac{\langle \alpha, x \rangle}{\|\alpha\|^2} \alpha. \quad (1.1)$$

A finite set  $R \subset \mathbb{R}^d \setminus \{0\}$  is called a root system if  $R \cap \mathbb{R}^d \cdot \alpha = \{\alpha, -\alpha\}$  and  $\sigma_\alpha R = R$  for all  $\alpha \in R$ . We assume that it is normalized by  $\|\alpha\|^2 = 2$  for all  $\alpha \in R$ . For a given root system  $R$  the reflections  $\sigma_\alpha, \alpha \in R$ , generate a finite group  $W \subset O(d)$ , the reflection group associated with  $R$ . All reflections in  $W$  correspond to suitable pairs of roots. For a given  $\beta \in \mathbb{R}_{reg}^d = \mathbb{R}^d \setminus \cup_{\alpha \in R} H_\alpha$ , we fix the positive subsystem  $R_+ = \{\alpha \in R \mid \langle \alpha, \beta \rangle > 0\}$ , then for each  $\alpha \in R$  either  $\alpha \in R_+$  or  $-\alpha \in R_+$ .

A function  $k : R \rightarrow \mathbb{C}$  on a root system  $R$  is called a multiplicity function if it is invariant under the action of the associated reflection group  $W$ . If one regards  $k$  as a function on the corresponding reflections, this means that  $k$  is constant on the conjugacy classes of reflections in  $W$ . For abbreviation, we introduce the index

$$\gamma = \gamma(R) = \sum_{\alpha \in R_+} k(\alpha). \quad (1.2)$$

Moreover, let  $\omega_k$  denotes the weight function

$$\omega_k(x) = \prod_{\alpha \in R_+} |\langle \alpha, x \rangle|^{2k(\alpha)}, \quad (1.3)$$

which is  $W$ -invariant and homogeneous of degree  $2\gamma$ .

For  $d = 1$  and  $W = \mathbb{Z}_2$ , the multiplicity function  $k$  is a single parameter denoted  $\gamma > 0$  and

$$\forall x \in \mathbb{R}, \omega_k(x) = |x|^{2\gamma}. \quad (1.4)$$

We introduce the Mehta-type constant

$$c_k = \left( \int_{\mathbb{R}^d} \exp(-\|x\|^2) \omega_k(x) dx \right)^{-1}, \quad (1.5)$$

which is known for all Coxeter groups  $W$ . (See [4][8][15]).

For an integrable function on  $\mathbb{R}^d$  with respect to the measure  $\omega_k(x) dx$  we have the relation

$$\int_{\mathbb{R}^d} f(x)\omega_k(x) dx = \int_0^{+\infty} \left( \int_{S^{d-1}} f(r\beta)\omega_k(\beta) d\sigma(\beta) \right) r^{2\gamma+d-1} dr, \quad (1.6)$$

where  $d\sigma$  is the normalized surface measure on the unit sphere  $S^{d-1}$  of  $\mathbb{R}^d$ . In particular if  $f$  is radial (i.e.  $SO(d)$ -invariant), then there exists a function  $F$  on  $[0, +\infty[$ , such that  $f(x) = F(\|x\|) = F(r)$ , with  $\|x\| = r$ , and the relation (1.6) takes the form

$$\int_{\mathbb{R}^d} f(x)\omega_k(x) dx = d_k \int_0^{+\infty} F(r)r^{2\gamma+d-1} dr, \quad (1.7)$$

where

$$d_k = \int_{S^{d-1}} \omega_k(\beta) d\sigma(\beta) = \frac{2}{c_k \Gamma(\gamma + 2d)}. \quad (1.8)$$

## 1.2 Dunkl operators and Dunkl kernel

The Dunkl operators  $T_j$   $j = 1, \dots, d$ , on  $\mathbb{R}^d$  associated with the finite reflection group  $W$  and multiplicity function  $k$  are given for a function  $f$  of class  $C^1$  on  $\mathbb{R}^d$  by

$$T_j f(x) = \frac{\partial}{\partial x_j} f(x) + \sum_{\alpha \in R_+} k(\alpha) \alpha_j \frac{f(x) - f(\sigma_\alpha(x))}{\langle \alpha, x \rangle}. \quad (1.9)$$

In the case  $k = 0$ , the  $T_j$ ,  $j = 1, \dots, d$ , reduce to the corresponding partial derivatives. In this paper, we will assume throughout that  $k \geq 0$  and  $\gamma \geq 0$ .

For  $f$  of class  $C^1$  on  $\mathbb{R}^d$  with compact support and  $g$  of class  $C^1$  on  $\mathbb{R}^d$  we have

$$\int_{\mathbb{R}^d} T_j f(x) g(x) \omega_k(x) dx = - \int_{\mathbb{R}^d} T_j g(x) f(x) \omega_k(x) dx. \quad (1.10)$$

For  $y \in \mathbb{R}^d$ , the system

$$\begin{cases} T_j u(x, y) = y_j u(x, y), & j = 1, \dots, d, \\ u(0, y) = 1, \end{cases} \quad (1.11)$$

admits a unique analytic solution on  $\mathbb{R}^d$ , which will be denoted  $K(x, y)$  and called the Dunkl kernel.

This kernel has a unique holomorphic extension to  $\mathbb{C}^d \times \mathbb{C}^d$ .

**Examples.1.1**

1) If  $d = 1$  and  $W = \mathbb{Z}_2$ , the Dunkl kernel is given by

$$K(z, t) = j_{\gamma-\frac{1}{2}}(izt) + \frac{zt}{2\gamma+1}j_{\gamma+\frac{1}{2}}(izt), \quad z, t \in \mathbb{C}, \quad (1.12)$$

where for  $\alpha \geq \frac{-1}{2}$ ,  $j_\alpha$  is the normalized Bessel function defined by

$$j_\alpha(z) = 2^\alpha \Gamma(\alpha + 1) \frac{J_\alpha(z)}{z^\alpha} = \Gamma(\alpha + 1) \sum_{n=0}^{\infty} \frac{(-1)^n (\frac{z}{2})^{2n}}{n! \Gamma(\alpha + n + 1)}, \quad (1.13)$$

with  $J_\alpha$  is the Bessel function of first kind and index  $\alpha$ . (See [6]).

2) The Dunkl kernel of index  $\gamma = \sum_{l=1}^d \alpha_l$ ,  $\alpha_l > 0$ , associated with the reflection group  $\mathbb{Z}_2 \times \dots \times \mathbb{Z}_2$  on  $\mathbb{R}^d$  is given for all  $x, y \in \mathbb{R}^d$  by

$$K(x, y) = \prod_{l=1}^d K(x_l, y_l), \quad (1.14)$$

where  $K(x_l, y_l)$  is the function defined by (1.12).

The Dunkl kernel possesses the following properties.

i) For  $z, t \in \mathbb{C}^d$ , we have  $K(z, t) = K(t, z)$ ;  $K(z, 0) = 1$  and  $K(\lambda z, t) = K(z, \lambda t)$ , for  $\lambda \in \mathbb{C}$ .

ii) For all  $x, y \in \mathbb{R}^d$  we have

$$|K(ix, y)| \leq 1, \quad (1.15)$$

iii) The function  $K(x, z)$  admits for all  $x \in \mathbb{R}^d$  and  $z \in \mathbb{C}^d$  the following Laplace type integral representation

$$K(x, z) = \int_{\mathbb{R}^d} e^{\langle y, z \rangle} d\mu_x(y), \quad (1.16)$$

where  $\mu_x$  is the measure given by the relation (3) satisfying

- $supp \mu_x \cap \{y \in \mathbb{R}^d / \|y\| = \|x\|\} \neq \emptyset$ . (1.17)
- For each  $r > 0, w \in W$  and each Borel set  $E \subset \mathbb{R}^d$  we have

$$\mu_{rx}(E) = \mu_x(r^{-1}E), \text{ and } \mu_{wx}(E) = \mu_x(w^{-1}E), \quad (1.18)$$

(See [17]).

### Examples 1.2

1) When  $d = 1$  and  $W = \mathbb{Z}_2$ , for all  $x \in \mathbb{R} \setminus \{0\}$  and  $z \in \mathbb{C}$  the relation (1.16) is of the form

$$K(x, z) = \frac{\Gamma(\gamma + \frac{1}{2})}{\sqrt{\pi}\Gamma(\gamma)} |x|^{-2\gamma} \int_{-|x|}^{|x|} (|x| - y)^\gamma (|x| + y)^{\gamma-1} e^{yz} dy. \quad (1.19)$$

Then in this case for all  $x \in \mathbb{R} \setminus \{0\}$  the measure  $\mu_x$  is given by  $d\mu_x(y) = \mathcal{K}(x, y)dy$  with

$$\mathcal{K}(x, y) = \frac{\Gamma(\gamma + \frac{1}{2})}{\sqrt{\pi}\Gamma(\gamma)} |x|^{-2\gamma} (|x| - y)^\gamma (|x| + y)^{\gamma-1} 1_{]-|x|, |x|[}(y), \quad (1.20)$$

where  $1_{]-|x|, |x|[}$  is the characteristic function of the interval  $]-|x|, |x|[$ .

2) The Dunkl kernel of index  $\gamma = \sum_{l=1}^d \alpha_l$ ,  $\alpha_l > 0$ , associated with the

reflection group  $\mathbb{Z}_2 \times \dots \times \mathbb{Z}_2$  on  $\mathbb{R}^d$ , possesses for all  $x \in \mathbb{R}_{reg}^d = \mathbb{R}^d \setminus \bigcup_{l=1}^d H_l$ , with  $H_l = \{x \in \mathbb{R}^d / x_l = 0\}$ , and  $z \in \mathbb{C}^d$ , the integral representation

$$K(x, z) = \int_{\mathbb{R}^d} \mathcal{K}(x, y) e^{\langle y, z \rangle} dy, \quad (1.21)$$

where

$$\mathcal{K}(x, y) = \prod_{l=1}^d \mathcal{K}(x_l, y_l), \quad (1.22),$$

with  $\mathcal{K}(x_l, y_l)$  given by the relation (1.20).

## 2 The Dunkl intertwining operator and its dual

**Notations.** We denote by  $C(\mathbb{R}^d)$  (resp.  $C_c(\mathbb{R}^d)$ ) the space of continuous functions on  $\mathbb{R}^d$  (resp. with compact support).

The Dunkl intertwining operator  $V_k$  is defined on  $C(\mathbb{R}^d)$  by

$$\forall x \in \mathbb{R}^d, \quad V_k f(x) = \int_{\mathbb{R}^d} f(y) d\mu_x(y), \quad (2.1)$$

where  $\mu_x$  is the measure given by the relation (3). (See [17][23]p.364-366). It possesses many properties in particular we have

i) For all  $g$  in  $C(\mathbb{R}^d)$  the function  $V_k(g)$  belongs to  $C(\mathbb{R}^d)$ . Moreover for all  $x \in \mathbb{R}^d$  in the closed ball  $\overline{B}(o, a)$  of center  $o$  and radius  $a > 0$ , we have

$$|V_k(g)(x)| \leq \sup_{y \in \overline{B}(o, a)} |g(y)|. \quad (2.2)$$

ii) We have

$$V_k(g)(o) = g(o). \quad (2.3)$$

iii) We have

$$\forall x \in \mathbb{R}^d, \forall z \in \mathbb{C}^d, K(x, z) = V_k(e^{\langle \cdot, z \rangle})(x). \quad (2.4)$$

The operator  ${}^tV_k$  satisfying for  $f$  in  $C_c(\mathbb{R}^d)$  and  $g$  in  $C(\mathbb{R}^d)$  the relation

$$\int_{\mathbb{R}^d} {}^tV_k(f)(y)g(y)dy = \int_{\mathbb{R}^d} V_k(g)(x)f(x)\omega_k(x)dx, \quad (2.5)$$

is given by

$$\forall y \in \mathbb{R}^d, {}^tV_k(f)(y) = \int_{\mathbb{R}^d} f(x)d\nu_y(x), \quad (2.6)$$

where  $\nu_y$  is a positive measure on the  $\sigma$ -algebra of  $\mathbb{R}^d$ , satisfying

- $\text{supp } \nu_y \subset \{x \in \mathbb{R}^d / \|x\| \geq \|y\|\}$ .
- For each  $a > 0$ ,  $w \in W$  and each Borel set  $E \subset \mathbb{R}^d$  we have

$$\nu_{ay}(E) = a^{2\gamma}\nu_y(a^{-1}E) \text{ and } \nu_{wy}(E) = \nu_y(w^{-1}E). \quad (2.7)$$

The operator  ${}^tV_k$  is called the dual Dunkl intertwining operator. (See [23]p.358-364).

It admits many properties in particular we have

i) For all  $f$  in  $C_c(\mathbb{R}^d)$  we have

$$\int_{\mathbb{R}^d} {}^tV_k(f)(y)dy = \int_{\mathbb{R}^d} f(x)\omega_k(x)dx. \quad (2.8)$$

ii) For all  $f$  in  $C_c(\mathbb{R}^d)$  the function  ${}^tV_k(f)$  belongs to  $C_c(\mathbb{R}^d)$  and we have

$$\text{supp } f \subset \overline{B}(o, a) \iff \text{supp } {}^tV_k(f) \subset \overline{B}(o, a), \quad (2.9)$$

where  $\overline{B}(o, a)$  is the closed ball of center  $o$  and radius  $a > 0$ .

iii) For all  $f$  in  $C_c(\mathbb{R}^d)$  and  $r > 0$ , we have

$$\forall y \in \mathbb{R}^d, {}^tV_k(f)(ry) = r^{2\gamma} {}^tV_k(f_r)(y), \text{ with } f_r(x) = f(rx). \quad (2.10)$$

iν) For all  $a > 0$ , we have

$$\forall y \in \mathbb{R}^d, {}^tV_k(e^{-a\|x\|^2})(y) = \frac{e^{-a\|y\|^2}}{a^\gamma \pi^{d/2} c_k}. \quad (2.11)$$

The result of the following proposition has been given in [23] p. 363, without proof.

**Proposition 2.1.** For all  $y \in \mathbb{R}^d$  we have

$$\text{supp}\nu_y \cap \{x \in \mathbb{R}^d / \|x\| = \|y\|\} \neq \emptyset. \quad (2.12)$$

**Proof**

-1<sup>st</sup> case:  $y \in \mathbb{R}^d \setminus \{0\}$

Suppose to the contrary that  $\text{supp}\nu_y \cap \{x \in \mathbb{R}^d / \|x\| = \|y\|\} = \emptyset$  for some  $y$ . Then there exists a constant  $\sigma \in ]1, +\infty[$  such that  $\text{supp}\nu_y \subset \{x \in \mathbb{R}^d / \|x\| \geq \sigma\|y\|\}$ . Thus from (2.6) for all  $a > 0$  we have

$${}^tV_k(e^{-a\|x\|^2})(y) = \int_{\|x\| \geq \sigma\|y\|} e^{-a\|x\|^2} d\nu_y(x).$$

We put  $u = \frac{x}{\sigma}$  then

$${}^tV_k(e^{-a\|x\|^2})(y) = \int_{\|u\| \geq \|y\|} e^{-a\sigma^2\|u\|^2} d\nu_y(u).$$

Then

$${}^tV_k(e^{-a\|x\|^2})(y) = {}^tV_k(e^{-a\sigma^2\|x\|^2})(y).$$

By using the relation (2.11) we obtain

$$\frac{e^{-a\|y\|^2}}{a^\gamma \pi^{d/2} c_k} = \frac{e^{-a\sigma^2\|y\|^2}}{(a\sigma^2)^\gamma \pi^{d/2} c_k}.$$

thus

$$e^{-a\|y\|^2} = \frac{e^{-a\sigma^2\|y\|^2}}{\sigma^{2\gamma}}.$$

If we tends  $a$  to zero we obtain

$$\sigma^{2\gamma} = 1.$$

As  $\gamma > 0$ . Then  $\sigma = 1$ . Contradiction.

-2<sup>nd</sup> case:  $y = 0$

Suppose to the contrary that  $0 \notin \text{supp}\nu_o$ . Then there exists  $r > 0$  such that  $\text{supp}\nu_o$  is contained in  $B^c(o, r)$  the complementary of the open ball  $B(o, r)$  of center  $o$  and radius  $r$ .

Let  $C$  be a compact contained in  $B^c(o, r)$ . There exists  $R > 0$  such that

$$C \subset B(o, R).$$

We have

$$\nu_o(C) \leq \nu_o(B(o, R)).$$

By applying the relation (2.7) with  $a = \frac{R}{\varepsilon}$ , to the second member of the inequality we obtain

$$\nu_o(C) \leq \left(\frac{R}{\varepsilon}\right)^{2\gamma} \nu_o(B(o, \varepsilon)).$$

Thus  $\nu_o = 0$ . Impossible.

This completes the proof of the proposition.

**Theorem 2.1.** Let  $(\nu_y)_{y \in \mathbb{R}^d}$  be the family of measures defined in formula (2.6) and let  $f$  be an integrable function on  $\mathbb{R}^d$  with respect to the measure  $\omega_k(x)dx$ . Then for almost all  $y$  (with respect to the Lebesgue measure on  $\mathbb{R}^d$ ),  $f$  is  $\nu_y$ -intégrable, the function

$$y \mapsto \nu_y(f) = \int_{\mathbb{R}^d} f(x) d\nu_y(x),$$

which will also be denoted by  ${}^tV_k(f)$ , is defined almost everywhere on  $\mathbb{R}^d$  and is Lebesgue integrable. Moreover for all bounded continuous functions  $g$  on  $\mathbb{R}^d$ , we have the formula

$$\int_{\mathbb{R}^d} \nu_y(f)g(y)dy = \int_{\mathbb{R}^d} f(x)V_k(g)(x)\omega_k(x)dx. \quad (2.13)$$

(See [7]).

**Theorem 2.2.** Let  $(\mu_x)_{x \in \mathbb{R}^d}$  be the family of measures defined in formula (2.1) and let  $g$  be a measurable and bounded function on  $\mathbb{R}^d$ . Then for almost all  $x$  (with respect to the Lebesgue measure on  $\mathbb{R}^d$ ) the function

$$x \mapsto \mu_x(g) = \int_{\mathbb{R}^d} g(y) d\mu_x(y)$$

which also will be denoted by  $V_k(g)$ , is defined almost everywhere on  $\mathbb{R}^d$ , measurable and bounded. Moreover for all functions  $f$  in  $C_c(\mathbb{R}^d)$  we have the formula

$$\int_{\mathbb{R}^d} \mu_x(g) f(x) \omega_k(x) dx = \int_{\mathbb{R}^d} {}^tV_k(f)(y) g(y) dy. \quad (2.14)$$

**Proof**

We will divide the proof in three steps.

i) From the properties of the operator  $V_k$  we deduce that the family of measures  $(\mu_x)_{x \in \mathbb{R}^d}$  is weak-\* continuous. More precisely for all  $g$  in  $C(\mathbb{R}^d)$  the function

$$x \mapsto \mu_x(g) = V_k(g)(x) = \int_{\mathbb{R}^d} g(y) d\mu_x(y),$$

belongs to  $C(\mathbb{R}^d)$ .

ii) Let  $f$  be in  $C_c(\mathbb{R}^d)$ . From the relation (2.13) we deduce that for all bounded function  $g$  in  $C(\mathbb{R}^d)$  we have

$$\int_{\mathbb{R}^d} \mu_x(g) f(x) \omega_k(x) dx = \int_{\mathbb{R}^d} {}^tV_k(f)(y) g(y) dy. \quad (2.15)$$

iii) If  $g$  is a measurable and bounded function on  $\mathbb{R}^d$ . Then parts i), ii) and Bourbaki's integration of measures Theorem [1 , p.17] shows that the function  $x \rightarrow \mu_x(g)$  exists for almost all  $x \in \mathbb{R}^d$  with respect to the Lebesgue measure, is measurable and bounded, and the relation (2.15) is valid for this function  $g$ .

The following theorem gives the expression of  ${}^tV_k(f)$  when  $f$  is radial. (See [23]).

**Theorem 2.3.** For  $\gamma > 0$  and for all  $f$  in  $D(\mathbb{R}^d)$  radial, we have

$$\forall y \in \mathbb{R}^d, {}^tV_k(f)(y) = \frac{\Gamma(\gamma + \frac{d}{2})d_k}{\pi^{\frac{d}{2}}\Gamma(\gamma)} \int_{\|y\|}^{+\infty} F(t)(t^2 - \|y\|^2)^{\gamma-1}tdt, \quad (2.16)$$

where  $F$  is the function in  $\mathcal{D}(\mathbb{R}_+)$  given by

$$f(x) = F(\|x\|) = F(r), \text{ with } r = \|x\|.$$

### Examples 2.1

1) When  $d = 1$  and  $W = \mathbb{Z}_2$ , the Dunkl intertwining operator  $V_k$  is defined by (2.1) with for  $x \in \mathbb{R} \setminus \{0\}$  we have  $d\mu_x(y) = \mathcal{K}(x, y)dy$ , where  $\mathcal{K}$  given by the relation (1.20).

The dual Dunkl intertwining operator  ${}^tV_k$  is defined by (2.6) with for all  $y \in \mathbb{R}$  we have  $d\nu_y(x) = \mathcal{K}(x, y)\omega_k(x)dx$ , where  $\mathcal{K}$  and  $\omega_k$  given respectively by the relations (1.20) and (1.4).

2) The Dunkl intertwining operator  $V_k$  of index  $\gamma = \sum_{l=1}^d \alpha_l$ ,  $\alpha_l > 0$ , associated with the reflection group  $\mathbb{Z}_2 \times \dots \times \mathbb{Z}_2$  on  $\mathbb{R}^d$ , is given for all  $f$  in  $C(\mathbb{R}^d)$  and  $x \in \mathbb{R}_{reg}^d = \mathbb{R}^d \setminus \bigcup_{l=1}^d H_l$ , with  $H_l = \{x \in \mathbb{R}^d / x_l = 0\}$ , by

$$V_k(f)(x) = \int_{\mathbb{R}^d} \mathcal{K}(x, y)f(y)dy \quad (2.17)$$

where  $\mathcal{K}(x, y)$  is given by the relation (1.22). By change of variables we obtain

$$\begin{aligned} \forall x \in \mathbb{R}^d, V_k(f)(x) &= \left[ \prod_{l=1}^d \frac{\Gamma(\alpha_l + \frac{1}{2})}{\sqrt{\pi}\Gamma(\alpha_l)} \right] \int_{[-1,1]^d} f(t_1x_1, t_2x_2, \dots, t_dx_d) \\ &\times \prod_{l=1}^d (1 - t_l)^{\alpha_l} (1 + t_l)^{\alpha_l - 1} dt_1 \dots dt_d. \end{aligned} \quad (2.18)$$

(See [25] p.2964).

The dual Dunkl intertwining operator is given for all  $f$  in  $C_c(\mathbb{R}^d)$  by

$$\forall y \in \mathbb{R}^d, {}^tV_k(f)(y) = \int_{\mathbb{R}^d} \mathcal{K}(x, y)f(x)\omega_k(x)dx, \quad (2.19)$$

where  $\mathcal{K}(x, y)$  is defined by the relation (1.22) and

$$\omega_k(x) = \prod_{j=1}^d |x_j|^{2\alpha_j} \quad (2.20)$$

### 3 Absolute continuity of the representing measures of the Dunkl intertwining operator and of its dual

The example 2.1 shows that when  $d = 1$  and  $W = \mathbb{Z}_2$  the representing measures of the Dunkl intertwining operator and of its dual are absolutely continuous.

In this section we suppose that  $d \geq 2$ .

We give now the following remark which concerns the multiplicity function  $k$ .

**Remark 3.1.**

Let  $\mathcal{V}' \subset \mathbb{R}^d$  be the  $\mathbb{R}$ -linear space of the subsystem  $R' = \{\alpha \in R; k(\alpha) \neq 0\}$  (with  $\mathcal{V}' = \{0\}$  if  $R' = 0$ ) and  $\mathcal{V}'' = (\mathcal{V}')^\perp \neq \{0\}$ . We have  $\mathbb{R}^d = \mathcal{V}' \oplus \mathcal{V}''$ . Thus all  $x \in \mathbb{R}^d$  can be written in the form  $x = x' + x''$  with  $x' \in \mathcal{V}'$  and  $x'' \in \mathcal{V}''$  (see [12]).

From the relations (1,9), (1,11), (1,16) we have

$$\forall x, \lambda \in \mathbb{R}^d, \quad K(x, \lambda) = e^{\langle x'', \lambda'' \rangle} K(x', \lambda').$$

Using this relation and (2.13) we deduce that the measures  $\mu_x$  and  $\nu_y$  with  $x, y \in \mathbb{R}^d$ , of the integral representations of the Dunkl intertwining operator  $V_k$  and its dual  ${}^tV_k$  are of the form

$$\mu_x = \delta_{x''} \otimes \mu_{x'}, \tag{3.1}$$

$$\nu_y = \delta_{y''} \otimes \nu_{y'}, \tag{3.2}$$

where  $\delta_{z''}$  is the Dirac measure at the point  $z'' \in \mathcal{V}''$ .

Thus for  $x, y \in \mathbb{R}^d \setminus \mathcal{V}'$ , the measures  $\mu_x$  and  $\nu_y$  are not absolute continuous.

In this section we shall suppose that the multiplicity function  $k$  satisfies

$$\forall \alpha \in R, \quad k(\alpha) > 0. \tag{3.3}$$

#### 3.1 Absolute continuity of the measure $\mu_x$

M.F.E. de Jeu and M. Rösler have proved in [12] that for all  $x \in \mathbb{R}^d_{\text{reg}}$  the measure  $\mu_x$  of the integral representation (2.1) of the Dunkl intertwining operator  $V_k$  is continuous

The purpose of this subsection is to prove that for all  $x \in \mathbb{R}_{\text{reg}}^d$  the measure  $\mu_x$  is absolute continuous with respect to the Lebesgue measure on  $\mathbb{R}^d$ , and to present some applications of this result.

**Notations.** We denote by

- $m$  the Lebesgue measure on  $\mathbb{R}^d$ .
- $B(\xi, r)$  the open ball of center  $\xi$  and radius  $r > 0$ .

By applying [21] p. 341 and Theorem 8.6 of [20] p.166 to the measure  $\nu_y$ ,  $y \in \mathbb{R}^d$ , we deduce that there exist a positive function  $\mathcal{K}^o(., y)$  locally integrable on  $\mathbb{R}^d$  with respect to the Lebesgue measure, and a positive measure  $\nu_y^s$  on  $\mathbb{R}^d$  such that for every Borel set  $E$  we have

$$\nu_y(E) = \int_E \mathcal{K}^o(x, y) dx + \nu_y^s(E). \quad (3.4)$$

with

$$\mathcal{K}^o(x, y) = \lim_{r \rightarrow 0} \frac{\nu_y(B(x, r))}{m(B(x, r))}. \quad (3.5)$$

The measure  $\nu_y^s$  and the Lebesgue measure  $m$  are mutually singular.

**Remark 3.2**

When the multiplicity function satisfies

$$\forall \alpha \in R, \quad k(\alpha) = 0, \quad (3.6)$$

which is equivalent to say that the subset  $\mathcal{V}'$  of the Remark 3.1 is empty, then from (3.2), (3.5) we deduce that

$$\mathcal{K}^0(x, y) = \begin{cases} 0 & \text{if } x \neq y, \\ +\infty & \text{if } x = y. \end{cases} \quad (3.7)$$

Thus

$$\int_E \mathcal{K}^0(x, y) dx = 0. \quad (3.8)$$

For  $x_o, y \in \mathbb{R}^d$  and  $n \in \mathbb{N}^*$ , we consider the following sequence given by

$$\overline{\Delta}_n(x_o, y) = \sup_{0 < \frac{1}{p} < \frac{1}{n}} \left\{ \frac{\nu_y(B(x_o, \frac{1}{p}))}{m(B(x_o, \frac{1}{p}))} \right\}. \quad (3.9)$$

These functions and their properties have been given only in the French translation [19] of Rudin's book [20].

**Lemma 3.1.**

- i) The sequence  $\{\overline{\Delta}_n(x_o, y)\}_{n \in \mathbb{N}^*}$  is decreasing.
- ii) For  $x_o \in \mathbb{R}^d$  and  $n \in \mathbb{N}^*$ , the function  $\overline{\Delta}_n(x_o, \cdot)$  is measurable positive.
- iii) For  $x_o, y \in \mathbb{R}^d$  we have

$$\mathcal{K}^o(x_o, y) = \lim_{n \rightarrow \infty} \overline{\Delta}_n(x_o, y).$$

**Proof**

We deduce i),ii) and iii) from the definition of the function  $\overline{\Delta}_n(x_o, y)$ , (3.5), and the relation (3) of [19] p.147.

**Lemma 3.2.** For  $x_o \in \mathbb{R}^d$ ,  $r > 0$  and for all bounded continuous function  $g$  on  $\mathbb{R}^d$  we have

$$\int_{B(x_o, r)} V_k(g)(x) \omega_k(x) dx = \int_{\mathbb{R}^d} g(y) \nu_y(B(x_o, r)) dy. \quad (3.10)$$

**Proof**

We deduce (3.10) from the relation (2.13). **Proposition 3.1.** Let  $g$  be a bounded continuous function on  $\mathbb{R}^d$ . Then for  $x_o \in \mathbb{R}^d$  the function  $\mathcal{K}^o(x_o, \cdot)$  is integrable on  $\mathbb{R}^d$  with respect to the Lebesgue measure and we have

$$V_k(g)(x_o) \omega_k(x_o) = \int_{\mathbb{R}^d} \mathcal{K}^o(x_o, y) g(y) dy. \quad (3.11)$$

**Proof**

-By writing  $g = g^+ - g^-$ , we can suppose in the following that  $g$  is positive. From the relation (3.4), for  $x_o \in \mathbb{R}^d$  and  $r > 0$ , we have

$$\frac{1}{m(B(x_o, r))} \int_{B(x_o, r)} V_k(g)(x) \omega_k(x) dx = \int_{\mathbb{R}^d} g(y) \frac{\nu_y(B(x_o, r))}{m(B(x_o, r))} dy. \quad (3.12)$$

By using (3.9) and by applying the relation (2) of [20] p.168 to the first member, and Fatou Lemma to the second, we obtain when  $r$  tends to zero.

$$\int_{\mathbb{R}^d} \mathcal{K}^o(x_o, y) g(y) dy \leq V_k(g)(x_o) \omega_k(x_o). \quad (3.13)$$

We replace in this inequality the function  $g$  by the constant function equal to 1, and next we use the fact that

$$\forall x \in \mathbb{R}^d, V_k(1)(x) = 1,$$

we deduce that

$$\int_{\mathbb{R}^d} \mathcal{K}^o(x_0, y) dy \leq \omega_k(x_0) < +\infty. \quad (3.14)$$

Then the function  $\mathcal{K}^o(x_0, \cdot)$  is integrable on  $\mathbb{R}^d$  with respect to the Lebesgue measure.

- On the other hand from the relation (3.10) for  $x_0 \in \mathbb{R}^d$  and  $n, p \in \mathbb{N}^*$  with  $n < p$ , we have

$$\frac{1}{m(B(x_0, \frac{1}{p}))} \int_{B(x_0, \frac{1}{p})} V_k(g)(x) \omega_k(x) dx \leq \int_{\mathbb{R}^d} g(y) \overline{\Delta}_n(x_0, y) dy. \quad (3.15)$$

By using the relation (3.13) and Lemma 3.1, we deduce that the sequence  $\{g(y) \overline{\Delta}_n(x_0, y)\}_{n \in \mathbb{N}^*}$  satisfies the hypothesis of the other version of the monotonic convergence theorem (see Theorem 5.7.27 of [21] p.234-235). By applying this theorem to the second member of (3.15), and the relation (2) of [20] p.168, to the first member of the same relation, we obtain when  $n$  tends to infinity

$$\int_{\mathbb{R}^d} \mathcal{K}^o(x_0, y) g(y) dy \geq V_k(g)(x_0) \omega_k(x_0). \quad (3.16)$$

We deduce (3.11) from the relations (3.13) and (3.16).

### Remark 3.3

- i) If we suppose that the multiplicity function  $k$  satisfies (3.6), then from (1.3), (2.1) and (3.1) we have

$$\forall x \in \mathbb{R}^d, w_k(x) = 1,$$

and

$$V_k = Id.$$

A proof analogue to that of Proposition 3.1 and by using the relation (3.8), we obtain for  $x_0 \in \mathbb{R}^d$  and  $g$  a positive bounded continuous

function on  $\mathbb{R}^d$ , the following inequalities which are the analogue of (3.13) and (3.16) :

$$0 = \int_{\mathbb{R}^d} \mathcal{K}^0(x_0, y)g(y)dy \leq V_k(g)(x_0)w_k(x_0),$$

$$0 = \int_{\mathcal{K}^d} \mathcal{K}^0(x_0, y)g(y)dy \geq V_k(g)(x_0)w_k(x_0).$$

But the second member of the second inequality is positive. Hence we obtain a contradiction.

This proof shows that the relation (3.11) is not true in this case.

- ii) When the subset  $\mathcal{V}'$  of the Remark 3.1 is such that  $\mathcal{V}' \neq \emptyset$ , then by using (3.2) and the result of the preceding i) we deduce that the relation (3.11) is also not true in this case.

This remark implies that the relation (3.11) is true only under the assumption (3.3).

**Proposition 3.2.**

- i) For all  $x_0 \in \bigcup_{\alpha \in R_+} H_\alpha$ , we have for almost all  $y \in \mathbb{R}^d$ :

$$\mathcal{K}^o(x_0, y) = 0.$$

- ii) For  $x_0 \in \mathbb{R}^d$ , we have

$$\omega_k(x_0)d\mu_{x_0}(y) = \mathcal{K}^o(x_0, y)dy, \quad y \in \mathbb{R}^d, \quad (3.17)$$

where  $\mu_{x_0}$  is the measure given by the relation (2.1).

**Proof**

We deduce the results of this proposition from the relation (3.11), the fact that

$$\omega_k(x) = 0 \iff x \in \bigcup_{\alpha \in R_+} H_\alpha$$

and the properties of the measure  $\mu_{x_0}$ .

**Notation.** For all  $x \in \mathbb{R}_{reg}^d$  and  $y \in \mathbb{R}^d$ , we put

$$\mathcal{K}(x, y) = \omega_k^{-1}(x)\mathcal{K}^o(x, y). \quad (3.18)$$

**Corollary 3.1.** The function  $\mathcal{K}(\cdot, y)$ ,  $y \in \mathbb{R}^d$ , given by the relation (3.18) satisfies

$$\forall x \in \mathbb{R}_{reg}^d, \text{supp}\mathcal{K}(x, y) \subset \overline{B}(0, \|x\|), \quad (3.19)$$

where  $\overline{B}(0, \|x\|)$  is the closed ball of center 0 and radius  $\|x\|$ .

**Theorem 3.1.** The representing measure  $\mu_x$  of the Dunkl intertwining operator  $V_k$  satisfies

i) For  $x \in \mathbb{R}_{reg}^d$  we have

$$d\mu_x(y) = \mathcal{K}(x, y)dy, \quad (3.20)$$

where  $\mathcal{K}(x, y)$  is the function given by (3.18).

ii) For all  $h$  in  $C(\mathbb{R}^d)$  we have

$$\forall x \in \mathbb{R}^d, \omega_k(x)V_k(h)(x) = \int_{\mathbb{R}^d} \mathcal{K}^o(x, y)h(y)dy, \quad (3.21)$$

and

$$\forall x \in \mathbb{R}_{reg}^d, V_k(h)(x) = \int_{\mathbb{R}^d} \mathcal{K}(x, y)h(y)dy. \quad (3.22)$$

**Proof**

- i) The relations (3.17) (3.18) give the result.
- ii) we obtain (3.21),(3.22) from (3.17), (3.20) and (2.1).

**Remark 3.4.**

From this theorem and the relation (2.1) we deduce that for all  $x \in \mathbb{R}_{reg}^d$  the measure  $\mu_x$  is absolutely continuous with respect to the Lebesgue measure.

**Examples 3.1**

On  $\mathbb{R}^d$  with  $W = \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2$  the function  $\mathcal{K}(x, y)$  of (3.20) is given by (1.22) and in this case the relation (3.22) is true for all  $x \in \mathbb{R}_{reg}^d = \mathbb{R}^d \setminus \bigcup_{\ell=1}^d H_\ell$  with  $H_\ell = \{x \in \mathbb{R}^d / x_\ell = 0\}$ .

**Corollary 3.2.**

i) For all  $x \in \mathbb{R}^d$  and  $z \in \mathbb{C}^d$  we have

$$\omega_k(x)K(x, -iz) = \int_{\mathbb{R}^d} \mathcal{K}^o(x, y)e^{-i\langle y, z \rangle} dy. \quad (3.23)$$

ii) For all  $x \in \mathbb{R}_{\text{reg}}^d$  and  $z \in \mathbb{C}^d$  we have

$$K(x, -iz) = \int_{\mathbb{R}^d} \mathcal{K}(x, y) e^{-i\langle y, z \rangle} dy. \quad (3.24)$$

**Proof**

We deduce the relations (3.23),(3.24) from the relation (3) and Theorem 3.1, ii).

In the following proposition we give some other properties of the function  $\mathcal{K}(x, y)$ .

**Proposition 3.3**

i) For all  $x \in \mathbb{R}_{\text{reg}}^d$  we have

$$\int_{\mathbb{R}^d} \mathcal{K}(x, y) dy = 1. \quad (3.25)$$

ii) For all  $r > 0$ ,  $w \in W$  and for all  $x \in \mathbb{R}_{\text{reg}}^d$  we have

$$\mathcal{K}(wx, y) = \mathcal{K}(x, wy), a.e. y \in \mathbb{R}^d, \quad (3.26)$$

$$\mathcal{K}(rx, y) = r^{-d} \mathcal{K}(x, \frac{y}{r}), a.e. y \in \mathbb{R}^d. \quad (3.27)$$

**Proof**

We deduce these relations from Theorem 3.1 ii) and the relations (3.21),(1.18).

**Corollary 3.3.** The generalized Bessel function  $J_W$  defined for  $x \in \mathbb{R}^d$  and  $z \in \mathbb{C}^d$  by (see [23] p.355-356)

$$J_W(-ix, z) = \frac{1}{|W|} \sum_{w \in W} K(-ix, wz), \quad (3.28)$$

admits the following integral representations

$$\forall x \in \mathbb{R}^d, \omega_k(x) J_W(-ix, z) = \int_{\mathbb{R}^d} E_W(-iz, y) \mathcal{K}_W^o(x, y) dy, \quad (3.29)$$

and

$$\forall x \in \mathbb{R}_{\text{reg}}^d, \omega_k(x) J_W(-ix, z) = \int_{\mathbb{R}^d} E_W(-iz, y) \mathcal{K}_W(x, y) dy, \quad (3.30)$$

with

$$E_W(-iz, y) = \frac{1}{|W|} \sum_{w \in W} e^{-i\langle y, wz \rangle}, \quad (3.31)$$

$$\mathcal{K}_W^o(x, y) = \frac{\omega_k^{-1}(x)}{|W|} \sum_{w \in W} \mathcal{K}^o(wx, y), \quad (3.32)$$

$$\mathcal{K}_W(x, y) = \frac{1}{|W|} \sum_{w \in W} \mathcal{K}(wx, y). \quad (3.33)$$

**Proof**

We deduce (3.29), (3.30) from the definition of the function  $J_W$  and Corollary 3.2.

### 3.2 Absolute continuity of the measure $\nu_y$

The purpose of this subsection is to prove that for all  $y \in \mathbb{R}^d$  the measure  $\nu_y$  of the integral representation (2.6) of the dual Dunkl intertwining operator  ${}^tV_k$  is absolute continuous with respect to the measure  $\omega_k(x)dx$ .

**Lemma 3.3.** For  $x, y_o \in \mathbb{R}^d$ , we have

$$\lim_{r \rightarrow 0} \left\{ \frac{\mu_x(B(y_o, r))}{m(B(y_o, r))} \omega_k(x) \right\} = \mathcal{K}^o(x, y_o), \quad (3.34)$$

where  $\mathcal{K}^o(\cdot, y_o)$  is the function given by the relation (3.4).

**Proof**

From Theorem 2.2 and the relation (3.11) we have

$$\frac{\mu_x(B(y_o, r))}{m(B(y_o, r))} \omega_k(x) = \frac{1}{m(B(y_o, r))} \int_{B(y_o, r)} \mathcal{K}^o(x, y) dy.$$

We deduce (3.31) from the relation (2) of [20] p.168.

For  $x, y_o \in \mathbb{R}^d$  and  $n \in \mathbb{N}^*$ , we consider the following sequence given by

$$\bar{\Lambda}_n(x, y_o) = \sup_{0 < \frac{1}{p} < \frac{1}{n}} \left\{ \frac{\mu_x(B(y_o, \frac{1}{p}))}{m(B(y_o, \frac{1}{p}))} \omega_k(x) \right\}. \quad (3.35)$$

**Lemma 3.4.**

i) The sequence  $\{\bar{\Lambda}_n(x, y_o)\}_{n \in \mathbb{N}^*}$  is decreasing.

- ii) For  $y_o \in \mathbb{R}^d$ ,  $n \in \mathbb{N}^*$  the function  $\bar{\Lambda}_n(\cdot, y_o)$  is measurable positive.  
iii) For  $x, y_o \in \mathbb{R}^d$ , we have

$$\lim_{n \rightarrow +\infty} \bar{\Lambda}_n(x, y_o) = \mathcal{K}^o(x, y_o).$$

**Proof**

We obtain i),ii) and iii) from the definition of  $\bar{\Lambda}_n(x, y_o)$ , (3.34) and the relation (3) of [19] p.147.

**Lemma 3.5.** For  $y_o \in \mathbb{R}^d$ ,  $r > 0$ , and for all  $f$  in  $C_c(\mathbb{R}^d)$  we have

$$\int_{B(y_o, r)} {}^tV_k(f)(y)dy = \int_{\mathbb{R}^d} \mu_x(B(y_o, r))f(x)\omega_k(x)dx. \quad (3.36)$$

**Proof**

We deduce (3.36) from the relation (2.14).

**Proposition 3.4.** For all  $f$  in  $C_c(\mathbb{R}^d)$  and  $y_o \in \mathbb{R}^d$  we have

$${}^tV_k(f)(y_o) = \int_{\mathbb{R}^d} \mathcal{K}^o(x, y_o)f(x)dx. \quad (3.37)$$

**Proof**

-By writing  $f = f^+ - f^-$ , we can suppose in the following that  $f$  is positive.

From the relation (3.36), for  $y_o \in \mathbb{R}^d$  and  $r > 0$ , we have

$$\frac{1}{m(B(y_o, r))} \int_{B(y_o, r)} {}^tV_k(f)(y)dy = \int_{\mathbb{R}^d} f(x) \frac{\mu_x(B(y_o, r))}{m(B(y_o, r))} \omega_k(x)dx. \quad (3.38)$$

By applying the relation (2) of [20] p.168, to the first member of (3.38), and Fatou Lemma to the second member of the same relation, we obtain when  $r$  tends to zero.

$$\int_{\mathbb{R}^d} \mathcal{K}^o(x, y_o)f(x)dx \leq {}^tV_k(f)(y_o) < +\infty. \quad (3.39)$$

Thus the function  $\mathcal{K}^o(\cdot, y_o)$  is locally integrable on  $\mathbb{R}^d$  with respect to the Lebesgue measure.

- From (3.38) for  $y_o \in \mathbb{R}^d$  and  $n, p \in \mathbb{N}^*$  with  $n < p$  we have

$$\frac{1}{m(B(y_o, \frac{1}{p}))} \int_{B(y_o, \frac{1}{p})} {}^tV_k(f)(y)dy \leq \int_{\mathbb{R}^d} f(x)\bar{\Lambda}_n(x, y_o)dx. \quad (3.40)$$

From the relation (3.39) and Lemma 3.4 we deduce that the sequence  $\{f(x)\overline{\Lambda}_n(x, y_0)\}_{n \in \mathbb{N}^*}$  satisfies the hypothesis of the other version of the monotonic convergence theorem (see Theorem 5.7.27 of [21] p.234-235). By applying this theorem to the second member of (3.40), and the relation (2) of [20] p.168, to the first member of the same relation, we obtain when  $n$  tends to infinity.

$${}^tV_k(f)(y_0) \leq \int_{\mathbb{R}^d} \mathcal{K}^o(x_0, y) f(x) dx. \quad (3.41)$$

We deduce (3.37) from (3.39) and (3.41).

**Theorem 3.2.** There is a positive function  $\mathcal{K}(\cdot, y)$ ,  $y \in \mathbb{R}^d$ , locally integrable on  $\mathbb{R}^d$  with respect to the Lebesgue measure, such that for all  $f$  in  $C_c(\mathbb{R}^d)$  we have

$$\forall y \in \mathbb{R}^d, {}^tV_k(f)(y) = \int_{\mathbb{R}^d} \mathcal{K}(x, y) f(x) \omega_k(x) dx. \quad (3.42)$$

**Proof**

We deduce the results from Proposition 3.4 and the relation (3.18).

**Remark 3.5**

Theorem 3.2 shows that for all  $y \in \mathbb{R}^d$  the measure  $\nu_y$  is absolutely continuous with respect to the measure  $\omega_k(x) dx$ . More precisely for all  $y \in \mathbb{R}^d$  we have

$$d\nu_y(x) = \mathcal{K}(x, y) \omega_k(x) dx.$$

**Proposition 3.5.** For  $y \in \mathbb{R}^d$  and almost  $t > 0$ , we have

$$\frac{1}{d_k} \int_{S^{d-1}} \mathcal{K}(t\beta, y) \omega_k(\beta) d\sigma(\beta) = \frac{\Gamma(\gamma + \frac{d}{2}) d_k}{\pi^{\frac{d}{2}} \Gamma(\gamma)} t^{2-2\gamma-d} (t^2 - \|y\|^2)^{\gamma-1} 1_{\|y\|, +\infty[}(t), \quad (3.43)$$

where  $1_{\|y\|, +\infty[}$  is the characteristic function of the interval  $\|y\|, +\infty[$ .

**Proof.**

Let  $f$  be a radial function in  $\mathcal{D}(\mathbb{R}^d)$ . From (3.42) we have

$$\forall y \in \mathbb{R}^d, {}^tV_k(f)(y) = \int_{\mathbb{R}^d} \mathcal{K}(x, y) F(\|x\|) \omega_k(x) dx,$$

where  $F$  is the function in  $\mathcal{D}(\mathbb{R}_+)$  given by

$$\forall x \in \mathbb{R}^d, f(x) = F(\|x\|).$$

Using (1.6) and the fact that the support of  $\mathcal{K}(x, y)$  is contained in the set  $\{x \in \mathbb{R}^d / \|x\| \geq \|y\|\}$ , we obtain

$$\forall y \in \mathbb{R}^d, {}^tV_k(f)(y) = \int_{\|y\|}^{+\infty} \left( \int_{S^{d-1}} \mathcal{K}(t\beta, y) \omega_k(\beta) d\sigma(\beta) \right) F(t) t^{2\gamma+d-1} dt.$$

By applying Theorem 2.3 we deduce that for almost all  $t > 0$ :

$$t^{2\gamma+d-1} \int_{S^{d-1}} \mathcal{K}(t\beta, y) \omega_k(\beta) d\sigma(\beta) = \frac{\Gamma(\gamma + \frac{d}{2}) d_k}{\pi^{\frac{d}{2}} \Gamma(\gamma)} t^{2-2\gamma-d} (t^2 - \|y\|^2)^{\gamma-1} 1_{\|y\|, +\infty[}(t).$$

We obtain (3.43) from this relation.

### Example 3.2

From the relations (2.17), (2.18) we deduce that the relation (3.42) of the dual Dunkl intertwining operator  ${}^tV_k$  on  $\mathbb{R}^d$  with the reflection group  $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \dots \times \mathbb{Z}_2$  can also be written for all  $f$  in  $C_c(\mathbb{R}^d)$  in the form

$$\begin{aligned} \forall y \in \mathbb{R}^d, {}^tV_k(f)(y) &= \left[ \prod_{j=1}^d \frac{\Gamma(\alpha_j + \frac{1}{2})}{\sqrt{\pi} \Gamma(\alpha_j)} \right] \int_{|x_1| > |y_1|} \dots \int_{|x_d| > |y_d|} f(x_1, \dots, x_d) \\ &\quad \times \left[ \prod_{j=1}^d (|x_j| - y_j)^{\alpha_j} (|x_j| + y_j)^{\alpha_j - 1} \right] dx_1 \dots dx_d. \end{aligned}$$

## 4 Applications

In this section we suppose that the multiplicity function  $k$  satisfies the assumption (3.3).

### 4.1 First application

**Theorem 4.1.** We have

$$\forall x \in \mathbb{R}^d, \lim_{\|z\| \rightarrow +\infty} \{\omega_k(x) K(-ix, z)\} = 0. \quad (4.1)$$

and

$$\forall x \in \mathbb{R}_{reg}^d, \lim_{\|z\| \rightarrow +\infty} K(-ix, z) = 0. \quad (4.2)$$

**Proof**

- From Corollary 3.2 i), for all  $x \in \mathbb{R}^d$ , and  $z \in \mathbb{R}^d$  we have

$$\omega_k(x)K(-ix, z) = \int_{\mathbb{R}^d} \mathcal{K}^o(x, y)e^{-i\langle y, z \rangle} dy.$$

As for  $x \in \mathbb{R}^d$  the function  $\mathcal{K}^o(x, \cdot)$  is integrable with respect to the Lebesgue measure on  $\mathbb{R}^d$ , then we obtain the relation (4.1) from Riemann-Lebesgue Lemma for the usual Fourier transform on  $\mathbb{R}^d$ .

- Corollary 3.2 ii) and the same proof give the relation (4.2).

**Remark 4.1.**

Let  $C$  denotes the Weyl chamber attached with the positive subsystem  $R_+$ ,

$$C = \{x \in \mathbb{R}^d / \langle \alpha, x \rangle > 0, \text{ for all } \alpha \in R_+\},$$

and for  $\delta > 0$ ,

$$C_\delta = \{x \in C / \langle \alpha, x \rangle > \delta \|x\|, \text{ for all } \alpha \in R_+\}.$$

M.F.E de Jeu and M. Rösler have proved in [12] the following behaviour for the Dunkl kernel  $K(x, -iz)$ , uniform for the variable tending to infinity in cones  $C_\delta$  : There exists a constant non-zero vector  $v = \{v_w\}_{w \in W}$  such that for all  $x \in C, w \in W$  and each  $\delta > 0$ ,

$$\lim_{\|z\| \rightarrow +\infty, z \in C_\delta} \sqrt{\omega_k(x)\omega_k(z)} e^{i\langle wx, z \rangle} K(wx, -iz) = v_w. \quad (4.3)$$

**Corollary 4.1.** We have

$$\forall x \in \mathbb{R}^d, \lim_{\|z\| \rightarrow +\infty} \{\omega_k(x)J_W(-ix, z)\} = 0. \quad (4.4)$$

and

$$\forall x \in \mathbb{R}_{reg}^d, \lim_{\|z\| \rightarrow +\infty} J_W(-ix, z) = 0. \quad (4.5)$$

**Proof**

We deduce (4.4),(4.5) from Corollary 3.3 and Theorem 4.1.

## 4.2 Second application

**Theorem 4.2.** For all function  $h$  in  $C(\mathbb{R}^d)$  we have

$$\forall t > 0, \int_{S^{d-1}} V_k(h)(t\xi)\omega_k(\xi)d\sigma(\xi) = \frac{\Gamma(\gamma + \frac{d}{2})d_k}{\pi^{d/2}\Gamma(\gamma)} t^{2-2\gamma-d} \times \int_{B(0,t)} h(y)(t^2 - \|y\|^2)^{\gamma-1} dy, \quad (4.6)$$

where  $B(0, t)$  is the open ball of center 0 and radius  $t$ .

**Proof**

From Theorem 3.1 ii) the relation (3.43) and Fubini's theorem, for almost all  $t \in ]0, +\infty[$  we have

$$\begin{aligned} \int_{S^{d-1}} V_k(h)(t\xi)\omega_k(\xi)d\sigma(\xi) &= \int_{\mathbb{R}^d} \left[ \int_{S^{d-1}} \mathcal{K}(t\xi, y)d\sigma(\xi) \right] h(y)dy \\ &= ct^{2-2\gamma-d} \int_{\mathbb{R}^d} h(y)(t^2 - \|y\|^2)^{\gamma-1} 1_{\|y\|, +\infty[}(t)dy \end{aligned}$$

where

$$c = \frac{\Gamma(\gamma + \frac{d}{2})d_k}{\pi^{\frac{d}{2}}\Gamma(\gamma)}.$$

By using the spherical coordinates we obtain

$$\int_{S^{d-1}} V_k(h)(t\xi)\omega_k(\xi)d\sigma(\xi) = ct^{2-2\gamma-d} \int_0^t \left( \int_{S^{d-1}} h(\varrho\eta)d\sigma(\eta) \right) (t^2 - \varrho^2)^{\gamma-1} \varrho^{d-1} d\varrho.$$

Thus

$$\int_{S^{d-1}} V_k(h)(t\xi)\omega_k(\xi)d\sigma(\xi) = ct^{2-2\gamma-d} \int_{B(0,t)} h(y)(t^2 - \|y\|^2)^{\gamma-1} dy.$$

Using the properties of the operator  $V_k$  we deduce that the first member of this relation is continuous on  $]0, +\infty[$ . The second member possesses also the same property. Then this relation is true for all  $t \in ]0, +\infty[$ .

**Remarks 4.2.**

i) From the relations (2.1),(1.20) we deduce that for  $d = 1$  the analogue of the relation (4.6) is of the form

$$\forall x > 0, \frac{V_k(h)(x) + V_k(h)(-x)}{2} = \frac{\Gamma(\gamma + \frac{1}{2})}{\pi^{\frac{1}{2}}\Gamma(\gamma)} x^{1-2\gamma} \int_{-x}^x h(y)(x^2 - y^2)^{\gamma-1} dy.$$

ii) By using the  $\omega_k$ -harmonic polynomials Y.Xu has proved in [25] the relation (4.6) for  $t = 1$  and  $h$  a  $\omega_k$ -harmonic polynomial.

### 4.3 Third application

The generalized (or Dunkl) translation operators  $\tau_x$ ,  $x \in \mathbb{R}^d$ , are defined on  $\mathcal{E}(\mathbb{R}^d)$  by

$$\forall y \in \mathbb{R}^d, \tau_x f(y) = (V_k)_x (V_k)_y [(V_k)^{-1}(f)(x + y)].$$

(See [24] p.33-35).

They satisfies many properties, in particular we have

$$\tau_x f(o) = f(x), \tau_x f(y) = \tau_y f(x), \tau_x(1)(y) = 1. \quad (4.7)$$

At the moment an explicit formula for the generalized (or Dunkl) translation operators is known only in the following cases( see [18] [22]).

**1<sup>st</sup> cas** :  $d = 1$  and  $W = \mathbb{Z}_2$ .

For all  $f$  in  $C(\mathbb{R})$  and  $y \in \mathbb{R}$  we have

$$\begin{aligned} \tau_x f(y) &= \frac{1}{2} \int_{-1}^1 f(\sqrt{x^2 + y^2 - 2xyt}) \left(1 + \frac{x-y}{\sqrt{x^2 + y^2 - 2xyt}}\right) \Phi_k(t) dt \\ &+ \frac{1}{2} \int_{-1}^1 f(-\sqrt{x^2 + y^2 - 2xyt}) \left(1 - \frac{x-y}{\sqrt{x^2 + y^2 - 2xyt}}\right) \Phi_k(t) dt, \end{aligned} \quad (4.8)$$

where

$$\Phi_k(t) = \frac{\Gamma(k + \frac{1}{2})}{\sqrt{\pi}\Gamma(k)} (1+t)(1-t^2)^{k-1}.$$

If we consider  $f$  even and we make the change of variables  $u = yt$ , we obtain for all  $y \in \mathbb{R} \setminus \{o\}$ :

$$\tau_x f(y) = \frac{\Gamma(k + \frac{1}{2})}{\sqrt{\pi}\Gamma(k)} |y|^{-2k} \int_{-|y|}^{|y|} F(\sqrt{x^2 + y^2 - 2xu}) (|y| - u)^k (|y| + u)^{k-1} du, \quad (4.9)$$

where  $F$  is the restriction of  $f$  on  $[0, +\infty[$ .

By using the relations (3.22),(1.22) we deduce that

$$\forall y \in \mathbb{R} \setminus \{o\}, \tau_x f(y) = V_k[F(\sqrt{x^2 + y^2 - 2x.})](y). \quad (4.10)$$

**2<sup>nd</sup> cas:**  $d \geq 2$

For all  $f$  in  $\mathcal{E}(\mathbb{R}^d)$  radial, we have

$$\forall y \in \mathbb{R}^d, \tau_x f(y) = \int_{\mathbb{R}^d} F(\sqrt{\|x\|^2 + \|y\|^2 - 2\langle x, \eta \rangle}) d\mu_x(\eta), \quad (4.11)$$

where  $F$  is the function on  $[0, +\infty[$  given by  $f(x) = F(\|x\|)$ .

For  $f$  in  $\mathcal{E}(\mathbb{R}^d)$ , even for  $d = 1$  and radial for  $d \geq 2$ , the relations (4.10), (4.11) and (3.20) implies that for all  $x \in \mathbb{R}^d$  we have

$$\forall y \in \mathbb{R}_{reg}^d, \tau_x f(y) = \int_{\mathbb{R}^d} F(\sqrt{\|x\|^2 + \|y\|^2 - 2\langle x, \eta \rangle}) \mathcal{K}(y, \eta) d\eta. \quad (4.12)$$

**Theorem 4.3.** For all  $x \in \mathbb{R}^d \setminus \{o\}$  and  $y \in \mathbb{R}_{reg}^d$ , there exists a positive function  $\mathcal{W}(x, y, (t, t'))$  satisfying

$$\int_{[0, +\infty[ \times \mathbb{R}^{d-1}} \mathcal{W}(x, y, (t, t')) dt dt' = 1,$$

such that for all  $f$  in  $\mathcal{E}(\mathbb{R}^d)$ , even for  $d = 1$  and radial for  $d \geq 2$ , we have

$$\tau_x f(y) = \int_{[0, +\infty[ \times \mathbb{R}^{d-1}} F(t) \mathcal{W}(x, y, (t, t')) dt dt'. \quad (4.13)$$

**Proof**

We deduce the results of this theorem from the relation (4.12), the change of variables:

$$t = \sqrt{\|x\|^2 + \|y\|^2 - 2\langle x, \eta \rangle}, t'_1 = \eta_2, \dots, t'_{d-1} = \eta_d,$$

and the properties (4.7).

**Remark 4.3.**

Theorem 4.3 shows that the measure  $\varrho_{x,y}^k$  ( see Theorem 5.1 of [18]) of the integral representation of the generalized (or Dunkl) translation operators  $\tau_x$ ,  $x \in \mathbb{R}^d \setminus \{o\}$  corresponding to the preceding cases, are absolute continuous with respect to the Lebesgue measure.

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