

Short proofs of some results of algebraic independence in non-zero characteristic.

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Abstract. The aim of this note is to give short and almost elementary proofs of two theorems, by Papanikolas [11] and Chang-Yu [5], on algebraic independence of Carlitz logarithms and values of Carlitz-Goss zeta function, modifying and generalising arguments of Denis [8] which proved earlier special cases of these results. These proofs were sketched in the text [12] and this note is intended to accompany it, somewhat as an appendix, by giving full details to some few lines remarks.

1 Introduction.

In [11] Papanikolas proved the analog of a well known conjecture of algebraic independence of logarithms of algebraic numbers, for logarithms of algebraic points on the $\mathbb{F}_q[T]$ -module of Carlitz. The proof is a consequence of a variant of Grothendieck period conjecture for a certain tannakian category of t -motives, also due Papanikolas. Another application of this variant of Grothendieck conjecture is due to Chang and Yu [5]. They determine all the algebraic relations between values of Carlitz-Goss zeta function at integers $n \geq 1$, hence proving the analogue of a well known conjecture describing algebraic relations between values of the Riemann zeta function at integers greater than or equal to two. The proofs these author gave require the computation of Galois groups associated to certain linear systems of order one σ -difference equations, somewhat involved essentially due to the multitude of algebraic subgroups of \mathbb{G}_a^n in positive characteristic. Particular cases of these results are also contained in Denis work [8], where he applies so-called Mahler's method without appealing to Galois theory. In this text we will show how Mahler-Denis method extends to prove the results above mentioned directly.

1.1 Statement of the results.

Let $q = p^r$ be a power of a prime number, \mathbb{F}_q the field with q elements, T an indeterminate, and let us write $A = \mathbb{F}_q[T]$, $K = \mathbb{F}_q(T)$. The valuation opposite of the degree (in T) $v : K \rightarrow \mathbb{Z} \cup \{\infty\}$ defined by $v(a/b) = \deg b - \deg a$ defines the norm $|\cdot|$ by $|x| = q^{-v(x)}$.

Let us write $K_\infty = \mathbb{F}_q((T^{-1}))$ (completion of K for v). The degree of an algebraic closure $\overline{K_\infty}$ over K_∞ is infinite and $\overline{K_\infty}$ is not complete for the unique extension of $|\cdot|$.

Let C be the completion of $\overline{K_\infty}$ for $|\cdot|$. It is well known that it is algebraically closed with residual field $\overline{\mathbb{F}_q}$. We write \overline{K} for an algebraic closure of K and we fix embeddings $\overline{K} \subset \overline{K_\infty} \subset C$. Elements of C are said *numbers*, elements of \overline{K} are said *algebraic numbers*.

1.1.1 Carlitz's exponential and logarithm functions.

Let us write $[i] := T^{q^i} - T$ ($i \geq 1$) and $[0] := 1$, let us consider Carlitz's *exponential* and *logarithm*:

$$e_{\text{Car}}(z) = \sum_{i \geq 0} \frac{z^{q^i}}{[i][i-1]^q \cdots [1]^{q^{i-1}}[0]^{q^i}}, \quad \log_{\text{Car}}(z) = \sum_{i \geq 0} \frac{(-1)^i z^{q^i}}{[i][i-1] \cdots [1][0]}.$$

The series e_{Car} converges uniformly on every open ball with center in 0 to an \mathbb{F}_q -linear surjective function $e_{\text{Car}} : C \rightarrow C$. Its kernel can be written as $\bar{\pi}A$, where

$$\bar{\pi} = T(-T)^{1/(q-1)} \prod_{i=1}^{\infty} (1 - T^{1-q^i})^{-1}, \quad (1)$$

convergent product defined once a choice of a $(q-1)$ -th root of $-T$ is made (the number $\bar{\pi}$ is defined up to multiplication by an element of \mathbb{F}_q^\times). Carlitz's exponential function e_{Car} allows to endow C with an action of A by polynomials so that the short exact sequence of A -modules holds:

$$0 \rightarrow \bar{\pi}A \rightarrow C \rightarrow C \rightarrow 0,$$

drinfeldian analogue of the short exact sequence for the exponential function $\exp : \mathbb{C} \rightarrow \mathbb{C}^\times$, $\exp(0) = 1$ (the A -action in the middle is the usual multiplication). The formal series \log_{Car} , reciprocal of e_{Car} in 0, converges for $|z| < q^{q/(q-1)} = |\bar{\pi}|$.

The first Theorem we shall prove in an elementary way is the following:

Theorem 1 (Papanikolas) *Let $\ell_1, \dots, \ell_m \in C$ be such that $e_{\text{Car}}(\ell_i) \in \bar{K}$ ($i = 1, \dots, m$). If ℓ_1, \dots, ℓ_m are linearly independent over K , then they also are algebraically independent.*

1.1.2 Carlitz-Goss polylogarithms and zeta functions.

Let us write $A_+ = \{a \in A, a \text{ monic}\}$. In [9], Goss introduced a function ζ , defined over $C \times \mathbb{Z}_p$ with values in C , such that

$$\zeta(T^n, n) = \sum_{a \in A_+} \frac{1}{a^n} \in K_\infty, \quad n \geq 1.$$

In the following, we will write $\zeta(n)$ for $\zeta(T^n, n)$. For $n \in \mathbb{N}$, let us also write $\Gamma(n) := \prod_{i=0}^s D_i^{n_i} \in K$, $n_0 + n_1q + \cdots + n_sq^s$ being the expansion of $n-1$ in base q and D_i being the polynomial $[i][i-1]^q \cdots [1]^{q^{i-1}}$. It can be proved that $z/e_{\text{Car}}(z) = \sum_{n=0}^{\infty} B_n \frac{z^n}{\Gamma(n+1)}$ for certain $B_n \in K$. The so-called Bernoulli-Carlitz relations can be obtained by a computation involving the logarithmic derivative of $e_{\text{Car}}(z)$: for all $m \geq 1$,

$$\frac{\zeta(m(q-1))}{\bar{\pi}^{m(q-1)}} = \frac{B_m}{\Gamma(m(q-1)+1)} \in K. \quad (2)$$

In particular, one sees that

$$\bar{\pi}^{q-1} = (T^q - T) \sum_{a \in A_+} a^{1-q} \in K_\infty.$$

We also have the rather obvious relations:

$$\zeta(mp^k) = \zeta(m)^{p^k}, \quad m, k \geq 1. \quad (3)$$

The second theorem we are going to prove directly is:

Theorem 2 (Chang, Yu) *The algebraic dependence relations over K between the numbers*

$$\bar{\pi}, \zeta(1), \zeta(2), \dots$$

are generated by Bernoulli-Carlitz relations (2) and the relations (3).

1.2 Two propositions.

We consider a perfect field U of characteristic p containing \mathbb{F}_q and a \mathbb{F}_q -automorphism $\alpha : U \rightarrow U$. Let us also consider the ring $\mathcal{R} = U[X_1, \dots, X_N]$ and write, for a polynomial $P = \sum_{\underline{\lambda}} c_{\underline{\lambda}} X^{\underline{\lambda}} \in \mathcal{R}$, P^α as the polynomial $\sum_{\underline{\lambda}} \alpha(c_{\underline{\lambda}}) X^{\underline{\lambda}}$. Let A_1, \dots, A_N be elements of U^\times , B_1, \dots, B_N be elements of U and, for a polynomial $P \in \mathcal{R}$, let us write

$$\tilde{P} = P^\alpha(A_1 X_1 + B_1, \dots, A_N X_N + B_N).$$

We now prove the following two Propositions, which provide together the analogue in positive characteristic of a result of Kubota [10, Theorem 2] (see also [13, Lemma 6]).

Proposition 1.1 *Let $P \in \mathcal{R}$ be a non-constant polynomial such that $\tilde{P}/P \in \mathcal{R}$. Then there exists a polynomial $G \in \mathcal{R}$ of the form $G = \sum_i c_i X_i + B^p$ such that $\tilde{G}/G \in \mathcal{R}$, where $c_1, \dots, c_N \in U$ are not all vanishing and $B \in \mathcal{R}$. If W is the subfield generated by \mathbb{F}_q and the coefficients of P , then there exists $M \geq 1$ such that for each coefficient c of G , $c^{p^M} \in W$.*

Proof. If $P \in \mathcal{R}$ is such that $\tilde{P} = QP$ for $Q \in \mathcal{R}$ one sees, comparing the degrees of \tilde{P} and P , that $Q \in U$ and if P is non-zero, $Q \neq 0$. The subset of \mathcal{R} of these polynomials is a semigroup \mathcal{S} containing U . If $P \in \mathcal{S}$ satisfies $\tilde{P} = QP$, then $F := \partial P / \partial X_i$ belongs to \mathcal{S} since $\tilde{F} = A_i^{-1} Q F$. Similarly, if $P = F^p \in \mathcal{S}$ with $F \in \mathcal{S}$ then $F \in \mathcal{S}$ as one sees easily that in this case, $\tilde{F} = Q^{1/p} F$.

By hypothesis, \mathcal{S} contains a non-constant polynomial P . We now show that the polynomial $G \in \mathcal{S}$ as in the Proposition can be constructed by iterated applications of partial derivatives $\partial_1 = \partial / \partial X_1, \dots, \partial_N = \partial / \partial X_N$ and p -root extractions starting from P .

Let P be as in the hypotheses. We can assume that P is not a p -th power. We can write:

$$P = \sum_{\underline{\lambda}=(\lambda_1, \dots, \lambda_N) \in \{0, \dots, p-1\}^N} c_{\underline{\lambda}} X^{\underline{\lambda}}, \quad c_{\underline{\lambda}} \in \mathcal{R}^p.$$

Let $M := \max\{\lambda_1 + \dots + \lambda_N, c_{\underline{\lambda}} \neq 0\}$. We can write $P = P_1 + P_2$ with

$$P_1 := \sum_{\lambda_1 + \dots + \lambda_N = M} c_{\underline{\lambda}} X^{\underline{\lambda}}.$$

There exists $(\beta_1, \dots, \beta_N) \in \{0, \dots, p-1\}^N$ with $\beta_1 + \dots + \beta_N = M-1$ and

$$P' := \partial_1^{\beta_1} \dots \partial_N^{\beta_N} P = \sum_{i=1}^N c'_i X_i + c'_0 \in \mathcal{S} \setminus \{0\}, \quad c'_0, c'_1, \dots, c'_N \in \mathcal{R}^p,$$

where

$$\partial_1^{\beta_1} \dots \partial_N^{\beta_N} P_1 = \sum_{i=1}^N c'_i X_i, \quad \partial_1^{\beta_1} \dots \partial_N^{\beta_N} P_2 = c'_0.$$

If (case 1) the polynomials c'_1, \dots, c'_N are all in U , then we are done. Otherwise, (case 2), there exists i such that c'_i is non-constant (its degree in X_j is then $\geq p$ for some j). Now, $c'_i = \partial_i P'$ belongs to $(\mathcal{R}^p \cap \mathcal{S}) \setminus \{0\}$ and there exists $s > 0$ with $c'_i = P''^{p^s}$ with $P'' \in \mathcal{S}$ which is not a p -th power. We have constructed an element P'' of \mathcal{S} which is not a p -th power, whose degrees in X_j are all strictly smaller than those of P for all j (if the polynomial depends on X_j).

We can repeat this process with P'' at the place of P and so on. Since at each stage we get a polynomial P'' with partial degrees in the X_j strictly smaller than those of P for all j (if P'' depends on X_j), we eventually terminate with a polynomial P which has all the partial degrees $< p$ in the indeterminates on which it depends, for which the case 1 holds.

As for the statement on the field W , we remark that we have applied to P an algorithm which constructs G from P applying finitely many partial derivatives and p -th roots extractions successively, the only operations bringing out of the field W being p -root extractions. Hence, the existence of the integer M is guaranteed. □

We denote by U_0 the subfield of U whose elements are the $x \in U$ such that $\alpha(x) = x$. Let V be a subgroup of U^\times such that $V \setminus V^p \neq \emptyset$.

Proposition 1.2 *Under the hypotheses of Proposition 1.1, let us assume that for all $A \in V \setminus \{1\}$, the only solution $x \in U$ of $\alpha(x) = Ax$ is zero and that $A_1, \dots, A_N \in V \setminus V^p$. Then, the polynomial $G \in \mathcal{R}$ given by this Proposition is of the form $G = \sum_i c_i X_i + c_0$ with $c_1, \dots, c_N \in U_0$ and $c_0 \in U$. Moreover, if $c_i, c_j \neq 0$ for $1 \leq i < j \leq N$, then $A_i = A_j$. Let \mathcal{I} be the non-empty subset of $\{1, \dots, N\}$ whose elements i are such that $c_i \neq 0$, let $A_i = A$ for all $i \in \mathcal{I}$. Then,*

$$c_0 = \frac{\alpha(c_0)}{A} + \frac{1}{A} \sum_{i \in \mathcal{I}} c_i B_i.$$

Proof. Proposition 1.1 gives us a polynomial G with $\tilde{G}/G \in \mathcal{R}$, of the form $\sum_i c_i X_i + B^p$ with $c_i \in U$ not all vanishing and $B \in \mathcal{R}$. Let $dX^{p\lambda}$ be a monomial of maximal degree in B^p . Since $\tilde{G} = QG$ with $Q \in U^\times$, we have $\alpha(d) = (A_1^{\lambda_1} \dots A_N^{\lambda_N})^{-p} Qd$. Moreover, $\alpha(c_i) = A_i^{-1} Qc_i$ for all i . Hence, if i is such that $c_i \neq 0$, $r := d/c_i$ satisfies $\alpha(r) = A_i(A_1^{\lambda_1} \dots A_N^{\lambda_N})^{-p} r$. Now, $A_i(A_1^{\lambda_1} \dots A_N^{\lambda_N})^{-p} \neq 1$ (because $A_i \in V \setminus V^p$) and $r = 0$, that is $d = 0$. This shows that $B \in U$. Let us suppose that $1 \leq i, j \leq N$ are such that $i \neq j$ and

$c_i, c_j \neq 0$. Let us write $r = c_i/c_j$; we have $\alpha(r) = A_j/A_i r$, from which we deduce $r \in U_0$ in case $A_j/A_i = 1$ and $r = 0$ otherwise. The Proposition is proved dividing $\sum_i c_i X_i + B^p$ by c_j with $j \neq 0$ and by considering the relation $\tilde{P} = QP$, once observed that $Q = A$. \square

2 Direct proof of Theorem 1.

For $\beta \in \bar{K}$ such that $|\beta| < q^{q/(q-1)}$, we will use the formal series in $\bar{K}((t))$

$$L_\beta(t) = \beta + \sum_{i=1}^{\infty} \frac{(-1)^i \beta^{q^i}}{(T^q - t) \cdots (T^{q^i} - t)},$$

defining holomorphic functions for $|t| < q^q$ with $L_\beta(T) = \log_{\text{Car}} \beta$ ⁽¹⁾.

We denote by W one of the following fields: $\bar{K}, \bar{K}_\infty, C$. For $f = \sum_i c_i t^i \in W((t))$ and $n \in \mathbb{Z}$ we write $f^{(n)} := \sum_i c_i^{q^n} t^i \in W((t))$, so that $f^{(-1)} = \sum_i c_i^{1/q} t^i$. We have the functional equation:

$$L_\beta^{(-1)}(t) = \beta^{1/q} + \frac{L_\beta(t)}{t - T}.$$

This implies that L_β allows meromorphic continuation to the whole C , with simple poles at the points $T^q, T^{q^2}, \dots, T^{q^n}, \dots$ of residue

$$(\log_{\text{Car}} \beta)^q, \frac{(\log_{\text{Car}} \beta)^{q^2}}{T^{q^2} - T^q}, \dots, \frac{(\log_{\text{Car}} \beta)^{q^n}}{(T^{q^n} - T^q)(T^{q^n} - T^{q^2}) \cdots (T^{q^n} - T^{q^{n-1}})}, \dots \quad (4)$$

Let β_1, \dots, β_m be algebraic numbers with $|\beta_i| < q^{q/(q-1)}$, let us write $L_i = L_{\beta_i}$ for $i = 1, \dots, m$. Let us also consider the infinite product (once a choice of a $(q-1)$ -th root of $-T$ is performed):

$$\Omega(t) = (-T)^{-q/(q-1)} \prod_{i=1}^{\infty} (1 - t/T^{q^i}),$$

converging everywhere to a entire holomorphic function with zeros at T^q, T^{q^2}, \dots , and write $L_0 = -\Omega^{-1}$, satisfying the functional equation

$$L_0^{(-1)}(t) = \frac{L_0(t)}{t - T},$$

with $L_0(T) = \bar{\pi}$, meromorphic with simple poles at the points $T^q, T^{q^2}, \dots, T^{q^n}, \dots$, with residues

$$\bar{\pi}^q, \frac{\bar{\pi}^{q^2}}{T^{q^2} - T^q}, \dots, \frac{\bar{\pi}^{q^n}}{(T^{q^n} - T^q)(T^{q^n} - T^{q^2}) \cdots (T^{q^n} - T^{q^{n-1}})}, \dots \quad (5)$$

We now prove the following Proposition.

¹Papanikolas uses this series in [11]. It is also possible to work with the series $\sum_{i=0}^{\infty} e_{\text{Car}}((\log_{\text{Car}} \beta)/T^{i+1})t^i$.

Proposition 2.1 *If the functions L_0, L_1, \dots, L_m are algebraically dependent over $\overline{K}(t)$, then $\overline{\pi}, \log_{\text{Car}} \beta_1, \dots, \log_{\text{Car}} \beta_m$ are linearly dependent over K .*

Proof. The functions L_i having infinitely many poles, are transcendental. Without loss of generality, we may assume that $m \geq 1$ is minimal so that for all $0 \leq n \leq m$ the functions obtained from the family (L_0, L_1, \dots, L_m) discarding L_n are algebraically independent over $\overline{K}(t)$.

We now apply Propositions 1.1 and 1.2. We take $U := \bigcup_{n \geq 0} \overline{K}(t^{1/p^n})$, which is perfect, and $\alpha : U \rightarrow U$ the q -th root map on \overline{K} (inverse of the Frobenius map), such that $\alpha(t) = t$; this is an \mathbb{F}_q -automorphism. Moreover, we take $N = m + 1$, $A_1 = \dots = A_N = (t - T)^{-1}$,

$$(B_1, \dots, B_N) = (0, \beta_1^{1/q}, \dots, \beta_m^{1/q}),$$

and $V = (t - T)^{\mathbb{Z}}$.

Let $\mathcal{T} \subset C[[t]]$ be the subring of formal series converging for all $t \in C$ with $|t| \leq 1$, let \mathcal{L} be its fraction field. Let $f \in \mathcal{L}$ be non-zero. A variant of Weierstrass preparation theorem (see [1, Lemma 2.9.1]) yields a unique factorisation:

$$f = \lambda \left(\prod_{|a|_\infty \leq 1} (t - a)^{\text{ord}_a(f)} \right) \left(1 + \sum_{i=1}^{\infty} b_i t^i \right), \quad (6)$$

where $0 \neq \lambda \in C$, $\sup_i |b_i| < 1$, and $|b_i| \rightarrow 0$, the product being over a finite index set. Taking into account (6), it is a little exercise to show that $U_0 = \bigcup_{i \geq 0} \mathbb{F}_q(t^{1/p^i})$ and that for $A \in V \setminus \{1\}$, the solutions in U of $f^{(-1)} = Af$ are identically zero (for this last statement, use the transcendence over U of Ω).

Let $P \in \mathcal{R}$ be an irreducible polynomial such that $P(L_0, L_1, \dots, L_m) = 0$; we clearly have $\tilde{P} = QP$ with $Q \in U$ and Propositions 1.1 and 1.2 apply to give $c_1(t), \dots, c_m(t) \in U_0$ not all zero and $c \in U$ such that

$$c(t) = (t - T)c^{(-1)}(t) + (t - T) \sum_{i=1}^m c_i(t) \beta_i^{1/q}. \quad (7)$$

We get, for all $k \geq 0$:

$$\begin{aligned} c(t) &= - \sum_{i=1}^m c_i(t) \left(\beta_i + \sum_{h=1}^k \frac{(-1)^h \beta_i^{q^h}}{(T^q - t)(T^{q^2} - t) \dots (T^{q^h} - t)} \right) \\ &\quad + \frac{c^{(k+1)}(t)}{(T^q - t)(T^{q^2} - t) \dots (T^{q^{k+1}} - t)}. \end{aligned} \quad (8)$$

We endow \mathcal{L} with a norm $\|\cdot\|$ in the following way: if $f \in \mathcal{L}^\times$ factorises as in (6), then $\|f\| := |\lambda|$. Let g be a positive integer. Then $\|\cdot\|$ extends in a unique way to the subfield $\mathcal{L}_g := \{f : f^{p^g} \in \mathcal{L}\}$. If $(f_i)_{i \in \mathbb{N}}$ is a uniformly convergent sequence in \mathcal{L}_g (on a certain closed ball centered at 0) such that $\|f_i\| \rightarrow 0$, then $f_i \rightarrow 0$ uniformly.

We observe that there exists $g \geq 0$ such that $c(t), c_1(t), \dots, c_m(t) \in \mathcal{L}_g$. Hence $c_1(t), \dots, c_m(t) \in \mathbb{F}_q(t^{1/p^g})$ and $\|c_i\| = 1$ if $c_i \neq 0$. This implies that

$$\left\| \sum_{i=1}^m c_i \beta_i^{1/q} \right\| \leq \max_i \{|\beta_i^{1/q}|\} < q^{1/(q-1)}.$$

By (7), $\|c\| \leq q^{q/(q-1)}$. Indeed, two cases occur. The first case when $\|c^{(-1)}\| \leq \max_i \{|\beta_i^{1/q}|\}$; here we have $\|c\| < q^{q/(q-1)}$ because $\|c^{(-1)}\| = \|c\|^{1/q}$ by (6) and $\max_i \{|\beta_i|\} < q^{q/(q-1)}$ by hypothesis. The second case when $\|c^{(-1)}\| > \max\{|\beta_1^{1/q}|, \dots, |\beta_m^{1/q}|\}$. In this case, $\max\{\|c^{(-1)}(t-T)\|, \|(t-T) \sum_{i=1}^m c_i(t) \beta_i^{1/q}\|\} = \|c^{(-1)}(t-T)\|$ which yields $\|c\| = q^{q/(q-1)}$ by (7).

Going back to (8) we see that the sequence of functions

$$E_h(t) = \frac{c^{(h+1)}(t)}{(T^q - t)(T^{q^2} - t) \dots (T^{q^{h+1}} - t)}$$

converges uniformly in every closed ball included in $\{t : |t| < q^q\}$, as the series defining the functions L_i ($i = 1, \dots, m$) do. We want to compute the limit of this sequence: we have two cases.

First case. We assume that $\|c\| < q^{q/(q-1)}$; there exists $\epsilon > 0$ such that $\|c\| = q^{(q-\epsilon)/(q-1)}$. Then, for all $h \geq 0$, $\|c^{(h+1)}\| = \|c\|^{q^{h+1}} = q^{(q^{h+2} - \epsilon q^{h+1})/(q-1)}$. On the other side:

$$\begin{aligned} \|(T^q - t)(T^{q^2} - t) \dots (T^{q^{h+1}} - t)\| &= |T|^{q + \dots + q^{h+1}} \\ &= q^{q(q^{h+1}-1)/(q-1)}. \end{aligned}$$

Hence,

$$\|E_h\| = q^{\frac{q^{h+2} - \epsilon q^{h+1}}{q-1} - \frac{q^{h+2} - q}{q-1}} = q^{\frac{q - \epsilon q^{h+1}}{q-1}} \rightarrow 0,$$

which implies $E_h \rightarrow 0$ (uniformly on every ball as above). This means that $\sum_{i=1}^m c_i(t)L_i(t) + c(t) = 0$. Let g be minimal such that there exists a non-trivial linear relation as above, with $c_1, \dots, c_m \in U_0 \cap \mathcal{L}_g$; we claim that $g = 0$. Indeed, if $g > 0$, $c_1, \dots, c_m \notin \mathbb{F}_q$ and there exists a non-trivial relation $\sum_{i=1}^m d_i(t)L_i(t)^{p^g} + d(t) = 0$ with $d_1, \dots, d_m \in \mathbb{F}_q[t]$ not all zero, $d(t) \in \overline{K}(t)$ and $\max_i \{\deg_t d_i\}$ minimal, non-zero. But letting the operator d/dt act on this relation we get a non-trivial relation with strictly lower degree because $dF^p/dt = 0$, leading to a contradiction.

Hence, $g = 0$ and $c_1, \dots, c_m \in \mathbb{F}_q(t)$. This also implies that $c \in \overline{K}$; multiplying by a common denominator, we get a non-trivial relation $\sum_{i=1}^m c_i(t)L_i(t) + c(t) = 0$ with $c_1, \dots, c_m \in \mathbb{F}_q[t]$ and $c \in \overline{K}(t)$. The function c being algebraic, it has finitely many poles. This means that

$$\sum_{i=1}^m c_i(t)L_i(t)$$

has finitely many poles but for all i , L_i has poles at T^q, T^{q^2}, \dots with residues as in (4), which implies that $\sum_{i=1}^m c_i(t)L_i(t)$ has poles in T^q, T^{q^2}, \dots . Since the functions c_i belong

to $\mathbb{F}_q[t]$, they vanish only at points of absolute value 1, and the residues of the poles are multiples of $\sum_{i=0}^m c_i(T)q^k (\log_{\text{Car}} \beta_i)^{q^k}$ ($k \geq 1$) by non-zero factors in A . They all must vanish: this happens if and only if $\sum_{i=1}^m c_i(T) \log_{\text{Car}} \beta_i = 0$, where we also observe that $c_i(T) \in K$; the Proposition follows in this case.

Second case. Here we know that the sequence E_h converges, but not to 0 and we must compute its limit. Let ν be in C with $|\nu| = 1$. Then, there exists $\mu \in \overline{\mathbb{F}_q}^\times$, unique such that $|\nu - \mu| < 1$. Hence, if $\lambda \in C$ is such that $|\lambda| = q^{q/(q-1)}$, there exists $\mu \in \overline{\mathbb{F}_q}^\times$ unique with

$$|\lambda - \mu(-T)^{q/(q-1)}| < q^{q/(q-1)}. \quad (9)$$

We have:

$$c(t) = \lambda \prod_{|a| \leq 1} \left(t^{1/p^g} - a \right)^{\text{ord}_a c} \left(1 + \sum_{i \geq 1} b_i t^{i/p^g} \right),$$

with $\lambda \in C^\times$, the product being finite and $|b_i| < 1$ for all i so that $\|c\| = |\lambda|$.

Let $\mu \in \overline{\mathbb{F}_q}^\times$ be such that (9) holds, and write:

$$\begin{aligned} c_1(t) &= (\lambda - \mu(-T)^{q/(q-1)}) \prod_{|a| \leq 1} \left(t^{1/p^g} - a \right)^{\text{ord}_a c} \left(1 + \sum_{i \geq 1} b_i t^{i/p^g} \right), \\ c_2(t) &= \mu(-T)^{q/(q-1)} \prod_{|a| \leq 1} \left(t^{1/p^g} - a \right)^{\text{ord}_a c} \left(1 + \sum_{i \geq 1} b_i t^{i/p^g} \right), \end{aligned} \quad (10)$$

so that $c(t) = c_1(t) + c_2(t)$, $\|c_1\| < q^{q/(q-1)}$ and $\|c_2\| = q^{q/(q-1)}$. For all h , we also write:

$$E_{1,h}(t) = \frac{c_1^{(h+1)}(t)}{(T^q - t)(T^{q^2} - t) \cdots (T^{q^{h+1}} - t)}, \quad E_{2,h}(t) = \frac{c_2^{(h+1)}(t)}{(T^q - t)(T^{q^2} - t) \cdots (T^{q^{h+1}} - t)}.$$

Following the first case, we easily check that $E_{1,h}(t) \rightarrow 0$ on every closed ball of center 0 included in $\{t : |t| < q^q\}$. It remains to compute the limit of $E_{2,h}(t)$.

We look at the asymptotic behavior of the images of the factors in (10) under the operators $f \mapsto f^{(n)}$, $n \rightarrow \infty$. The sequence of functions $(1 + \sum_{i \geq 1} b_i t^{i/p^g})^{(n)}$ converges to 1 for $n \rightarrow \infty$ uniformly on every closed ball as above. Let \mathcal{E} be the finite set of the a 's involved in the finite product (10), take $a \in \mathcal{E}$. If $|a| < 1$, then $a^{(n)} \rightarrow 0$ and $(t^{1/p^g} - a)^{(n)} \rightarrow t^{1/p^g}$. If $|a| = 1$, there exists $\mu_a \in \overline{\mathbb{F}_q}^\times$ such that $|a - \mu_a| < 1$ and we can find $n_a > 0$ integer such that $\lim_{s \rightarrow \infty} a^{(sn_a)} = \mu_a$, whence $\lim_{s \rightarrow \infty} (t^{1/p^g} - a)^{(sn_a)} = t^{1/p^g} - \mu_a$.

Let us also denote by $\tilde{n} > 0$ the smallest positive integer such that $\mu^{q^{\tilde{n}}} = \mu$. Let N be the lowest common multiple of \tilde{n} and the n_a 's with a varying in \mathcal{E} . Then the sequence of functions:

$$\left(\left(\prod_{|a| \leq 1} \left(t^{1/p^g} - a \right)^{\text{ord}_a c} \right) \left(1 + \sum_{i \geq 1} b_i t^{i/p^g} \right) \right)^{(Ns)}, \quad s \in \mathbb{N}$$

converges to a non-zero element Z of $\overline{\mathbb{F}_q}(t^{1/p^g}) \in U$.

For $n \in \mathbb{N}$, let us write:

$$V_n(t) := \mu^{q^n} \frac{(-T)^{q^{n+1}/(q-1)}}{(T^q - t)(T^{q^2} - t) \cdots (T^{q^{n+1}} - t)}.$$

We have:

$$\begin{aligned} (-T)^{q/(q-1)} \prod_{i=1}^{n+1} \left(1 - \frac{t}{T^{q^i}}\right)^{-1} &= (-1)^{q/(q-1)} T^{q/(q-1)} T^{q(q+\cdots+q^{n+1})} \prod_{i=1}^{n+1} (T^{q^i} - t)^{-1} \\ &= (-1)^{q/(q-1)} T^{q^{n+2}/(q-1)} \prod_{i=1}^{n+1} (T^{q^i} - t)^{-1}. \end{aligned}$$

Hence $\lim_{n \rightarrow \infty} T^{q/(q-1)} / ((T^q - t)(T^{q^2} - t) \cdots (T^{q^{n+1}} - t))^{-1} = \Omega(t)^{-1}$ from which we deduce that $\lim_{s \rightarrow \infty} E_{2,sN}(t) = c_0(t)L_0(t)$ with $c_0 \in \overline{\mathbb{F}_q}^\times(t^{1/p^g})$. We have proved that for some $c_1, \dots, c_m \in \mathbb{F}_q(t^{1/p^g})$, $c_0 \in \overline{\mathbb{F}_q}(t^{1/p^g})^\times$ and $c \in \overline{K}(t^{1/p^g})$, $\sum_{i=0}^m c_i L_i + c = 0$. Applying the same tool used in the first case we can further prove that in fact, $g = 0$. If c_0 is not defined over \mathbb{F}_q , then applying the operator $f \mapsto f^{(-1)}$ we get another non-trivial relation $c'_0 + \sum_{i=1}^m c_i L_i = c'$ with $c' \in \overline{K}(t)$ and $c'_0 \in \overline{\mathbb{F}_q}^\times(t)$ not equal to c_0 ; subtracting it from the former relation yields $L_0 \in \overline{K}(t)$ which is impossible since Ω is transcendental over $C(t)$. Hence $c_0 \in \mathbb{F}_q(t)$ too. Multiplying by a common denominator in $\mathbb{F}_q[t]$ and applying arguments of the first case again (by using the explicit computation of the residues of the poles of L_0 at T^q, T^{q^2}, \dots), we find a non-trivial relation $c_0(T)\overline{\pi} + \sum_{i=1}^m c_i(T) \log_{\text{Car}} \beta_i = 0$. \square

Proof of Theorem 1. If $\ell \in C$ is such that $e_{\text{Car}}(\ell) \in \overline{K}$, then there exist $a, b \in A$, $\beta \in \overline{K}$ with $|\beta| < q^{q/(q-1)}$ such that $\ell = a \log_{\text{Car}} \beta + b\overline{\pi}$. This well known property (also used in [11], see Lemma 7.4.1), together with Theorem 3.1.1 of [3], implies Theorem 1. \square

3 Direct proof of Theorem 2.

Let $s \geq 1$ be an integer and let Li_n denote the s -th Carlitz's *polylogarithm*:

$$\text{Li}_s(z) = \sum_{k=0}^{\infty} \frac{(-1)^{ks} z^{q^k}}{([k][k-1] \cdots [1])^s},$$

so that $\text{Li}_1(z) = \log_{\text{Car}}(z)$ (the series $\text{Li}_s(z)$ converges for $|z| < q^{sq/(q-1)}$).

For $\beta \in \overline{K} \cap K_\infty$ such that $|\beta| < q^{sq/(q-1)}$, we will use as in [8] the series

$$F_{s,\beta}(x) = \tilde{\beta}(x) + \sum_{i=1}^{\infty} \frac{(-1)^{is} \tilde{\beta}(x)^{q^i}}{(x^q - T)^s \cdots (x^{q^i} - T)^s},$$

where $\tilde{\beta}(x)$ is the formal series in $\mathbb{F}_q((1/x))$ obtained from the formal series of $\beta \in \mathbb{F}_q((1/T))$ by replacing T with x , an independent indeterminate. These series define

holomorphic functions for $|x| > q^{1/q}$ allowing meromorphic continuations to the whole $C \setminus \{0\}$, with poles at the points T^{1/q^i} .

We have the functional equations:

$$F_{s,\beta}(x^q) = (x^q - T)^s (F_{s,\beta}(x) - \tilde{\beta}(x)),$$

moreover,

$$F_{s,\beta}(T) = \beta_j + \sum_{i=1}^{\infty} \frac{(-1)^i \beta^{q^i}}{(T^q - T)^s \dots (T^{q^i} - T)^s} = \text{Li}_s(\beta).$$

Let \mathcal{J} be a finite non-empty subset of $\{1, 2, \dots\}$ such that if $n \in \mathcal{J}$, p does not divide n . Let us consider, for all $s \in \mathcal{J}$, an integer $l_s \geq 1$ and elements $\beta_{s,1}, \dots, \beta_{s,l_s} \in \overline{K} \cap K_{\infty}$ with $|\beta_{s,i}| < q^{qs/(q-1)}$ ($i = 1, \dots, l_s$). We remark that if s is divisible by $q-1$ then, for all $r > q^{1/q}$ the product:

$$(-x)^{sq/(q-1)} \prod_{i=1}^{\infty} \left(1 - \frac{T}{x^{q^i}}\right)^{-s}$$

converges uniformly in the region $\{x \in C, |x| \geq r\}$ to a holomorphic function $F_{s,0}(x)$, which is the $(q-1)$ -th power of a formal series in $K((1/(-x)^{1/(q-1)}))$, hence in $K((1/x))$. Moreover, $F_{s,0}(T) = \pi^s$.

Proposition 3.1 *If the functions $(F_{s,\beta_{s,1}}, \dots, F_{s,\beta_{s,l_s}})_{s \in \mathcal{J}}$ are algebraically dependent over $\overline{K}(x)$, there exists $s \in \mathcal{J}$ and a non-trivial relation*

$$\sum_{i=1}^{l_s} c_i F_{s,\beta_{s,i}}(x) = f(x) \in \overline{K}(x)$$

with $c_1, \dots, c_{l_s} \in \overline{K}$ if $q-1$ does not divide s , or a non-trivial relation:

$$\sum_{i=1}^{l_s} c_i F_{s,\beta_{s,i}}(x) + \lambda F_{s,0}(x) = f(x) \in \overline{K}(x)$$

with $c_1, \dots, c_{l_s}, \lambda \in \overline{K}$ if $q-1$ divides s . In both cases, non-trivial relations can be found with $c_1, \dots, c_{l_s}, \lambda \in A$.

Proof. Without loss of generality, we may assume that \mathcal{J} is minimal so that for all $n \in \mathcal{J}$ and $i \in \{1, \dots, l_n\}$ the functions obtained from the family $(F_{s,\beta_{s,1}}, \dots, F_{s,\beta_{s,l_s}})_{s \in \mathcal{J}}$ discarding $F_{n,\beta_{n,1}}$ are algebraically independent over $\overline{K}(x)$.

We want to apply Propositions 1.1 and 1.2. We take $U := \bigcup_{n \geq 0} \overline{K}(x^{1/p^n})$, which is perfect, and $\alpha : U \rightarrow U$ the identity map on \overline{K} and such that $\alpha(x) = x^q$. We also take:

$$\begin{aligned} (X_1, \dots, X_N) &= (Y_{s,1}, \dots, Y_{s,l_s})_{s \in \mathcal{J}}, \\ (A_1, \dots, A_N) &= \underbrace{((x^q - T)^s, \dots, (x^q - T)^s)}_{l_s \text{ times}}_{s \in \mathcal{J}}, \\ (B_1, \dots, B_N) &= (\widetilde{\beta_{s,1}}(x), \dots, \widetilde{\beta_{s,l_s}}(x))_{s \in \mathcal{J}}. \end{aligned}$$

We take $V = (x^q - T)^{\mathbb{Z}}$. We have $U_0 = \overline{K}$ and for $A \in V \setminus \{1\}$, the solutions of $f(x^q) = Af(x)$ are identically zero as one sees easily writing down a formal power series for $f \in U$ at infinity.

Let $P \in \mathcal{R}$ be an irreducible polynomial such that $P((F_{s,\beta_{s,1}}, \dots, F_{s,\beta_{s,l_s}})_{s \in \mathcal{J}}) = 0$; we clearly have $\tilde{P} = QP$ with $Q \in U$ and Propositions 1.1 and 1.2 apply. They give $s \in \mathcal{J}$, $c_1, \dots, c_{l_s} \in \overline{K}$ not all zero and $c_0 \in U$ such that

$$c_0(x) = \frac{c_0(x^q)}{(x^q - T)^s} - \frac{1}{(x^q - T)^s} \sum_{i=1}^{l_s} c_i \widetilde{\beta_{s,i}}(x). \quad (11)$$

We now inspect this relation in more detail. To ease the notations, we write $l_s = m$ and $F_i(x) := F_{s,\beta_{s,i}}$ ($i = 1, \dots, m$). Since $\tilde{\beta}(x)^q = \tilde{\beta}(x^q)$ for all $\beta \in K$, from (11) we get, for all $k \geq 0$:

$$c_0(x) = - \sum_{i=1}^m c_i \left(\tilde{\beta}_i(x) + \sum_{h=1}^k \frac{(-1)^{hs} \tilde{\beta}_i(x)^{q^h}}{((x^q - T)(x^{q^2} - T) \cdots (x^{q^{h+1}} - T))^s} \right) + \frac{c_0(x^{q^{k+1}})}{((x^q - T)(x^{q^2} - T) \cdots (x^{q^{k+1}} - T))^s}. \quad (12)$$

By Proposition 1.1, there exists $M > 0$ such that $c_0(x)^{q^M} \in \overline{K}(x)$, which implies that $c_0(x^{q^M}) \in \overline{K}(x)$. Indeed, the expansion of $c_0(x)^{q^M}$ in $\overline{K}((1/x))$ is ultimately periodic if and only if the expansion of $c_0(x^{q^M})$ is ultimately periodic. By equation (12) we see that $c_0(x) \in \overline{K}(x)$.

We write $c_0(x) = \sum_{i \geq i_0} d_i x^{-i}$ with $d_i \in \overline{K}$. The sequence $(|d_i|)_i$ is bounded; let κ be an upper bound. If $x \in C$ is such that $|x| \geq r > q^{1/q}$ with r independent on x , then $|c_0(x)| = \sup_i |d_i| |x|^{-i} \leq \kappa \sup_i |x|^{-i} \leq \kappa |x|^{\deg_x c_0}$. Moreover, for $|x| > r$ with r as above, $|x|^{q^s} > |T| = q$ for all $s \geq 1$ so that $|x^{q^s} - T| = \max\{|x|^{q^s}, |T|\} = |x|^{q^s}$. Hence we get:

$$|(x^q - T)(x^{q^2} - T) \cdots (x^{q^{k+1}} - T)| = |x|^{q+q^2+\cdots+q^{k+1}} = |x|^{\frac{q(k+1)-1}{q-1}}.$$

Let us write:

$$R_k(x) := \frac{c_0(x^{q^{k+1}})}{((x^q - T)(x^{q^2} - T) \cdots (x^{q^{k+1}} - T))^s}.$$

We have, for $|x| \geq r > q^{1/q}$ and for all k :

$$|R_k(x)| \leq \kappa |x|^{q^{k+1} \deg_x c_0 - \frac{sq(q^{k+1}-1)}{q-1}}. \quad (13)$$

Since $|\beta_i| < q^{sq/(q-1)}$, $\deg_x \tilde{\beta}_i(x) < sq/(q-1)$ for all i . In (11) we have two cases: one if $\deg_x (c_0(x^q)/(T - x^q)^s) \leq \max_i \{\deg_x \tilde{\beta}_i\}$, one if $\deg_x (c_0(x^q)/(T - x^q)^s) > \max_i \{\deg_x \tilde{\beta}_i\}$. In the first case we easily see that $\deg_x c_0 < sq/(q-1)$ (notice that $\deg_x c_0(x^q) = q \deg_x c_0$). In the second case, $\deg_x c_0 = q \deg_x c_0 - sq$ which implies $\deg_x c_0 = sq/(q-1)$.

First case. Here, there exists $\epsilon > 0$ such that $\deg_x c_0 = (sq - \epsilon)/(q - 1)$. We easily check (assuming that $|x| \geq r > q^{1/q}$):

$$\begin{aligned} |R_k(x)| &\leq \kappa |x|^{\frac{sq-\epsilon}{q-1}q^{k+1} - \frac{sq(q^{k+1}-1)}{q-1}} \\ &\leq \kappa |x|^{\frac{sq-\epsilon q^{k+1}}{q-1}} \end{aligned}$$

and the sequence of functions $(R_k(x))_k$ converges uniformly to zero in the domain $\{x, |x| \geq r\}$ for all $r > q^{1/q}$. Letting k tend to infinity in (12), we find $\sum_i c_i F_i(x) + c_0(x) = 0$; that is what we expected.

Second case. In this case, the sequence $|R_k(x)|$ is bounded but does not tend to 0. Notice that this case does not occur if $q - 1$ does not divide s , because $c_0 \in \overline{K}(x)$ and its degree is a rational integer. Hence we suppose that $q - 1$ divides s .

Let us write:

$$c_0(x) = \lambda x^{sq/(q-1)} + \sum_{i > sq/(q-1)} d_i x^i,$$

with $\lambda \in \overline{K}^\times$. We have

$$\lim_{k \rightarrow \infty} \frac{\sum_{i > sq/(q-1)} d_i x^{q^k i}}{((x^q - T)(x^{q^2} - T) \cdots (x^{q^{k+1}} - T))^s} = 0$$

(uniformly on $|x| > r > q^{1/q}$), as one checks easily by following the first case.

For all $k \geq 0$ we have:

$$\begin{aligned} (-x)^{sq/(q-1)} \prod_{i=1}^{k+1} \left(1 - \frac{T}{x^{q^i}}\right)^{-s} &= (-1)^{sq/(q-1)} x^{sq/(q-1)} x^{s(q+\cdots+q^{k+1})} \prod_{i=1}^{k+1} (x^{q^i} - T)^{-s} \\ &= (-1)^{sq/(q-1)} x^{sq^{k+2}/(q-1)} \prod_{i=1}^{k+1} (x^{q^i} - T)^{-s}. \end{aligned}$$

Hence we have $\lim_{k \rightarrow \infty} \lambda x^{sq/(q-1)} / ((x^q - T)(x^{q^2} - T) \cdots (x^{q^{k+1}} - T))^s = \lambda F_{s,0}(x)$ and $\sum_i c_i F_i(x) + \lambda F_{s,0}(x) + c_0(x) = 0$.

We now prove the last statement of the Proposition: this follows from an idea of Denis. The proof is the same in both cases and we work with the first only. There exists $a \geq 0$ minimal such that the p^a -th powers of c_1, \dots, c_{l_s} are defined over the separable closure K^{sep} of K . The trace $K^{\text{sep}} \rightarrow K$ can be extended to formal series $K^{\text{sep}}((1/x)) \rightarrow K((1/x))$; its image does not vanish. We easily get, multiplying by a denominator in A , a non-trivial relation

$$\sum_i b_i F_i(x)^{q^a} + b_0(x) = 0$$

with $b_i \in A$ and $b_0(x) \in \overline{K}(x)$. If the coefficients b_i are all in \mathbb{F}_q , this relation is the p^a -th power of a linear relation as we are looking for. If every relation has at least one of the coefficients b_i not belonging to \mathbb{F}_q , the one with $\max_i \{\deg_T b_i\}$ and a minimal has in fact $a = 0$ (otherwise, we apply the operator d/dT to find one with smaller degree, because $dg^{p^a}/dT = 0$ if $a > 0$). \square

We need a criterion of algebraic independence by Denis in [6, 7] that we quote here for convenience of the reader.

Theorem 3 *Let $L \subset \overline{K}$ be a finite extension of K . We consider f_1, \dots, f_m holomorphic functions in a domain $|x| > r \geq 1$ with Taylor's expansions in $L((1/x))$. Let us assume that there exist elements $a_i, b_i \in L(x)$ ($i = 1, \dots, m$) such that*

$$f_i(x) = a_i(x)f_i(x^q) + b_i(x), \quad 1 \leq i \leq m.$$

Let $x_0 \in L$, $|x_0| > r$, such that for all n , $x_0^{q^n}$ is not a zero of any of the functions a_i and is not a pole of any of the functions b_i .

If the series f_1, \dots, f_m are algebraically independent over $\overline{K}(x)$, then the numbers $f_1(x_0), \dots, f_m(x_0)$ are algebraically independent.

The next step is the following Proposition.

Proposition 3.2 *If the numbers $(\text{Li}_s(\beta_{s,1}), \dots, \text{Li}_s(\beta_{s,l_s}))_{s \in \mathcal{J}}$ are algebraically dependent over \overline{K} , there exists $s \in \mathcal{J}$ and a non-trivial relation*

$$\sum_{i=1}^{l_s} c_i \text{Li}_s(\beta_{s,i}) = 0$$

with $c_1, \dots, c_{l_s} \in A$ if $q-1$ does not divide s , or a non-trivial relation:

$$\sum_{i=1}^{l_s} c_i \text{Li}_s(\beta_{s,i}) + \lambda \overline{\pi}^s = 0$$

with $c_1, \dots, c_{l_s}, \lambda \in A$ if $q-1$ divides s .

Proof. By Theorem 3, the functions $F_{s,i}$ ($s \in \mathcal{J}, 1 \leq i \leq l_s$) are algebraically dependent over $\overline{K}(t)$. Proposition 3.1 applies and gives $s \in \mathcal{J}$ as well as a non-trivial linear dependence relation. If $q-1$ does not divide s , by Proposition 3.1 there exists a non-trivial relation

$$\sum_{i=1}^{l_s} c_i F_{s,\beta_{s,i}}(x) = f(x) \in \overline{K}(x)$$

with $c_1, \dots, c_{l_s} \in A$. We substitute $x = T$ in this relation:

$$\sum_{i=1}^{l_s} c_i \text{Li}_s(\beta_{s,i}) = f \in \overline{K}.$$

After [2] pp.172-176, for all $x \in C$ such that $|x| < q^{qs/(q-1)}$, there exist

$$v_1(x), \dots, v_{s-1}(x) \in C$$

such that

$$\begin{pmatrix} 0 \\ \vdots \\ 0 \\ x \end{pmatrix} = \exp_s \begin{pmatrix} v_1(x) \\ \vdots \\ v_{s-1}(x) \\ \text{Li}_s(x) \end{pmatrix},$$

\exp_s being the exponential function associated to the s -th twist of Carlitz's module. Moreover:

$$\exp_s \begin{pmatrix} c_j v_1(\beta_{s,j}) \\ \vdots \\ c_j v_{s-1}(\beta_{s,j}) \\ c_j \text{Li}_s(\beta_{s,j}) \end{pmatrix} = \phi_{\text{Car}}^{\otimes s}(c_j) \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \beta_{s,j} \end{pmatrix} \in \overline{K}^s, \quad j = 1, \dots, n_s,$$

where $\phi_{\text{Car}}^{\otimes s}(c_j)$ denotes the action on the s -th twist of Carlitz's module. By \mathbb{F}_q -linearity, there exist numbers $w_1, \dots, w_{s-1} \in C$ such that

$$\exp_s \begin{pmatrix} w_1 \\ \vdots \\ w_{s-1} \\ c \end{pmatrix} \in \overline{K}^s.$$

Yu's sub- t -module Theorem (in [14]) implies the following analogue of Hermite-Lindemann's Theorem. Let $G = (\mathbb{G}_a^s, \phi)$ be a regular t -module with exponential function e_ϕ , with $\phi(g) = a_0(g)\tau^0 + \dots$, for all $g \in A$. Let $u \in C^s$ be such that $e_\phi(u) \in \mathbb{G}_a^s(\overline{K})$. Let V the smallest vector subspace of C^s containing u , defined over \overline{K} , stable by multiplication by $a_0(g)$ for all $g \in A$. Then the \mathbb{F}_q -subspace $e_\phi(V)$ of C^s equals $H(C)$ with H sub- t -module of G .

This result with G the s -th twist of Carlitz's module and $e_\phi = \exp_s$ implies the vanishing of c and the K -linear dependence of the numbers

$$\text{Li}_s(\beta_{s,1}), \dots, \text{Li}_s(\beta_{s,l_s}).$$

If $q-1$ divides s then by Proposition 3.1 there exists a non-trivial relation

$$\sum_{i=1}^{l_s} c_i F_{s,\beta_{s,i}}(x) + \lambda F_{s,0}(x) = f(x) \in \overline{K}(x)$$

with $c_1, \dots, c_{l_s}, \lambda \in K$. We substitute $x = T$ in this relation:

$$\sum_{i=1}^{l_s} c_i \text{Li}_s(\beta_{s,i}) + \lambda \overline{\pi}^s = f \in \overline{K}.$$

The Proposition follows easily remarking that, after [2] again, there exist

$$v_1, \dots, v_{s-1} \in C$$

such that

$$\begin{pmatrix} 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix} = \exp_s \begin{pmatrix} v_1 \\ \vdots \\ v_{s-1} \\ \overline{\pi}^s \end{pmatrix}.$$

□

Proof of Theorem 2. To deduce Theorem 2 from Proposition 3.2 we quote Theorem 3.8.3 p. 187 of Anderson-Thakur in [2]. For all $i \leq nq/(q-1)$ there exists $h_{n,i} \in A$ such that if we set

$$P_n := \sum_i \phi_{\text{Car}}^{\otimes n}(h_{n,i}) \begin{pmatrix} 0 \\ \vdots \\ 0 \\ T^i \end{pmatrix},$$

then the last coordinate P_n is equal to $\Gamma(n)\zeta(n)$ (where $\Gamma(n)$ denotes Carlitz's arithmetic Gamma function). Moreover, there exists $a \in A \setminus \{0\}$ with $\phi_{\text{Car}}^{\otimes n}(a)P_n = 0$ if and only if $q-1$ divides n . This implies that

$$\Gamma(n)\zeta(n) = \sum_{i=0}^{[nq/(q-1)]} h_{n,i} \text{Li}_n(T^i).$$

The numbers $h_{n,i}$ are explicitly determined in [2]. In particular, one has

$$\zeta(s) = \text{Li}_s(1), \quad s = 1, \dots, q-1.$$

We apply Theorem 3 and Proposition 3.2 with $\mathcal{J} = \mathcal{J}^\# \cup \{q-1\}$, $\mathcal{J}^\#$ being the set of all the integers $n \geq 1$ with $p, q-1$ not dividing n , $l_{q-1} = 1$, $\beta_{q-1,1} = 1$, and for $s \in \mathcal{J}^\#$,

$$(\beta_1, \dots, \beta_{l_s}) = (T^{i_0}, \dots, T^{i_{m_s}}),$$

where the exponents $0 \leq i_0 < \dots < i_{m_s} \leq sq/(q-1)$ are chosen so that

$$\zeta(s) \in K\text{Li}_s(1) + \dots + K\text{Li}_s(T^{[sq/(q-1)]}) = K\text{Li}_s(T^{i_0}) \oplus \dots \oplus K\text{Li}_s(T^{i_{m_s}}).$$

□

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