A Nice Labelling for Tree-Like Event Structures of Degree 3

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Abstract. We address the problem of finding nice labellings for event structures of degree 3. We develop a minimum theory by which we prove that the labelling number of an event structure of degree 3 is bounded by a linear function of the height. The main theorem we present in this paper states that event structures of degree 3 whose causality order is a tree have a nice labelling with 3 colors. Finally, we exemplify how to use this theorem to construct upper bounds for the labelling number of other event structures of degree 3.

1 Introduction

Event structures, introduced in [1], are nowadays a widely recognized model of true concurrent computation and have found many uses since then. They are an intermediate abstract model that make it possible to relate other more concrete models such as Petri Nets or higher dimensional automata [2]. They provide formal semantics of process calculi [3,4]. More recently, logicians became interested in event structures with the aim of constructing models of proof systems that are invariant under the equalities induced by the cut elimination procedure [5,6].

Our interest for event structures stems from the fact that they combine distinct approaches to the modeling of concurrent computation. On one side, language theorists have developed the theory of partially commutative monoids [7] as the basic language to approach concurrency. On the other hand, the framework of domain theory and, ultimately, order theoretic ideas have often been proposed as the proper tools to handle concurrency, see for example [8]. In this paper we pursue a combinatorial problem that lies at the intersection of these two approaches. It is the problem of finding nice labellings for event structures of fixed degree. To our knowledge, this problem has not been investigated any longer since it was posed and partially solved in [9].

Let us recall that an event structure is made up of a set of local events E which is ordered by a causality relation $\leq$. Moreover, a concurrency relation $\frown$, that may only relate causally independent events, is given. A global state of the computation is modeled as a clique of the concurrency relation. Global states may be organized into a poset, the coherent domain of an event structure, which represents all the concurrent non-deterministic executions of a system. Roughly speaking, the nice labelling problem consists in representing the coherent domain of an event structure as a poset of traces or, more precisely, pomsets. That is, such a domain should be reconstructed using the standard ingredients of trace theory: an alphabet Σ, a local independence relation I, and a prefix closed subset of the free monoid L, see [10,11]. By the general theory relating traces to ordered sets, the problem always has a solution (Σ, I, L). We are asked to find a solution with the cardinality of the alphabet Σ minimal. The problem is actually equivalent to a graph coloring problem in that we can associate to an event structure a graph, of which we are asked to compute the chromatic number. The degree of an event structure is the maximal number of upper covers of some elements in the associated domain. Under the graph theoretic translation of the problem, the degree coincides with the clique number, and therefore it is a lower bound for the cardinality of a solution. A main contribution in [9] was to prove that event structures of degree 2 have a nice labelling with 2 letters, i.e. they have a solution (Σ, I, L) with $\mathrm{card}(\Sigma) = 2$. On the other hand, it was proved there that event structures of higher degrees may require more letters than the degree.

The labelling problem may be thought to be a generalization of the problem of covering a poset by disjoint chains. Dilworth's Theorem [12] states that the minimal cardinality of such a cover equals the maximal cardinality of an antichain. This theorem and the results of [9] constitute the few knowledge on the problem presently available to us. For example, we cannot state that there is some fixed $k > n$ for which every event structure of degree n has a nice labelling with at most k letters. In light of standard graph theoretic results [13], the above statement should not be taken for granted.

We present here our first results on the nice labelling problem for event structures of degree 3. We develop a minimum theory that shows that the graph of a degree 3 event structure, when restricted to an antichain, is almost acyclic and can be colored with 3 letters. This observation allows to construct an upper bound to the labelling number of such event structure as a linear function of its height. We prove then our main theorem stating that event structures of degree 3, whose causality order is a tree, have a nice labelling with 3 letters. Let us just say that such an event structure may represent a concurrent system where some processes are only allowed to fork or to take local nondeterministic choices. Finally, we exemplify how to use this theorem to construct upper bounds for the labelling number of other event structures of degree 3. In some simple cases, we obtain constant upper bounds to the labelling number, i.e. upper bounds that are functions of no parameter.

While these results do not answer the general problem, that of computing the labelling number of degree 3 event structures, we are aware that graph coloring problems may be difficult to answer. Thus we decided to present these results and share the knowledge so far acquired and also to encourage other researchers to pursue the problem. Let us mention why we believe that this and other problems in the combinatorics of concurrency deserve to be deeply investigated. The theory of event structures is now being applied within verification. A model checker, POEM, presently developed in Marseilles, makes explicit use of trace theory and of the theory of partial orders to represent the state space of a concurrent system [14]. The combinatorics of posets is there exploited to achieve an the efficient exploration of the global states of concurrent systems [15]. Thus, having a solid theoretical understanding of such combinatorics is, for us, a prerequisite and a complement for designing efficient algorithms for these kind of tools.

The paper is structured as follows. After recalling the order theoretic concepts we shall use, we introduce event structures and the nice labelling problem in section 2. In section 3 we develop the first properties of event structures of degree 3. As a result, we devise an upper bound for the labelling number of such event structures as a linear function of the height. In section 4 we present our main result stating that event structures whose underlying order is a tree may be labeled with 3 colors. In section 5 we develop a general approach to construct upper bounds to the labelling number of event structures of degree 3. Using this approach and the results of the previous section, we compute a constant upper bound for a class of degree 3 event structures that have some simplifying properties and which are consequently called simple.

Order Theoretic Preliminaries. We shall introduce event structures in the next section. For the moment being let us anticipate that part of an event structure is a set E of events which is partially ordered by a causality relation $\leq$. In this paper we shall heavily make use of order theoretic concepts. We introduce them here together with the notation that shall be used. All these concepts will apply to the poset $\langle E, \leq \rangle$ of an event structure.

A finite poset is a pair $\langle P, \leq \rangle$ where P is a finite set and $\leq$ is a reflexive, transitive and antisymmetric relation on P. A subset $X \subseteq P$ is a *lower set* if $y \leq x \in X$ implies $y \in X$. If $Y \subseteq P$, then we denote by $\downarrow Y$ the least lower set containing Y. The explicit formula for $\downarrow Y$ is

$$\downarrow Y = \{ x \in P \mid \exists y \in Y \text{ s.t. } x \leq y \} .$$

Two elements $x, y \in P$ are *comparable* if and only if either $x \leq y$ or $y \leq x$. We write $x \simeq y$ to mean that x, y are comparable. A *chain* is sequence $x_0, \ldots, x_n$ of elements of P such that $x_0 < x_1 < \ldots < x_n$. The integer n is the length of the chain. The *height* of an element $x \in P$, noted $\mathrm{h}(x)$, is the length of the longest chain in $\downarrow \{ x \}$. The height of P is $\max\{ \mathrm{h}(x) \mid x \in P \}$. An *antichain* is a subset $X \subseteq P$ such that $x \not\simeq y$ for each pair $x, y \in X$. The *width* of $\langle P, \leq \rangle$, noted $\mathrm{w}(P, \leq)$, is the integer $\max\{ \mathrm{card}(A) \mid A \text{ is an antichain} \}$. If the interval $\{ z \in P \mid x \leq z \leq y \}$ is the two elements set $\{ x, y \}$, then we say that x is a *lower cover* of y or that y is *an upper cover* of x. We denote this relation by $x \prec y$. The Hasse diagram of $\langle P, \leq \rangle$ is the directed graph $\langle P, \prec \rangle$. For $x \in P$, the *degree* of x, noted $\deg(x)$, is the number of upper covers of x. That is, the degree of x is the outdegree of x in the Hasse diagram. The degree of $\langle P, \leq \rangle$, noted $\deg(P, \leq)$, is the integer $\max\{ \deg(x) \mid x \in P \}$. We shall denote by $\mathrm{f}(x)$ the number of lower covers of x (i.e. the indegree of x in the Hasse diagram). The poset $\langle P, \leq \rangle$ is *graded* if $x \prec y$ implies $\mathrm{h}(y) = \mathrm{h}(y) + 1$.

2 Event Structures and the Nice Labelling Problem

Event structures are a basic model of concurrency introduced in [1]. The definition we present here is from [2].

Definition 1. *An* event structure *is a triple* $\mathcal{E} = \langle E, \leq, \mathcal{C} \rangle$ *such that*

- $\langle E, \leq \rangle$ *is a poset, such that for each* $x \in E$ *the lower set* $\downarrow \{ x \}$ *is finite,*
- $\mathcal{C}$ *is a collection of subsets of* E *such that:*
  1. $\{ x \} \in \mathcal{C}$ *for each* $x \in E$*,*
  2. $X \subseteq Y \in \mathcal{C}$ *implies* $X \in \mathcal{C}$*,*
  3. $X \in \mathcal{C}$ *implies* $\downarrow X \in \mathcal{C}$*.*

The order $\leq$ of an event structure $\mathcal{E}$ is known as the *causality* relation between events. The collection $\mathcal{C}$ is known as the set of configurations of $\mathcal{E}$. A configuration $X \in \mathcal{C}$ of causally unrelated events – that is, an antichain w.r.t. $\leq$ – is a sort of snapshot of the global state of some distributed computation. A snapshot X may be transformed into a description of the computation that takes into account its history. This is done by adding to X the events that causally have determined events in X. That is, the history aware description is the lower set $\downarrow X$ generated by X.

Two elements $x, y \in E$ are said to be *concurrent* if $x \not\simeq y$ and there exists $X \in \mathcal{C}$ such that $x, y \in X$. Two concurrent elements will be thereby noted by $x \frown y$. It is useful to introduce a weakened version of the concurrency relation where we allow elements to be comparable: $x \overset{\frown}{\simeq} y$ if and only if $x \frown y$ or $x \simeq y$. Equivalently, $x \overset{\frown}{\simeq} y$ if and only if there exists $X \in \mathcal{C}$ such that $x, y \in X$. In many concrete models the set of configurations is completely determined by the concurrency relation.

Definition 2. *An event structure* $\mathcal{E}$ *is* coherent *if* $\mathcal{C}$ *is the set of cliques of the weak concurrency relation:* $X \in \mathcal{C}$ *if and only if* $x \overset{\frown}{\simeq} y$ *for every pair of elements* $x, y \in X$*.*

Coherent event structures are also known as event structures with binary conflict. To understand the naming let us explicitely introduce the conflict relation and two other derived relations:

- *Conflict*: $x \smile y$ if and only if $x \not\simeq y$ and $x \not\overset{\frown}{\simeq} y$.
- *Minimal conflict*: $x \overset{\smile}{=} y$ if and only (i) $x \smile y$, (ii) $x' < x$ implies $x' \overset{\frown}{\simeq} y$, and (iii) $y' < y$ implies $x \overset{\frown}{\simeq} y'$.
- *Orthogonality*: $x \overset{\frown}{=} y$ if and only if $x \overset{\smile}{=} y$ or $x \frown y$.

A coherent event structure is completely described by a triple $\langle E, \leq, \frown \rangle$ where the latter is a symmetric relation subject to the following condition: $x \frown y$ and $z \leq x$ implies $z \frown y$ or $z \leq y$. Similarly, a coherent event structure is completely determined by the order and the conflict relation. In this paper we shall deal with coherent event structures only and, from now on, event structure will be a synonym for coherent event structure.

We shall focus mainly on the orthogonality relation. Let us observe that two orthogonal elements are called independent in [9]. We prefer however not to use this naming: we shall frequently make use of standard graph theoretic language and argue about cliques, not on their dual, independent sets. The orthogonality relation clearly is symmetric and moreover it inherits from the concurrency relation the following property: if $x \frown y$ and $z \leq x$, then $z \frown y$ or $z \leq y$.

Definition 3. *A* nice labelling *of an event structure $\mathcal{E}$ is a pair (λ, Σ), where Σ is a finite alphabet and $\lambda : E \longrightarrow \Sigma$ is such that $\lambda(x) \neq \lambda(y)$ whenever $x \frown y$.*

That is, if we let $\mathcal{G}(\mathcal{E})$ – the graph of $\mathcal{E}$ – be the pair $\langle E, \frown \rangle$, a labelling of $\mathcal{E}$ is a coloring of the graph $\mathcal{G}(\mathcal{E})$. For a graph G, let $\gamma(G)$ denote its chromatic number. Let us say then that the *labelling number* of $\mathcal{E}$ is $\gamma(\mathcal{G}(\mathcal{E}))$. The *nice labelling problem* for a class $\mathcal{K}$ of event structures amounts to computing the number

$$\gamma(\mathcal{K}) = \max\{\, \gamma(\mathcal{G}(\mathcal{E})) \mid \mathcal{E} \in \mathcal{K} \,\}\,.$$

To understand the origins of this problem, let us recall the definition of the domain of an event structure.

Definition 4. *The domain $\mathcal{D}(\mathcal{E})$ of an event structure $\mathcal{E} = \langle E, \leq, \mathcal{C} \rangle$ is the collection of lower sets in $\mathcal{C}$, ordered by subset inclusion.*

Following a standard axiomatization in theoretical computer science [2] $\mathcal{D}(\mathcal{E})$ is a stable domain which is coherent if $\mathcal{E}$ is coherent. Stable means that $\mathcal{D}(\mathcal{E})$ is essentially a distributive lattice. As a matter of fact, if $\mathcal{E}$ is finite, then a possible alternative axiomatization of the poset $\mathcal{D}(\mathcal{E})$ is as follows. It is easily seen that the collection $\mathcal{D}(\mathcal{E})$ is closed under binary intersections, hence it is a finite meet semilattice without a top element, a chopped lattice in the sense of [16, Chapter 4]. Also the chopped lattice is distributive, meaning that whenever $X, Y, Z \in \mathcal{D}(\mathcal{E})$ and $X \cup Y \in \mathcal{D}(\mathcal{E})$, then $Z \cap (X \cup Y) = (Z \cap X) \cup (Z \cap Y)$. It can be shown that every distributive chopped lattice is isomorphic to the domain of a finite – not necessarily coherent – event structure.

Lemma 5. *A set $\{\, x_1, \ldots, x_n \,\}$ is a clique in the graph $\mathcal{G}(\mathcal{E})$ iff there exists $I \in \mathcal{D}(\mathcal{E})$ such that $I \cup \{\, x_i \,\}$, $i = 1, \ldots, n$, are distinct upper covers of I in the domain $\mathcal{D}(\mathcal{E})$.*

The Lemma shows that a nice labelling $\lambda : E \longrightarrow \Sigma$ allows to label the edges of the Hasse diagram of $\mathcal{D}(\mathcal{E})$ so that: (i) outgoing edges from the same source vertex have distinct labels, (ii) perspective edges – i.e. edges $I_0 \prec I_1$ and $J_0 \prec J_1$ such that $I_0 = I_1 \cap J_0$ and $J_1 = I_1 \cup J_0$ – have the same label. These conditions are necessary and sufficient to show that $\mathcal{D}(\mathcal{E})$ is order isomorphic to a consistent (but not complete) set of P-traces (or pomsets) on the alphabet Σ in the sense of [10].

The *degree* of an event structure $\mathcal{E}$ is the degree of the domain $\mathcal{D}(\mathcal{E})$, that is, the maximum number of upper covers of a lower set I within the poset $\mathcal{D}(\mathcal{E})$.

Lemma 5 shows that the degree of $\mathcal{E}$ is equal to the size of a maximal clique in $\mathcal{G}(\mathcal{E})$, i.e. to the clique number of $\mathcal{G}(\mathcal{E})$. Henceforth, the degree of $\mathcal{E}$ is a lower bound to $\gamma(\mathcal{G}(\mathcal{E}))$. The following Theorems state the few results on the nice labelling problem that are available in the literature.

Theorem 6 (see [12]). *Let $\mathcal{NC}_n$ be the class of event structures of degree at most n with empty conflict relation. Then $\gamma(\mathcal{NC}_n) = n$.*

Theorem 7 (see [9]). *Let $\mathcal{K}_n$ be the class of event structures of degree at most n. Then $\gamma(\mathcal{K}_n) = n$ if $n \leq 2$ and $\gamma(\mathcal{K}_n) \geq n + 1$ otherwise.*

The last theorem has been our starting point for investigating the nice labelling problem for event structures of degree 3.

3 Cycles and Antichains

From now own, in this and the following sections, $\mathcal{E} = \langle E, \leq, \mathcal{C} \rangle$ will denote a coherent event structure of degree at most 3. We begin our investigation of the nice labelling problem by studying the restriction to an antichain of the graph $\mathcal{G}(\mathcal{E})$. The main tool we shall use is the following Lemma. It is a straightforward generalization of [9, Lemma 2.2] to degree 3. In [17] we proposed generalizations of this Lemma to higher degrees, pointing out their strong geometrical flavor.

Lemma 8. *Let $\{ x_0, x_1, x_2 \}, \{ x_1, x_2, x_3 \}$ be two size 3 cliques in the graph $\mathcal{G}(\mathcal{E})$ sharing the same face $\{ x_1, x_2 \}$. Then x_0, x_3 are comparable.*

Proof. Let us suppose that x_0, x_3 are not comparable. It is not possible that $x_0 \,\overline{\frown}\, x_3$, since then we have a size 4 clique in the graph $\mathcal{G}(\mathcal{E})$. Thus $x_0 \smile x_3$ and we can find $x'_0 \leq x_0$ and $x'_3 \leq x_3$ such that $x'_0 = x'_3$. We claim that $\{ x'_0, x_1, x_2, x'_3 \}$ is a size 4 clique in $\mathcal{G}(\mathcal{E})$, thus reaching a contradiction.

If $x'_0 \not\overline{\frown}\, x_1$, then $x'_0 \leq x_1$, but this, together with $x_1 \,\overline{\frown}\, x_3$, contradicts $x'_0 = x'_3$. Similalry, $x'_0 \,\overline{\frown}\, x_2$, $x'_3 \,\overline{\frown}\, x_1$, $x'_3 \,\overline{\frown}\, x_2$. $\square$

We are going to improve on the previous Lemma. To this goal, let us say that a sequence $x_0 x_1 \ldots x_{n-1} x_n$ is a *straight cycle* if $x_n = x_0$, $x_i \,\overline{\frown}\, x_{i+1}$ for $i = 0, \ldots, n-1$, $x_i \not\leq x_j$ whenever $i, j \in \{ 0, \ldots, n-1 \}$ and $i \neq j$. As usual, the integer n is the length of the cycle. Observe that a straight cycle is simple, i.e., a part from the endpoints of the cycle, it does not visit twice the same vertex . The height of a straight cycle $C = x_0 x_1 \ldots x_n$ is the integer

$$\mathrm{h}^{+}(C) = \sum_{i=0,\ldots,n-1} \mathrm{h}(x_i) + 1 \,.$$

The definition of h^{+} implies that if C' is a subcycle of C induced by a chord, then $\mathrm{h}^{+}(C') < \mathrm{h}^{+}(C)$.

Proposition 9. *The graph $\mathcal{G}(\mathcal{E})$ does not contain a straight cycle of length strictly greater than 3.*

Proof. Let $\mathcal{SC}_{\geq 4}$ be the collection of straight cycles in $\mathcal{G}(\mathcal{E})$ whose length is at least 4. We shall show that if $C \in \mathcal{SC}_{\geq 4}$, then there exists $C' \in \mathcal{SC}_{\geq 4}$ such that $\mathrm{h}^{+}(C') < \mathrm{h}^{+}(C)$.

Let C be the straight cycle $x_0 \mathrel{\underline{\frown}} x_1 \mathrel{\underline{\frown}} x_2 \ldots x_{n-1} \mathrel{\underline{\frown}} x_n = x_0$ where $n \geq 4$. Let us suppose that this cycle has a chord. It follows, by Lemma 8, that $n > 4$. Hence the chord divides the cycle into two straight cycles, one of which has still length at least 4 and whose height is less than the height of C, since it contains a smaller number of vertices.

Otherwise C has no chord and $x_0 \not\mathrel{\underline{\frown}} x_2$. This means that either there exists $x_0' < x_0$ such that $x_0' \not\frown x_0$, or there exists $x_2' < x_2$ such that $x_0 \not\frown x_2'$. By symmetry, we can assume the first case holds. As in the proof of Lemma 8 $\{ x_0', x_1, x_2, x_3 \}$ form an antichain, and $x_0' x_1 x_2 x_3$ is a path. Let C' be the set $\{ x_0' x_1, \ldots x_{n-1} x_0' \}$. If C' is an antichain, then C' is a straight cycle such that $\mathrm{h}^{+}(C') < \mathrm{h}^{+}(C)$. Otherwise the set $\{ j \in \{ 4, \ldots, n-1 \} \mid x_j \geq x_0' \}$ is not empty. Let i be the minimum in this set, and observe that $x_{i-1} \mathrel{\underline{\frown}} x_i$ and $x_0' \leq x_i$ but $x_0' \not\leq x_{i-1}$ implies $x_{i-1} \mathrel{\underline{\frown}} x_0'$. Thus $\tilde{C} = x_0' x_1 x_2 x_3 \ldots x_{i-1} x_0'$ is a straight cycle of lenght at least 4 such that $\mathrm{h}^{+}(\tilde{C}) < \mathrm{h}^{+}(C)$. □

Corollary 10. *Any subgraph of $\mathcal{G}(\mathcal{E})$ induced by an antichain can be colored with 3 colors.*

Proof. Since the only cycles have length at most 3, such an induced graph is chordal and its clique number is 3. It is well known that the chromatic number of chordal graphs equals their clique number [18]. □

In the rest of this section we exploit the previous observations to construct upper bounds for the labelling number of $\mathcal{E}$. We remark that these upper bounds might appear either too abstract, or too trivial. On the other hand, we believe that they well illustrate the kind of problems that arise when trying to build complex event structures that might have labelling number greater than 4.

A *stratifying function* for $\mathcal{E}$ is a function $h : E \longrightarrow \mathbb{N}$ such that, for each $n \geq 0$, the set $\{ x \in E \mid h(x) = n \}$ is an antichain. The height function is a stratifying function. Also $\varsigma(x) = \mathrm{card}\{ y \in E \mid y < x \}$ is a stratifying function. With respect to a stratifying function h the h-skewness of $\mathcal{E}$ is defined by

$$\mathrm{skew}_h(\mathcal{E}) = \max\{ |h(x) - h(y)| \mid x \mathrel{\underline{\frown}} y \} .$$

More generally, the skewness of $\mathcal{E}$ is defined by

$$\mathrm{skew}(\mathcal{E}) = \min\{ \mathrm{skew}_h(\mathcal{E}) \mid h \text{ is a stratifying function } \} .$$

Proposition 11. *If $\mathrm{skew}(\mathcal{E}) < n$ then $\gamma(\mathcal{G}(\mathcal{E})) \leq 3n$.*

Proof. Let h be a stratifying function such that $|h(x) - h(y)| < n$ whenever $x \mathrel{\underline{\frown}} y$. For each $k \geq 0$, let $\lambda_k : \{ x \in E \mid h(x) = k \} \longrightarrow \{ a, b, c \}$ be a coloring of the graph induced by $\{ x \in E \mid h(x) = k \}$. Define $\lambda : E \longrightarrow \{ a, b, c \} \times \{ 0, \ldots, n-1 \}$ as follows:

$$\lambda(x) = (\lambda_{h(x)}(x), h(x) \bmod n) .$$

Let us suppose that $x \mathrel{\underline{\frown}} y$ and $h(x) \geq h(y)$, so that $0 \leq h(x) - h(y) < n$. If $h(x) = h(y)$, then by construction $\lambda_{h(x)}(x) = \lambda_{h(y)}(x) \neq \lambda_{h(y)}(y)$. Otherwise $h(x) > h(y)$ and $0 \leq h(x) - h(y) < n$ implies $h(x) \bmod n \neq h(y) \bmod n$. In both cases we obtain $\lambda(x) \neq \lambda(y)$. □

An immediate consequence of Proposition 11 is the following upper bound for the labelling number of $\mathcal{E}$:

$$\gamma(\mathcal{G}(\mathcal{E})) \leq 3(\mathrm{h}(\mathcal{E}) + 1)\,.$$

To appreciate the upper bound, consider that another approximation to the labelling number of $\mathcal{E}$ is provided by Dilworth's Theorem [12], stating that $\gamma(\mathcal{G}(\mathcal{E})) \leq \mathrm{w}(\mathcal{E})$. To compare the two bounds, consider that there exist event structures of degree 3 whose width is an exponential function of the height.

4 An Optimal Nice Labelling for Trees and Forests

We prove in this section the main contribution of this paper. Assuming that $\langle E, \leq \rangle$ is a tree or a forest, then we define a labelling with 3 colors, and prove it is a nice labelling. Since clearly we can construct a tree which needs at least three colors, such a labelling is optimal. Before defining the labelling, we shall develop a small amount of observations.

Definition 12. *We say that two distinct events are* twins *if they have the same set of lower covers.*

Clearly if x, y are twins, then $z < x$ if and only if $z < y$. More importantly, if x, y are twins, then the relation $x \mathrel{\underline{\frown}} y$ holds. As a matter of fact, if $x' < x$ then $x' < y$, hence $x' \stackrel{\frown}{\simeq} y$. Similarly, if $y' < y$ then $y' \stackrel{\frown}{\simeq} x$. It follows that a set of events having the same lower covers form a clique in $\mathcal{G}(\mathcal{E})$, hence it has at most the degree of an event structure, 3 in the present case. To introduce the next Lemmas, if $x \in Y \subseteq E$, define

$$O_x^Y = \{\, z \mid z \mathrel{\underline{\frown}} x \text{ and } y \not\leq z, \text{ forall } y \in Y \,\}\,.$$

If $Y = \{\, x, y \,\}$, then we shall abuse of notation and write $O_x^{x,y}$, $O_x^{y,x}$ as synonyms of O_x^Y. Thus $z \in O_x^{x,y}$ if and only if $z \mathrel{\underline{\frown}} x$ and $y \not\leq z$.

Lemma 13. *If x, y, z are parwise distinct twins, then $O_x^{\{x,y,z\}} = \emptyset$.*

Proof. Let us suppose that $w \in O_x^{\{x,y,z\}}$. If $w \mathrel{\underline{\frown}} y$, then $w \simeq z$ by Lemma 8. Since $z \not\leq w$, then $w < z$. However this implies $w < x$, contradicting $w \mathrel{\underline{\frown}} x$. Hence $w \not\mathrel{\underline{\frown}} y$ and we can find $w' \leq w$, $y' \leq y$ such that $w' \mathrel{\underline{\frown}} y'$. It cannot be the case that $y' < y$, otherwise $y' < x$ and the pair (w', y'), properly covered by the pair (w, x), cannot be a minimal conflict. Thus $w' < w$, and y' equals to y. We claim that $w' \in O^{\{x,y,z\}}$. As a matter of fact, w' cannot be above any of the elements in $\{\, x, y, z \,\}$, otherwise w would have the same property. From

$w \,\overline{\frown}\, x$ and $w' < w$, we deduce that $w' \,\overline{\frown}\, x$ or $w' \leq x$. If the latter, then $w' < x$, so that $w' < y$, contradicting $w' = y$. Therefore $w' \,\overline{\frown}\, x$ and $\{ w', x, y \}, \{ x, y, z \}$ are two 3-cliques sharing the same face $\{ x, y \}$. As before, $w' \simeq z$, leading to a contradiction. □

Lemma 14. *If x, y are twins, then $O_x^{x,y}, O_y^{x,y}$ are comparable w.r.t. set inclusion and $O_x^{x,y} \cap O_y^{x,y}$ is a linear order.*

Proof. We observe first that if $z \in O_x^{x,y}$ and $w \in O_y^{x,y}$ then $z \simeq w$. As a consequence $O_x^{x,y} \cap O_y^{x,y}$ is linearly ordered.

Let us suppose that there exists $z \in O_x^{x,y}$ and $w \in O_y^{x,y}$ such that $z \not\simeq w$. Observe then that $\{ z, x, y, w \}$ is an antichain: $y \not\leq z$, and $z < y$ implies $z < x$, which is not the case due to $z \,\overline{\frown}\, x$. Thus $z \not\simeq y$ and similarly $w \not\simeq x$.

Since there cannot be a length 4 straight cycle, we deduce $z \neq w$. Let $z' \leq z$ and $w' \leq w$ be such that $z' = w'$. We claim first that $z' \,\overline{\frown}\, x$. Otherwise, $z' \leq x$ and $z' < x$, since $z' = x$ implies $x \leq z$. The relation $z' < x$ in turn implies $z' < y$, which contradicts $z' = w'$. Also it cannot be the case that $y \leq z'$, since otherwise $y \leq z$. Thus, we have argued that $z' \in O_x^{x,y}$. Similarly $w' \in O_y^{x,y}$. As before $\{ z', x, y, w' \}$ is an antichain, hence z', x, y, w' also form a length 4 straight cycle, a contradiction.

Observe now that $w \leq z \in O_x^{x,y}$ and $w \not\leq x$ implies $w \in O_x^{x,y}$. From $w \leq z \,\overline{\frown}\, x$ deduce $w \,\overline{\frown}\, x$ or $w \leq x$. Since $w \not\leq x$, then $w \,\overline{\frown}\, x$. Also, if $y \leq w$ then $y \leq z$, which is not the case.

Let $z \in O_x^{x,y} \setminus O_y^{x,y}$, pick any $w \in O_y^{x,y}$ and recall that z, w are comparable. We cannot have $z \leq w$ since $z \not\leq y$ implies then $z \in O_y^{x,y}$. Hence $w < z \in O_x^{x,y}$ and $w \not\leq x$ imply $w \in O_x^{x,y}$ by the previous observation. □

The following Lemma will prove to be the key observation in defining later a nice labelling.

Lemma 15. *Let (x, y) (z, w) be two pairs of pairwise distinct twins such that $z \in O_x^{x,y} \cap O_y^{x,y}$ and $w \not\leq x$. Then $O_z^{w,z} \supset O_w^{w,z}$.*

Proof. If $O_z^{w,z} \not\supset O_w^{w,z}$, then $O_z^{w,z} \subseteq O_w^{w,z}$ by Lemma 14. Since $w \not\leq x$ and $w \neq y$, then $w \not\leq y$. We have shown that $x, y \in O_z^{w,z}$, hence $x, y \in O_w^{w,z}$. It follows that $\{ x, y, z, w \}$ is a size 4 clique, a contradiction. □

We come now to discuss some subsets of E for which we shall prove that there exists a nice labelling with 3 letters. The intuitive reason for that is the presence of many twins.

Definition 16. *A subset $T \subseteq E$ is a* tree *if and only if*

- *each $x \in T$ has exactly one lower cover $\pi(x) \in E$,*
- *T is convex: $x, z \in T$ and $x < y < z$ implies $y \in T$,*
- *if x, y are minimal in T, then $\pi(x) = \pi(y)$.*

If T is a tree and $x \in T$, the height of x in T, noted $\mathrm{h}_T(x)$, is the cardinality of the set $\{\, y \in T \mid y < x \,\}$. A linear ordering $\lhd$ on T is said to be compatible with the height if it satisfies

$$\mathrm{h}_T(x) < \mathrm{h}_T(y) \text{ implies } x \lhd y\,. \qquad \text{(HEIGHT)}$$

It is not difficult to see that such a linear ordering always exists. With respect to such linear ordering, define

$$O_{\lhd}(x) = \{\, y \in T \mid y \mathrel{\overline{\frown}} x \text{ and } y \lhd x \,\}\,, \quad x \in T.$$

We shall represent $O_{\lhd}(x)$ as the disjoint union of $C_{\lhd}(x)$ and $L_{\lhd}(x)$ where

$$C_{\lhd}(x) = \{\, y \in O_{\lhd}(x) \mid z \prec x \text{ implies } z \leq y \,\}\,, \qquad L_{\lhd}(x) = O_{\lhd}(x) \setminus C_{\lhd}(x)\,.$$

With respect these sets $C_{\lhd}(x), L_{\lhd}(x)$, $x \in T$, we develop a series of observations.

Lemma 17. *If $y \in C_{\lhd}(x)$ then x, y are twins. Consequently there can be at most two elements in $C_{\lhd}(x)$.*

Proof. If $y \in C_{\lhd}(x)$, then $y \lhd x$ and $\mathrm{h}_T(y) \leq \mathrm{h}_T(x)$. Since y is above any lower cover of x, and distinct from such a lower cover, then $\mathrm{h}_T(x) \leq \mathrm{h}_T(y)$. It follows that $\mathrm{h}_T(x) = \mathrm{h}_T(y)$, hence if z is a lower cover of x, then it is also a lower cover of y. Since x, y have exactly one lower cover, it follows that x, y are twins. □

Lemma 18. *If x, y are twins, then $L_{\lhd}(x) \subseteq O_x^{x,y}$. If $z \in L_{\lhd}(x)$ and $z' \in O_x^{x,y}$ is such that $z' \leq z$, then $z' \in L_{\lhd}(x)$. That is, $L_{\lhd}(x)$ is a lower set of $O_x^{x,y}$.*

Proof. Let $z \in L_{\lhd}(x)$, so that $z \mathrel{\overline{\frown}} x$ and $z \mathrel{\overline{\frown}} \pi(x)$. The relation $y \leq z$ implies that $\pi(x) = \pi(y) \leq y \leq z$, and hence contradicts $z \mathrel{\overline{\frown}} \pi(x)$. Hence $y \not\leq z$ and $z \in O_x^{x,y}$. Let us suppose that $z' < z$ and $z' \mathrel{\overline{\frown}} x$. Then $\mathrm{h}(z') < \mathrm{h}(z)$ and $z' \lhd z$, so that $z' \in C_{\lhd}(x)$. Since $z' \mathrel{\overline{\frown}} x$ then either $z' \mathrel{\overline{\frown}} \pi(x)$, or $\pi(x) \leq z'$. However, the latter property implies $\pi(x) \leq z$, which is not the case. Therefore $z' \mathrel{\overline{\frown}} \pi(x)$ and $z' \in L_{\lhd}(x)$. □

Lemma 19. *If x, y, z are pairwise distinct twins, then $L_{\lhd}(x) = \emptyset$ and z is the minimal element of $O_x^{x,y} \cap O_y^{x,y}$. In particular $O_{\lhd}(x) = C_{\lhd}(x) \subseteq \{\, y, z \,\}$.*

Proof. By the previous observation $L_{\lhd}(x) \subseteq O_x^{x,y}$ and, similarly, $L_{\lhd}(x) \subseteq O_x^{x,z}$. Hence $L_{\lhd}(x) \subseteq O_x^{x,y} \cap O_x^{x,z} = O_x^{x,y,z} = \emptyset$, by Lemma 13. Since $x \mathrel{\overline{\frown}} z$ and $y \mathrel{\overline{\frown}} z$ then $z \in O_x^{x,y} \cap O_y^{x,y}$. If $z' < z$ then $z' < x$ and $z' < y$ hence $z' \notin O_x^{x,y} \cup O_y^{x,y}$.

Finally, the relation $C_{\lhd}(x) \subseteq \{\, y, z \,\}$ follows from Lemma 17. □

The previous observations motivate us to introduce the next Definition.

Definition 20. *Let us say that $x, y \in T$ are a* proper pair of twins *if they are distinct and $\{\, z \mid \pi(z) = \pi(x) \,\} = \{\, x, y \,\}$. We say that a linear order $\lhd$ on T is compatible with proper pair of twins if it satisfies* (HEIGHT) *and moreover*

$$O_x^{x,y} \supset O_y^{x,y} \text{ implies } x \lhd y\,, \qquad \text{(TWINS)}$$

for each proper pair of twins x, y.

Again is not difficult to see that such a linear order always exists and in the following we shall assume that $\lhd$ satisfies both (HEIGHT) and (TWINS).

We are ready to define a partial labelling of the event structure $\mathcal{E}$ whose domain is T. W.r.t. $\lhd$ let us say that $x \in T$ is *principal* if $C_{\lhd}(x) = \emptyset$. Let $\Sigma = \{ a_0, a_1, a_2 \}$ be a three elements totally ordered alphabet. The labelling $\lambda : T \longrightarrow \Sigma$ is defined by induction on $\lhd$ as follows:

1. If $x \in T$ is principal and $\mathrm{h}_T(x) = 0$, then we let $\lambda(x) = a_0$.
2. If $x \in T$ is principal and $\mathrm{h}_T(x) \geq 1$, let $\pi(x)$ be its unique lower cover. Since $\pi(x) \in T$ and $\pi(x) \lhd x$, $\lambda(\pi(x))$ is defined and we let $\lambda(x) = \lambda(\pi(x))$.
3. If x is not principal and $L_{\lhd}(x) = \emptyset$, then, by Lemma 17, we let $\lambda(x)$ be the least symbol not in $\lambda(C_{\lhd}(x))$.
4. If x is not principal and $L_{\lhd}(x) \neq \emptyset$ then:
   - by Lemma 19 $C_{\lhd}(x) = \{ y \}$ is a singleton and x, y is a proper pair of twins,
   - by Lemma 18 $L_{\lhd}(x)$ is a lower set of $O_x^{x,y}$. By the condition (TWINS), $O_x^{x,y} \subseteq O_y^{x,y}$, so that $O_x^{x,y}$ is a linear order. Let therefore z_0 be the common least element of $L_{\lhd}(x)$ and $O_x^{x,y}$.

   We let $\lambda(x)$ be the unique symbol not in $\lambda(\{ y, z_0 \})$.

Proposition 21. *For each $x, y \in T$, if $x \frown y$ then $\lambda(x) \neq \lambda(y)$.*

Proof. It suffices to prove that $\lambda(y) \neq \lambda(x)$ if $y \in O_{\lhd}(x)$. The statement is proved by induction on $\lhd$. Let us suppose the statement is true for all $z \lhd x$.

(i) If $\mathrm{h}_T(x) = 0$ then x is minimal in T, so that $O_{\lhd}(x) = C_{\lhd}(x)$. If moreover x is principal then $O_{\lhd}(x) = C_{\lhd}(x) = \emptyset$, so that the statement holds trivially.

(ii) If x is principal and $\mathrm{h}_T(x) \geq 1$, then its unique lower cover $\pi(x)$ belongs to T. Observe that $O_{\lhd}(x) = L_{\lhd}(x) = \{ y \in T \mid y \lhd x \text{ and } y \frown \pi(x) \}$, so that if $y \in O_{\lhd}(x)$, then $y \frown \pi(x)$. Since $y \lhd x$ and $\pi(x) \lhd x$, and either $y \in O_{\lhd}(\pi(x))$ or $\pi(x) \in O_{\lhd}(y)$, it follows that $\lambda(x) = \lambda(\pi(x)) \neq \lambda(y)$ from the inductive hypothesis.

(iii) If x is not principal and $L_{\lhd}(x) = \emptyset$, then $O_{\lhd}(x) = C_{\lhd}(x)$ and, by construction, $\lambda(y) \neq \lambda(x)$ whenever $y \in O_{\lhd}(x)$.

(iv) If x is not principal and $L_{\lhd}(x) \neq \emptyset$, then let $C_{\lhd}(x) = \{ y \}$ and let z_0 be the common least element of $L_{\lhd}(x)$ and $O_x^{x,y}$. Since by construction $\lambda(x) \neq \lambda(y)$, to prove that the statement holds for x, it is enough to pick $z \in L_{\lhd}(x)$ and argue that $\lambda(z) \neq \lambda(x)$. We claim that each element $z \in L_{\lhd}(x) \setminus \{ z_0 \}$ is principal. If the claim holds, then $\lambda(z) = \lambda(\pi(z))$, so that $\lambda(z) = \lambda(z_0)$ is inductively deduced.

Suppose therefore that there exists $z \in L_{\lhd}(x)$ which is not principal and let $w \in C_{\lhd}(z)$. Observe that x, y form a proper pair of twins, since otherwise $L_{\lhd}(x) = \emptyset$ by Lemma 19. Similarly w, z form a proper pair of twins: otherwise, if z, w, u are pairwise distinct twins, then either $w \leq x$ or $u \leq x$ by Lemma 13. However this is not possible, since for example $z_0 \leq \pi(x) < u \leq x$ contradicts $z_0 \frown x$.

Since $y \lhd x$, condition (TWINS) implies $O_x^{x,y} \subseteq O_y^{x,y}$, and hence $z \in O_x^{x,y} \cap O_y^{x,y}$. If $w \in C_{\lhd}(z)$, then we cannot have $w \leq x$ or $w = y$, since again we would deduce $z_0 \leq x$. Thus Lemma 15 implies $O_z^{w,z} \supset O_w^{w,z}$. On the other hand, $w \lhd z$ and condition (TWINS) implies $O_z^{w,z} \subseteq O_w^{w,z}$.

Thus, we have reached a contradiction by assuming $C_{\lhd}(z) \neq \emptyset$. It follows that z is principal. □

The obvious corollary of Proposition 21 is that if $\mathcal{E}$ is already a sort of tree, then it has a nice labelling with 3 letters. We state this fact as the following Theorem, after we having made precise the meaning of the phrase "$\mathcal{E}$ is a sort of tree."

Definition 22. *Let us say that* $\mathcal{E}$ *is a* forest *if every element has at most one lower cover. Let* $\mathcal{F}_3$ *be the class of event structures of degree* 3 *that are forests.*

Theorem 23. *The labelling number of the class* $\mathcal{F}_3$ *is* 3.

As a matter of fact, let $\mathcal{E}$ be a forest, and consider the event structure $\mathcal{E}_\perp$ obtained from $\mathcal{E}$ by adding a new bottom element $\perp$. Remark that the graph $\mathcal{G}(\mathcal{E}_\perp)$ is the same graph as $\mathcal{G}(\mathcal{E})$ apart from the fact that an isolated vertex $\perp$ has been added. The set of events E is a tree within $\mathcal{E}_\perp$, hence the graph induced by E in $\mathcal{G}(\mathcal{E}_\perp)$ can be colored with three colors. But this graph is exactly $\mathcal{G}(\mathcal{E})$.

5 More Upper Bounds

The results presented in the previous section exemplify a remarkable property of event structures of degree 3: many types of subsets of events induce a subgraph of $\mathcal{G}(\mathcal{E})$ that can be colored with 3 colors. These include antichains by Corollary 10, trees by Proposition 21, and lower sets in $\mathcal{C}$, that is configurations of $\mathcal{E}$. As a matter fact, if $X \in \mathcal{C}$, then $\mathrm{w}(X) \leq 3$, so that such a subset can be labeled with 3 letters by Dilworth's Theorem. Also, recall that the star of an event $x \in E$ is the subgraph of $\mathcal{G}(\mathcal{E})$ induced by the subset $\{\, x \,\} \cup \{\, y \in E \,|\, y \mathrel{\overline{\frown}} x \,\}$. A star can also be labeled with 3 letters. To understand the reason, let $\mathcal{N}_x$ be the event structure $\langle \{\, y \,|\, y \mathrel{\overline{\frown}} x \,\}, \leq_x, \frown_x \rangle$, where $\leq_x$ and $\frown_x$ are the restrictions of the causality and concurrency relations to the set of events of $\mathcal{N}_x$.

Lemma 24. *The degree of* $\mathcal{N}_x$ *is strictly less than* $\deg(\mathcal{E})$.

Proof. The lemma follows since if $y \mathrel{\overline{\frown}}_x z$ in $\mathcal{N}_x$, then $y \mathrel{\overline{\frown}} z$ in $\mathcal{E}$. As a matter of fact, let us suppose that $y \mathrel{\overline{\frown}}_x z$ in $\mathcal{N}_x$ and $y' < y$. If $y' \in \mathcal{N}_x$, then $y' \mathrel{\underline{\frown}} z$. If $y' \notin \mathcal{N}_x$, then $y' < x$. It follows then from $z \mathrel{\overline{\frown}} x$ and $y' < x$ that $y' \mathrel{\underline{\frown}} x$. Similarly, if $z' < z$ then $z' \mathrel{\underline{\frown}} x$. □

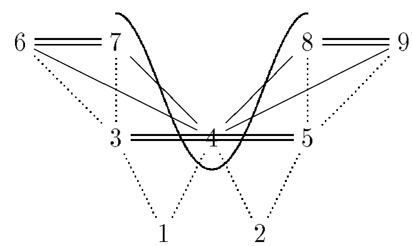
Hence, if $\deg(\mathcal{E}) = 3$, then $\deg(\mathcal{N}_x) \leq 2$, and it can be labeled with 2 letters, by [9]. It follows that star of x can be labeled with 3 letters.

It might be asked whether this property can be exploited to construct nice labellings. The positive answer comes from a standard technique in graph theory [19]. Consider a partition $\mathcal{P} = \{\, [z] \mid z \in E \,\}$ of the set of events such that each equivalence class $[z]$ has a labelling with 3 letters. Define the quotient graph $\mathcal{G}(\mathcal{P}, \mathcal{E})$ as follows: its vertexes are the equivalence classes of $\mathcal{P}$ and $[x] \mathrel{\overline{\frown}} [y]$ if and only if there exists $x' \in [x]$, $y' \in [y]$ such that $x' \mathrel{\overline{\frown}} y'$.

Proposition 25. *If the graph $\mathcal{G}(\mathcal{P}, \mathcal{E})$ is n-coloriable, then $\mathcal{E}$ has a labelling with $3n$ colors.*

Proof. For each equivalence class $[x]$ choose a labelling $\lambda_{[x]}$ of $[x]$ with an alphabet with 3 letters. Let λ_0 a coloring of the graph $\mathcal{G}(\mathcal{P}, \mathcal{E})$ and define $\lambda(x) = (\lambda_{[x]}(x), \lambda_0([x]))$. Then λ is a labelling of $\mathcal{E}$: if $x \mathrel{\underline{\frown}} y$ and $[x] = [y]$, then $\lambda_{[x]}(x) = \lambda_{[y]}(x) \neq \lambda_{[y]}(y)$ and otherwise, if $[x] \neq [y]$, then $[x] \mathrel{\underline{\frown}} [y]$ so that $\lambda_0([x]) \neq \lambda_0([y])$. □

The reader should remark that Proposition 25 generalizes Proposition 11. The Proposition also suggests that a finite upper bound for the labelling number of event structures of degree 3 might indeed exist.



We conclude the paper by exemplifying how to use the Labelling Theorem on trees and the previous Lemma to construct a finite upper bound for the labelling number of event structures that we call simple due to their additional simplifying properties. Consider the event structure on the right and name it $\mathcal{S}$. In this picture we have used dotted lines for the edges of the Hasse diagram of $\langle E, \leq \rangle$, simple lines for maximal concurrent pairs, and double lines for minimal conflicts. Concurrent pairs $x \frown y$ that are not maximal, i.e. for which there exists x', y' such that $x' \mathrel{\underline{\frown}} y'$ and either $x < x'$ or $y < y'$, are not drawn. We leave the reader to verify that a nice labelling of $\mathcal{S}$ needs at least 4 letters. On the other hand, it shouldn't be difficult to see that a nice labelling with 4 letters exists. To obtain it, take apart events with at most 1 lower cover from the others, as suggested in the picture. Use then the results of the previous section to label with three letters the elements with at most one lower cover, and label the only element with two lower covers with a forth letter.

A formalization of this intuitive method leads to the following Definition and Proposition.

Definition 26. *We say that an event structure is* simple *if*

1. *it is graded, i.e.* $\mathrm{h}(x) = \mathrm{h}(y) - 1$ *whenever* $x \prec y$,
2. *every size* 3 *clique of* $\mathcal{G}(\mathcal{E})$ *contains a minimal conflict.*

The event structure $\mathcal{S}$ is simple and proves that even simple event structures cannot be labeled with just 3 letters.

Proposition 27. *Every simple event structure of degree* 3 *has a nice labelling with* 12 *letters.*

Proof. Recall that $\mathrm{f}(x)$ is the number of lower covers of x and let $E_n = \{ x \in E \mid \mathrm{f}(x) = n \}$. Observe that a simple $\mathcal{E}$ is such that $E_3 = \emptyset$: if $x \in E_3$, then its three lower covers form a clique of concurrent events. Also, by considering the lifted event structure $\mathcal{E}_\perp$, introduced at the end of section 4, we can assume that $\mathrm{card}(E_0) = 1$, i.e. $\mathcal{E}$ has just one minimal element which necessarily is isolated in the graph $\mathcal{G}(\mathcal{E})$.

Let $\lhd$ be a linear ordering of E compatible with the height. W.r.t. this linear ordering we shall use a notation analogous to the one of the previous section: we let

$$Q_\lhd(x) = \{\, y \in E \mid y \lhd x \text{ and } y \mathrel{\underline{\frown}} x \,\}\,, \quad C(x) = \{\, y \in E \mid y' \prec y \text{ implies } y' \prec x \,\}\,.$$

Claim. The subgraph of $\mathcal{G}(\mathcal{E})$ induced by E_2 can be colored with 3-colors.

We claim first that if $x \in E_2$ then $Q_\lhd(x) \subseteq C(x)$. Let $y \in Q_\lhd(x)$ and let x_1, x_2 be the two lower covers of x. From $x_i < x \mathrel{\underline{\frown}} y$ it follows $x_i < y$ or $x_i \frown y$. If $x_i \frown y$ for $i = 1, 2$, then y, x_1, x_2 is a clique of concurrent events. Therefore, at least one lower cover of x is below y, let us say $x_1 < y$. It follows that $\mathrm{h}(y) \geq \mathrm{h}(x)$, and since $y \lhd x$ implies $\mathrm{h}(y) \leq \mathrm{h}(x)$, then x, y have the same height. We deduce that $x_1 \prec y$. If y has a second lower cover y' which is distinct from x_1, then y', x_1, x_2 is a clique of concurrent events. Hence, if such y' exists, then $y' = x_2$. Second, we remark that if $y, z \in C(x)$ and $x \in E_2$ then $y \mathrel{\underline{\frown}} z$: if $y' < y$ then $y' \leq x$ so that $x \mathrel{\underline{\frown}} z$ implies $y' \mathrel{\underset{=}{\frown}} z$, and symmetrically. It follows that for $x \in E_2$, $C(x)$ may have at most 2 elements. In particular, the restriction of $\lhd$ to E_2 is a 2-elimination ordering. $\square$ *Claim*

For $x \in E_1$ let $\rho(x) = \max\{\, z \in E \mid z \leq x, z \notin E_1 \,\}$ and $[x] = \{\, y \in E_1 \mid \rho(y) = \rho(x) \,\}$. Let $\mathcal{P}$ be the partition $\{\, E_0 \,\} \cup \{\, [x] \mid x \in E_1 \,\} \cup \{\, E_2 \,\}$. Since each $[x]$, $x \in E_1$, is a tree, the partition $\mathcal{P}$ is such that each equivalence class induces a 3-colorable subgraph of $\mathcal{G}(\mathcal{E})$.

Claim. The graph $\mathcal{G}(\mathcal{P}, \mathcal{E})$ is 4-colorable.

Since E_0 is isolated in $\mathcal{G}(\mathcal{P}, \mathcal{E})$, is it enough to prove that the subgraph of $\mathcal{G}(\mathcal{P}, \mathcal{E})$ induced by the trees $\{\, [x] \mid x \in E_1 \,\}$ is 3-colorable. Transport the linear ordering $\lhd$ to a linear ordering on the set of trees: $[y] \lhd [x]$ if and only if $\rho(y) \lhd \rho(x)$. Define $Q_\lhd([x])$ as usual, we claim that $Q_\lhd([x])$ may contain at most two trees.

We define a function $f : Q_\lhd([x]) \longrightarrow C(\rho(x))$ as follows. If $[y] \mathrel{\underline{\frown}} [x]$ and $[y] \lhd [x]$ then we can pick $y' \in [y]$ and $x' \in [x]$ such that $y' \mathrel{\underline{\frown}} x'$. We notice also that $y' \mathrel{\underline{\frown}} \rho(x)$: from $\rho(x) \leq x' \mathrel{\underline{\frown}} y'$, we deduce $\rho(x) \mathrel{\underline{\frown}} y'$ or $\rho(x) \leq y'$. The latter, however, implies $\rho(x) \leq \rho(y)$, by the definition of ρ, and this relation contradicts $\rho(y) \lhd \rho(x)$. Thus we let

$$f([y]) = \min\{\, z \mid \rho(y) \leq z \leq y' \text{ and } z \not\leq \rho(x) \,\}\,.$$

By definition, $f([y]) \mathrel{\underline{\frown}} \rho(x)$ and every lower cover of $f([y])$ is a lower cover of x. This clearly holds if $f([y]) \neq \rho(y)$, and if $f([y]) = \rho(y)$ then it holds since $\rho(y) \lhd \rho(x)$ implies $f([y]) \in Q_\lhd(x) \subseteq C(x)$ by the previous Claim. Thus the set $f(Q_\lhd(x))$ has cardinality at most 2 and, moreover, we claim that f is injective. Let us suppose that $f([y]) = f([z])$. If $f([y]) = \rho(y)$, then $f([z]) = \rho(z)$ as well and

$[y] = [x]$. Otherwise $f([y]) = f([x])$ impliees $\rho(y) = \rho(f([y])) = \rho(f([z])) = \rho(z)$ and $[y] = [z]$. □ *Claim*

Thus, by applying Proposition 25, we deduce that $\mathcal{G}(\mathcal{E})$ has a labelling with 12 letters. □

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