

# Linear elliptic differential equations

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## 1 Introduction

### 1.1 Motivation: A model problem

Many physical problems can be modeled by partial differential equations. Let us consider for example the case of an elastic membrane  $\Omega$ , with fixed boundary  $\Gamma$ , subject to pressure forces  $f$ . The vertical membrane displacement is represented by a real valued function  $u$ , which solves the equation

$$-\Delta u(x) = f(x), \quad x = (x_1, x_2) \in \Omega, \quad (1.1)$$

where the Laplace operator  $\Delta$  is defined, in two dimensions, by

$$\Delta u = \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2}.$$

As the membrane is glued to the curve  $\Gamma$ ,  $u$  satisfies the condition

$$u(x) = 0, \quad x \in \Gamma. \tag{1.2}$$

The system (1.1)-(1.2) is the homogeneous Dirichlet problem for the Laplace operator. It enters the more general framework of (linear) elliptic boundary value problems, which consist of a (linear) partial differential equation (in the example above, of order two: the highest order in the derivatives) inside an open set  $\Omega$  of the whole space  $\mathbb{R}^N$ , satisfying some “elliptic” property, completed by (linear) conditions on the boundary  $\Gamma$  of  $\Omega$ , called “boundary conditions”. In the sequel, we only consider the linear case.

Our aim is to answer the following questions: does this problem admit a solution? in which space? is this solution unique? does it depend continuously on the given data  $f$ ? In case of positive answers, we say that the problem is “well posed” in the Hadamard sense. But other questions can also be raised, such as the sign of the solution, for example, or its regularity. We give a full survey of linear elliptic problems in a bounded or in an exterior domain with a sufficiently smooth boundary and in the whole space. In the general theory of the elliptic problem we consider only smooth coefficients. We survey the standard theory, which can be found in the several well-known monographs of 1960s. The new trends in investigation of the elliptic problems is to consider more general domains with nonsmooth boundaries and nonsmooth coefficients. On the other hand the regularity results for elliptic systems have not been improved during last thirty years. New trends also require employment of more general function spaces and more general functional background.

The number of references (Further reading) is strictly limited here; there are only some of the most important publications. The basic facts can usually be found in more places and sometimes we do not mention the particular reference. To the very basic references throughout belong [F], [GT], [DL], [Ho], [LU], [LM], [RR], [W]; of course there are many others.

## 1.2 The method

To answer the above questions, we generally use, for such elliptic problems, an approach based on what is called a “variational formulation” (see section 3 below): the boundary value problem is first transformed into a variational problem of lower order, which is solved in a Hilbertian frame with help of the Lax-Milgram Theorem (based on the representation theorem). All questions are then solved (existence, uniqueness, continuity in terms of the data, regularity). But this variational formalism does not necessary allow to treat all the situations and it is limited to the Hilbertian case. Other strategies can then be developed, based on a priori estimates and duality arguments for the existence problem, or maximum principle for the question of unicity. Without forgetting the particular cases where an explicit Green kernel is computable (the Laplacean operator in the whole space case for example).

Moreover, the study of linear elliptic equations is directly linked to the function spaces background. It is the reason why we first deal with Sobolev spaces—both of the integer and fractional order and we survey their basic properties, imbedding and trace theorems. We pay attention to the Riesz and Bessel potentials and we define weighted Sobolev spaces important in the context of unbounded opens. Second, we present the variational approach and the Lax-Milgram theorem as a key point to solve a large class of boundary value problems. We give examples: the Dirichlet and Neumann problems for the Poisson equation, the Newton problem for more general second order operators; we also investigate mixed boundary conditions and present an example of a problem of fourth order. Then, we briefly present the arguments for studying general elliptic problems and concentrate on second order elliptic problems; we recall the weak and strong maximum principle, formulate the Fredholm alternative and tackle the regularity questions. Moreover, we are interested in the existence and uniqueness of solution of the Laplace equation in the whole space and in exterior opens. Finally, we present some particular examples arising from physical problems, either in fluid mechanics (the Stokes system) or in elasticity.

## 2 Sobolev and other type of spaces

Throughout,  $\Omega \subset \mathbb{R}^N$  will generally be an open subset of the  $N$ -dimensional Euclidean space  $\mathbb{R}^N$ . A *domain* will be an open and connected subset of

$\mathbb{R}^N$ . We shall use standard notations for the spaces  $L^p(\Omega)$ ,  $C^\infty(\Omega)$  etc. and their norms. Let us agree that  $C^{k,r}(\Omega)$ ,  $k \in \mathbb{N}$ ,  $r \in (0, 1)$ , denote the space of functions  $f$  in  $C^k(\Omega)$ , whose derivatives  $D^\alpha f$ ,  $\alpha = (\alpha_1, \dots, \alpha_N) \in \mathbb{N}^N$  of order  $|\alpha| = \sum_{i=1}^N \alpha_i = k$  are all  $r$ -Hölder continuous. By  $\overline{\Omega}$  in the notations for some of these spaces we mean that the functions have the corresponding property on  $\Omega$  and that they can be continuously extended to  $\overline{\Omega}$ .

Let us recall several fundamental concepts. The space  $\mathcal{D}(\Omega)$  of the *test functions* in  $\Omega$  consists of all infinitely differentiable  $\varphi$  with a compact support in  $\Omega$ . A locally convex topology can be introduced here. The elements of the dual space  $\mathcal{D}'(\Omega)$  are called the *distributions*. If  $f \in L^1_{\text{loc}}(\Omega)$  (i.e.  $f \in L^1(K)$  for all compact subset  $K$  of  $\Omega$ ), then  $f$  is a *regular distribution*; the duality is represented by  $\int_\Omega f(x)\varphi(x)dx$ . If  $f \in \mathcal{D}'(\Omega)$ , we define the *distributional* or the *weak derivative*  $D^\alpha$  of  $f$  as the distribution  $\varphi \mapsto (-1)^{|\alpha|} \langle f, D^\alpha \varphi \rangle$ . Plainly, if  $f \in L^1_{\text{loc}}$  has “classical” partial derivatives in  $L^1_{\text{loc}}$ , then it coincides with the corresponding weak derivative.

If  $\Omega = \mathbb{R}^N$ , it is sometimes more suitable to work with the *tempered distributions*. The role of  $\mathcal{D}(\Omega)$  is played by the space  $\mathcal{S}(\mathbb{R}^N)$  of  $C^\infty$  functions with finite pseudonorms  $\sup |D^\alpha f(x)|(1 + |x|)^k$ ,  $|\alpha|, k = 0, 1, 2, \dots$ . Recall that the Fourier transform  $\mathcal{F}$  maps  $\mathcal{S}(\mathbb{R}^N)$  into itself and the same is true for the space of the tempered distributions  $\mathcal{S}'(\mathbb{R}^N)$ .

## 2.1 Sobolev spaces of positive order

The *Sobolev space*  $W^{k,p}(\Omega)$ ,  $1 \leq p \leq \infty$ ,  $k \in \mathbb{N}$ , is the space of all  $f \in L^p(\Omega)$ , whose weak derivatives up to order  $k$  are regular distributions belonging to  $L^p(\Omega)$ ; in  $W^{k,p}(\Omega)$  we introduce the norm

$$\|f\|_{W^{k,p}(\Omega)} = \left( \sum_{|\alpha| \leq k} \int_\Omega |D^\alpha f(x)|^p dx \right)^{1/p} \quad (2.1)$$

when  $p < \infty$  and  $\max_{|\alpha| \leq k} \sup_{x \in \Omega} \text{ess} |D^\alpha f(x)|$  if  $p = \infty$ . The space  $W^{k,p}(\Omega)$  is a Banach space, separable for  $p < \infty$ , reflexive for  $1 < p < \infty$ ; it is a Hilbert space for  $p = 2$ , more simply denoted  $H^m(\Omega)$ . In the following we shall consider only the range  $p \in (1, \infty)$ .

The link with the classical derivatives is given by this well-known fact: A function  $f$  belongs to  $W^{1,p}(\Omega)$  if and only if it is a.e. equal to a function  $\tilde{u}$ , absolutely continuous on almost all line segments in  $\Omega$  parallel to the

coordinate axes, whose (classical) derivatives belong to  $L^p(\Omega)$  (the Beppo-Levi theorem).

For  $1 < p < \infty$  and noninteger  $s > 0$  the Sobolev space  $W^{s,p}(\Omega)$  of order  $s$  is defined as the space of all  $f$  with the finite norm

$$\|f\|_{W^{s,p}(\Omega)} = \left( \|f\|_{W_p^{[s]}(\Omega)}^p + \sum_{|\alpha|=[s]} \int_{\Omega} \int_{\Omega} \frac{|D^{\alpha} f(x) - D^{\alpha} f(y)|^p}{|x-y|^{N+p(s-[s])}} \right)^{1/p},$$

where  $[s]$  is the integer part of  $s$ . Details e.g. in [Ad], [Z].

## 2.2 Imbedding theorems

One of the most useful and important features of the functions in Sobolev spaces is an improvement of their integrability properties and the compactness of various imbeddings. Theorems of this type were first proved by Sobolev and Kondrashev. Let us agree that the symbol  $\hookrightarrow$  and  $\hookrightarrow\hookrightarrow$  stands for an imbedding and for a compact imbedding, respectively.

**Theorem 2.1.** *Let  $\Omega$  be a Lipschitz open. Then*

- (i) *If  $sp < N$ , then  $W^{s,p}(\Omega) \hookrightarrow L^{p^*}(\Omega)$  with  $p^* = Np/(N - ps)$  (the Sobolev exponent). If  $|\Omega| < \infty$ , then the target space is any  $L^r(\Omega)$  with  $0 < r \leq p^*$ .*

*If  $\Omega$  is bounded, then  $W^{s,p}(\Omega) \hookrightarrow\hookrightarrow L^q(\Omega)$  for all  $1 \leq q < p^*$ .*

- (ii) *If  $sp > N$ , then  $W^{j+s,p}(\Omega) \hookrightarrow C^j(\Omega)$ . If  $\Omega$  has the Lipschitz boundary, then  $W^{j+s,p}(\Omega) \hookrightarrow C^{j,\mu}(\overline{\Omega})$ .*

*If  $sp > N$ , then  $W^{j+s,p}(\Omega) \hookrightarrow\hookrightarrow C^j(\Omega)$  and  $W^{j+s,p}(\Omega) \hookrightarrow\hookrightarrow W_q^j(\Omega)$  for all  $1 \leq q \leq \infty$ . If, moreover,  $\Omega$  has the Lipschitz boundary, then the target space can be replaced by  $C^{j,\mu}(\overline{\Omega})$  provided  $sp > N > (s-1)p$  and  $0 < \mu < s - N/p$ .*

Note that if the imbedding  $W^{s,p}(\Omega) \hookrightarrow L^q(\Omega)$  is compact for some  $q \geq p$ , then  $|\Omega| < \infty$ . Moreover, if  $\limsup_{r \rightarrow \infty} |\{x \in \Omega; r \leq |x| < r+1\}| > 0$ , then  $W^{s,p}(\Omega) \hookrightarrow L^q(\Omega)$  cannot be compact.

### 2.3 Traces and Sobolev spaces of negative order

Let  $s > 0$  and let  $\Omega$  be, for simplicity, a bounded open subset of  $\mathbb{R}^N$  with boundary  $\Gamma$  of class  $C^{[s],1}$ . Then with help of local coordinates, we can define Sobolev spaces  $W^{s,p}(\Gamma)$  (also denoted  $H^s(\Gamma)$  for  $p = 2$ ) on  $\Gamma = \partial\Omega$  (see e.g. [N], [Ad] for details). If  $f \in C(\overline{\Omega})$ , then  $f|_\Gamma$  has sense. Introducing the space  $\mathcal{D}(\overline{\Omega})$  of restrictions in  $\Omega$  of functions in  $\mathcal{D}(\mathbb{R}^N)$ , one can show that if  $f \in \mathcal{D}(\overline{\Omega})$ , we have  $\|f|_\Gamma\|_{W^{1-1/p,p}(\Gamma)} \leq C\|f\|_{W^{1,p}(\Omega)}$  so that, in view of the density of  $\mathcal{D}(\overline{\Omega})$  in  $W^{1,p}(\Omega)$ , the restriction of  $f$  to  $\Gamma$  can be uniquely extended to the whole  $W^{1,p}(\Omega)$ . The result is the bounded *trace operator*  $\gamma_0 : W^{1,p}(\Omega) \rightarrow W^{1-1/p,p}(\Gamma)$ . Moreover, every  $g \in W^{1-1/p,p}(\Gamma)$  can be extended to a (non unique) function  $f \in W^{1,p}(\Omega)$  and this extension operator is bounded with respect to the corresponding norms.

More generally, let us suppose  $\Gamma$  is of class  $C^{k-1,1}$  and define the operator  $\text{Tr}_n$  for any  $f \in \mathcal{D}(\overline{\Omega})$  by  $\text{Tr}_n f = (\gamma_0 f, \gamma_1 f, \dots, \gamma_{k-1} f)$ , where

$$\gamma_j f(x) = \frac{\partial^j f}{\partial n^j}(x) = \sum_{|\alpha|=j} \frac{j!}{\alpha!} (\partial^\alpha f(x) / \partial x^\alpha) n^\alpha, \quad x \in \Gamma,$$

is the  $j$ -th order derivative of  $f$  with respect to the outer normal  $n$  at  $x \in \Gamma$ ; by density, this operator can be uniquely extended to a continuous linear mapping defined on the space  $W^{k,p}(\Omega)$ ; moreover,  $\gamma_0(W^{k,p}(\Omega)) = W^{k-1/p,p}(\Gamma)$ .

The kernel of this mapping is the space  $\mathring{W}^{k,p}(\Omega)$  (denoted by  $H_0^k(\Omega)$  for  $p = 2$ ), where  $\mathring{W}^{s,p}(\Omega)$  is defined as the closure of  $\mathcal{D}(\Omega)$  in  $W^{s,p}(\Omega)$  ( $s > 0$ ). For  $1 < p < \infty$  the following holds:  $\mathring{W}^{s,p}(\mathbb{R}^N) = W^{s,p}(\mathbb{R}^N)$ ,  $\mathring{W}^{s,p}(\Omega) = W^{s,p}(\Omega)$  provided  $0 < s \leq 1/p$ . If  $s < 0$ , then the space  $W^{s,p}(\Omega)$  is defined as the dual to  $\mathring{W}^{-s,p'}(\Omega)$ , where  $p' = p/(p-1)$  (see e.g. [Tr1], [Tr2]). Observe that for an arbitrary  $\Omega$  a function  $f \in W^{1,p}(\Omega)$  has the zero trace if and only if  $f(x)/\text{dist}(x, \Gamma)$  belongs to  $L^p(\Omega)$ .

For  $p = 2$ , we simply denote by  $H^{-k}(\Omega)$  the dual space of  $H_0^k(\Omega)$ . In the case of bounded opens, we recall the following useful Poincaré-Friedrichs inequality (for simplicity, we state it here in the Hilbert frame):

**Theorem 2.2.** *Let  $\Omega$  be bounded (at least in one direction of the space). Then there exists a positive constant  $C_P(\Omega)$  such that*

$$\|v\|_{L_2(\Omega)} \leq C_P(\Omega) \|\nabla v\|_{[L_2(\Omega)]^N} \quad \text{for all } v \in H_0^1(\Omega). \quad (2.2)$$

## 2.4 The whole space case: Riesz and Bessel potentials

The Riesz potentials  $\mathcal{I}_\alpha$  naturally occur when one defines the formal powers of the Laplace operator  $\Delta$ . Namely, if  $f \in \mathcal{S}(\mathbb{R}^N)$  and  $\alpha > 0$ , then  $\mathcal{F}[(-\Delta)^{\alpha/2}f](\xi) = |\xi|^\alpha \mathcal{F}f(\xi)$ . This can be taken formally as a definition of the *Riesz potential*  $\mathcal{I}_\alpha$  on  $\mathcal{S}'(\mathbb{R}^N)$ ,

$$\mathcal{I}_\alpha f(\cdot) = \mathcal{F}^{-1} [|\xi|^{-\alpha} \mathcal{F}f(\xi)] (\cdot)$$

for any  $\alpha \in \mathbb{R}$ . If  $0 < \alpha < N$ , then  $\mathcal{I}_\alpha f(x) = (I_\alpha * f)(x)$ , where  $I_\alpha$  is the inverse Fourier transform of  $|\xi|^{-\alpha}$ ,

$$I_\alpha(x) = C_\alpha |x|^{\alpha-N}, \quad C_\alpha = \Gamma((N-\alpha)/2) (\pi^{N/2} 2^\alpha \Gamma(\alpha/2))^{-1}$$

( $\Gamma$  is the Gamma function) is the *Riesz kernel*. The following formula is also true:

$$I_\alpha(x) = C_\alpha \int_0^\infty t^{\frac{\alpha-N}{2}} e^{-\frac{\pi|x|^2}{t}} \frac{dt}{t}.$$

Recall that every  $f \in \mathcal{S}(\mathbb{R}^N)$  can be represented as the *Riesz potential*  $\mathcal{I}_\alpha g$  of a suitable function  $g \in \mathcal{S}(\mathbb{R}^N)$ , namely,  $g = (-\Delta)^{\alpha/2} f$ ; we get the representation formula

$$f(x) = \mathcal{I}_\alpha g(x) = C_\alpha \int_{\mathbb{R}^N} \frac{g(y)}{|x-y|^{N-\alpha}} dy.$$

The standard density argument implies then an appropriate statement for functions in  $W^{k,p}(\mathbb{R}^N)$  with an integer  $k$  and for the *Bessel potential spaces*  $\mathcal{H}^{\alpha,p}(\mathbb{R}^N)$ —see below for their definition. The original Sobolev imbedding theorem comes from the combination of this representation and the basic continuity property of  $I_\alpha$ ,  $\alpha p < N$ ,

$$I_\alpha : L^p(\mathbb{R}^N) \rightarrow L_q(\mathbb{R}^N), \quad \frac{1}{q} = \frac{1}{p} - \frac{\alpha}{N}.$$

To get an isomorphic representation of a Bessel potential space (of a Sobolev space with positive integer smoothness in particular) it is more convenient to consider the *Bessel potentials* (of order  $\alpha \in \mathbb{R}$ ),

$$\mathcal{G}_\alpha f(x) = (G_\alpha * f)(x) = \mathcal{F}^{-1} ([1 + |\xi|^2]^{-\alpha/2} \mathcal{F}f(\xi)) (x)$$

(with a slight abuse of the notations); the following formula for the *Bessel kernel*  $G_\alpha$  is well known:

$$G_\alpha(x) = c_\alpha^{-1} \int_0^\infty t^{\frac{\alpha-N}{2}} e^{-\frac{\pi|x|^2}{t} - \frac{t}{4\pi}} \frac{dt}{t}$$

(cf. the analogous formula for  $I_\alpha$ ), where  $c_\alpha = (4\pi)^{\alpha/2} \Gamma(\alpha/2)$ . The kernels  $G_\alpha$  can alternatively be expressed with help of Bessel or Macdonald functions.

Now we can define the *Bessel potential spaces*. For  $s \in \mathbb{R}$  and  $1 < p < \infty$ , let  $\mathcal{H}^{s,p}(\mathbb{R}^N)$  be the space of all  $f \in \mathcal{S}'(\mathbb{R}^N)$  with the finite norm

$$\|f\|_{\mathcal{H}^{s,p}(\mathbb{R}^N)} = \left( \int_{\mathbb{R}^N} \mathcal{F}^{-1} \left( (1 + |\xi|^2)^{s/2} \mathcal{F}f(\xi) \right)^p d\xi \right)^{1/p}.$$

In other words, the spaces  $\mathcal{H}^{s,p}(\mathbb{R}^N)$  are isomorphic copies of  $L^p(\mathbb{R}^N)$ .

For  $k = 0, 1, 2, \dots$ , plainly  $\mathcal{H}^{k,2}(\mathbb{R}^N) = W^{k,2}(\mathbb{R}^N)$  by virtue of the Plancherel theorem. But it is true also for integer  $s$  and general  $1 < p < \infty$  (see e.g. [Tr1]).

**Remark 2.3.** Much more comprehensive theory of general Besov and Lizorkin-Triebel spaces in  $\mathbb{R}^N$  has been established in last decades, relying on the the Littlewood-Paley theory. Spaces on opens can be defined as restrictions of functions in the corresponding space on the whole  $\mathbb{R}^N$ , allowing to derive their properties from those valid for functions on  $\mathbb{R}^N$ . The justification for that are extension theorems. In particular, there exists a universal extension operator for the Lipschitz open, working for all the spaces mentioned up to now. We refer to [Tr1], [Tr2].

## 2.5 Unbounded opens and weighted spaces

The study of the elliptic problems in unbounded opens is usually carried out with use of suitable Sobolev weighted space. The Poisson equation

$$-\Delta u = f \quad \text{in } \mathbb{R}^N, \quad N \geq 2, \quad (2.3)$$

is the typical example; the Poincaré inequality (2.2) is not true here and it is suitable to introduce Sobolev spaces with weights.

Let  $m \in \mathbb{N}$ ,  $1 < p < \infty$ ,  $\alpha \in \mathbb{R}$ ,  $k = m - N/p - \alpha$  if  $N/p + \alpha \in \{1, \dots, m\}$  and  $k = -1$  elsewhere. For an open  $\Omega \subset \mathbb{R}^N$  we define

$$W_\alpha^{m,p}(\Omega) = \left\{ v \in \mathcal{D}'(\Omega), \begin{array}{l} 0 \leq |\lambda| \leq k, \quad \rho^{\alpha-m-|\lambda|} (\log \rho)^{-1} D^\lambda u \in L^p(\Omega), \\ k+1 \leq |\lambda| \leq m, \quad \rho^{\alpha-m+|\lambda|} D^\lambda u \in L^p(\Omega) \end{array} \right\},$$

where  $\rho(x) = (1 + |x|^2)^{1/2}$ ,  $\log \rho = \log(2 + |x|^2)$ . Note that  $W_\alpha^{m,p}$  is a reflexive Banach space for the norm  $\|\cdot\|_{W_\alpha^{m,p}}$  defined by

$$\|u\|_{W_\alpha^{m,p}}^p = \sum_{0 \leq |\lambda| \leq k} \|\rho^{\alpha-m+|\lambda|} (\log \rho)^{-1} D^\lambda u\|_{L^p(\Omega)}^p + \sum_{k+1 \leq |\lambda| \leq m} \|\rho^{\alpha-m+|\lambda|} D^\lambda u\|_{L^p(\Omega)}^p.$$

We also introduce the following seminorm

$$|u|_{W_\alpha^{m,p}} = \left( \sum_{|\lambda|=m} \|\rho^\alpha D^\lambda u\|_{L^p(\Omega)}^p \right)^{1/p}.$$

Let

$$\mathring{W}_\alpha^{m,p}(\Omega) = \{v \in W_\alpha^{m,p}; \gamma_0(v) = \dots = \gamma_{m-1}(v) = 0\}.$$

If  $\Omega$  is a Lipschitz domain, then  $\mathring{W}_\alpha^{m,p}(\Omega)$  is the closure of  $\mathcal{D}(\Omega)$  in  $W_\alpha^{m,p}(\Omega)$ , while  $\mathcal{D}(\overline{\Omega})$  is dense in  $W_\alpha^{m,p}(\Omega)$ . We denote by  $W_\alpha^{-m,p'}(\Omega)$  the dual of  $\mathring{W}_\alpha^{m,p}(\Omega)$ . We note that these spaces contain also polynomials,

$$P_j \subset W_\alpha^{m,p}(\Omega) \Leftrightarrow \begin{cases} j = [m - \frac{N}{p} - \alpha] & \text{if } \frac{N}{p} + \alpha \notin Z \\ j = m - \frac{N}{p} - \alpha & \text{elsewhere,} \end{cases}$$

where  $[s]$  is the integer part of  $s$  and  $P_{[s]} = \{0\}$  if  $[s] < 0$ . The fundamental property of functions belonging to these spaces is that they satisfy the Poincaré weighted inequality. A open  $\Omega$  is an *exterior domain* if it is the complement of a closure of a bounded domain in  $R^N$ .

**Theorem 2.4.** *Suppose that  $\Omega$  is an exterior domain or  $\Omega = \mathbb{R}_+^N$  or  $\Omega = \mathbb{R}^N$ . Then*

- (i) *the seminorm  $|\cdot|_{W_\alpha^{m,p}(\Omega)}$  is a norms on  $W_\alpha^{m,p}(\Omega)/P_j$ , equivalent to the quotient norm with  $j' = \min(m-1, j)$ ,*
- (ii) *the seminorm  $|\cdot|_{W_\alpha^{m,p}(\Omega)}$  is equivalent to the full norm on  $\mathring{W}_\alpha^{m,p}(\Omega)$ .*

### 3 Variational approach

Let us first describe the method on the model problem (1.1)-(1.2), supposing  $f \in L^2(\Omega)$  and  $\Omega$  bounded. We first suppose that this problem admits a sufficiently smooth function  $u$ . Let  $v$  be any arbitrary (smooth) function; we

multiply equation (1.1) by  $v(x)$  and integrate with respect to  $x$  over  $\Omega$ ; this gives

$$\int_{\Omega} -(\Delta uv)(x) dx = \int_{\Omega} (fv)(x) dx.$$

Using the following Green's formula ( $d\Gamma(x)$  denotes the measure on  $\Gamma = \partial\Omega$  and  $\frac{\partial u}{\partial n}(x) = \nabla u(x) \cdot n(x)$ , where  $n(x)$  is the unit normal at point  $x$  of  $\Gamma$  oriented towards the exterior of  $\Omega$ )

$$\int_{\Omega} (\Delta uv)(x) dx = - \int_{\Omega} (\nabla u \cdot \nabla v)(x) dx + \int_{\Gamma} \left(\frac{\partial u}{\partial n} v\right)(\sigma) d\sigma, \quad (3.1)$$

we get, since  $v|_{\Gamma} = 0$ :  $\mathcal{A}(u, v) = L(v)$ , where we have set

$$\mathcal{A}(u, v) = \int_{\Omega} \nabla u(x) \cdot \nabla v(x) dx, \quad L(v) = \int_{\Omega} f(x)v(x) dx. \quad (3.2)$$

The idea is to study in fact this new problem (showing first its equivalence with the boundary value problem), noting that it makes sense for far less regular functions  $u, v$  (and also  $f$ ), in fact  $u, v \in H_0^1(\Omega)$  (and  $f \in H^{-1}(\Omega)$ ).

### 3.1 The Lax-Milgram theorem

The general form of a variational problem is

$$\text{to find } u \in V \text{ such that } \mathcal{A}(u, v) = L(v) \text{ for all } v \in V, \quad (3.3)$$

where  $V$  is a Hilbert space,  $\mathcal{A}$  a bilinear continuous form defined on  $V \times V$  and  $L$  a linear continuous form defined on  $V$ . We say moreover that  $\mathcal{A}$  is *V-elliptic* if there exists a positive constant  $\alpha$  such that

$$\mathcal{A}(u, u) \geq \alpha \|u\|_V^2 \quad \text{for all } u \in V. \quad (3.4)$$

The following theorem is due to Lax and Milgram.

**Theorem 3.1.** *Let  $V$  be a Hilbert space. We suppose that  $\mathcal{A}$  is a bilinear continuous form on  $V \times V$  which is  $V$ -elliptic and that  $L$  is a linear continuous form on  $V$ . Then the variational problem (3.3) has a unique solution  $u$  on  $V$ . Moreover, if  $\mathcal{A}$  is symmetric,  $u$  is characterized as the minimum value on  $V$  of the quadratic functional  $E$  defined by*

$$\text{for all } v \in V, \quad E(v) = \frac{1}{2} \mathcal{A}(v, v) - L(v). \quad (3.5)$$

**Remark 3.2.** (i) We have the following “energy estimate”:  $\|u\|_V \leq \frac{1}{\alpha} \|L\|_{V'}$ . In the particular case of our model problem, this inequality shows the continuity of the solution  $u \in H_0^1(\Omega)$  with respect to the data  $f \in L^2(\Omega)$  (that can be weakened by choosing  $f \in H^{-1}(\Omega)$ ).

(ii) Theorem 3.1 can be extended to sesquilinear continuous forms  $\mathcal{A}$  defined on  $V \times V$ ; such form is called  $V$ -elliptic if there exists a positive constant  $\alpha$  such that

$$\operatorname{Re}\mathcal{A}(u, u) \geq \alpha \|u\|_V^2 \quad \text{for all } u \in V. \quad (3.6)$$

Finally,  $V'$  is the dual of  $V$ .

(iii) Denoting by  $A$  the linear operator defined on the space  $V$  by  $\mathcal{A}(u, v) = \langle Au, v \rangle_{V', V}$ , for all  $v \in V$ , the Lax-Milgram theorem shows that  $A$  is an isomorphism from  $V$  onto its dual space  $V'$ , and the problem (3.3) is equivalent to solving the equation  $Au = L$ .

(iv) Let us do some remarks concerning the numerical aspects. First, this variational formulation is the starting point of the well known finite element method: The idea is to compute a solution of an approximate variational problem stated on a finite subspace of  $V$  (leading to the resolution of a linear system), with a precise control of the error with the exact solution  $u$ . Second, the equivalence with a minimization problem allows the use of other numerical algorithms.

Let us now present some classical examples of second order elliptic problems than can be solved with help of the variational theory.

### 3.2 The Dirichlet problem for the Poisson equation

We consider the problem on a bounded Lipschitz open  $\Omega \subset \mathbb{R}^N$

$$\begin{aligned} -\Delta u &= f && \text{in } \Omega, \\ u &= u_0 && \text{on } \Gamma = \partial\Omega, \end{aligned} \quad (3.7)$$

with  $u_0 \in H^{1/2}(\Gamma)$ , so that there exists  $U_0 \in H^1(\Omega)$  satisfying  $\gamma_0(U_0) = u_0$ . The variational formulation of problem (3.7) is

$$\text{to find } u \in U_0 + H_0^1(\Omega) \text{ such that for all } v \in H_0^1(\Omega), \quad \mathcal{A}(u, v) = L(v), \quad (3.8)$$

with  $\mathcal{A}$  given by (3.2) and a more general  $L$  with  $f \in H^{-1}(\Omega)$ , defined by

$$L(v) = \langle f, v \rangle_{H^{-1}(\Omega), H_0^1(\Omega)}. \quad (3.9)$$

The existence and uniqueness of a solution of (3.8) follows from Theorem 3.6 (and Poincaré inequality (2.2)). Conversely, thanks to the density of  $\mathcal{D}(\Omega)$  in  $H_0^1(\Omega)$ , we can show that  $u$  satisfies (3.7). More precisely, we get:

**Theorem 3.3.** *Let us suppose  $f \in H^{-1}(\Omega)$  and  $u_0 \in H^{1/2}(\Gamma)$ ; let  $U_0 \in H^1(\Omega)$  satisfy  $\gamma_0(U_0) = u_0$ . Then the boundary value problem (3.7) has a unique solution  $u$  such that  $u - U_0 \in H_0^1(\Omega)$ . This is also the unique solution of the variational problem (3.8). Moreover, there exists a positive constant  $C = C(\Omega)$  such that*

$$\|u\|_{H^1(\Omega)} \leq C (\|f\|_{H^{-1}(\Omega)} + \|u_0\|_{H^{1/2}(\Gamma)}), \quad (3.10)$$

which shows that  $u$  depends continuously on the data  $f$  and  $u_0$ .

Moreover, using technics of Nirenberg's differential quotients, we have the following regularity result (see e.g. [Gri]):

**Theorem 3.4.** *Let us suppose that  $\Omega$  is a bounded open subset of  $\mathbb{R}^N$  with a boundary of class  $\mathcal{C}^{1,1}$  and let satisfy  $f \in L^2(\Omega)$ ,  $u_0 \in H^{3/2}(\Gamma)$ . Then  $u \in H^2(\Omega)$  and each equation in (3.7) is satisfied almost everywhere (on  $\Omega$  for the first one and on  $\Gamma$  for the boundary condition). Moreover, there exists a positive constant  $C = C(\Omega)$  such that*

$$\|u\|_{H^2(\Omega)} \leq C [\|f\|_{L^2(\Omega)} + \|g\|_{H^{3/2}(\Gamma)}]. \quad (3.11)$$

By induction, if the data are more regular, i.e.  $f \in H^k(\Omega)$  and  $u_0 \in H^{k+3/2}(\Gamma)$  (with  $k \in \mathbb{N}$ ), and if  $\Gamma$  is of class  $\mathcal{C}^{k+1,1}$ , we get  $u \in H^{k+2}(\Omega)$ .

**Remark 3.5.** Let us point out the importance of the open geometry. For example, if  $\Omega$  is a bounded plane polygon, one can find  $u \in H_0^1(\Omega)$  with  $\Delta u \in \mathcal{C}^\infty(\overline{\Omega})$ , such that  $u \notin H^{1+\pi/w}(\Omega)$ , where  $w$  is the biggest value of the interior angles of the polygon. In particular, if the polygon is not convex, the solution of the Dirichlet problem (3.7) cannot be in  $H^2(\Omega)$ .

### 3.3 The Neumann problem for the Poisson equation

We consider the problem ( $n$  is the unit outer normal on  $\Gamma$ )

$$\begin{aligned} -\Delta u &= f && \text{in } \Omega \\ \frac{\partial u}{\partial n} &= h && \text{on } \Gamma. \end{aligned} \quad (3.12)$$

Setting  $E(\Delta) = \{v \in H^1(\Omega); \Delta v \in L^2(\Omega)\}$ , the space  $\mathcal{D}(\overline{\Omega})$  is a dense subspace, and we have the following Green formula for all  $u \in E(\Delta)$  and  $v \in H^1(\Omega)$ :

$$\int_{\Omega} \Delta u(x)v(x) dx = - \int_{\Omega} \nabla u(x) \cdot \nabla v(x) dx + \left\langle \frac{\partial u}{\partial n}, \gamma_0 v \right\rangle_{H^{-1/2}(\Gamma), H^{1/2}(\Gamma)}.$$

If  $u \in H^1(\Omega)$  satisfies (3.12) with  $f \in L^2(\Omega)$  and  $h \in H^{-1/2}(\Gamma)$ , then for any function  $v \in H^1(\Omega)$ , we have, by virtue of the above Green formula,

$$\mathcal{A}(u, v) = \tilde{L}(v), \quad \tilde{L}v = \int_{\Omega} (fv)(x) dx + \langle h, \gamma_0 v \rangle_{H^{-1/2}(\Gamma), H^{1/2}(\Gamma)}.$$

But here, the form  $\mathcal{A}$  is not  $H^1(\Omega)$ -elliptic; in fact, one can check that, if problem (3.12) has a solution, then we have necessarily (take  $v = 1$  above)

$$\int_{\Omega} f(x) dx + \langle h, 1 \rangle_{H^{-1/2}(\Gamma), H^{1/2}(\Gamma)} = 0. \quad (3.13)$$

Moreover, we note that if  $u$  is a solution, then  $u + C$ , where  $C$  is an arbitrary constant, is also a solution. So the variational problem is not well posed on  $H^1(\Omega)$ . It can be, however, solved in the quotient space  $H^1(\Omega)/\mathbb{R}$  which is a Hilbert space for the quotient norm

$$\|\dot{v}\|_{H^1(\Omega)/\mathbb{R}} = \inf_{k \in \mathbb{R}} \|v + k\|_{1, \Omega}, \quad (3.14)$$

but also for the semi-norm  $v \mapsto |v|_{H^1(\Omega)} = \sqrt{\mathcal{A}(v, v)}$ , which is an equivalent norm on this quotient space, see [N].

Then, supposing that the data  $f$  and  $h$  satisfy the ‘‘compatibility condition’’ (3.13), we can apply the Lax-Milgram theorem to the variational problem

$$\text{to find } \dot{u} \in H \text{ such that } \mathcal{A}(\dot{u}, \dot{v}) = \tilde{L}(\dot{v}) \text{ for all } \dot{v} \in V \quad (3.15)$$

with  $V = H^1(\Omega)/\mathbb{R}$ . We get the following result (see e.g. [N]):

**Theorem 3.6.** *Let us suppose that  $\Omega$  is connected and that the data  $f \in L^2(\Omega)$  and  $h \in H^{-1/2}(\Gamma)$  satisfy (3.13). Then the variational problem (3.15) has a unique solution  $\dot{u}$  in the space  $H^1(\Omega)/\mathbb{R}$  and this solution is continuous with respect to the data, i.e. there exists a positive constant  $C = C(\Omega)$  such that*

$$|u|_{H^1(\Omega)} \leq C (\|f\|_{L^2(\Omega)} + \|h\|_{H^{-1/2}(\Gamma)}) \quad \text{for all } u \in \dot{u}.$$

Moreover, if  $\Gamma$  is of class  $\mathcal{C}^{1,1}$  and if the data satisfy  $f \in L^2(\Omega)$ ,  $g \in H^{1/2}(\Gamma)$ , then every  $u \in \dot{u}$  is such that  $u \in H^2(\Omega)$  and it satisfies each equation in 3.12 almost everywhere.

### 3.4 Problem with mixed boundary conditions

Here we consider more general boundary conditions: the Dirichlet conditions on a closed subset  $\Gamma_1$  of  $\Gamma = \partial\Omega$ , and the Neumann, or more generally the “Robin”, conditions on the other part  $\Gamma_2 = \Gamma - \Gamma_1$ . We seek  $u$  such that ( $f \in L^2(\Omega)$ ,  $h \in L^2(\Gamma_2)$ ,  $a \in L^\infty(\Gamma_2)$ )

$$\begin{aligned} -\Delta u &= f && \text{in } \Omega, \\ u &= 0 && \text{on } \Gamma_1, \quad au + \frac{\partial u}{\partial n} = h && \text{on } \Gamma_2. \end{aligned} \quad (3.16)$$

Let  $V = \{v \in H^1(\Omega); \gamma_0 v = 0 \text{ on } \Gamma_1\}$ . Then (3.3) is the variational formulation of this problem with

- (i)  $\mathcal{A}(u, v) = \int_{\Omega} \nabla u(x) \cdot \nabla v(x) dx + \int_{\Gamma_2} (a\gamma_0 u \gamma_0 v)(\sigma) d\sigma$ ,
- (ii)  $L(v) = \int_{\Omega} f(x)v(x) dx + \int_{\Gamma_2} (h\gamma_0 v)(\sigma) d\sigma$ .

Supposing for example  $a \geq 0$ , we get a unique solution  $u \in V$  for this variational problem by virtue of the Lax-Milgram theorem. Moreover, if  $u \in H^2(\Omega)$ , then  $u$  is the unique solution in  $H^2(\Omega) \cap V$  of the problem (3.16).

### 3.5 The Newton problem for more general operators

Let  $\Omega$  be a bounded open subset of  $\mathbb{R}^n$ . We now consider more general second order operators of the form  $v \mapsto -\nabla \cdot (M\nabla v) + b \cdot \nabla v + cv$ , where  $b \in [W^{1,\infty}(\Omega)]^N$ ,  $c \in L^\infty(\Omega)$ ,  $M$  is an  $N \times N$  square matrix with entries  $M_{ij}$ , and  $\nabla \cdot (M\nabla v)$  stands for  $\sum_{i,j=1}^N \frac{\partial}{\partial x_i} [M_{ij} \frac{\partial v}{\partial x_j}]$ . We also assume that there is a positive constant  $\alpha_M$  such that

$$\sum_{i,j=1}^N M_{ij}(x) \xi_i \xi_j \geq \alpha_M \sum_{i=1}^N \xi_i^2 \text{ for a.e. } x \in \Omega \text{ and } \xi = (\xi_1, \dots, \xi_N) \in \mathbb{R}^N.$$

For given data  $f \in L^2(\Omega)$ ,  $h \in L^2(\Gamma)$ , we look for  $u$  solution of the problem

$$\begin{aligned} -\nabla \cdot (M\nabla u) + b \cdot \nabla u + cu &= f && \text{in } \Omega, \\ au + n \cdot (M\nabla u) &= h && \text{on } \Gamma. \end{aligned} \quad (3.17)$$

We assume that  $a \in L^\infty(\Gamma)$ . The variational formulation of this problem is still (3.3), with  $V = H^1(\Omega)$  and

$$\mathcal{A}(u, v) = \int_{\Omega} M \nabla u \cdot \nabla v \, dx + \int_{\Omega} [b \cdot \nabla u + cu]v \, dx + \int_{\Gamma} a \gamma_0 u \gamma_0 v \, d\sigma, \quad (3.18)$$

$$L(v) = \int_{\Omega} f(x)v(x) \, dx + \int_{\Gamma} (h \gamma_0 v)(\sigma) \, d\sigma, \quad (3.19)$$

If the following conditions

$$c - \frac{1}{2} \nabla \cdot b \geq C_0 \geq 0 \quad \text{a.e. on } \Omega, \quad a + \frac{1}{2} b \cdot \nu g e C_1 \geq 0 \quad \text{a.e. on } \Gamma$$

are fulfilled, with  $(C_0, C_1) \neq (0, 0)$ , then the bilinear form  $\mathcal{A}$  is V-elliptic and the Lax-Milgram theorem applies.

### 3.6 A biharmonic problem

We consider the Dirichlet problem for the operator of fourth order: ( $c \in L^\infty(\Omega)$ ):

$$\Delta^2 u + cu = f \quad \text{in } \Omega, \quad (3.20)$$

$$u = u_0 \quad \text{on } \Gamma, \quad \frac{\partial u}{\partial n} = h \quad \text{on } \Gamma. \quad (3.21)$$

**Theorem 3.7.** *Let us suppose that  $\Omega$  has a boundary of class  $\mathcal{C}^{1,1}$  and that the data satisfy  $f \in H^{-2}(\Omega)$ ,  $u_0 \in H^{3/2}(\Gamma)$ ,  $h \in H^{1/2}(\Gamma)$ . Let  $U_0 \in H^2(\Omega)$  be such that  $\gamma_0(U_0) = u_0$ ,  $\gamma_1(U_0) = h$ . Then, if  $c \geq 0$  a.e. in  $\Omega$ , the boundary value problem (3.20)-(3.21) has a unique solution  $u$  such that  $u - U_0 \in H_0^2(\Omega)$ , and  $u$  is also the unique solution of the variational problem*

$$\text{to find } u \in U_0 + H_0^2(\Omega) \text{ such that } \mathcal{A}(u, v) = l(v) \text{ for all } v \in H_0^2(\Omega), \quad (3.22)$$

where  $l(v) = \langle f, v \rangle_{H^{-2}(\Omega), H_0^2(\Omega)}$  and

$$\mathcal{A}(u, v) = \int_{\Omega} \Delta u(x) \Delta v(x) \, dx + \int_{\Omega} (cu v)(x) \, dx. \quad (3.23)$$

Moreover, there exists a positive constant  $C = C(\Omega)$  such that

$$\|u\|_{H^2(\Omega)} \leq C [\|f\|_{H^{-2}(\Omega)} + \|u_0\|_{H^{3/2}(\Gamma)} + \|h\|_{H^{1/2}(\Gamma)}], \quad (3.24)$$

which shows that  $u$  depends continuously upon the data  $f$ ,  $u_0$  and  $h$ .

**Remark 3.8.** The Hilbert space choice  $V$  is of crucial importance for the  $V$ -ellipticity. In fact, let us consider for example the problem (3.20), (3.25), with

$$\Delta u = 0 \quad \text{on } \Gamma, \quad \frac{\partial \Delta u}{\partial n} = 0 \quad \text{on } \Gamma. \quad (3.25)$$

In fact, the associated bilinear form is not  $V$ -elliptic for  $V = H^2(\Omega)$  but it is  $V$ -elliptic for  $V = \{v \in L^2(\Omega); \Delta v \in L^2(\Omega)\}$ .

## 4 General elliptic problems

Here  $\Omega$  will be a bounded and sufficiently regular open subset of  $\mathbb{R}^N$ . Let us consider a general linear differential operator of the form

$$A(x, D)u = \sum_{|\mu| \leq l} a_\mu(x) D^\mu u, \quad a_\mu(x) \in \mathbb{C} \quad (4.1)$$

Setting  $A_0(x, \xi) = \sum_{|\mu|=l} a_\mu(x) \xi^\mu$ , we say that the operator  $A$  is *elliptic* at a point  $x$  if  $A_0(x, \xi) \neq 0$  for all  $\xi \in \mathbb{R}^N - \{0\}$ . One can show that, if  $N \geq 3$ ,  $l$  is even, i.e.  $l = 2m$ ; the same result holds for  $N = 2$  if the coefficients  $a_\mu$  are real. Moreover, for  $N \geq 3$ , every elliptic operator is *properly elliptic*, in the following sense: For any independent vectors  $\xi, \xi'$  in  $\mathbb{R}^N$ , the polynomial  $\tau \mapsto A_0(\cdot, \xi + \tau \xi')$  has  $m$  roots with positive imaginary part.

The aim here is to study boundary value problems of the following type:

$$Au = f \quad \text{in } \Omega, \quad (4.2)$$

$$B_j u = g_j \quad \text{on } \Gamma, \quad j = 0, \dots, m-1, \quad (4.3)$$

where  $A$  is properly elliptic on  $\overline{\Omega}$ , with sufficiently regular coefficients, and the operators  $B_j$  are boundary operators, of order  $m_j \leq 2m - 1$ , that must satisfy some compatibility conditions with respect to the operator  $A$  (see [RR] for details; these conditions were introduced by Agmon, Douglas and Nirenberg). For example,  $A = (-1)^m \Delta^m$  and  $B_j = \frac{\partial^j}{\partial n^j}$  is a convenient choice.

In order to show that problem (4.2)-(4.3) has a solution  $u \in H^{2m+r}(\Omega)$  ( $r \in \mathbb{N}$ ), the idea is to show that the operator  $\mathcal{P}$  defined by  $u \mapsto \mathcal{P}(u) = (Au, B_0 u, \dots, B_{m-1} u)$  is an index operator from  $H^{2m+r}(\Omega)$  into  $G = H^r(\Omega) \times \prod_{j=0}^{m-1} H^{2m+r-m_j-1/2}(\Gamma)$  and to express the compatibility conditions through the adjoint problem.

We recall that a linear continuous operator  $\mathcal{P}$  is an index operator if

- (i)  $\dim \text{Ker } \mathcal{P} < \infty$ , and  $\text{Im } \mathcal{P}$  closed
- (ii)  $\text{codim Im } \mathcal{P} < \infty$ .

Then the index  $\chi(\mathcal{P})$  is given by  $\chi(\mathcal{P}) = \dim \text{Ker } \mathcal{P} - \text{codim Im } \mathcal{P}$ . We recall the following Peetre's theorem:

**Theorem 4.1.** *Let  $E, F$  and  $G$  be three reflexive Banach spaces such that  $E \hookrightarrow F$ , and  $\mathcal{P}$  a linear continuous operator from  $E$  to  $G$ . Then condition (i) is equivalent to*

*(iii) there exists  $C \geq 0$ , such that for all  $u \in E$ , we have*

$$\|u\|_E \leq C (\|\mathcal{P}u\|_G + \|u\|_F.)$$

Applying this theorem to our problem (4.2)-(4.3), condition (i) results from a priori estimates of the following type:

$$\|u\|_{H^{2m+r}(\Omega)} \leq C (\|\mathcal{P}u\|_G + \|u\|_{H^{2m+r-1}(\Omega)})$$

and condition (ii) by similar a priori estimates for the dual problem.

## 5 Second order elliptic problems

We consider a second order differential operator of the “divergence form”

$$Au = - \sum_{i,j=1}^N (a^{ij}(x)u_{x_i})_{x_j} + \sum_{i=1}^N b^i(x)u_{x_i} + c(x)u \quad (5.1)$$

with given coefficient functions  $a^{ij}, b^i, c$  ( $i, j = 1, \dots, N$ ), and where we have used the notation  $u_{x_i} = \frac{\partial u}{\partial x_i}$ . Such operators are said *uniformly strongly elliptic* in  $\Omega$  if there exists  $\alpha > 0$  such that

$$\sum_{|i|=|j|=1} a^{ij}(x)\xi^i \xi^j \geq \alpha|\xi|^2 \quad \text{for all } x \in \Omega, \xi \in \mathbb{R}^N.$$

**Remark 5.1.** There exists elliptic problems for which the associated variational problem does not necessarily satisfy the ellipticity condition. Let us consider the following example, due to Seeley: Let  $\Omega = \{(r, \theta) \in (\pi, 2\pi) \times [0, 2\pi]\}$  and  $A = -(e^{i\theta} \frac{\partial}{\partial \theta})^2 - e^{2i\theta}(1 + \frac{\partial^2}{\partial r^2})$ . One can check that, for all  $\lambda \in \mathbb{C}$ , the problem  $Au + \lambda u = f$  in  $\Omega$  and  $u = 0$  on  $\Gamma$  admits nonzero solutions  $u$  which are given by (with  $\mu$  such that  $\mu^2 = \lambda$ )  $u = \sin r \cos(\mu e^{-i\theta})$  and  $u = \sin r \sin(\mu e^{-i\theta})$  for  $\lambda \neq 0$ ;  $u = \sin r$  and  $u = \sin \theta e^{-i\theta}$  for  $\lambda = 0$ .

Most of results concerning existence, unicity, regularity for second order elliptic problems can be established thanks to a maximum principle. There exist different types of maximum principles, that we now present.

## 5.1 Maximum principle

**Theorem 5.2 (weak maximum principle).** *Let  $A$  be a uniformly strongly elliptic operator of the form (5.1) in a bounded open  $\Omega \subset \mathbb{R}^N$ , with  $a^{ij}, b^i, c \in L^\infty(\Omega)$  and  $c \geq 0$ . Let  $u \in C^2(\Omega) \cap C(\overline{\Omega})$  and*

$$Au \geq 0 \text{ [resp. } Au \leq 0] \text{ in } \Omega.$$

Then

$$\inf_{\Omega} u \geq \inf_{\partial\Omega} u^- \text{ [resp. } \sup_{\Omega} u \leq \sup_{\partial\Omega} u^+],$$

where  $u^+ = \max(u, 0)$  and  $u^- = -\min(u, 0)$ . If  $c = 0$  in  $\Omega$ , one can replace  $u^-$  [resp.  $u^+$ ] by  $u$ .

**Theorem 5.3 (strong principle maximum).** *Under the assumptions of the above theorem, if  $u$  is not a constant function in  $C^2(\Omega) \cap C(\overline{\Omega})$  such that  $Au \geq 0$  [ resp.  $Au \leq 0$  ], then  $\inf_{\Omega} u < u(x)$  [resp.  $\sup_{\Omega} u > u(x)$ ], for all  $x \in \Omega$ .*

**Remark 5.4.** These two maximum principles can be adapted to elliptic operators in nondivergence form, i.e.,

$$Au = - \sum_{i,j=1}^N a^{ij}(x)u_{x_i x_j} + \sum_{i=1}^N b^i(x)u_{x_i} + c(x)u. \quad (5.2)$$

## 5.2 Fredholm alternative

We now present some existence results which are based on on the Fredholm alternative rather than on the variational method.

Let us consider two Hilbert spaces  $V$  and  $H$ , where  $V$  is a dense subspace of  $H$  and  $V \hookrightarrow H$ . Denoting by  $V'$  the dual space of  $V$ , and identifying  $H$  with its dual space, we have the following imbeddings:  $V \hookrightarrow H \hookrightarrow V'$ . Let  $\mathcal{A}$  be a sesquilinear form on  $V \times V$ ,  $V$ -coercive with respect to  $H$ , that is, there exists  $\lambda_0 \in \mathbb{R}$  and  $\alpha > 0$  such that

$$\operatorname{Re}(\mathcal{A}(v, v)) + \lambda_0 \|v\|_H^2 \geq \alpha \|v\|_V^2 \quad \text{for all } v \in V.$$

Denoting by  $A$  the operator associated with the bilinear form  $\mathcal{A}$  (see item (iii) of Remark 3.1), the equation  $Au = f$  is equivalent to  $u - \lambda_0 Tu = g$ , with  $T = (A + \lambda_0 Id)^{-1}$  and  $g = Tf$ . Note that  $T$  is an isomorphism from  $H$  onto  $D(A) = \{u \in H; Au \in H\}$ .

The operator  $T : H \rightarrow H$  is compact and, thanks to the Fredholm alternative, there are two situations:

- (i) either  $\text{Ker } A = 0$  and  $A$  is an isomorphism from  $D(A)$  onto  $H$
- (ii) or  $\text{Ker } A \neq 0$ ; then  $\text{Ker } A$  is of finite dimension, and the problem  $Au = f$  with  $f \in H$  admits a solution if and only if  $f \in \text{Im } A = [\text{Ker}(A^*)]^\perp$ .

We now give another example in a non Hilbertian frame. Let us consider the problem [Gri]:  $Au = f$  in  $\Omega$  and  $Bu = g$  on  $\Gamma$  where  $\Gamma$  is of class  $C^{1,1}$ ,  $A$ , which is defined by (5.1), is uniformly strongly elliptic with  $a^{ij} = a^{ji} \in C^{0,1}(\overline{\Omega})$ ,  $b^i, c \in L^\infty(\Omega)$  and  $Bu = \gamma_0(u)$  or  $Bu = \gamma_1(u)$ . One can show that the operator  $u \mapsto (Au, Bu)$  is a Fredholm operator of index zero from  $W^{2,p}(\Omega)$  in  $L^p(\Omega) \times W^{2-d-1/p,p}(\Gamma)$  (with  $d = 0$  if  $Bu = \gamma_0(u)$  and  $d = 1$  if  $Bu = \gamma_1(u)$ ).

### 5.3 Regularity

Assume that  $\Omega$  is a bounded open. Suppose that  $u \in H_0^1(\Omega)$  is a weak solution of the equation

$$\begin{aligned} Au &= f && \text{in } \Omega, \\ u &= 0 && \text{on } \Gamma, \end{aligned} \tag{5.3}$$

where  $A$  has the divergence form (5.1). We now address the question whether  $u$  is in fact smooth: this is the *regularity problem* for weak solutions.

**Theorem 5.5 ( $H^2$ -regularity).** *Let  $\Omega$  be open, of class  $C^{1,1}$ ,  $a^{ij} \in C^1(\overline{\Omega})$ ,  $b^i, c \in L^\infty(\Omega)$ ,  $f \in L^2(\Omega)$ . Suppose furthermore that  $u \in H^1(\Omega)$  is a weak solution of (5.3). Then  $u \in H^2(\Omega)$  and we have the estimate*

$$\|u\|_{H^2(\Omega)} \leq C(\|f\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)}),$$

where the constant  $C$  depends only on  $\Omega$  and on the coefficients of  $A$ .

**Theorem 5.6 (higher regularity).** *Let  $m$  be a nonnegative integer,  $\Omega$  be open, of class  $C^{m+1,1}$  and assume that  $a^{ij} \in C^{m+1}(\overline{\Omega})$ ,  $b^i, c \in C^{m+1}(\overline{\Omega})$ ,*

$f \in H^m(\Omega)$ . Suppose furthermore that  $u \in H^1(\Omega)$  is a weak solution of (5.3). Then  $u \in H^{m+2}(\Omega)$  and

$$\|u\|_{H^{m+2}(\Omega)} \leq C(\|f\|_{H^m(\Omega)} + \|u\|_{L^2(\Omega)}),$$

where the constant  $C$  depends only on  $\Omega$  and on the coefficients of  $A$ . In particular, if  $m > N/2$ , then  $u \in C^2(\overline{\Omega})$ . Moreover if  $\Omega$  is of  $C^\infty$  class and  $f \in C^\infty(\overline{\Omega})$ ,  $a^{ij} \in C^\infty(\overline{\Omega})$ ,  $b^i, c \in C^\infty(\overline{\Omega})$ , then  $u \in C^\infty(\overline{\Omega})$ .

**Remark 5.7.** (i) If  $u \in H_0^1(\Omega)$  is the unique solution of (5.3), one can omit the  $L^2$  norm of  $u$  in the right hand side of the above estimate.

(ii) Moreover, let us suppose the coefficients  $a^{ij}, b^i$  and  $c$  are all  $C^\infty$  and  $f \in C^\infty(\Omega)$ ; then, if  $u \in H^1(\Omega)$  satisfies  $Au = f$ ,  $u \in C^\infty(\Omega)$ ; this is due to the ‘‘hypoellipticity’’ property satisfied by the operator  $A$ .

We have a similar result in the  $L^p$  frame [Gri]:

**Theorem 5.8 ( $W^{2,p}$ -regularity).** Let  $\Omega$  be open, of class  $C^{1,1}$ ,  $a^{ij} \in C^1(\overline{\Omega})$ ,  $b^i, c \in L^\infty(\Omega)$ . Suppose furthermore that  $b^i = 0, 1 \leq i \leq N$  and  $c \geq 0$  a.e or  $c \geq \beta > 0$  a.e. Then for every  $f \in L^p(\Omega)$  there exists a unique solution  $u \in W^{2,p}(\Omega)$  of (5.3).

## 6 Unbounded open

### 6.1 The whole space

Note in passing that we shall work with the weighted Sobolev spaces  $W_\alpha^{m,p}(\Omega)$  defined in Subsection 2.5.

**Theorem 6.1.** *The following claims hold true:*

(i) Let  $f \in W_0^{-1,p}(\mathbb{R}^N)$  satisfy the compatibility condition

$$\langle f, 1 \rangle_{W_0^{-1,p}(\mathbb{R}^N) \times W_0^{1,p'}(\mathbb{R}^N)} = 0 \quad \text{if } p' \geq N.$$

Then the problem (2.3) has a solution  $u \in W_0^{1,p}(\mathbb{R}^N)$ , which is unique up to an element in  $\mathcal{P}_{[1-N/p]}$  and satisfies the estimate

$$\|u\|_{W_0^{1,p}(\mathbb{R}^N)/\mathcal{P}_{[1-N/p]}} \leq C\|f\|_{W_0^{-1,p}(\mathbb{R}^N)}.$$

Moreover, if  $1 < p < N$ , then  $u = E * f$ .

- (ii) If  $f \in L^p(\mathbb{R}^N)$ , then the problem (2.3) has a solution  $u \in W_0^{2,p}(\mathbb{R}^N)$ , which is unique up to an element in  $\mathcal{P}_{[2-N/p]}$  and if  $1 < p < N/2$ , then  $u = E * f$ .

The Calderón-Zygmund inequality

$$\left\| \frac{\partial^2 \varphi}{\partial x_i \partial x_j} \right\|_{L^p(\mathbb{R}^N)} \leq C(N, p) \|\Delta \varphi\|_{L^p(\mathbb{R}^N)}, \quad \varphi \in \mathcal{D}(\mathbb{R}^N),$$

and Theorem 2.4 are crucial for establishing Theorem 6.1. Further, point (i) means that the Riesz potential of second order satisfies

$$I_2 : W_0^{-1,p}(\mathbb{R}^N) \perp \mathcal{P}_{[1-N/p]} \rightarrow W_0^{1,p}(\mathbb{R}^N) / \mathcal{P}_{[1-\frac{N}{p}]}$$

(where the initial space is the orthogonal complement of  $\mathcal{P}_{[1-N/p]}$  in  $W_0^{-1,p}(\mathbb{R}^N)$ ) and it is an isomorphism.

Note that here

$$W_0^{1,p}(\mathbb{R}^N) = \{v \in L^{p^*}(\mathbb{R}^N); \nabla v \in L^p(\mathbb{R}^N)\}$$

for  $1 < p < N$  and  $1/p^* = 1/p - 1/N$ . And for  $1 < r < \frac{N}{2}$ , we have also the continuity property

$$I_2 : L^r(\mathbb{R}^N) \rightarrow L^q(\mathbb{R}^N), \quad \text{for } \frac{1}{q} = \frac{1}{r} - \frac{2}{N}.$$

**Remark 6.2.** The problem

$$u - \Delta u = f \quad \text{in } \mathbb{R}^N \tag{6.1}$$

is of a completely different nature than the problem (2.3). The class of function spaces appropriate for the problem (6.1) are the classical Sobolev spaces. With help of the Calderón-Zygmund theory one can prove that if  $f \in L^p(\mathbb{R}^N)$ , then the unique solution of (6.1) belongs to  $W^{2,p}(\mathbb{R}^N)$  and can be represented as the Bessel potential of second order (see [S]):  $u = G * f$ , where  $G$  is the appropriate Bessel kernel, that is,  $G$ , for which  $\widehat{G}(\xi) \sim (1 + |\xi|^2)^{-1/2}$ . Recall that in particular  $G(x) \sim |x|^{-1} e^{-|x|}$  for  $N = 3$ . In the Hilbert case  $f \in L^2(\mathbb{R}^N)$  we get

$$(1 + |\xi|^2) \widehat{u} \in L^2(\mathbb{R}^N),$$

which, by Plancherel's theorem, implies that  $u \in H^2(\mathbb{R}^N)$ . For  $f \in W^{-1,p}(\mathbb{R}^N)$ , the problem 6.1 has a unique solution  $u \in W^{1,p}(\mathbb{R}^N)$  satisfying the estimate

$$\|u\|_{W^{1,p}(\mathbb{R}^N)} \leq C(p, n) \|f\|_{W^{-1,p}(\mathbb{R}^N)}.$$

## 6.2 Exterior domain

We consider the problem in an exterior domain with the Dirichlet boundary condition

$$\begin{aligned} -\Delta u &= f && \text{in } \Omega, \\ u &= g && \text{on } \partial\Omega, \end{aligned} \tag{6.2}$$

where  $f \in W_0^{-1,p}(\Omega)$  and  $g \in W^{1-1/p,p}(\partial\Omega)$ . Invoking the results for  $\mathbb{R}^N$  and bounded domains one can prove the existence of a solution  $u \in W_0^{1,p}(\Omega)$  which is unique up to an element of the kernel  $A_0^p(\Omega) = \{z \in W_0^{1,p}(\Omega); \Delta z = 0\}$  provided that  $f$  satisfies the compatibility condition

$$\langle f, \varphi \rangle = \left\langle g, \frac{\partial \varphi}{\partial n} \right\rangle \quad \text{for all } \varphi \in A_0^{p'}(\Omega).$$

The kernel can be characterized in the following way: It is reduced to  $\{0\}$  if  $p = 2$  or  $p < N$  and if not, then

$$A_0^p(\Omega) = \{C(\lambda - 1); C \in \mathbb{R}\} \quad \text{if } p \geq N \geq 3,$$

where  $\lambda$  is (unique) solution in  $W_0^{1,2}(\Omega) \cap W_0^{1,p}(\Omega)$  of the problem  $\Delta \lambda = 0$  in  $\Omega$  and  $\lambda = 1$  on  $\partial\Omega$ , and

$$A_0^p(\Omega) = \{C(\mu - u_0); C \in \mathbb{R}\} \quad \text{if } p > N = 2,$$

where  $u_0(x) = (2\pi|\Gamma|)^{-1} \int_{\Gamma} \log|y - x| d\sigma_y$  and  $\mu$  is the only solution in  $W_0^{1,2}(\Omega) \cap W_0^{1,p}(\Omega)$  of the problem  $\Delta \mu = 0$  in  $\Omega$  and  $\mu = u_0$  on  $\Gamma$ .

**Remark 6.3.** Similar results exist for the Neumann problem in an exterior domain (see [AGG]). The framework of the spaces  $W_{\alpha}^{m,p}(\mathbb{R}_+^N)$  also for the Dirichlet problem in  $\mathbb{R}_+^N$  was considered in the literature, too. For a more general theory see [KM].

## 7 Elliptic systems

### 7.1 The Stokes system

The Stokes problem is a classical example in the fluid mechanics. This system models the slow motion with the field of the velocity  $\vec{u}$  and the pressure  $\pi$ ,

satisfying

$$\begin{aligned} -\nu\Delta\vec{u} + \nabla\pi &= \vec{f} & \text{in } \Omega, \\ \operatorname{div}\vec{u} &= h & \text{in } \Omega, \\ \vec{u} &= \vec{g} & \text{on } \Gamma = \partial\Omega, \end{aligned} \tag{S}$$

where  $\nu > 0$  denotes the viscosity,  $\vec{f}$  is an exterior force,  $\vec{g}$  is the velocity of the fluid on the domain boundary and  $h$  measures the compressibility of the fluids (if  $h = 0$ , it is an incompressible fluid). The functions  $h$  and  $g$  must satisfy the compatibility condition

$$\int_{\Omega} h(x) dx = \int_{\Gamma} \vec{g} \cdot \vec{n} d\sigma. \tag{7.1}$$

**Theorem 7.1.** *Let  $\Omega$  be a Lipschitz bounded domain in  $\mathbb{R}^N$ ,  $N \geq 2$ . Let  $\vec{f} \in H^{-1}(\Omega)^N$ ,  $h \in L^2(\Omega)$  and  $g \in H^{1/2}(\Gamma)^N$  satisfy (7.1). Then the problem (S) has a unique solution  $(\vec{u}, \pi) \in H^1(\Omega)^N \times L^2(\Omega)/\mathbb{R}$  satisfying the a priori estimate*

$$\|\vec{u}\|_{H^1(\Omega)} + \|\pi\|_{L^2(\Omega)/\mathbb{R}} \leq C(\|\vec{f}\|_{H^{-1}(\Omega)} + \|h\|_{L^2(\Omega)} + \|g\|_{H^{1/2}(\Gamma)}).$$

In order to prove Theorem 7.1 one can start with a homogeneous problem. The procedure of finding  $\vec{u}$  is a simple application of the Lax-Milgram theorem. Application of De Rham's theorem gives the pressure  $\pi$ . We introduce the space

$$\mathcal{V} = \{\vec{v} \in \mathcal{D}(\Omega)^N; \operatorname{div}\vec{v} = 0\}$$

and define  $\vec{F} \in H^{-1}(\Omega)^N$  by

$$\langle \vec{F}, \vec{v} \rangle_{H^{-1} \times \dot{H}^1} = 0 \quad \text{for all } \vec{v} \in \mathcal{V}.$$

Moreover, there exists  $\pi \in L^2(\Omega)$ , unique up to an additive constant, and such that  $\vec{F} = \nabla\pi$ . The problem (S), which we transform to the homogeneous case ( $h = 0, g = 0$ ), can be formulated on an abstract level. Let  $X$  and  $M$  be two real Hilbert spaces and consider the following variational problem: Given  $\vec{L} \in X'$  and  $X \in M'$ , find  $(\vec{u}, \pi) \in X \times M$  such that

$$\begin{aligned} \mathcal{A}(\vec{u}, \vec{v}) + B[\vec{v}, \pi] &= \vec{L}(\vec{v}), & v \in X, \\ B[\vec{u}, q] &= X(q), & q \in M. \end{aligned} \tag{7.2}$$

where the bilinear forms  $\mathcal{A}$ ,  $B$  and the linear form  $\vec{L}$  are defined by

$$\mathcal{A}(\vec{u}, \vec{v}) = \int_{\Omega} \nabla\vec{u} \cdot \nabla\vec{v}, \quad B[\vec{v}, q] = - \int_{\Omega} [q\nabla \cdot \vec{v}], \quad L(\vec{v}) = \int_{\Omega} \vec{f} \cdot \vec{v}.$$

**Theorem 7.2.** *If the bilinear form  $\mathcal{A}$  is coercive in the space*

$$V = \{\vec{v} \in X; B[\vec{v}, q] = 0 \text{ for all } q \in M,$$

*i.e. if there exists  $\alpha > 0$  such that*

$$\mathcal{A}(\vec{v}, \vec{v}) \geq \alpha \|\vec{v}\|_X^2, \quad \vec{v} \in V,$$

*then the problem (7.2) has a unique solution  $(\vec{u}, \pi)$  if and only if the bilinear form  $\mathcal{B}$  satisfies the “inf-sup” condition:*

$$\text{there exists } \beta > 0 \text{ such that } \inf_{q \in M} \sup_{\vec{v} \in X} \frac{\mathcal{B}(\vec{v}, q)}{\|\vec{v}\|_X \|q\|_M} \geq \beta.$$

As for the Dirichlet problem, the regularity result is the following:

**Theorem 7.3.** *Let  $\Omega$  be a bounded domain in  $\mathbb{R}^N$ , of the class  $C^{m+1,1}$  if  $m \in \mathbb{N}$  and  $C^{1,1}$  if  $m = -1$ . Let  $f \in W^{m,p}(\Omega)^N$ ,  $h \in W^{m+1,p}(\Omega)$  and  $\vec{g} \in W^{m+2-1/p,p}(\Gamma)^N$  satisfy condition (7.1). Then the problem (S) has a unique solution  $(\vec{u}, \pi) \in W^{m+2,p}(\Omega)^N \times W^{m+1,p}(\Omega)/\mathbb{R}$ .*

**Remark 7.4.** It is possible to solve (S) under weaker assumption, for instance if  $\vec{f} \in W^{-1/p}(\Omega')$ ,  $h = 0$  and  $\vec{g} \in W^{-1/p,p}(\Gamma)^N$ . We can prove that then  $(\vec{u}, \pi) \in L^p(\Omega)^N \times W^{-1,p}(\Omega)$ .

## 7.2 The linearized elasticity

The equations governing the displacement  $\vec{u} = (u_1, u_2, u_3)$  of a three dimensional structure subjected to an external force field  $\vec{f}$  are written as ( $\Omega$  is a bounded open subset of  $\mathbb{R}^3$  and  $\Gamma = \partial\Omega$ )

$$\begin{aligned} -\mu\Delta\vec{u} - (\lambda + \mu)\nabla(\nabla \cdot \vec{u}) &= \vec{f} && \text{in } \Omega, \\ \vec{u} &= 0 && \text{on } \Gamma_0, \\ \sum_{j=1}^3 \sigma_{ij}(\vec{u})\nu_j &= \vec{g}_i && \text{on } \Gamma_1 = \Gamma - \Gamma_0, \end{aligned}$$

where  $\lambda > 0$  and  $\mu > 0$  are two material characteristic constants, called the *Lamé coefficients*, and  $(\vec{v} = (v_1, v_2, v_3))$

$$\sigma_{ij}(\vec{v}) = \sigma_{ji}(\vec{v}) = \lambda\delta_{ij} \sum_{k=1}^3 \varepsilon_{kk}(\vec{v}) + 2\mu\varepsilon_{ij}(\vec{v}) \quad (7.3)$$

$$\text{with } \varepsilon_{ij}(\vec{v}) = \varepsilon_{ji}(\vec{v}) = \frac{1}{2}(\partial_j v_i + \partial_i v_j),$$

where  $\delta_{ij}$  denotes the Kronecker symbol, i.e.  $\delta_{ij} = 1$ , for  $i = j$  and  $\delta_{ij} = 0$ , for  $i \neq j$ . These equations describe the equilibrium of an elastic homogeneous isotropic body that cannot move along  $\Gamma_0$ ; along  $\Gamma_1$ , surface forces of density  $\vec{g} = (g_1, g_2, g_3)$  are given. The case  $\Gamma_1 = \emptyset$  physically corresponds to clamped structures. The matrix with entries  $\varepsilon_{ij}(\vec{u})$  is the linearized strain tensor while  $\sigma_{ij}(\vec{u})$  represents the linearized stress tensor; the relationship (7.3) between these tensors is known as *Hooke's law*. We refer for example to [CL], [NH] (and to the references herein) for most of the results stated in this paragraph. The variational formulation of this problem is

$$\text{to find } \vec{u} \in V \text{ such that } \mathcal{A}(\vec{u}, \vec{v}) = L(\vec{v}) \text{ for all } \vec{v} \in V, \quad (7.4)$$

where the bilinear form  $\mathcal{A}$  and the linear form  $L$  are given by

$$\mathcal{A}(\vec{u}, \vec{v}) = \int_{\Omega} [\lambda(\nabla \cdot \vec{u})(\nabla \cdot \vec{v}) + 2\mu \sum_{i,j=1}^3 \varepsilon_{ij}(\vec{u})\varepsilon_{ij}(\vec{v})](x) dx, \quad (7.5a)$$

$$L(\vec{v}) = \int_{\Omega} \vec{f}(x) \cdot \vec{v}(x) dx + \int_{\Gamma_1} \vec{g}(x) \cdot \vec{v}(\sigma) d\sigma; \quad (7.5b)$$

The functional space  $V$  is defined as

$$V = \{\vec{v} = (v_1, v_2, v_3) \in [H^1(\Omega)]^3; \gamma_0 v_i = 0 \text{ on } \Gamma_0, 1 \leq i \leq 3\}.$$

To prove the ellipticity of  $\mathcal{A}$ , one needs the following Korn inequality: There exists a positive constant  $C(\Omega)$  such that, for all  $\vec{v} = (v_1, v_2, v_3) \in [H^1(\Omega)]^3$ , we have

$$\|\vec{v}\|_{1,\Omega} \leq C(\Omega) \left[ \sum_{i,j=1}^3 \|\varepsilon_{ij}(\vec{v})\|_{L^2(\Omega)}^2 + \sum_{i=1}^3 \|v_i\|_{L^2(\Omega)}^2 \right]^{1/2}. \quad (7.6)$$

The following result holds true:

**Theorem 7.5.** *Let  $\Omega$  be a bounded open in  $\mathbb{R}^3$  with a Lipschitz boundary, and let  $\Gamma_0$  be a measurable subset of  $\Gamma$ , whose measure (with respect to the surface measure  $d\Gamma(x)$ ) is positive. Then the mapping*

$$\vec{v} \mapsto \left[ \sum_{i,j=1}^3 \|\varepsilon_{ij}(\vec{v})\|_{L^2(\Omega)}^2 \right]^{1/2}$$

*is a norm on  $V$ , equivalent to the usual norm  $\|\cdot\|_{1,\Omega}$ .*

As a consequence, we get:

**Theorem 7.6.** *Under the above assumptions, there exists a unique  $u \in V$  solving the variational problem (7.4)-(7.5). This solution is also the unique one which minimizes the energy functional*

$$E(\vec{v}) = \frac{1}{2} \int_{\Omega} [\lambda(\nabla \cdot \vec{v})^2 + 2\mu \sum_{i,j=1}^3 [\varepsilon_{ij}(\vec{v})]^2](x) dx \\ - \left[ \int_{\Omega} \vec{f}(x) \cdot \vec{v}(x) dx + \int_{\Gamma_1} \vec{g}(x) \cdot \vec{v}(\sigma) d\sigma \right]$$

over the space  $V$ .

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