

Reproductive strong solutions of Navier-Stokes equations with non homogeneous boundary conditions

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Abstract

The object of the present paper is to show the existence and the uniqueness of a reproductive strong solution of the Navier-Stokes equations, i.e. the solution \mathbf{u} belongs to $\mathbf{L}^\infty(0, T; V) \cap \mathbf{L}^2(0, T; \mathbf{H}^2(\Omega))$ and satisfies the property $\mathbf{u}(\mathbf{x}, T) = \mathbf{u}(\mathbf{x}, 0) = \mathbf{u}_0(\mathbf{x})$. One considers the case of an incompressible fluid in two dimensions with nonhomogeneous boundary conditions, and external forces are neglected.

Key Words: Navier-Stokes equations, incompressible fluid, reproductive solution, nonhomogeneous boundary conditions.

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1 Introduction and notations

Let Ω be an open and bounded domain of \mathbb{R}^2 , with a sufficiently smooth boundary Γ ; and let us consider the Navier-Stokes equations:

$$\begin{cases} \frac{\partial \mathbf{v}}{\partial t} - \nu \Delta \mathbf{v} + \mathbf{v} \cdot \nabla \mathbf{v} + \nabla p = 0 & \text{in } Q_T = \Omega \times]0, T[, \\ \operatorname{div} \mathbf{v} = 0 & \text{in } Q_T, \\ \mathbf{v} = \mathbf{g} & \text{on } \Sigma_T = \Gamma \times]0, T[, \\ \mathbf{v}(0) = \mathbf{v}_0 & \text{in } \Omega. \end{cases} \quad (1)$$

where \mathbf{g} , \mathbf{v}_0 and $T > 0$ are given. We suppose that :

$$\operatorname{div} \mathbf{v}_0 = 0 \quad \text{in } \Omega, \quad \mathbf{v}_0 \cdot \mathbf{n} = 0 \quad \text{on } \Gamma, \quad (2)$$

and

$$\mathbf{g} \cdot \mathbf{n} = 0 \quad \text{on } \Sigma_T. \quad (3)$$

One is interested on one hand by the existence of strong solutions of system (1). On the other hand, one seeks data conditions to establish the existence of a reproductive solution generalizing the concept of a periodic solution. Kaniel and

Shinbrot [5] showed the existence of these solutions for system (1) in dimensions 2 and 3 with external forces but zero boundary condition i.e. $\mathbf{g} = 0$. With another approach using semigroups, one can also point out the work of Takeshita [10] in dimension 2.

We need to introduce the following functional spaces, with r and s positive numbers:

$$\mathbf{H}^{r,s}(Q_T) = \mathbf{L}^2([0, T[; \mathbf{H}^r(\Omega)) \cap \mathbf{H}^s([0, T[; \mathbf{L}^2(\Omega))$$

These are Hilbert spaces for the norm

$$\|\mathbf{v}\|_{\mathbf{H}^{r,s}(Q_T)} = \left(\int_0^T \|\mathbf{v}(t)\|_{\mathbf{H}^r(\Omega)}^2 dt + \|\mathbf{v}\|_{\mathbf{H}^s([0, T[; \mathbf{L}^2(\Omega))}^2 \right)^{1/2}.$$

Let us recall that for $s = 1$, for example,

$$\|\mathbf{v}\|_{\mathbf{H}^1([0, T[; \mathbf{L}^2(\Omega))} = \left[\int_0^T \left(\|\mathbf{v}(t)\|_{\mathbf{L}^2(\Omega)}^2 + \left\| \frac{\partial \mathbf{v}}{\partial t} \right\|_{\mathbf{L}^2(\Omega)}^2 \right) dt \right]^{1/2}.$$

In the same manner one defines spaces $\mathbf{H}^{r,s}(\Sigma_T)$.

We now introduce the following spaces:

$$\begin{aligned} \mathcal{V} &= \{ \mathbf{v} \in \mathcal{D}(\Omega)^2; \operatorname{div} \mathbf{v} = 0 \text{ in } \Omega \}, \\ \mathbf{H} &= \{ \mathbf{v} \in \mathbf{L}^2(\Omega); \operatorname{div} \mathbf{v} = 0 \text{ in } \Omega, \mathbf{v} \cdot \mathbf{n} = 0 \text{ on } \Gamma \}, \\ V &= \{ \mathbf{v} \in \mathbf{H}_0^1(\Omega); \operatorname{div} \mathbf{v} = 0 \text{ in } \Omega \}, \end{aligned}$$

Let us recall that \mathcal{V} is dense in \mathbf{H} and V for their respective topologies.

Here, $\mathcal{D}(\Omega)$ is the class of \mathcal{C}^∞ functions with compact support in Ω . The notations (\cdot, \cdot) et $((\cdot, \cdot))$ indicate the scalar products in $\mathbf{L}^2(\Omega)$ and in $\mathbf{H}_0^1(\Omega)$ respectively, and $|\cdot|$ et $\|\cdot\|$ the associated norms.

In the order to solve problem (1), we will have to remove boundary condition \mathbf{g} . and consider a new problem with zero boundary condition. We note that if $\mathbf{v} \in \mathbf{H}^{2,1}(Q_T)$ is solution of (1), then thanks to the Aubin compactness lemma (see J.L. Lions [8], R. Temam [11]) one will have

$$\mathbf{v} \in \mathcal{C}^0([0, T]; \mathbf{H}^1(\Omega)) \hookrightarrow \mathcal{C}^0([0, T]; \mathbf{H}^{1/2}(\Gamma))$$

So that a necessary condition for \mathbf{v} to exist is that:

$$\mathbf{g}(\mathbf{x}, 0) = \mathbf{v}_0(\mathbf{x}), \quad \mathbf{x} \in \Gamma. \quad (4)$$

Combining (2)-(4), one has:

$$\mathbf{g} \cdot \mathbf{n} = 0 \quad \text{on } \Gamma \times [0, T[.$$

The following lemma allows us to state hypotheses on \mathbf{g} (voir Lions-Magenes [7]).

Lemma 1.1. *Suppose that (4) takes place and let*

$$\mathbf{g} \in \mathbf{H}^{3/2, 3/4}(\Sigma_T), \quad \mathbf{v}_0 \in \mathbf{H}^1(\Omega). \quad (5)$$

Then there exists a function $\mathbf{R} \in \mathbf{H}^{2,1}(Q_T)$ such that

$$\mathbf{R} = \mathbf{g} \quad \text{on } \Sigma_T \quad \text{et} \quad \mathbf{R}(0) = \mathbf{v}_0 \quad \text{in } \Omega, \quad (6)$$

and satisfying the estimates

$$\|\mathbf{R}\|_{\mathbf{H}^{2,1}(Q_T)} \leq C \left(\|\mathbf{g}\|_{\mathbf{H}^{3/2, 3/4}(\Sigma_T)} + \|\mathbf{v}_0\|_{\mathbf{H}^1(\Omega)} \right). \square \quad (7)$$

We now consider the problem:

For a given \mathbf{g} verifying (5), one seeks (\mathbf{u}, q) which satisfies

$$\begin{cases} \frac{\partial \mathbf{u}}{\partial t} - \nu \Delta \mathbf{u} + \nabla q = 0 & \text{in } Q_T, \\ \operatorname{div} \mathbf{u} = \operatorname{div} \mathbf{R} & \text{in } Q_T, \\ \mathbf{u} = 0 & \text{on } \Sigma_T, \\ \mathbf{u}(0) = \mathbf{0} & \text{in } \Omega. \end{cases} \quad (8)$$

The following proposition holds (see Dautray-Lions [2], O. A. Ladyzhenskaya [6], V.A. Solonnikov [9]) :

Proposition 1.2. *We suppose that (5) holds,*

$$\operatorname{div} \mathbf{v}_0 = 0 \quad \text{on } \Omega, \quad \mathbf{v}_0 \cdot \mathbf{n} = 0 \quad \text{in } \Gamma, \quad \text{and } \mathbf{g} \cdot \mathbf{n} = 0 \quad \text{in } \Sigma_T. \quad (9)$$

Then problem (8) has an unique solution (\mathbf{u}, q) such that

$$\mathbf{u} \in \mathbf{H}^{2,1}(Q_T), \quad q \in L^2(0, T; H^1(\Omega)^2)$$

with the estimates

$$\|\mathbf{u}\|_{\mathbf{H}^{2,1}(Q_T)} + \|q\|_{L^2(0, T; H^1(\Omega)^2)} \leq C \left(\|\mathbf{g}\|_{\mathbf{H}^{3/2, 3/4}(\Sigma_T)} + \|\mathbf{v}_0\|_{\mathbf{H}^1(\Omega)} \right). \square \quad (10)$$

Thus the function defined by

$$\mathbf{G} = \mathbf{R} - \mathbf{u} \quad \text{in } Q_T \quad (11)$$

satisfies the estimates (7) and

$$\operatorname{div} \mathbf{G} = 0 \quad \text{in } Q_T, \quad (12)$$

$$\mathbf{G} = \mathbf{g} \quad \text{on } \Sigma_T, \quad (13)$$

$$\mathbf{G}(\mathbf{x}, 0) = \mathbf{v}(\mathbf{x}, 0) \quad \mathbf{x} \in \Omega. \quad (14)$$

This yields the following lemma:

Lemma 1.3. *Let \mathbf{g} and \mathbf{v}_0 satisfy (4), (5) and (9). Then there exists $\mathbf{G} \in \mathbf{H}^{2,1}(Q_T)$ satisfying (12)-(14) and the estimate*

$$\|\mathbf{G}\|_{\mathbf{H}^{2,1}(Q_T)} \leq C \left(\|\mathbf{g}\|_{\mathbf{H}^{3/2,3/4}(\Sigma_T)} + \|\mathbf{v}_0\|_{\mathbf{H}^1(\Omega)} \right). \square$$

Moreover, one has the next lemma

Lemma 1.4. *Let $\varepsilon > 0$, and let \mathbf{g} and \mathbf{v}_0 satisfy the hypotheses of lemma 1.3. Then there exists $\mathbf{G}_\varepsilon \in \mathbf{H}^{2,1}(Q_T)$ such that*

$$\operatorname{div} \mathbf{G}_\varepsilon = 0 \quad \text{in } Q_T,$$

$$\mathbf{G}_\varepsilon = \mathbf{g} \quad \text{on } \Sigma_T,$$

$$\|\mathbf{G}_\varepsilon(\cdot, 0)\|_{\mathbf{H}^1(\Omega)} \leq C_\varepsilon \|\mathbf{G}(\cdot, 0)\|_{\mathbf{H}^1(\Omega)}$$

and

$$\forall \mathbf{v} \in V, \quad |b(\mathbf{v}, \mathbf{G}_\varepsilon(t), \mathbf{v})| \leq \beta(\varepsilon, t) \|\nabla \mathbf{v}\|_{\mathbf{L}^2(\Omega)}^2$$

with

$$\sup_{t \in [0, T]} \beta(\varepsilon, t) \rightarrow 0 \quad \text{when } \varepsilon \rightarrow 0.$$

Moreover, there exists an increasing function $L : \mathbb{R}^+ \rightarrow \mathbb{R}^+$, not depending on ε , such that

$$\|\mathbf{G}_\varepsilon\|_{\mathbf{H}^{2,1}(Q_T)} \leq L \left(\frac{\varepsilon}{\|\mathbf{g}\|_{\mathbf{H}^{3/2,3/4}(\Sigma_T)} + \|\mathbf{v}_0\|_{\mathbf{H}^1(\Omega)}} \right) \left(\|\mathbf{g}\|_{\mathbf{H}^{3/2,3/4}(\Sigma_T)} + \|\mathbf{v}_0\|_{\mathbf{H}^1(\Omega)} \right).$$

Proof.

i) Step 1 : One takes up again the Hopf construction (see Girault & Raviart [4], Temam [11], Lions [8], Galdi [3]).

ii) *Step 2* : The open domain Ω being smooth, and since $\operatorname{div} \mathbf{G}_\varepsilon = 0$ in Q_T and $\mathbf{G} \cdot \mathbf{n} = 0$ on $\Gamma \times [0, T[$, there exists, for all $t \in [0, T[$, a function ψ depending on \mathbf{x} and t , such that

$$\mathbf{G} = \operatorname{rot} \psi \quad \text{in} \quad \Omega \times [0, T]$$

with $\psi = 0$ on $\Gamma \times [0, T[$, $\psi \in \mathbf{L}^2(0, T; \mathbf{H}^3(\Omega))$, $\frac{\partial \psi}{\partial t} \in \mathbf{L}^2(0, T; \mathbf{H}^1(\Omega))$ and satisfying the estimate

$$\|\psi\|_{\mathbf{L}^2(0, T; \mathbf{H}^3(\Omega))} + \|\psi_t\|_{\mathbf{L}^2(0, T; \mathbf{H}^1(\Omega))} \leq C \|\mathbf{G}\|_{\mathbf{H}^{2,1}(Q_T)}. \quad (15)$$

iii) *Step 3* : Let

$$\mathbf{G}^\varepsilon = \operatorname{rot}(\theta_\varepsilon \psi).$$

One deduces from the properties of θ_ε , for $j = 1, 2$:

$$|\mathbf{G}_j^\varepsilon(x, t)| \leq C \left(\frac{\varepsilon}{\rho(x)} |\psi(x, t)| + |\nabla \psi(x, t)| \right) \quad \text{if} \quad \rho(x) \leq 2\delta(\varepsilon)$$

and $\mathbf{G}_j^\varepsilon = 0$ if $\rho(x) > 2\delta(\varepsilon)$.

We note that

$$\psi \in C([0, T]; \mathbf{H}^2(\Omega)) \leftrightarrow C([0, T]; \mathbf{L}^\infty(\Omega)).$$

Therefore,

$$|\mathbf{G}_j^\varepsilon(x, t)| \leq C \left(\frac{\varepsilon}{\rho(x)} + |\nabla \psi(x, t)| \right) \quad \text{if} \quad \rho(x) \leq 2\delta(\varepsilon).$$

Thus, for all $\mathbf{v} \in \mathbf{H}_0^1(\Omega)$,

$$\|\mathbf{v}_i \mathbf{G}_j^\varepsilon\|_{\mathbf{L}^2(\Omega)} \leq C \left[\varepsilon \left\| \frac{\mathbf{v}_i}{\rho} \right\|_{\mathbf{L}^2(\Omega)} + \left(\int_{\rho(x) \leq 2\delta(\varepsilon)} \mathbf{v}_i^2 |\nabla \psi|^2 dx \right)^{1/2} \right]$$

$$\|\mathbf{v}_i \mathbf{G}_j^\varepsilon\|_{\mathbf{L}^2(\Omega)} \leq C\varepsilon \|\nabla \mathbf{v}_i\|_{\mathbf{L}^2(\Omega)} + C \|\nabla \mathbf{v}_i\|_{\mathbf{L}^2(\Omega)} \times \left(\int_{\rho(x) \leq 2\delta(\varepsilon)} |\nabla \psi|^3 dx \right)^{1/3}$$

Setting

$$\beta(\varepsilon, t) = \left(\int_{\rho(x) \leq 2\delta(\varepsilon)} |\nabla\psi|^3 dx \right)^{1/3},$$

it's clear that

$$\lim_{\varepsilon \rightarrow 0} \beta(\varepsilon, t) = 0 \text{ uniformly on } [0, T].$$

The second inequality of lemma 1.4 is a consequence of Hölder inequality. The first inequality follows from Hardy inequality for $\mathbf{H}_0^1(\Omega)$ -functions and properties of θ_ε . \square

2 Existence of strong solutions

Let us make a change of the unknown function in problem (1), by setting

$$\mathbf{u} = \mathbf{v} - \mathbf{G}_\varepsilon, \quad \mathbf{u}_0 = \mathbf{v}_0 - \mathbf{G}_\varepsilon(\cdot, 0),$$

where \mathbf{G}_ε is the function given by lemma 1.4. Problem (1) then becomes:

$$\begin{cases} \frac{\partial \mathbf{u}}{\partial t} - \nu \Delta \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{G}_\varepsilon + \mathbf{G}_\varepsilon \cdot \nabla \mathbf{u} + \nabla p = \mathbf{f}_\varepsilon & \text{in } Q_T \\ \operatorname{div} \mathbf{u} = 0 & \text{in } Q_T \\ \mathbf{u} = 0 & \text{on } \Sigma_T \\ \mathbf{u}(0) = \mathbf{u}_0^\varepsilon & \text{in } \Omega \end{cases} \quad (16)$$

with

$$\mathbf{f}_\varepsilon = - \frac{\partial \mathbf{G}_\varepsilon}{\partial t} + \nu \Delta \mathbf{G}_\varepsilon - \mathbf{G}_\varepsilon \cdot \nabla \mathbf{G}_\varepsilon \quad \text{and} \quad \mathbf{u}_0^\varepsilon = \mathbf{v}_0 - \mathbf{G}_\varepsilon(\cdot, 0). \quad (17)$$

We note that $\mathbf{u}_0^\varepsilon \in V$ and

$$\|\mathbf{u}_0^\varepsilon\|_{\mathbf{H}^1(\Omega)} \leq C_\varepsilon \left(\|\mathbf{g}\|_{\mathbf{H}^{3/2, 3/4}(\Sigma_T)} + \|\mathbf{v}_0\|_{\mathbf{H}^1(\Omega)} \right). \quad (18)$$

Moreover, $\mathbf{f}_\varepsilon \in \mathbf{L}^2(0, T; \mathbf{L}^2(\Omega))$ and

$$\|\mathbf{f}_\varepsilon\|_{\mathbf{L}^2(0, T; \mathbf{L}^2(\Omega))} \leq C_\varepsilon \left(\|\mathbf{g}\|_{\mathbf{H}^{3/2, 3/4}(\Sigma_T)} + \|\mathbf{v}_0\|_{\mathbf{H}^1(\Omega)} \right). \quad (19)$$

Now we are able to announce and to establish the following theorem :

Theorem 2.1. *Let \mathbf{v}_0 and \mathbf{g} satisfy the hypotheses of lemma 1.3. Then problem (16) has a unique solution (\mathbf{u}, p) such that*

$$\mathbf{u} \in \mathbf{L}^2(0, T; \mathbf{H}^2(\Omega)) \cap \mathbf{L}^\infty(0, T; V), \quad \frac{\partial \mathbf{u}}{\partial t} \in \mathbf{L}^2(0, T; \mathbf{H}), \quad p \in \mathbf{L}^2(0, T; H^1(\Omega)),$$

p being unique up to an $\mathbf{L}^2(0, T)$ -function of the single variable t .

Proof.

2.1 Approximate solutions

We use the Galerkin method. Let $m \in \mathbb{N}^*$ and $\mathbf{u}_{0m} \in \langle \mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_m \rangle$ such that

$$\mathbf{u}_{0m} \rightarrow \mathbf{u}_0^\varepsilon \text{ in } V, \text{ if } m \rightarrow \infty,$$

where \mathbf{w}_j are the Stokes operator eigenfunctions. For each m , one defines an approximate solution of (16) by :

$$\left\{ \begin{array}{l} \mathbf{u}_m(t) = \sum_{j=1}^m g_{jm}(t) \mathbf{w}_j \\ (\mathbf{u}_m'(t), \mathbf{w}_j) + \nu ((\mathbf{u}_m(t), \mathbf{w}_j)) + b(\mathbf{u}_m(t), \mathbf{u}_m(t), \mathbf{w}_j) \\ + b(\mathbf{u}_m(t), \mathbf{G}_\varepsilon(t), \mathbf{w}_j) + b(\mathbf{G}_\varepsilon(t), \mathbf{u}_m(t), \mathbf{w}_j) = (\mathbf{f}_\varepsilon(t), \mathbf{w}_j) \\ \mathbf{u}_m(0) = \mathbf{u}_{0m}, \quad j = 1, \dots, m \end{array} \right. \quad (20)$$

This is a nonlinear differential system of m equations in m unknowns g_{jm} , $j = 1, \dots, m$:

$$\begin{aligned} & \sum_{i=1}^m (\mathbf{w}_i, \mathbf{w}_j) g'_{im}(t) + \nu \sum_{i=1}^m ((\mathbf{w}_i, \mathbf{w}_j)) g_{im}(t) + \sum_{i,l=1}^m b(\mathbf{w}_i, \mathbf{w}_l, \mathbf{w}_j) g_{im}(t) g_{lm}(t) + \\ & + \sum_{i=1}^m [b(\mathbf{w}_i, \mathbf{G}_\varepsilon(t), \mathbf{w}_j) g_{im}(t) + b(\mathbf{G}_\varepsilon(t), \mathbf{w}_i, \mathbf{w}_j) g_{im}(t)] = (\mathbf{f}_\varepsilon(t), \mathbf{w}_j), \\ & j = 1, \dots, m \end{aligned}$$

2.2 Estimates I

Let us multiply (20) by $g_{jm}(t)$ and sum over j :

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} |\mathbf{u}_m(t)|^2 + \nu \|\mathbf{u}_m(t)\|^2 &= -b(\mathbf{u}_m(t), \mathbf{G}_\varepsilon(t), \mathbf{u}_m(t)) + (\mathbf{f}_\varepsilon(t), \mathbf{u}_m(t)) \\ &\leq |\mathbf{f}_\varepsilon(t)| \|\mathbf{u}_m(t)\| + |b(\mathbf{u}_m(t), \mathbf{G}_\varepsilon(t), \mathbf{u}_m(t))| \end{aligned}$$

One deduces from lemma 1.4 that :

$$\frac{1}{2} \frac{d}{dt} |\mathbf{u}_m(t)|^2 + \frac{\nu}{2} \|\mathbf{u}_m(t)\|^2 \leq \frac{1}{2\nu C^2(\Omega)} |\mathbf{f}_\varepsilon(t)|^2 + \beta(\varepsilon, t) \|\mathbf{u}_m(t)\|^2.$$

As $\sup_{t \in [0, T]} \beta(\varepsilon, t) \rightarrow 0$ when $\varepsilon \rightarrow 0$, for a fixed and small $\varepsilon > 0$, one has:

$$\frac{d}{dt} |\mathbf{u}_m(t)|^2 + \frac{\nu}{2} \|\mathbf{u}_m(t)\|^2 \leq \frac{1}{\nu C^2(\Omega)} |\mathbf{f}_\varepsilon(t)|^2. \quad (21)$$

Integrating (21) from 0 to s, one deduces that:

$$\begin{aligned} |\mathbf{u}_m(s)|^2 &\leq |\mathbf{u}_{0m}|^2 + \frac{1}{\nu C^2(\Omega)} \int_0^s |\mathbf{f}_\varepsilon(t)|^2 dt \\ &\leq |\mathbf{u}_0^\varepsilon|^2 + \frac{1}{\nu C^2(\Omega)} \|\mathbf{f}_\varepsilon\|_{\mathbf{L}^2(0,T;\mathbf{L}^2(\Omega))}^2 \\ &\leq C_\varepsilon \left(\|\mathbf{g}\|_{\mathbf{H}^{3/2,3/4}(\Sigma_T)}^2 + \|\mathbf{v}_0\|_{\mathbf{H}^1(\Omega)}^2 \right) \end{aligned}$$

according to (18) and (20). Therefore

$$\mathbf{u}_m \in \mathbf{L}^\infty(0, T; \mathbf{H}), \quad (22)$$

and $\{\mathbf{u}_m\}$ is an equibounded sequence in $\mathbf{L}^\infty(0, T; \mathbf{H})$.

Next, thanks to (21), one has:

$$\mathbf{u}_m \in \mathbf{L}^2(0, T; V), \quad (23)$$

and the sequence $\{\mathbf{u}_m\}$ is equibounded in $\mathbf{L}^2(0, T; V)$.

2.3 Estimates II

Let us multiply (20) by $\lambda_j g_{jm}(t)$ and sum over j :

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\mathbf{u}_m(t)\|^2 + \nu |A\mathbf{u}_m(t)|^2 + b(\mathbf{u}_m(t), \mathbf{u}_m(t), A\mathbf{u}_m(t)) + \\ b(\mathbf{G}_\varepsilon(t), \mathbf{u}_m(t), A\mathbf{u}_m(t)) + b(\mathbf{u}_m(t), \mathbf{G}_\varepsilon(t), A\mathbf{u}_m(t)) = (\mathbf{f}_\varepsilon, A\mathbf{u}_m(t)) \end{aligned} \quad (24)$$

where A is the Stokes operator. Let us begin by considering the nonlinear terms.

For the first term, thanks to the Gagliardo-Nirenberg inequality one has

$$\begin{aligned} |b(\mathbf{u}_m(t), \mathbf{u}_m(t), A\mathbf{u}_m(t))| &\leq \|\mathbf{u}_m(t)\|_{\mathbf{L}^4(\Omega)} \|\nabla \mathbf{u}_m(t)\|_{\mathbf{L}^4(\Omega)} |A\mathbf{u}_m(t)| \\ &\leq C |\mathbf{u}_m(t)|^{1/2} \|\mathbf{u}_m(t)\| |A\mathbf{u}_m(t)|^{3/2} \\ &\leq C \|\mathbf{u}_m(t)\|^4 + \frac{\nu}{8} |A\mathbf{u}_m(t)|^2. \end{aligned}$$

In the same way,

$$\begin{aligned} |b(\mathbf{G}_\varepsilon(t), \mathbf{u}_m(t), A\mathbf{u}_m(t))| &\leq \|\mathbf{G}_\varepsilon(t)\|_{\mathbf{L}^4(\Omega)} \|\nabla \mathbf{u}_m(t)\|_{\mathbf{L}^4(\Omega)} |A\mathbf{u}_m(t)| \\ &\leq C \|\mathbf{G}_\varepsilon(t)\|_{\mathbf{H}^1(\Omega)} \|\mathbf{u}_m(t)\|^{1/2} |A\mathbf{u}_m(t)|^{3/2} \\ &\leq C \|\mathbf{G}_\varepsilon(t)\|_{\mathbf{H}^1(\Omega)}^4 \|\mathbf{u}_m(t)\|^2 + \frac{\nu}{8} |A\mathbf{u}_m(t)|^2. \end{aligned}$$

We remark that, according to lemma 1.4, one has:

$$\|\mathbf{G}_\varepsilon\|_{\mathbf{L}^\infty(0,T;\mathbf{H}^1(\Omega))} \leq C \left(\|\mathbf{g}\|_{\mathbf{H}^{3/2,3/4}(\Sigma_T)} + \|\mathbf{v}_0\|_{\mathbf{H}^1(\Omega)} \right).$$

So that

$$|b(\mathbf{G}_\varepsilon(t), \mathbf{u}_m(t), A\mathbf{u}_m(t))| \leq C \|\mathbf{u}_m(t)\|^2 + \frac{\nu}{8} |A\mathbf{u}_m(t)|^2.$$

Finally,

$$\begin{aligned} |b(\mathbf{u}_m(t), \mathbf{G}_\varepsilon(t), A\mathbf{u}_m(t))| &\leq \|\mathbf{u}_m(t)\|_{\mathbf{L}^4(\Omega)} \|\nabla \mathbf{G}_\varepsilon(t)\|_{\mathbf{L}^4(\Omega)} |A\mathbf{u}_m(t)| \\ &\leq C \|\mathbf{u}_m(t)\|^2 \|\mathbf{G}_\varepsilon(t)\|_{\mathbf{H}^2(\Omega)}^2 + \frac{\nu}{8} |A\mathbf{u}_m(t)|^2. \end{aligned}$$

Hence,

$$\frac{d}{dt} \|\mathbf{u}_m(t)\|^2 + \nu |A\mathbf{u}_m(t)|^2 \leq \frac{C}{\nu} |\mathbf{f}_\varepsilon(t)|^2 + C \left[\|\mathbf{u}_m(t)\|^4 + \|\mathbf{u}_m(t)\|^2 \left(1 + \|\mathbf{G}_\varepsilon(t)\|_{\mathbf{H}^2(\Omega)}^2\right) \right].$$

Let

$$\sigma_m(t) = C \left[\|\mathbf{u}_m(t)\|^2 + \left(1 + \|\mathbf{G}_\varepsilon(t)\|_{\mathbf{H}^2(\Omega)}^2\right) \right].$$

One knows that

$$\sigma_m(t) \in \mathbf{L}^1(0, T);$$

so that, according to the Gronwall lemma and (24), one has:

$$\mathbf{u}_m \in \mathbf{L}^\infty(0, T; V) \cap \mathbf{L}^2(0, T; \mathbf{H}^2(\Omega)), \quad (25)$$

and $\{\mathbf{u}_m\}$ is an equibounded sequence in $\mathbf{L}^\infty(0, T; V) \cap \mathbf{L}^2(0, T; \mathbf{H}^2(\Omega))$.

2.4 Estimates III

Let us multiply (20) by $g'_{jm}(t)$ and sum over j from 1 to m . Then

$$\begin{aligned} |\mathbf{u}'_m(t)|^2 &= \nu (A\mathbf{u}_m(t), \mathbf{u}'_m(t)) - b(\mathbf{u}_m(t), \mathbf{u}_m(t), \mathbf{u}'_m(t)) \\ &\quad - b(\mathbf{G}_\varepsilon(t), \mathbf{u}_m(t), \mathbf{u}'_m(t)) - b(\mathbf{u}_m(t), \mathbf{G}_\varepsilon(t), \mathbf{u}'_m(t)) + (\mathbf{f}_\varepsilon, \mathbf{u}'_m(t)). \end{aligned}$$

From this, one deduces that

$$\begin{aligned} |\mathbf{u}'_m(t)|^2 &\leq \nu |A\mathbf{u}_m(t)| |\mathbf{u}'_m(t)| + C \|\mathbf{u}_m(t)\|_{\mathbf{L}^4(\Omega)} \|\nabla \mathbf{u}_m(t)\|_{\mathbf{L}^4(\Omega)} |\mathbf{u}'_m(t)| \\ &\quad + C \|\mathbf{G}_\varepsilon(t)\|_{\mathbf{L}^4(\Omega)} \|\nabla \mathbf{u}_m(t)\|_{\mathbf{L}^4(\Omega)} |\mathbf{u}'_m(t)| \\ &\quad + C \|\mathbf{u}_m(t)\|_{\mathbf{L}^4(\Omega)} \|\nabla \mathbf{G}_\varepsilon(t)\|_{\mathbf{L}^4(\Omega)} |\mathbf{u}'_m(t)| + |\mathbf{f}_\varepsilon(t)| |\mathbf{u}'_m(t)| \end{aligned}$$

Using the Gagliardo-Nirenberg inequality, estimates (25) and (19), and lemma 1.4 giving the estimate of \mathbf{G}_ε , one deduces that

$$\mathbf{u}'_m \in \mathbf{L}^2(0, T; \mathbf{H}), \quad (26)$$

and $\{\mathbf{u}'_m\}$ is an equibounded sequence in $\mathbf{L}^2(0, T; H)$.

2.5 Taking the limit.

It is a consequence of the above estimates that the sequence \mathbf{u}_m has a subsequence \mathbf{u}_m , the same notation being used to avoid unnecessary notation overload:

$$\mathbf{u}_m \rightharpoonup \mathbf{u} \text{ weakly}^* \quad \text{in } \mathbf{L}^\infty(0, T; V), \quad (27)$$

$$\mathbf{u}_m \rightharpoonup \mathbf{u} \text{ weakly} \quad \text{in } \mathbf{L}^2(0, T; \mathbf{H}^2(\Omega)), \quad (28)$$

$$\mathbf{u}'_m \rightharpoonup \mathbf{u}' \text{ weakly} \quad \text{in } \mathbf{L}^2(0, T; \mathbf{H}). \quad (29)$$

But we have a compact embedding

$$\{\mathbf{v} \in \mathbf{L}^2(0, T; \mathbf{H}^2(\Omega) \cap V), V' \in \mathbf{L}^2(0, T; \mathbf{H})\} \xrightarrow{\text{compact}} \mathbf{L}^2(0, T; V)$$

So that

$$\mathbf{u}_m \rightarrow \mathbf{u} \text{ strongly} \quad \text{in } \mathbf{L}^2(0, T; V) \text{ and a.e. in } Q_T \quad (30)$$

Let m_0 be fixed and $\mathbf{v} \in \langle \mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_{m_0} \rangle$. Let m tend towards $+\infty$ in (20). Then

$$\begin{aligned} (\mathbf{u}'(t), \mathbf{v}) + \nu((\mathbf{u}(t), \mathbf{v})) &+ b(\mathbf{u}(t), \mathbf{u}(t), \mathbf{v}) + b(\mathbf{u}(t), \mathbf{G}_\varepsilon(t), \mathbf{v}) \\ &+ b(\mathbf{G}_\varepsilon(t), \mathbf{u}(t), \mathbf{v}) = (\mathbf{f}_\varepsilon(t), \mathbf{v}), \end{aligned}$$

This last relation being valid for all m_0 , it remains true for all $\mathbf{v} \in \langle \mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_m \rangle$, $\forall m \in \mathbb{N}^*$.

Finally let $\mathbf{v} \in V$. There exists $\mathbf{v}_m \in \langle \mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_m \rangle$ such that $\mathbf{v}_m \rightarrow \mathbf{v}$ in V and

$$\begin{aligned} (\mathbf{u}'(t), \mathbf{v}) + \nu((\mathbf{u}(t), \mathbf{v})) + b(\mathbf{u}(t), \mathbf{u}(t), \mathbf{v}) \\ + b(\mathbf{u}(t), \mathbf{G}_\varepsilon(t), \mathbf{v}) + b(\mathbf{G}_\varepsilon(t), \mathbf{u}(t), \mathbf{v}) = (\mathbf{f}_\varepsilon(t), \mathbf{v}) \end{aligned} \quad (31)$$

Now let us note that for all $t \in [0, T]$,

$$\mathbf{u}_m(t) \rightharpoonup \mathbf{u}(t) \quad \text{weakly in } V,$$

and thus

$$\mathbf{u}_m(0) = \mathbf{u}_{0m} \rightharpoonup \mathbf{u}(0) \quad \text{weakly in } V.$$

Since

$$\mathbf{u}_{0m} \rightarrow \mathbf{u}_0^\varepsilon \quad \text{in } V,$$

we have:

$$\mathbf{u}(0) = \mathbf{u}_0^\varepsilon.$$

2.6 Existence of pressure.

From (31), one has, for all $\mathbf{v} \in V$,

$$\langle \mathbf{u}' - \nu \Delta \mathbf{u} + B(\mathbf{u}, \mathbf{u}) + B(\mathbf{u}, \mathbf{G}_\varepsilon) + B(\mathbf{G}_\varepsilon, \mathbf{u}) - \mathbf{f}_\varepsilon, \mathbf{v} \rangle_{\mathbf{H}^{-1}(\Omega) \times \mathbf{H}_0^1(\Omega)} = 0.$$

Consequently, there exists a unique function p of $L^2(0, T)$ satisfying (16) and such that :

$$p \in L^2(0, T; \mathbf{H}^1(\Omega)).$$

This ends the proof of theorem 2.1. \square

3 Uniqueness Theorem

Theorem 3.1 *Problem (16) has a unique solution.*

Proof.

Let \mathbf{u} and \mathbf{v} be two solutions satisfying the hypotheses of theorem 2.1 and let $\mathbf{w} = \mathbf{u} - \mathbf{v}$. Then one has

$$\frac{\partial \mathbf{w}}{\partial t} - \nu \Delta \mathbf{w} + \mathbf{w} \cdot \nabla \mathbf{u} + \mathbf{v} \cdot \nabla \mathbf{w} + \mathbf{w} \cdot \nabla \mathbf{G}_\varepsilon + \mathbf{G}_\varepsilon \cdot \nabla \mathbf{w} = \mathbf{0}$$

Multiplying by \mathbf{w} , we obtain

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} |\mathbf{w}(t)|^2 + \nu \|\mathbf{w}(t)\|^2 = & -(\mathbf{w} \cdot \nabla \mathbf{u}, \mathbf{w}) - (\mathbf{v} \cdot \nabla \mathbf{w}, \mathbf{w}) \\ & -(\mathbf{w} \cdot \nabla \mathbf{G}_\varepsilon, \mathbf{w}) - (\mathbf{G}_\varepsilon \cdot \nabla \mathbf{w}, \mathbf{w}) \end{aligned}$$

But $b(\mathbf{v}, \mathbf{w}, \mathbf{w}) = 0$ and $b(\mathbf{G}_\varepsilon, \mathbf{w}, \mathbf{w}) = 0$. This yields

$$\frac{1}{2} \frac{d}{dt} |\mathbf{w}(t)|^2 + \nu \|\mathbf{w}(t)\|^2 = -b(\mathbf{w}, \mathbf{u}, \mathbf{w}) - b(\mathbf{w}, \mathbf{G}_\varepsilon, \mathbf{w}).$$

One then integrates with respect to t and we get

$$\frac{1}{2} |\mathbf{w}(t)|^2 + \nu \int_0^t \|\mathbf{w}(s)\|^2 ds = - \int_0^t b(\mathbf{w}, \mathbf{u}, \mathbf{w}) ds - \int_0^t b(\mathbf{w}, \mathbf{G}_\varepsilon, \mathbf{w}) ds.$$

Since

$$\begin{aligned} \left| \int_0^t b(\mathbf{w}, \mathbf{u}, \mathbf{w}) ds \right| & \leq C_1 \int_0^t \|\mathbf{w}(s)\|_{\mathbf{L}^4(\Omega)} \|\mathbf{u}(s)\|_{\mathbf{L}^2(\Omega)} ds \\ & \leq C_2 \int_0^t |\mathbf{w}(s)| \|\mathbf{w}(s)\| \|\mathbf{u}(s)\| ds \\ & \leq \frac{\nu}{2} \int_0^t \|\mathbf{w}(s)\|^2 ds + C_3 \int_0^t |\mathbf{w}(s)|^2 \|\mathbf{u}(s)\|^2 ds. \end{aligned}$$

and, by the same way,

$$\int_0^t b(\mathbf{w}, \mathbf{G}_\varepsilon, \mathbf{w}) ds \leq \frac{\nu}{2} \int_0^t \|\mathbf{w}(s)\|^2 ds + C_4 \int_0^t |\mathbf{w}(s)|^2 |\nabla \mathbf{G}_\varepsilon(s)|^2 ds.$$

it follows that

$$|\mathbf{w}(t)|^2 \leq C_5 \int_0^t |\mathbf{w}(s)|^2 \left(|\nabla \mathbf{G}_\varepsilon(s)|^2 + \|\mathbf{u}(s)\|^2 \right) ds$$

Thanks to the Gronwall lemma, one deduces $\mathbf{w} = 0$. \square

4 Existence of strong reproductive solution

We first recall results obtained by Kaniel et Shinbrot [5] in the study of the following problem :

$$\begin{cases} \frac{\partial \mathbf{u}}{\partial t} - \nu \Delta \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla p = \mathbf{f} & \text{in } Q_T \\ \operatorname{div} \mathbf{u} = 0 & \text{in } Q_T \\ \mathbf{u} = 0 & \text{on } \Sigma_T \\ \mathbf{u}(0) = \mathbf{u}_0 & \text{in } \Omega \end{cases} \quad (32)$$

where Ω is an open and bounded domain of \mathbb{R}^3 , with a smooth boundary Γ .

The following result establishes the property of a reproductive solution

Theorem 4.1. *Let $T > 0$, and $\mathbf{f} \in \mathcal{B}_{R,T}$ with \mathbf{f} small enough. Then, there exists an unique function \mathbf{u}_0 , independent of t , with $\nabla \mathbf{u}_0 \in \mathcal{B}_{R,T}$ and such that the solution of (32) reproduces its initial value at $t = T$:*

$$\mathbf{u}(\mathbf{x}, T) = \mathbf{u}(\mathbf{x}, 0) = \mathbf{u}_0(\mathbf{x}),$$

where

$$\mathcal{B}_{R,T} = \left\{ \mathbf{u} \in \mathbf{L}^\infty(0, T; \mathbf{L}^2(\Omega)) : \|\mathbf{u}\|_{\mathbf{L}^\infty(0, T; \mathbf{L}^2(\Omega))} \leq R \right\}.$$

We begin by recalling the following lemma.

Lemma 4.2. *If*

$$\mathbf{u} \in \mathbf{L}^2(0, T; \mathbf{H}^2(\Omega) \cap V) \text{ and } \mathbf{u}' \in \mathbf{L}^2(0, T; \mathbf{H})$$

then

$$\mathbf{u} \in C([0, T]; V)$$

and

$$\frac{d}{dt} \|\mathbf{u}(t)\|^2 = -2(\mathbf{u}'(t), \Delta \mathbf{u}(t)). \square$$

Now, let

$$\mathbf{v}_0 \in \mathbf{H}^1(\Omega) \cap \mathbf{H}, \quad \mathbf{w}_0 \in \mathbf{H}^1(\Omega) \cap \mathbf{H}, \quad \mathbf{g} \in \mathbf{H}^{3/2, 3/4}(\Sigma_T) \quad (33)$$

with

$$\mathbf{g} \cdot \mathbf{n} = 0 \quad \text{on} \quad \Sigma_T \quad \text{and} \quad \mathbf{v}_0(\mathbf{x}) = \mathbf{w}_0(\mathbf{x}) = \mathbf{g}(\mathbf{x}, 0) \quad \mathbf{x} \in \Gamma. \quad (34)$$

With these assumptions, it follows from theorem 2.1 that system (1), with data $(\mathbf{v}_0, \mathbf{g})$, (respectively $(\mathbf{w}_0, \mathbf{g})$), has an unique solution

$$\mathbf{v} \in \mathbf{L}^2(0, T; \mathbf{H}^2(\Omega) \cap \mathbf{H}) \cap \mathbf{L}^\infty(0, T; \mathbf{H}^1(\Omega)) \quad \text{and} \quad \mathbf{v}' \in \mathbf{L}^2(0, T; \mathbf{H}),$$

(respectively

$$\mathbf{w} \in \mathbf{L}^2(0, T; \mathbf{H}^2(\Omega) \cap \mathbf{H}) \cap \mathbf{L}^\infty(0, T; \mathbf{H}^1(\Omega)) \quad \text{and} \quad \mathbf{w}' \in \mathbf{L}^2(0, T; \mathbf{H}).$$

Let us now set $\mathbf{z} = \mathbf{v} - \mathbf{w}$. Then

$$\begin{cases} \frac{\partial \mathbf{z}}{\partial t} - \nu \Delta \mathbf{z} + \mathbf{w} \cdot \nabla \mathbf{z} + \mathbf{z} \cdot \nabla \mathbf{v} + \nabla r = \mathbf{0} & \text{in} \quad Q_T, \\ \operatorname{div} \mathbf{z} = 0 & \text{in} \quad Q_T, \\ \mathbf{z} = 0 & \text{on} \quad \Sigma_T, \\ \mathbf{z}(0) = \mathbf{v}_0 - \mathbf{w}_0 & \text{in} \quad \Omega. \end{cases} \quad (35)$$

where $r = p - q$ (q being the pressure corresponding to \mathbf{w}).

Lemma 4.3. *If*

$$\max \left(\|\mathbf{v}\|_{\mathbf{L}^\infty(0, T; \mathbf{H}^1(\Omega))}, \|\mathbf{w}\|_{\mathbf{L}^\infty(0, T; \mathbf{H}^1(\Omega))} \right) \leq M \quad (36)$$

under the assumptions (33) and (34) with $0 < M \ll 1$, then

$$\frac{d}{dt} \|\mathbf{z}(t)\|^2 + \nu \|\mathbf{z}(t)\|^2 \leq 0 \quad (37)$$

and thus, for all $t \in [0, T]$,

$$\|\mathbf{v}(t) - \mathbf{w}(t)\| \leq \|\mathbf{v}_0 - \mathbf{w}_0\| \exp(-\nu t). \quad (38)$$

Proof.

Let $P: \mathbf{L}^2(\Omega) \rightarrow \mathbf{H}$, be the orthogonal projection operator. Then

$$\forall \varphi \in \mathbf{H}, (\nabla r, \varphi) = 0.$$

In particular, let us multiply (35) by $P \Delta \mathbf{z} = A\mathbf{z}$:

$$\frac{1}{2} \frac{d}{dt} \|\mathbf{z}(t)\|^2 + \nu |A\mathbf{z}|^2 = -(\mathbf{w} \cdot \nabla \mathbf{z}, A\mathbf{z}) - (\mathbf{z} \cdot \nabla \mathbf{v}, A\mathbf{z})$$

But

$$\begin{aligned} |(\mathbf{w} \cdot \nabla \mathbf{z}, A\mathbf{z})| &\leq \|\mathbf{w}\|_{\mathbf{L}^4(\Omega)} \|\nabla \mathbf{z}\|_{\mathbf{L}^4(\Omega)} |A\mathbf{z}| \\ &\leq C \|\mathbf{w}\| |A\mathbf{z}|^2 \end{aligned}$$

and

$$\begin{aligned} |(\mathbf{z} \cdot \nabla \mathbf{v}, A\mathbf{z})| &\leq \|\mathbf{z}\|_{\mathbf{L}^\infty(\Omega)} \|\mathbf{v}\| |A\mathbf{z}| \\ &\leq C \|\mathbf{v}\| |A\mathbf{z}|^2. \end{aligned}$$

So that if

$$C \left(\|\mathbf{v}\|_{\mathbf{L}^\infty(0,T;\mathbf{H}^1(\Omega))} + \|\mathbf{w}\|_{\mathbf{L}^\infty(0,T;\mathbf{H}^1(\Omega))} \right) \leq \frac{\nu}{2}$$

then

$$\frac{d}{dt} \|\mathbf{z}(t)\|^2 + \nu \|\mathbf{z}(t)\|^2 \leq 0$$

and one deduces (38). \square

4.1 The main result

Lemma 4.4. *Suppose that \mathbf{g} and \mathbf{v}_0 satisfy hypotheses (4)-(5) and (9). Let us suppose moreover that $\mathbf{f}_\varepsilon \in \mathbf{L}^\infty(0,T;\mathbf{L}^2(\Omega))$ and that*

$$\|\mathbf{g}\|_{\mathbf{H}^{3/2,3/4}(\Sigma_T)} + \|\mathbf{v}_0\|_{\mathbf{H}^1(\Omega)} \leq \alpha \quad (39)$$

$$\|\mathbf{f}_\varepsilon\|_{\mathbf{L}^\infty(0,T;\mathbf{L}^2(\Omega))} \leq K \quad (40)$$

with $\alpha > 0$ and $0 < K \ll 1$. Then, if \mathbf{u} is the solution given by theorem 2.1, one has:

$$\sup_{t \in [0,T]} \|\nabla \mathbf{u}(t)\|_{\mathbf{L}^2(\Omega)} \leq M \quad (41)$$

Remark 4.5. Let us recall that

$$\mathbf{u}_0 = \mathbf{v}_0 - \mathbf{G}_\varepsilon(\cdot, 0)$$

Consequently, if hypothesis (39) takes place, one has from lemma 1.4 :

$$\begin{aligned} \|\mathbf{u}_0\| &\leq \|\mathbf{u}_0\|_{\mathbf{H}^1(\Omega)} \leq \|\mathbf{v}_0\|_{\mathbf{H}^1(\Omega)} + \|\mathbf{G}_\varepsilon(\cdot, 0)\|_{\mathbf{H}^1(\Omega)} \\ &\leq \|\mathbf{v}_0\|_{\mathbf{H}^1(\Omega)} + L \left(\|\mathbf{g}\|_{\mathbf{H}^{3/2,3/4}(\Sigma_T)} + \|\mathbf{v}_0\|_{\mathbf{H}^1(\Omega)} \right) \\ &\leq \alpha(L+1) = M. \square \end{aligned}$$

Proof of lemma 4.4. (see Batchi [5])

Let us multiply (16) by $\mathbf{A}\mathbf{u}$ and integrate on Ω :

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\mathbf{u}\|^2 + \nu |\mathbf{A}\mathbf{u}|^2 \leq & \int_{\Omega} \mathbf{f}_{\varepsilon} \cdot \mathbf{A}\mathbf{u} dx - \int_{\Omega} (\mathbf{u} \cdot \nabla \mathbf{u}) \cdot \mathbf{A}\mathbf{u} dx \\ & - \int_{\Omega} (\mathbf{u} \cdot \nabla \mathbf{G}_{\varepsilon}) \cdot \mathbf{A}\mathbf{u} dx - \int_{\Omega} (\mathbf{G}_{\varepsilon} \cdot \nabla \mathbf{u}) \cdot \mathbf{A}\mathbf{u} dx \end{aligned}$$

But

$$\begin{aligned} \left| \int_{\Omega} (\mathbf{u} \cdot \nabla \mathbf{u}) \cdot \mathbf{A}\mathbf{u} dx \right| & \leq \|\mathbf{u}\|_{\mathbf{L}^{\infty}(\Omega)} \|\mathbf{u}\| |\mathbf{A}\mathbf{u}| \\ & \leq C_1 \|\mathbf{u}\| |\mathbf{A}\mathbf{u}|^2, \end{aligned}$$

where C_1 is such that $\|\mathbf{u}\|_{\mathbf{L}^{\infty}(\Omega)} \leq C_1 |\mathbf{A}\mathbf{u}|$.

In the same way, one also has

$$\left| \int_{\Omega} (\mathbf{u} \cdot \nabla \mathbf{G}_{\varepsilon}) \cdot \mathbf{A}\mathbf{u} dx \right| \leq C_1 \|\nabla \mathbf{G}_{\varepsilon}\|_{\mathbf{L}^2(\Omega)} |\mathbf{A}\mathbf{u}|^2$$

But thanks to the lemma 1.4, one knows that

$$\mathbf{G}_{\varepsilon} \in \mathbf{L}^{\infty}(0, T; \mathbf{H}^1(\Omega))$$

and

$$\begin{aligned} \|\nabla \mathbf{G}_{\varepsilon}\|_{\mathbf{L}^2(\Omega)} & \leq C_2 \|\mathbf{G}_{\varepsilon}\|_{\mathbf{H}^{2,1}(Q_T)} \\ & \leq C_2 L \left(\|\mathbf{g}\|_{\mathbf{H}^{3/2, 3/4}(\Sigma_T)} + \|\mathbf{v}_0\|_{\mathbf{H}^1(\Omega)} \right) \\ & \leq C_3 \alpha. \end{aligned}$$

It then follows that

$$\begin{aligned} \left| \int_{\Omega} (\mathbf{G}_{\varepsilon} \cdot \nabla \mathbf{u}) \cdot \mathbf{A}\mathbf{u} dx \right| & \leq \|\mathbf{G}_{\varepsilon}\|_{\mathbf{L}^4(\Omega)} \|\nabla \mathbf{u}\|_{\mathbf{L}^4(\Omega)} |\mathbf{A}\mathbf{u}| \\ & \leq C_4 \|\mathbf{G}_{\varepsilon}\|_{\mathbf{H}^1(\Omega)} |\mathbf{A}\mathbf{u}| \|\nabla \mathbf{u}\|_{\mathbf{L}^2(\Omega)}^{1/2} \|\nabla^2 \mathbf{u}\|_{\mathbf{L}^2(\Omega)}^{1/2} \\ & \leq C_5 \alpha \|\mathbf{u}\|^{1/2} |\mathbf{A}\mathbf{u}|^{3/2} \\ & \leq C_5 \alpha \sqrt{C_6} |\mathbf{A}\mathbf{u}|^2, \end{aligned}$$

with $\|\mathbf{u}\| \leq C_6 |\mathbf{A}\mathbf{u}|$.

Thus,

$$\frac{1}{2} \frac{d}{dt} \|\mathbf{u}\|^2 + \nu |\mathbf{A}\mathbf{u}|^2 \leq |\mathbf{f}_{\varepsilon}| |\mathbf{A}\mathbf{u}| + C_1 \|\mathbf{u}\| |\mathbf{A}\mathbf{u}|^2 + C_1 C_3 \alpha |\mathbf{A}\mathbf{u}|^2 + C_5 \alpha \sqrt{C_6} |\mathbf{A}\mathbf{u}|^2. \quad (42)$$

Let $\varphi(t) = \|\mathbf{u}(t)\|$

i) Let us first suppose that $\|\mathbf{u}_0\| < M$.

Let $t_0 > 0$ be the smallest $t > 0$ such that $\varphi(t_0) = M$. According to (41), one then has

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\mathbf{u}(t)\|_{t=t_0}^2 + \nu |\mathbf{A}\mathbf{u}(t_0)|^2 \leq & K |\mathbf{A}\mathbf{u}(t_0)| + C_1 M |\mathbf{A}\mathbf{u}(t_0)|^2 \\ & + C_1 C_3 \alpha |\mathbf{A}\mathbf{u}(t_0)|^2 + C_5 \alpha \sqrt{C_6} |\mathbf{A}\mathbf{u}(t_0)|^2. \end{aligned}$$

Let us choose α sufficiently small and K such that

$$K = \frac{\nu}{8} \frac{1}{C_6} M, \quad (C_1 M + C_1 C_3 \alpha + C_5 \alpha \sqrt{C_6}) \leq \frac{3\nu}{8}$$

Then

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\mathbf{u}(t)\|_{t=t_0}^2 + \nu |A\mathbf{u}(t_0)|^2 &\leq \frac{\nu}{8} \frac{1}{C_6} M |A\mathbf{u}(t_0)| + \frac{3\nu}{8} |A\mathbf{u}(t_0)|^2 \\ \frac{1}{2} \frac{d}{dt} \|\mathbf{u}(t)\|_{t=t_0}^2 + \nu |A\mathbf{u}(t_0)|^2 &\leq \frac{\nu}{8} \frac{1}{C_6} \|\mathbf{u}(t_0)\| |A\mathbf{u}(t_0)| + \frac{3\nu}{8} |A\mathbf{u}(t_0)|^2 \\ \frac{1}{2} \frac{d}{dt} \|\mathbf{u}(t)\|_{t=t_0}^2 + \nu |A\mathbf{u}(t_0)|^2 &\leq \frac{\nu}{2} |A\mathbf{u}(t_0)|^2. \end{aligned}$$

Thus

$$\frac{d}{dt} \|\mathbf{u}(t)\|_{t=t_0}^2 + \nu |A\mathbf{u}(t_0)|^2 \leq 0$$

which implies that

$$\frac{d}{dt} \|\mathbf{u}(t)\|_{t=t_0}^2 \leq 0$$

Consequently, there exists $t^* \in [0, t_0[$ such that

$$\varphi(t^*) > \varphi(t_0), \text{ in contradiction with the definition of } t_0.$$

Therefore

$$\forall t \in [0, T], \varphi(t) < M.$$

ii) Suppose now that $\|\mathbf{u}_0\| = M$.

According to the above calculations, one verifies that $\varphi'(0) < 0$ and thus there exists $t^* > 0$ such that

$$\forall t \in]0, t^*], \varphi(t) < M.$$

Repeating the reasoning made in i), one shows that on $[t^*, T]$, $\varphi(t) < M$, and this ends the proof. \square

Remark 4.6. From now on, we assume that \mathbf{g} does not depend on time. More precisely, it is supposed that

$$\mathbf{g} \in \mathbf{H}^{3/2}(\Gamma), \quad \mathbf{g} \cdot \mathbf{n} = 0 \text{ on } \Gamma. \quad (43)$$

One recalls that $\mathbf{v}_0 \in \mathbf{H}^1(\Omega)$ satisfies

$$\operatorname{div} \mathbf{v}_0 = 0 \text{ in } \Omega, \quad \mathbf{v}_0 \cdot \mathbf{n} = 0 \text{ on } \Gamma \quad (44)$$

and that

$$\mathbf{v}_0 = \mathbf{g} \quad \text{on } \Gamma. \quad (45)$$

One knows that there exists $\mathbf{G} \in \mathbf{H}^2(\Omega)$ such that

$$\begin{cases} \operatorname{div} \mathbf{G} = 0 & \text{in } \Omega, \\ \mathbf{G} = \mathbf{g} & \text{on } \Gamma, \end{cases} \quad (46)$$

with

$$\|\mathbf{G}\|_{\mathbf{H}^2(\Omega)} \leq C \|\mathbf{g}\|_{\mathbf{H}^{3/2}(\Gamma)}. \quad (47)$$

Processing as in lemma 1.4, one shows the existence, for all $\varepsilon > 0$, of $\mathbf{G}_\varepsilon \in \mathbf{H}^2(\Omega)$ satisfying (44)-(47) and the estimates:

$$\forall \mathbf{v} \in \mathbf{V}, |b(\mathbf{v}, \mathbf{G}_\varepsilon, \mathbf{v})| \leq \varepsilon \|\mathbf{g}\|^2 \quad (48)$$

The right side \mathbf{f}_ε in system (16) then becomes independent of time and satisfies

$$\mathbf{f}_\varepsilon \in L^\infty\left(0, T; L^2(\Omega)^2\right) \quad (49)$$

In the same way, \mathbf{u}_0^ε becomes

$$\mathbf{u}_0^\varepsilon = \mathbf{v}_0 - \mathbf{G}_\varepsilon \quad (50)$$

with \mathbf{G}_ε depends only on \mathbf{g} . \square

4.2 Reproductive solution result

With these assumptions on \mathbf{g} and \mathbf{v}_0 , lemma 4.2 remains naturally valid and one is able to establish the theorem which follows :

Theorem 4.7. *Let $\mathbf{g} \in \mathbf{H}^{3/2}(\Gamma)$ such that $\mathbf{g} \cdot \mathbf{n} = 0$ on Γ and*

$$\|\mathbf{g}\|_{\mathbf{H}^{3/2}(\Gamma)} \leq \alpha \quad (51)$$

with $0 < \alpha \ll 1$. Then, there exists $\mathbf{v}_0 \in \mathbf{H}^1(\Omega)$ such that $\operatorname{div} \mathbf{v}_0 = 0$ in Ω and $\mathbf{v}_0 = \mathbf{g}$ on Γ , and such that the solution $\mathbf{v} = \mathbf{u} + \mathbf{G}_\varepsilon$ where \mathbf{u} is given by theorem 2.1, is reproductive:

$$\mathbf{v}(T) = \mathbf{v}(0) = \mathbf{v}_0.$$

Proof. Let $\mathbf{G}_\varepsilon \in \mathbf{H}^2(\Omega)$ be the extension of \mathbf{g} satisfying (45)-(47) and

$$\mathbf{f}_\varepsilon = \nu \Delta \mathbf{G}_\varepsilon - \mathbf{G}_\varepsilon \cdot \nabla \mathbf{G}_\varepsilon$$

Let $\mathbf{u}_0^\varepsilon = \mathbf{v}_0 - \mathbf{G}_\varepsilon \in V$ and $\mathbf{u} \in L^2(0, T; \mathbf{H}^2(\Omega)) \cap L^\infty(0, T; V)$ be the unique solution of (16). We note that the function $\mathbf{v} = \mathbf{u} + \mathbf{G}_\varepsilon$ is the unique solution

of the initial problem (1). As in the proof of lemma 4.3, it is clear that if $\|\mathbf{u}_0^\varepsilon\| < M$, then

$$\sup_{t \in [0, T]} \|\mathbf{u}(t)\| \leq M$$

provided that $\|\mathbf{f}_\varepsilon\|_{\mathbf{L}^2(\Omega)}$ is sufficiently small, which follows from (49).

Let us define the application

$$\begin{aligned} \mathbf{L} : \quad \mathbf{u}_0^\varepsilon &\longrightarrow \mathbf{u}(\cdot, T) \\ B_M &\longrightarrow B_M \end{aligned}$$

where $B_M = \{\mathbf{z} \in \mathbf{V}, \|\mathbf{z}\| \leq M\}$;

$\mathbf{u}(\cdot, T)$ being the unique solution of (16) at $t = T$.

Moreover, as in remark 4.5, it is clear that if $\|\mathbf{v}_0\| \leq \alpha$ and $\|\mathbf{w}_0\| \leq \alpha$ then

$$\|\mathbf{u}_0^\varepsilon\| \leq M \quad \text{and} \quad \|\mathbf{w}_0^\varepsilon\| \leq M,$$

with $\mathbf{y}_0^\varepsilon = \mathbf{w}_0 - \mathbf{G}_\varepsilon$.

So that

$$\begin{aligned} \mathbf{L}\mathbf{u}_0^\varepsilon(t) - \mathbf{L}\mathbf{y}_0^\varepsilon(t) &= \mathbf{u}(t) - \mathbf{y}(t) \\ &= \mathbf{u}(t) - \mathbf{G}_\varepsilon - (\mathbf{y}(t) - \mathbf{G}_\varepsilon) \\ &= \mathbf{v}(t) - \mathbf{w}(t), \end{aligned}$$

and, according to lemma 4.2

$$\begin{aligned} \|\mathbf{L}\mathbf{u}_0^\varepsilon(t) - \mathbf{L}\mathbf{y}_0^\varepsilon(t)\| &= \|\mathbf{v}(T) - \mathbf{w}(T)\| \\ &\leq \|\mathbf{v}_0 - \mathbf{w}_0\| \exp(-\nu T) \\ &\leq \|\mathbf{u}_0^\varepsilon - \mathbf{y}_0^\varepsilon\| \exp(-\nu T) \end{aligned}$$

Thus \mathbf{L} is a contraction and has a fixed point. \square

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