

Navier-Stokes equations with periodic boundary conditions and pressure loss

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Abstract

We present in this note the existence and uniqueness results for the Stokes and Navier-Stokes equations which model the laminar flow of an incompressible fluid inside a two-dimensional channel of periodic sections. The data of the pressure loss coefficient enables us to establish a relation on the pressure and to thus formulate an equivalent problem.

Keywords : Navier-Stokes equations, incompressible fluid, bidimensional channel, periodic boundary conditions, pressure loss.

1 Introduction

The problem which one proposes to study here is that modelling a laminar flow inside a two-dimensional plane channel with periodic section. Let Ω be an open bounded connected lipschtzian of \mathbb{R}^2 (see figure hereafter), where $\Gamma_0 = \{0\} \times]-1, 1[$ and $\Gamma_1 = \{1\} \times]-1, 1[$.

One defines the space

$$V = \left\{ \mathbf{v} \in \mathbf{H}^1(\Omega); \operatorname{div} \mathbf{v} = 0, \mathbf{v} = \mathbf{0} \text{ on } \Gamma_2, \mathbf{v}|_{\Gamma_0} = \mathbf{v}|_{\Gamma_1} \right\}$$

and for $\lambda \in \mathbb{R}$ given, one considers the problem

$$(\mathcal{S}) \left\{ \begin{array}{l} \text{Find } \mathbf{u} \in V, \text{ such that} \\ \forall \mathbf{v} \in V, \int_{\Omega} \nabla \mathbf{u} \cdot \nabla \mathbf{v} \, d\mathbf{x} = \lambda \int_{-1}^{+1} v_1(1, y) \, dy. \end{array} \right.$$

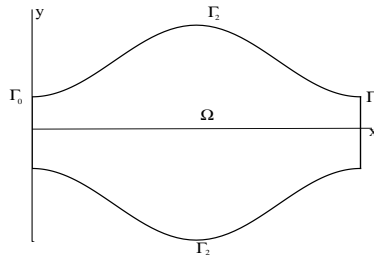


Figure 1. Geometry of channel

2 Resolution of the problem (\mathcal{S})

Initially one proposes to study the problem (\mathcal{P}) . One has it

Theorem 2.1 *Problem (S) has an unique solution $\mathbf{u} \in V$. Moreover, there is a constant $C(\Omega) > 0$ such that:*

$$\|\mathbf{u}\|_{\mathbf{H}^1(\Omega)} \leq \lambda C(\Omega). \quad (1)$$

Proof: Let us note initially that space V provided the norm $H^1(\Omega)^2$ being a closed subspace of $H^1(\Omega)^2$ is thus an Hilbert space. Let us set

$$a(\mathbf{u}, \mathbf{v}) = \int_{\Omega} \nabla \mathbf{u} \cdot \nabla \mathbf{v} \, d\mathbf{x}, \quad l(\mathbf{v}) = \lambda \int_{-1}^{+1} v_1(1, y) \, dy.$$

It is clear, thanks to the Poincaré inequality, that the bilinear continuous form is V -coercive. It is easy to also see that $l \in V'$. One deduces from Lax-Milgram Theorem the existence and uniqueness of \mathbf{u} solution of (S). Moreover,

$$\int_{\Omega} |\nabla \mathbf{u}|^2 \, d\mathbf{x} \leq \lambda \sqrt{2} \left(\int_{-1}^{+1} |u_1(1, y)|^2 \, dy \right)^{1/2},$$

i.e.

$$\begin{aligned} \|\nabla \mathbf{u}\|_{L^2(\Omega)}^2 &\leq \lambda \sqrt{2} \|\mathbf{u}\|_{L^2(\Gamma)} \leq \lambda \sqrt{2} \|\mathbf{u}\|_{H^{1/2}(\Gamma)} \\ &\leq \lambda C_1(\Omega) \|\mathbf{u}\|_{H^1(\Omega)} \end{aligned}$$

Thus there is the estimate (1). \square

We now will give an interpretation of the problem (S). One introduces the space

$$\mathcal{V} = \left\{ \mathbf{v} \in \mathcal{D}(\Omega)^2; \quad \operatorname{div} \mathbf{v} = 0 \right\}.$$

Let \mathbf{u} be the solution of (S). Then, for all $\mathbf{v} \in \mathcal{V}$, one has :

$$\langle -\Delta \mathbf{u}, \mathbf{v} \rangle_{\mathcal{D}'(\Omega) \times \mathcal{D}(\Omega)} = 0.$$

So that thanks to De Rham Theorem, there exists $p \in \mathcal{D}'(\Omega)$ such that

$$-\Delta \mathbf{u} + \nabla p = 0 \text{ in } \Omega. \quad (2)$$

Moreover, since $\nabla p \in H^{-1}(\Omega)^2$, it is known that there exists $q \in L^2(\Omega)$ such that (see [1])

$$\nabla q = \nabla p \quad \text{in } \Omega. \quad (3)$$

The open Ω being connected, there exists $C \in \mathbb{R}$ such that $p = q + C$, what means that $p \in L^2(\Omega)$. Let us recall that (see [1])

$$\inf_{K \in \mathbb{R}} \|p + K\|_{L^2(\Omega)} \leq C \|\nabla p\|_{H^{-1}(\Omega)^2}.$$

One deduces from the estimate (1) and from (2) that

$$\inf_{K \in \mathbb{R}} \|p + K\|_{L^2(\Omega)} \leq C \|\Delta \mathbf{u}\|_{H^{-1}(\Omega)^2} \leq C \|\mathbf{u}\|_{H^1(\Omega)^2} \leq \lambda C(\Omega).$$

Since $\mathbf{u} \in H^1(\Omega)^2$ and $\mathbf{0} = -\Delta \mathbf{u} + \nabla p \in L^2(\Omega)^2$, it is shown that $-\frac{\partial \mathbf{u}}{\partial \mathbf{n}} + p\mathbf{n} \in H^{-1/2}(\Gamma)^2$ and one has the Green formula: for all $\mathbf{v} \in V$

$$\int_{\Omega} (-\Delta \mathbf{u} + \nabla p) \cdot \mathbf{v} \, d\mathbf{x} = \int_{\Omega} \nabla \mathbf{u} \cdot \nabla \mathbf{v} \, d\mathbf{x} + \left\langle -\frac{\partial \mathbf{u}}{\partial \mathbf{n}} + p\mathbf{n}, \mathbf{v} \right\rangle, \quad (4)$$

where the bracket represents the duality product $H^{-1/2}(\Gamma) \times H^{1/2}(\Gamma)$. Moreover, as $p \in L^2(\Omega)$ and $\Delta p = 0$ in Ω , one has $p \in H^{-1/2}(\Gamma)$. Consequently, one has therefore $\frac{\partial \mathbf{u}}{\partial \mathbf{n}} \in H^{-1/2}(\Gamma)^2$. The function \mathbf{u} being solution of (\mathcal{S}) , for all $\mathbf{v} \in V$ one has:

$$\left\langle \frac{\partial \mathbf{u}}{\partial \mathbf{n}} - p\mathbf{n}, \mathbf{v} \right\rangle = \lambda \int_{-1}^{+1} v_1(1, y) \, dy, \quad (5)$$

i.e.

$$\left\langle \frac{\partial \mathbf{u}}{\partial x} - p\mathbf{e}_1, \mathbf{v} \right\rangle_{\Gamma_1} + \left\langle -\frac{\partial \mathbf{u}}{\partial x} + p\mathbf{e}_1, \mathbf{v} \right\rangle_{\Gamma_0} = \langle \lambda \mathbf{e}_1, \mathbf{v} \rangle_{\Gamma_1}. \quad (6)$$

where $\mathbf{e}_1 = (1, 0)$.

i) Let $\mu \in H_{00}^{1/2}(\Gamma_1)$ and let us set

$$\mu_2 = \begin{cases} \mu \text{ on } \Gamma_0 \cup \Gamma_1 \\ 0 \text{ on } \Gamma_2 \end{cases} \quad \text{and} \quad \boldsymbol{\mu} = \begin{pmatrix} 0 \\ \mu_2 \end{pmatrix}$$

where (see [2])

$$H_{00}^{1/2}(\Gamma_1) = \left\{ \varphi \in \mathbf{L}^2(\Gamma_1); \exists \mathbf{v} \in \mathbf{H}^1(\Omega), \text{ with } \mathbf{v}|_{\Gamma_2} = \mathbf{0}, \mathbf{v}|_{\Gamma_0 \cup \Gamma_1} = \varphi \right\}.$$

It is checked easily that

$$\boldsymbol{\mu} \in H^{1/2}(\Gamma)^2 \quad \text{and} \quad \int_{\Gamma} \boldsymbol{\mu} \cdot \mathbf{n} \, d\sigma = 0.$$

So that there exists $\mathbf{v} \in H^1(\Omega)^2$ satisfying (see [3])

$$\operatorname{div} \mathbf{v} = 0 \quad \text{in} \quad \Omega \quad \text{and} \quad \mathbf{v} = \boldsymbol{\mu} \quad \text{on} \quad \Gamma.$$

In particular $\mathbf{v} \in V$ and according to (6), one has

$$\left\langle \frac{\partial u_2}{\partial x}, \mu \right\rangle_{\Gamma_1} = \left\langle \frac{\partial u_2}{\partial x}, \mu \right\rangle_{\Gamma_0},$$

which means that

$$\frac{\partial u_2}{\partial x} \Big|_{\Gamma_1} = \frac{\partial u_2}{\partial x} \Big|_{\Gamma_0}. \quad (7)$$

One deduces now from (6) that for all $\mathbf{v} \in V$,

$$\left\langle \frac{\partial u_1}{\partial x} - p, v_1 \right\rangle_{\Gamma_1} + \left\langle -\frac{\partial u_1}{\partial x} + p, v_1 \right\rangle_{\Gamma_0} = \langle \lambda, v_1 \rangle_{\Gamma_1}. \quad (8)$$

But, $\operatorname{div} \mathbf{u} = 0$ and $u_2|_{\Gamma_1} = u_2|_{\Gamma_0}$, one thus has

$$\frac{\partial u_2}{\partial y} \Big|_{\Gamma_1} = \frac{\partial u_2}{\partial y} \Big|_{\Gamma_0} \quad \text{and} \quad \frac{\partial u_1}{\partial x} \Big|_{\Gamma_1} = \frac{\partial u_1}{\partial x} \Big|_{\Gamma_0}. \quad (9)$$

Consequently, thanks to (8) one has:

$$\langle -p, v_1 \rangle_{\Gamma_1} + \langle p, v_1 \rangle_{\Gamma_0} = \langle \lambda, v_1 \rangle_{\Gamma_1} \quad (10)$$

ii) Let $\nu \in H_{00}^{1/2}(\Gamma_1)$ and let us set

$$\nu_1 = \begin{cases} \nu & \text{on } \Gamma_0 \cup \Gamma_1 \\ 0 & \text{on } \Gamma_2 \end{cases} \quad \text{and} \quad \boldsymbol{\nu} = \begin{pmatrix} \nu_1 \\ 0 \end{pmatrix}.$$

One easily checks that

$$\boldsymbol{\nu} \in H^{1/2}(\Gamma)^2 \quad \text{and} \quad \int_{\Gamma} \boldsymbol{\nu} \cdot \mathbf{n} \, d\sigma = 0.$$

So that there exists $\mathbf{v} \in H^1(\Omega)^2$ satisfying

$$\operatorname{div} \mathbf{v} = 0 \quad \text{in } \Omega \quad \text{and} \quad \mathbf{v} = \boldsymbol{\nu} \quad \text{on } \Gamma.$$

In particular $\mathbf{v} \in V$ and according to (14), one has

$$\langle -p, \boldsymbol{\nu} \rangle_{\Gamma_1} + \langle p, \boldsymbol{\nu} \rangle_{\Gamma_0} = \langle \lambda, \boldsymbol{\nu} \rangle_{\Gamma_1}$$

i.e.

$$p|_{\Gamma_1} = p|_{\Gamma_0} - \lambda \tag{11}$$

where the equality takes place with the $H^{1/2}$ sense. In short, if $\mathbf{u} \in H^1(\Omega)^2$ is solution of (\mathcal{S}) , then there exists $p \in L^2(\Omega)$, unique with an additive constant such that:

$$-\Delta \mathbf{u} + \nabla p = \mathbf{0} \quad \text{in } \Omega, \tag{12}$$

$$\operatorname{div} \mathbf{u} = 0 \quad \text{in } \Omega, \tag{13}$$

$$\mathbf{u} = \mathbf{0} \quad \text{on } \Gamma_2, \quad \mathbf{u}|_{\Gamma_1} = \mathbf{u}|_{\Gamma_0}, \tag{14}$$

$$\frac{\partial \mathbf{u}}{\partial x}|_{\Gamma_1} = \frac{\partial \mathbf{u}}{\partial x}|_{\Gamma_0}, \tag{15}$$

$$p|_{\Gamma_1} = p|_{\Gamma_0} - \lambda. \tag{16}$$

It is clear that if $(\mathbf{u}, p) \in H^1(\Omega)^2 \times L^2(\Omega)$ checks (12)-(16), then \mathbf{u} is solution

of (\mathcal{S}) . Thus it

Theorem 2.2 *The problem (12)-(16) has an unique solution $(\mathbf{u}, p) \in H^1(\Omega)^2 \times L^2(\Omega)$ up to an additive constant for p . Moreover, \mathbf{u} verifies (\mathcal{S}) and*

$$\|\mathbf{u}\|_{\mathbf{H}^1(\Omega)} + \|p\|_{L^2(\Omega)/\mathbb{R}} \leq \lambda C(\Omega). \quad \square$$

Remark 1 *The pressure verifies the relation (16), which means that p satisfies the relation of Patankar et al.[5].*

3 Navier-Stokes Equations

One takes again the assumptions of the Stokes problem given above. For $\lambda \in \mathbb{R}$ given, the one considers the following problem

$$(\mathcal{NS}) \left\{ \begin{array}{l} \text{Find } \mathbf{u} \in V, \text{ such that} \\ \forall \mathbf{v} \in V, \int_{\Omega} \nabla \mathbf{u} \cdot \nabla \mathbf{v} \, d\mathbf{x} + b(\mathbf{u}, \mathbf{u}, \mathbf{v}) = \lambda \int_{-1}^{+1} v_1(1, y) \, dy \end{array} \right.$$

with

$$b(\mathbf{u}, \mathbf{v}, \mathbf{w}) = \int_{\Omega} (\mathbf{u} \cdot \nabla) \mathbf{v} \cdot \mathbf{w} \, d\mathbf{x}$$

With an aim of establishing the existence of the solutions of the problem (\mathcal{NS}) , one uses the Brouwer fixed point theorem (see [4], [6]). One will show it

Theorem 3.1 *The problem (\mathcal{NS}) has at least a solution $\mathbf{u} \in V$. Moreover, \mathbf{u} checks the estimate (1).*

Proof: To show the existence of \mathbf{u} , one constructs the approximate solutions of the problem (\mathcal{NS}) by the Galerkin method and then thanks to the arguments of compactness, one makes a passage to the limit.

i) For each fixed integer $m \geq 1$, one defines an approximate solution \mathbf{u}_m of (\mathcal{NS}) by

$$\mathbf{u}_m = \sum_{i=1}^m g_{im} \mathbf{w}_i, \quad \text{with } g_{im} \in \mathbb{R} \quad (17)$$

$$((\mathbf{u}_m, \mathbf{w}_i)) + b(\mathbf{u}_m, \mathbf{u}_m, \mathbf{w}_i) = \langle \lambda \mathbf{n}, \mathbf{w}_i \rangle_{\Gamma_1}, i = 1, \dots, m$$

where $V_m = \langle \mathbf{w}_1, \dots, \mathbf{w}_m \rangle$ vector spaces spanned by the vectors $\mathbf{w}_1, \dots, \mathbf{w}_m$ and $\{\mathbf{w}_i\}$ is an Hilbertian basis of V which is separable. Let us note that (17) is equivalent to:

$$\forall \mathbf{v} \in V_m, ((\mathbf{u}_m, \mathbf{v})) + b(\mathbf{u}_m, \mathbf{u}_m, \mathbf{v}) = \lambda \int_{-1}^{+1} v_1(1, y) dy. \quad (18)$$

With an aim to establish the existence of the solutions of the problem \mathbf{u}_m , the operator as follows is considered

$$P_m : V_m \rightarrow V_m$$

$$\mathbf{u} \longmapsto P_m(\mathbf{u})$$

defined by

$$\forall \mathbf{u}, \mathbf{v} \in V_m, ((P_m(\mathbf{u}), \mathbf{v})) = ((\mathbf{u}, \mathbf{v})) + b(\mathbf{u}, \mathbf{u}, \mathbf{v}) - \lambda \int_{-1}^{+1} v_1(1, y) dy.$$

Let us note initially that P_m is continuous and

$$\forall \mathbf{u} \in V, b(\mathbf{u}, \mathbf{u}, \mathbf{u}) = 0.$$

Indeed, thanks to the Green formula, one has

$$b(\mathbf{u}, \mathbf{u}, \mathbf{u}) = -\frac{1}{2} \int_{\Omega} |\mathbf{u}|^2 \operatorname{div} \mathbf{u} \, d\mathbf{x} + \frac{1}{2} \int_{\Gamma} (\mathbf{u} \cdot \mathbf{n}) |\mathbf{u}|^2 \, d\sigma = 0.$$

But, $\operatorname{div} \mathbf{u} = 0$ in Ω and

$$\int_{\Gamma} (\mathbf{u} \cdot \mathbf{n}) |\mathbf{u}|^2 \, d\sigma = \int_{\Gamma_0} (\mathbf{u} \cdot \mathbf{n}) |\mathbf{u}|^2 \, d\sigma + \int_{\Gamma_1} (\mathbf{u} \cdot \mathbf{n}) |\mathbf{u}|^2 \, d\sigma.$$

since the external normal to Γ_0 is opposed to that of Γ_1 and $\mathbf{u} \in V$.

Thanks to Brouwer Theorem, there exists \mathbf{u}_m satisfying (18) and

$$\|\mathbf{u}_m\|_{\mathbf{H}^1(\Omega)} \leq \lambda C(\Omega). \quad (19)$$

ii) We can extract a subsequence \mathbf{u}_ν such that

$$\mathbf{u}_\nu \rightharpoonup \mathbf{u} \quad \text{weakly in } V,$$

and thanks to the compact imbedding of V in $L^2(\Omega)^2$, we obtain

$$\forall \mathbf{v} \in V, ((\mathbf{u}, \mathbf{v})) + b(\mathbf{u}, \mathbf{u}, \mathbf{v}) = \lambda \int_{-1}^{+1} v_1(1, y) \, dy.$$

As for the Stokes problem, one shows the existence of $p \in L^2(\Omega)$, unique except for an additive constant, such that

$$\left\{ \begin{array}{ll} -\Delta \mathbf{u} + (\mathbf{u} \cdot \nabla) \mathbf{u} + \nabla p = \mathbf{0} & \text{in } \Omega, \\ \operatorname{div} \mathbf{u} = 0 & \text{in } \Omega, \\ \mathbf{u} = \mathbf{0} & \text{on } \Gamma_2, \\ \mathbf{u}|_{\Gamma_1} = \mathbf{u}|_{\Gamma_0}. & \end{array} \right.$$

It is checked finally that

$$\frac{\partial \mathbf{u}}{\partial x} \Big|_{\Gamma_1} = \frac{\partial \mathbf{u}}{\partial x} \Big|_{\Gamma_0} ,$$

$$p|_{\Gamma_1} = p|_{\Gamma_0} - \lambda.$$

Remark 2 i) Theorem 3.1 of problem (\mathcal{NS}) takes place in three dimension.

ii) One can show that the solution (\mathbf{u}, p) belongs to $H^2(\Omega)^2 \times H^1(\Omega)$.

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