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A new class of rank one transformations with singular spectrum *

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* Dedicated to Professor J. De Sam Lazaro

Abstract. We introduce a new tool to study the spectral type of rank one transformations using the method of central limit theorem for trigonometric sums. We get some new applications.

1. *Introduction*

The purpose of this paper is to bring a new tool in the study of the spectral type of rank one transformations. Rank one transformations have simple spectrum and in [**O**] D.S. Ornstein, using a random procedure, produced a family of mixing rank one transformations. It follows that the Ornstein's class of transformations may possibly contain a candidate for Banach's well-known problem whether there exists a dynamical system $(\Omega, \mathcal{A}, \mu, T)$ with simple Lebesgue spectrum. But, in 1993, J. Bourgain in [**B**] proved that almost surely Ornstein's transformations have singular spectrum. Subsequently, using the same method, I. Klemes [**Kl**] and I. Klemes & K. Reinhold [**K-R**] obtain that mixing staircase transformations of Adams [**A**] and Adams & Friedman [**AF**] have singular spectrum. They conjecture that all rank one transformations have singular spectrum.

Here we shall exhibit a new class of rank one transformations with singular spectrum. Our assumption include some new class of Ornstein transformations and a class of Creutz-Silva rank one transformations [**C-S**]. Our proof is based on techniques introduced by J. Bourgain [**B**] in the context of rank one transformations and developed by Klemes [**Kl**], Klemes-Rienhold [**K-R**], Dooley-Eigen [**E-D**], together with some ideas from the proof of the central limit theorem for trignometric sums. The fundamental key, as noted by Klemes [**Kl**], is the estimation of the L^1-norm of a certain trigonometric polynomial $(|P_m|^2 - 1)$. We shall use the method of central limit theorem for trignometric sums to produce an estimate of this L^1-norm.

2. Rank One Transformation by Construction

Using the cutting and stacking method described in [Fr1], [Fr2], one can construct inductively a family of measure preserving transformations, called rank one transformations, as follows

Let B_0 be the unit interval equipped with the Lebesgue measure. At stage one we divide B_0 into p_0 equal parts, add spacers and form a stack of height h_1 in the usual fashion. At the k^{th} stage we divide the stack obtained at the $(k-1)^{th}$ stage into p_{k-1} equal columns, add spacers and obtain a new stack of height h_k. If during the k^{th} stage of our construction the number of spacers put above the j^{th} column of the $(k-1)^{th}$ stack is $a_j^{(k-1)}$, $0 \leq a_j^{(k-1)} < \infty$, $1 \leq j \leq p_k$, then we have

$$h_k = p_{k-1}h_{k-1} + \sum_{j=1}^{p_{k-1}} a_j^{(k-1)}.$$

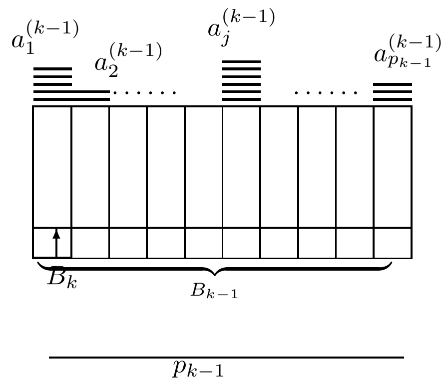

Figure 1 : k^{th}–tower.

Proceeding in this way we get a rank one transformation T on a certain measure space $(X, \mathcal{B}, \nu)$ which may be finite or $\sigma-$finite depending on the number of spacers added.

The construction of a rank one transformation thus needs two parameters, $(p_k)_{k=0}^{\infty}$ (parameter of cutting and stacking), and $((a_j^{(k)})_{j=1}^{p_k})_{k=0}^{\infty}$ (parameter of spacers). Put

$$T \stackrel{def}{=} T_{(p_k,(a_j^{(k)})_{j=1}^{p_k})_{k=0}^{\infty}}$$

In [**C-N1**] and [**K-R**] it is proved that the spectral type of this transformation is given (upto possibly some discrete measure) by

$$d\sigma = W^* \lim \prod_{k=1}^{n} |P_k|^2 \, d\lambda, \tag{1.1}$$

where

$$P_k(z) = \frac{1}{\sqrt{p_k}} \left(\sum_{j=0}^{p_k-1} z^{-(jh_k + \sum_{i=1}^{j} a_i^{(k)})} \right),$$

λ denotes the normalized Lebesgue measure on the circle group $\mathbb{T}$ and $W^* \lim$ denotes weak*limit in the space of bounded Borel measures on $\mathbb{T}$.

The principal result of this paper is the following:

Theorem 2.1. *Let $T = T_{(p_k,(a_j^{(k)})_{j=1}^{p_k})_{k=0}^{\infty}}$ be a rank one transformation such that,*

$$\begin{aligned}
&(i) \quad a_{j+1}^{(k)} \geq 2s_k(j), j = 0, \cdots, p_k - 1, \\
&\text{ with } \quad s_k(j) = a_1^{(k)} + \ldots + a_j^{(k)}, s_k(0) = 0, \\
&(ii) \quad \frac{s_k(p_k)}{h_k} < \frac{1}{2}
\end{aligned}$$

then the spectrum of T is singular.

We remark that the spectrum of rank one transformation is always singular if the cutting parameter p_k is bounded. In fact, Klemes-Reinhold prove that if $\displaystyle\sum_{k=0}^{\infty} \frac{1}{{p_k}^2} = \infty$ then the associated rank one transformation has singular spectrum. Henceforth, we assume that the series $\displaystyle\sum_{k=0}^{\infty} \frac{1}{{p_k}^2}$ converges.

We point out also that the condition (ii) of theorem 2.1 holds in the case of rank one transformations satisfying a restricted growth condition provided that $\min_{1 \leq j \leq p_k} (a_j^{(k)}) = 0$. Following Creutz-Silva [**C-S**], we say that a rank one transformation $T = T_{(p_k,(a_j^{(k)})_{j=1}^{p_k})_{k=0}^{\infty}}$ has restricted growth if

$$\frac{\displaystyle\sum_{i=1}^{p_k} \left(a_j^{(k)} - \min_{1 \leq j \leq p_k} (a_j^{(k)}) \right)}{h_k + \displaystyle\min_{1 \leq j \leq p_k} (a_j^{(k)})} \xrightarrow[k \to +\infty]{} 0,$$

The proof of our main result is based on the method of J. Bourgain in [**B**]. We shall recall the main ideas of this method.

Proposition 2.2. *The following are equivalent*

(i) $\sigma \perp \lambda$

(ii) $\inf\{\int \int \prod_{l=1}^{k} |P_{n_l}(z)|\, d\lambda,\ k \in \mathbb{N},\ n_1 < n_2 < \ldots < n_k\} = 0.$

Now fix some subsequence $\mathcal{N} = \{n_1 < n_2 < \ldots < n_k\}$, $k \in \mathbb{N}$, $m > n_k$ and put

$$Q(z) = \prod_{i=1}^{k} |P_{n_i}(z)|\,.$$

One can show, using the same arguments as in [**EA**], the following lemma.

Lemma 2.3.

$$\int Q(z)\,|P_m(z)|\,d\lambda(z) \leq$$
$$\frac{1}{2}\left(\int Q d\lambda + \int Q(z)\,|P_m(z)|^2\,d\lambda(z)\right) - \frac{1}{8}\left(\int Q\left||P_m(z)|^2 - 1\right| d\lambda(z)\right)^2.$$

Proposition 2.4. $\lim_{m\to\infty} \int Q\,|P_m(z)|^2\,d\lambda(z) = \int Q\ d\lambda(z).$

Proof : The sequence of probability measures $|P_m(z)|^2\,d\lambda(z)$ converges weakly to the Lebsegue measure. ∎

We have also the following proposition:

Proposition 2.5. *There exist a subsequence of the sequence* $(||P_m(z)| - 1|)$ *which converge weakly to some non-negative function* ϕ *which satisfies* $\phi \leq 2$*, almost surely with respect to the Lebesgue measure.*

Proof : The sequence $||P_m(z)| - 1|$ is bounded in L^2. It follows that there exist a subsequence which converges weakly to some non-negative L^2 function ϕ. Let ω be a non-negative continuous function, then we have

$$\begin{aligned}\int \omega\,||P_m(z)| - 1|\,d\lambda(z) &\leq \int \omega\,|P_m(z)|\,d\lambda(z) + \int \omega d\lambda(z)\\ &\leq \left(\int \omega d\lambda(z)\right)^{\frac{1}{2}}\left(\int \omega\,|P_m(z)|^2\,d\lambda(z)\right)^{\frac{1}{2}} + \int \omega d\lambda(z).\end{aligned}$$

Hence

$$\int \omega\phi d\lambda \leq 2\int \omega d\lambda.$$

Since this holds for all non-negative continuous ω, we have $\phi \leq 2$ a.e. ∎

Put

$$\alpha = \phi\ \ d\lambda.$$

We shall prove the following:

Proposition 2.6. $\alpha \perp \sigma$.

For the proof of the proposition 2.6 we need the following classical lemma [**K-S**].

Lemma 2.7. *Let ρ, τ be two nonnegative finite measure on a measurable space X. Then the following properties are equivalent :*
(i) $\rho \perp \tau$
(ii) *Given $\varepsilon > 0$, there exists a nonnegative measurable function f on X such that $f > 0$, $\tau - a.e.$ and such that*

$$\Big(\int f d\rho\Big)\Big(\int \frac{d\tau}{f}\Big) < \varepsilon.$$

Proof of Proposition 2.6 : Let $\beta_1 = \inf\{\int Q d\alpha, \ : \ Q = \prod_{i=1}^{k} |P_{n_i}(z)|, k \in \mathbb{N}, n_1 < n_2 < \cdots < n_k\}$ and $\beta_2 = \inf\{\int Q d\lambda, \ : \ Q = \prod_{i=1}^{k} |P_{n_i}(z)|, k \in \mathbb{N}, n_1 < n_2 < \cdots < n_k\}$. Then, for any Q, we have

$$\int Q d\alpha \geq \beta_1, \quad \liminf \int Q|P_m| d\lambda \geq \beta_2$$

Combine lemma 2.3, proposition 2.4 and proposition 2.5 and take the inf over all Q to obtain:

$$\beta_2 \leq \beta_2 - \frac{1}{8}\beta_1^2$$

It follows that

$$\beta_1 = 0.$$

We claim: $\beta_1 = 0$ implies $\int \prod_{j=1}^{k} |P_j| d\alpha \xrightarrow[k\to\infty]{} 0$. In fact, let $\mathcal{N} = \{n_1 < n_2 < \cdots < n_k\} \subset \mathbb{N}^*$ and $N > n_k$. Then, by Cauchy-Schwarz inequality, we have

$$\begin{aligned} \int \prod_{j=1}^{N} |P_j| d\alpha &= \int \sqrt{\prod_{j\in\mathcal{N}} |P_j|}\sqrt{\prod_{j\in\mathcal{N}} |P_j|} \prod_{j\notin\mathcal{N}} |P_j| d\alpha \\ &\leq \Big(\int \prod_{j\in\mathcal{N}} |P_j| d\alpha\Big)^{\frac{1}{2}} \Big(\int \prod_{j\in\mathcal{N}} |P_j| d\alpha \prod_{j\notin\mathcal{N}} |P_j|^2 d\alpha\Big)^{\frac{1}{2}} \end{aligned} \tag{1.2}$$

But

$$\begin{aligned}\int \prod_{j\in\mathcal{N}} |P_j| \prod_{j\notin\mathcal{N}} |P_j|^2 d\alpha &\leq 2\int \prod_{j\in\mathcal{N}} |P_j| \prod_{j\notin\mathcal{N}} |P_j|^2 d\lambda \\ &\leq 2\int \prod_{j=1}^{N} |P_j| \prod_{j\notin\mathcal{N}} |P_j| d\lambda \\ &\leq 2\left(\int \prod_{j=1}^{N} |P_j|^2 d\lambda\right)^{\frac{1}{2}} \left(\int \prod_{j\notin\mathcal{N}} |P_j|^2 d\lambda\right)^{\frac{1}{2}} \\ &\leq 2 \qquad (1.3)\end{aligned}$$

Combine (1.2) and (1.3) to get the claim. The proof of the proposition follows from lemma 2.7. ■

3. Estimation of L_1-Norm of $(|P_m(z)| - 1)$ and the Central Limit Theorem.
In this section we assume that for m sufficiently large

$$\begin{aligned}&(i) \quad a_{j+1}^{(m)} \geq 2s_m(j), j = 0, \cdots, p_m - 1, \\ &\text{with} \quad s_m(j) = a_1^{(m)} + \ldots + a_j^{(m)}, s_m(0) = 0, \\ &(ii) \quad \frac{s_m(p_m)}{h_m} < \frac{1}{2}\end{aligned}$$

Under the above assumptions, we shall proof the following:

Proposition 3.1. $\alpha \geq K\lambda$, for some positive constant K.

The proof of the proposition is based on an estimate of $\int_A ||P_m| - 1| d\lambda$, where A is a Borel set with $\lambda(A) > 0$. More precisely we shall study the stochastic behavior of the sequence $|P_m|$. For that our principal strategy is based on the method of the central limit theorem for trigonometric sums. A nice account can be found in [**K**]. It is well-known that Hadamard lacunary trigonometric seires satisfies the central limit theorem [**Z, p.263-264**]. The central limit theorem for trigonometric sums has been studied by many authors, Zygmund and Salem [**Z, p.263-264**], Erdös [**E**], J.P. Kahane [**Ka**], Berkers [**Be**], Murai [**Mu**], Takahashi [**T**], Fukuyama and Takahashi [**F-T**], and many others. The same method is used to study the asymptotic behavior of Riesz-Raikov sums [**P**].

Here, we shall prove the following:

Proposition 3.2. If (i) and (ii) hold then for any Borel subset A of $\mathbb{T}$ with $\lambda(A) > 0$, the distribution of the sequence of random variables $\frac{\sqrt{2}}{\sqrt{p_m}} \sum_{j=0}^{p_m-1} \cos((jh_m + s_m(j))t)$ converges to the Gauss distribution. That is

$$\frac{1}{\lambda(A)}\left\{t \in A \quad : \quad \frac{\sqrt{2}}{\sqrt{p_m}} \sum_{j=0}^{p_m-1} \cos((jh_m + s_m(j))t) \leq x\right\}$$
$$\xrightarrow[m\to\infty]{} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{1}{2}t^2} dt \stackrel{\text{def}}{=} \mathcal{N}\left(]-\infty, x]\right). \tag{1.4}$$

The proof of proposition 3.2 is based on the idea of the proof of martingale central limit theorem due to McLeish [**Mc**]. The main ingredient is the following .

Lemma 3.3. For $n \geq 1$, let U_n, T_n be random variables such that

1. $U_n \longrightarrow a$ in probability.
2. $\{T_n\}$ is uniformly integrable.
3. $\{|T_n U_n|\}$ is uniformly integrable.
4. $\mathbb{E}(T_n) \longrightarrow 1$.

Then $\mathbb{E}(T_n U_n) \longrightarrow a$.

Proof : Write $T_n U_n = T_n(U_n - a) + aT_n$. As $\mathbb{E}(T_n) \longrightarrow 1$, we need to show that $\mathbb{E}(T_n(U_n - a)) \longrightarrow 0$.
Since $\{T_n\}$ is uniformly integrable, we have $T_n(U_n - a) \longrightarrow 0$ in probability. Also, both $T_n U_n$ and aT_n are uniformly integrable, and so the combination $T_n(U_n - a)$ is uniformly integrable. Hence, $\mathbb{E}(T_n(U_n - a)) \longrightarrow 0$. ■

Let us recall the following expansion

$$\exp(ix) = (1 + ix)\exp(-\frac{x^2}{2} + r(x)),$$

where $|r(x)| \leq |x|^3$, for real x.

Theorem 3.4. Let $\{X_{nj} \quad : \quad 1 \leq j \leq k_n, n \geq 1\}$ be a triangular array of random variables. $S_n = \sum_{j=1}^{k_n} X_{nj}$, $T_n = \prod_{j=1}^{k_n}(1 + itX_{nj})$, and $U_n = \exp\left(-\frac{t^2}{2}\sum_j X_{nj}^2 + \sum_j r(tX_{nj})\right)$. Suppose that

1. $\{T_n\}$ is uniformly integrable.
2. $\mathbb{E}(T_n) \longrightarrow 1$.
3. $\sum_j X_{nj}^2 \longrightarrow 1$ in probability.
4. $\max|X_{nj}| \longrightarrow 0$ in probability.

Then $\mathbb{E}(\exp(itS_n)) \longrightarrow \exp(-\frac{t^2}{2})$.

Proof : Let t be fixed. From condition (3) and (4),

$$|\sum_j r(tX_{nj})| \leq |t|^3 \sum_j |X_{nj}|^3 \leq |t|^3 \max_j |X_{nj}| \sum_j X_{nj}^2 \xrightarrow[n\to\infty]{} 0 \text{ in probability.}$$

$U_n = \exp\left(-\frac{t^2}{2}\sum_j X_{nj}^2 + \sum_j r(tX_{nj})\right) \longrightarrow \exp(-\frac{t^2}{2})$ in probability as $n \longrightarrow +\infty$. This verifies the condition (1) of Lemma 3.3 with $a = \exp(-\frac{t^2}{2})$. It is easy to check that the conditions (2) ,(3) and (4) of the lemma 3.3 hold. Thus $E\left(\exp(itS_n)\right) = E\left(T_nU_n\right) \longrightarrow \exp(-\frac{t^2}{2})$. ■

Let m be a positive integer and put

$$W_m \stackrel{\text{def}}{=} \{(\sum_{i\in I} \varepsilon_i p_i)h_m + \sum_{i\in I} \varepsilon_i s_m(p_i) \ : \ \varepsilon_i \in \{0,-1,1\}, I \subset \{0,\cdots,p_m-1\}, p_i \in \{0,\cdots,p_m-1\}\}.$$

The element $w = (\sum_{i\in I} \varepsilon_i p_i)h_m + \sum_{i\in I} \varepsilon_i s_m(p_i)$ is called a word.

We shall need the following combinatorial lemma.

Lemma 3.5. Under the assumptions (i) and (ii) of theorem 2.1, all the words of W_m are distinct.

Proof : Let $w, w' \in W_m$, write

$$\begin{aligned} w &= (\sum_{i\in I} \varepsilon_i p_i)h_m + \sum_{i\in I} \varepsilon_i s_m(p_i) \\ w' &= (\sum_{i\in I'} \varepsilon'_i p'_i)h_m + \sum_{i\in I} \varepsilon'_i s_m(p'_i). \end{aligned}$$

Then $w = w'$ implies

$$\{(\sum_{i\in I} \varepsilon_i p_i) - (\sum_{i\in I'} \varepsilon'_i p'_i)\}h_m = \sum_{i\in I'} \varepsilon'_i s_m(p'_i) - \sum_{i\in I} \varepsilon_i s_m(p_i)$$

But the modulus of LHS is greater than h_m and the modulus of RHS is less than $2\sum_{j=0}^{p_m-1} a_j^{(m)}$. It follows that

$$\begin{aligned} \sum_{i\in I} \varepsilon_i p_i &= (\sum_{i\in I'} \varepsilon'_i p'_i) \quad \text{and} \\ \sum_{i\in I} \varepsilon_i s_m(p_i) &= \sum_{i\in I'} \varepsilon'_i s_m(p'_i). \end{aligned}$$

Since from (i) we have $s_m(p+1) \geq 3s_m(p)$, we get that the representation in the form $\sum_{i\in I} \varepsilon_i s_m(p_i)$ is unique and the proof of the lemma is complete. ■

Proof of proposition 3.2 : Let A a Borel set, $\lambda(A) > 0$. Using the Helly theorem we may assume that the sequence $\frac{\sqrt{2}}{\sqrt{p_m}}\sum_{j=0}^{p_m-1}\cos((jh_m+s_m(j))t)$ converge in distribution. As is well-known, it is sufficient to show that for every real number x,

$$\frac{1}{\lambda(A)}\int_A \exp\left\{-ix\frac{\sqrt{2}}{\sqrt{p_m}}\sum_{j=0}^{p_m-1}\cos((jh_m+s_m(j))t)\right\}dt \xrightarrow[m\to\infty]{} \exp(-\frac{x^2}{2}).$$

To this end we apply theorem 3.4. in the following context. The measure space is the given Borel set A of positive Lebesgue measure in the circle with the normalised measure and the random variables are given by

$$X_{mj} = \frac{\sqrt{2}}{\sqrt{p_m}}\cos((jh_m+s_m(j))t), \qquad \text{where} \quad 0 \le j \le p_m-1,\ m \in \mathbb{N}.$$

It is easy to check that the variables $\{X_{mj}\}$ satisfy conditions (1) and (4). Further, condition (3) follows from fact that

$$\int_0^{2\pi}\left|\sum_{j=0}^{p_m-1}X_{mj}^2 - 1\right|^2 dt \xrightarrow[m\to\infty]{} 0.$$

It remains to verify comdition (2) of theorem 3.4. it is sufficient to show that

$$\int_A \prod_{j=0}^{p_m-1}\left(1 - ix\frac{\sqrt{2}}{\sqrt{p_m}}\cos((jh_m+s_m(j))t)\right)dt \xrightarrow[m\to\infty]{} \lambda(A). \tag{1.5}$$

Write

$$\begin{aligned}\Theta_m(x,t) &= \prod_{j=0}^{p_m-1}\left(1 - ix\frac{\sqrt{2}}{\sqrt{p_m}}\cos((jh_m+s_m(j))t\right)\\ &= 1+\sum_{w=1}^{N_m}\rho_w{}^{(m)}(x)\cos(wt),\end{aligned}$$

where $\rho_w = 0$ if w is not of the form $(\sum_{i\in I}\varepsilon_i p_i)h_m + \sum_{i\in I}\varepsilon_i s_m(p_i)$, $N_m = \left\{\frac{p_m(p_m-1)}{2}\right\}h_m + s_m(p_m-1) + s_m(p_m-2) + \cdots + 1$.

We claim that it is sufficient to prove the following:

$$\int_0^{2\pi} R(t)\prod_{j=0}^{p_m-1}\left(1 - ix\frac{\sqrt{2}}{\sqrt{p_m}}\cos((jh_m+s_m(j))t)\right)dt \xrightarrow[m\to\infty]{} \int_0^{2\pi} R(t)dt, \tag{1.6}$$

for any trigonometric polynomial R. In fact, assume that (1.6) holds and let $\epsilon > 0$. Then, by the density of trigonometric polynomials, one can find a trigonometric polynomial R_ϵ such that

$$||\chi_A - R_\epsilon||_1 < \epsilon,$$

where χ_A is indicator function of A. But

$$\left|\prod_{j=0}^{p_m-1}\left(1-ix\frac{\sqrt{2}}{\sqrt{p_m}}\cos((jh_m+s_m(j))t)\right)\right| \leq \left\{\prod_{j=0}^{p_m-1}\left(1+\frac{2x^2}{p_m}\right)\right\}^{\frac{1}{2}},$$

Since $1+u \leq e^u$, we get

$$\mid \Theta_m(x,t) \mid \leq e^{x^2}. \tag{1.7}$$

Hence, according to (1.6), for m sufficiently large, we have

$$\left|\int_A \Theta_m(x,t)dt - \lambda(A)\right| = \mid \int_A \Theta_m(x,t)dt - \int_0^{2\pi} \Theta_m(x,t)R_\epsilon(t)dt +$$
$$\int_0^{2\pi} \Theta_m(x,t)R_\epsilon(t)dt - \int_0^{2\pi} R_\epsilon(t)dt + \int_0^{2\pi} R_\epsilon(t)dt - \lambda(A)\mid < e^{x^2}\epsilon + 2\epsilon.$$

The proof of the claim is complete. It still remains to prove (1.6). Observe that

$$\int_0^{2\pi} \Theta_m(x,t)R(t)dt = \int_0^{2\pi} R(t)dt + \sum_{w=1}^{N_m} {\rho_w}^{(m)}(x)\int_0^{2\pi} R(t)\cos(wt)dt$$

and for $w = p_{i_1}h_m + s_m(p_{i_1}) + \sum_{j=1}^r \varepsilon_j\{(p_{i_j}h_m) + s_m(p_{i_j})\}$, we have

$$|{\rho_w}^{(m)}(x)| \leq \frac{2^{1-r}|x|^r}{p_m^{\frac{r}{2}}},$$

hence

$$\max_{w\in W} |{\rho_w}^{(m)}(x)| \leq \frac{|x|}{p_m^{\frac{1}{2}}} \xrightarrow[m\to\infty]{} 0.$$

Since $\displaystyle\sum_{w\in\mathbb{Z}} \mid \int_0^{2\pi} e^{-iwt}R(t)dt\mid$ is bounded, we deduce

$$\mid \sum_{w=1}^{N_m} {\rho_w}^{(m)}(x)\int_0^{2\pi} R(t)\cos(wt)dt\mid \leq \frac{|x|}{p_m^{\frac{1}{2}}}\sum_{w\in\mathbb{Z}} \mid \int_0^{2\pi} e^{-iwt}R(t)dt\mid \xrightarrow[m\to\infty]{} 0.$$

The proof of the proposition 3.2. is complete. ■

Proof of proposition 3.1 : Let A be a Borel subset of $\mathbb{T}$, and $x \in]1,+\infty[$, then, for any positive integer m, we have

$$\begin{aligned}\int_A ||P_m(\theta)|-1|d\lambda(\theta) &\geq \int_{\{\theta\in A \;:\; |P_m(\theta)|>x\}} ||P_m|-1|d\lambda(\theta)\\ &\geq (x-1)\lambda\{\theta\in A \;:\; |P_m(\theta)|>x\}\\ &\geq (x-1)\lambda\{\theta\in A \;:\; |\Re(P_m(\theta))|>x\}\end{aligned}$$

Let m go to infinity and use propositions 2.5 and 3.2 to get

$$\int_A \phi d\lambda \geq (x-1)\{1-\mathcal{N}([-\sqrt{2}x,\sqrt{2}x])\}\lambda(A).$$

Put $K=(x-1)\{1-\mathcal{N}([-\sqrt{2}x,\sqrt{2}x])\}$. Hence $\alpha(A) \geq K\lambda(A)$, for any Borel subset A of $\mathbb{T}$ which proves the proposition. ∎

Now, We give the proof of our main result.

Proof of theorem 2.1 : Follows easily from the proposition 2.6 combined with proposition 3.1. ∎

Let us mention that the same proof works for the following more general statement.

Theorem 3.6. *Let $T = T_{(p_k,(a_j^{(k)})_{j=1}^{p_k})_{k=0}^{\infty}}$ be a rank one transformation such that,*

$$(i)\quad a_{j+1}^{(k)} \geq 2s_k(j), j=0,\cdots,p_k-1,$$
$$\text{with}\quad s_k(j)=a_1^{(k)}+\ldots+a_j^{(k)}, s_k(0)=0,$$
$$(ii)\quad \frac{s_k(p_k)-p_k\min_{1\leq j\leq p_k}(a_j^{(k)})}{h_k+\min_{1\leq j\leq p_k}(a_j^{(k)})} < \frac{1}{2}$$

then the spectrum of T is singular.

Remark. We note that, in [**B**] and [**Kl**], the strategy of the authors is to show that the absolutely continuous measure β, obtained as the limit of some subsequence of the sequence $(||P_m|^2-1|d\lambda)_{m\geq 0}$, is equivalent to Lebesgue measure, in fact the authors prove that

$$\beta \geq K\lambda, \qquad (E)$$

for some $K>0$. In the case of Ornstein transformations, the relation (E) hold almost surely.

4. Simple Proof Of Bourgain Theorem.

Bourgain Theorem deals with Ornstein transformations. In Ornstein's construction, the p_k's are rapidly increasing, and the number of spacers, $a_i^{(k)}$, $1 \leq i \leq p_k-1$, are chosen randomly. This may be organized in different ways as pointed out by J. Bourgain in [**B**]. Here we suppose that we are given two sequences (t_k), (p_k) of positive integers and a sequence (ξ_k) of probability measure such that the support of each ξ_k is a subset of $X_k = \{-\frac{t_k}{2},\cdots,\frac{t_k}{2}\}$. We choose now independently, according to ξ_k, the numbers $(x_{k,i})_{i=1}^{p_k-1}$, and x_{k,p_k} is chosen deterministically in $\mathbb{N}$. We put, for $1 \leq i \leq p_k$,

$$a_i^{(k)} = t_k + x_{k,i} - x_{k,i-1}, \quad \text{with} \quad x_{k,0} = 0.$$

We have

$$h_{k+1} = p_k(h_k + t_k) + x_{k,p_k}.$$

So the deterministic sequences of positive integers $(p_k)_{k=0}^{\infty}$, $(t_k)_{k=0}^{\infty}$ and $(x_{k,p_k})_{k=0}^{\infty}$ determine completely the sequence of heights $(h_k)_{k=0}^{\infty}$. The total measure of the resulting measure space is finite if

$$\sum_{k=0}^{\infty} \frac{t_k}{h_k} + \sum_{k=0}^{\infty} \frac{x_{k,p_k}}{p_k h_k} < \infty. \tag{1.8}$$

We will assume that this requirement is satisfied.
We thus have a probability space of Ornstein transformations $\Omega = \prod_{l=0}^{\infty} X_l^{p_l-1}$ equipped with the natural probability measure $\mathbb{P} \stackrel{\text{def}}{=} \otimes_{l=1}^{\infty} P_l$, where $P_l \stackrel{\text{def}}{=} \otimes_{j=1}^{p_l-1} \xi_j$; ξ_j is the probability measure on X_j. We denote this space by $(\Omega, \mathcal{A}, \mathbb{P})$. So $x_{k,i}$, $1 \leq i \leq p_k - 1$, is the projection from Ω onto the i^{th} co-ordinate space of $\Omega_k \stackrel{\text{def}}{=} X_k^{p_k-1}$, $1 \leq i \leq p_k - 1$. Naturally each point $\omega = (\omega_k = (x_{k,i}(\omega))_{i=1}^{p_k-1})_{k=0}^{\infty}$ in Ω defines the spacers and therefore a rank one transformation $T_{\omega,x}$, where $x = (x_{k,p_k})$.
This construction is more general than the construction due to Ornstein [**O**] which corresponds to the case $t_k = h_{k-1}$, ξ_k is the uniform distribution on X_k and $p_k >> h_{k-1}$.

We recall that Ornstein in [**O**] proved that there exists a sequence $(p_k, x_{k,p_k})_{k\in\mathbb{N}}$ such that, $T_{\omega,x}$ is almost surely mixing. Later in [**Pr**] Prikhod'ko obtains the same result for some special choice of the sequence of the distribution (ξ_m) and recently, using the idea of D. Creutz and C. E. Silva [**C-S**] one can extend this result to a large family of probability measures associated to Ornstein construction. In our general construction, according to (1.1) the spectral type of each T_ω , up to a discrete measure, is given by

$$\sigma_{T_\omega} = \sigma_{\chi_{B_0}}^{(\omega)} = \sigma^{(\omega)} = W^* \lim \prod_{l=1}^{N} \frac{1}{p_l} \left| \sum_{p=0}^{p_l-1} z^{p(h_l+t_l)+x_{l,p}} \right|^2 d\lambda.$$

With the above notations, we state Bourgain theorem in the following form:

Theorem 4.1 ([EA-P-P]). *For every choice of* $(p_k), (t_k), (x_{k,p_k})$ *and for any family of probability measures* ξ_k *on* $X_k = \{-t_k, \cdots, t_k\}$ *of* $\mathbb{Z}$, $k \in \mathbb{N}^*$, *for which*

$$\sum_{s\in X_m} \xi(s)^2 \xrightarrow[m\to\infty]{} 0,$$

the associated generalized Ornstein transformations has almost surely singular spectrum. i.e.,

$$\mathbb{P}\{\omega \ : \ \sigma^{(\omega)} \perp \lambda\} = 1.$$

where $\mathbb{P} \stackrel{\text{def}}{=} \otimes_{l=0}^{\infty} \otimes_{j=1}^{p_l-1} \xi_l$ *is the probability measure on* $\Omega = \prod_{l=0}^{\infty} X_l^{p_l-1}$.

In the context of Ornstein's constrution, we state proposition 2.5 in the following form:

Proposition 4.2. *There exist a subsequence of the sequence* $(||P_m(z)|-1|)$ *which converge weakly to some non-negative function* $\phi(\omega,\theta)$ *which satisfies* $\phi \leq 2$*, almost surely with respect the* $\mathbb{P} \bigotimes \lambda$.

Proof : Easy exercise. ∎

Put, for any $\omega \in \Omega$

$$\alpha_\omega = \phi(\omega,\theta)d\lambda.$$

We shall prove that α_ω is equivalent to the Lebesgue measure for almost all ω. In fact, we have the following proposition:

Proposition 4.3. *There exist an absolutely positive constant* K *such that for almost all* ω *we have*

$$\alpha_\omega \geq K\lambda.$$

The proof is based on the following two lemmas.

Lemma 4.4. $lim \displaystyle\int\int ||P_m| - |P'_m||d\theta d\mathbb{P} \xrightarrow[n\to\infty]{} 0$, *where* $P'_m(\theta) = P_m(\theta) - \displaystyle\int_\Omega P_m(\theta)d\mathbb{P}$.

Proof : For any $m \in \mathbb{N}$, we have

$$\begin{aligned}
\int\int ||P_m| - |P'_m||d\theta d\mathbb{P} &\leq \int\int |P_m - P'_m|d\theta d\mathbb{P} \\
&= \int |\int P_m d\mathbb{P}|d\theta \\
&\leq \int |\sum_{p=0}^{p_m-1} \frac{1}{\sqrt{p_m}} z^{p(h_m+t_m))}|| \sum_{s\in X_m} \xi_m(s) z^s|dz.
\end{aligned}$$

Hence by Cauchy-Schwarz inequality

$$\int\int ||P_m| - |P'_m||d\theta d\mathbb{P} \leq \sum_{s\in X_m} \xi(s)^2 \xrightarrow[m\to\infty]{} 0.$$

The proof of the lemma is complete. ∎

Now observe that we have

$$\int_T |\sum_{s\in X_m} \xi_m(s) z^s|^2 dz = \int_T |\sum_{s\in X_m} \xi_m(s) z^{2s}|^2 dz = \sum_{s\in X_m} \xi(s)^2 \xrightarrow[m\to\infty]{} 0.$$

So, we may extract a subsequence (m_k) for which, for almost all $t \in [0, 2\pi)$, we have

$$\sum_{s\in X_{m_k}} \xi_{m_k}(s) e^{ist} \xrightarrow[k\to\infty]{} 0 \text{ and } \sum_{s\in X_{m_k}} \xi_{m_k}(s) e^{is2t} \xrightarrow[k\to\infty]{} 0$$

Define

$$\Theta \stackrel{def}{=} \{\theta \ : \ \sum_{s\in X_{m_k}} \xi_{m_k}(s) e^{is\theta} \xrightarrow[k\to\infty]{} 0 \text{ and } \sum_{s\in X_{m_k}} \xi_{m_k}(s) e^{is2t} \xrightarrow[k\to\infty]{} 0\}$$

Choose $m \in \{m_k\}$, $t \in \Theta$ and put, for $j \in \{0, \cdots, p_m - 1\}$,

$$Y_{m,j}(\omega) = \cos((j(h_m + t_m) + x_{m,j}(\omega))t) - \int \cos((j(h_m + t_m) + x_{m,j}(\omega))t) d\mathbb{P},$$

$$Z_{m,j}(\omega) = \sqrt{\frac{2}{p_m}} Y_{m,j}(\omega).$$

Lemma 4.5. *For any fixed $t \in \Theta$, the distribution of the sequence of random variables $\sum_{j=0}^{p_m-1} Z_{m,j}(\omega)$ converges to the Gauss distribution. That is,*

$$\mathbb{P}\left\{\omega \in \Omega \ : \ \sum_{j=0}^{p_m-1} Z_{m,j}(\omega) \leq x\right\} \xrightarrow[m\to\infty]{} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{1}{2}u^2} du \stackrel{\text{def}}{=} \mathcal{N}(]-\infty, x]).$$

Proof : Since the random variables are independent, centred and uniformly bounded by $\frac{2\sqrt{2}}{\sqrt{p_m}}$, the conditions (1), (2) and (4) of the theorem 3.4 are satisfied. We have also the following:

$$\mathbb{E}\left\{\left(\sum_{j=1}^{p_m-1} Z_{m,j}^2 - 1\right)^2\right\}$$
$$= \frac{4(p_m-1)}{p_m^2}\mathbb{E}(Y_{m,1}^4) + \frac{(p_m-1)(p_m-2)}{p_m^2}(\mathbb{E}(2Y_{m,1}^2))^2 - 2\frac{p_m-1}{p_m}\mathbb{E}(2Y_{m,1}^2) + 1.$$

But $\mathbb{E}(2Y_{m,1}^2) \xrightarrow[m\to\infty]{} 1$, it follows that the variables $\{Z_{m,j}\}$ satisfy condition (3) of theorem 3.4. Thus all the conditions of theorem 3.4 hold and we conclude that the distribution of $\sum_{j=0}^{p_m-1} Z_{m,j}(\omega)$ converges to normal distribution. ■

Proof of proposition 4.3 : Let A be a Borel subset of $\mathbb{T}$, C a cylinder set in Ω, and $x \in]1, +\infty[$, then, for any positive integer m, we have

$$
\begin{aligned}
&\int_{A\times C} ||P_m(\theta)| - 1| d\lambda(\theta) d\mathbb{P} \\
\geq\ & \mathbb{P}(C) \int_{A\times\Omega} ||P'_m(\theta)| - 1| d\lambda(\theta) d\mathbb{P} - \int ||P_m| - |P'_m|| d\mathbb{P} d\lambda \\
\geq\ & \mathbb{P}(C) \int_{\{|P'_m|>x\}\bigcap A\times\Omega} ||P'_m| - 1| d\lambda(\theta) d\mathbb{P} - \int ||P_m| - |P'_m|| d\mathbb{P} d\lambda \\
\geq\ & \mathbb{P}(C)(x-1) \int_A \mathbb{P}\{|\Re(P'_m(\theta))| > x\} d\lambda - \int ||P_m| - |P'_m|| d\mathbb{P} d\lambda
\end{aligned}
$$

Let m go to infinity and combine lemmas 4.4 and 4.5 to get

$$
\int_{A\times C} \phi d\lambda d\mathbb{P} \geq (x-1)\{1 - \mathcal{N}([-\sqrt{2}x, \sqrt{2}x])\}\lambda(A)\mathbb{P}(C).
$$

Put $K = (x-1)\{1 - \mathcal{N}([-\sqrt{2}x, \sqrt{2}x])\}$. Hence, for almost all ω, we have, for any Borel set $A \subset \mathbb{T}$, $\alpha_\omega(A) \geq K\lambda(A)$, and the proof of the proposition is complete. ■

Proof of theorem 4.1 : Follow easily from the proposition 4.3 combined with the proposition 2.6. ■

Remark. We point out that there exist a rank one mixing transformations on the space with finite measure satisfying the condition of theorem 3.6, In fact, following the notations of section 4, one may define the spacers in the Ornstein construction by

$$
a_j^{(k)} = 3^j t_k + x_{k,j} - x_{k,j-1},
$$

and choose the sequence $(t_k)_{k\in\mathbb{N}}$ such that the measure of dynamical system is finite. Thus the condition of theorem 3.6 hold and the class is mixing almost surely.

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References

[A] T. R. Adams, *On Smorodinsky conjecture,* Proc. Amer. Math. Soc. 126 (1998), 739–744.

[AF] T. R. Adams & N. A. Friedman, *Mixing staircase,* Preprint, 1992.

[Be] I. Berkes, *On the central limit theorem for lacunary trigonometric series,* Anal. Math. 4 (1978), no. 3, 159–180.

[B] J. Bourgain, *On the spectral type of Ornstein class one transfor-mations, Isr. J. Math.* ,**84** (1993), 53-63.

[C-N1] J. R. Choksi and M. G. Nadkarni , *The maximal spectral type of rank one transformation, Can. Math. Bull.,* **37 (1)** (1994), 29-36.

[C-N3] J. R. Choksi and M. G. Nadkarni, *On the question of transformations with simple Lebesgue spectrum.*, Lie groups and ergodic theory (Mumbai, 1996), 33–57, Tata Inst. Fund. Res. Stud. Math., 14, Tata Inst. Fund. Res., Bombay, 1998.

[C-S] D. Creutz and C. E. Silva, *Mixing on a class of rank-one transformations,* Ergod. Th. & Dynam. Sys. 24 (2004), no. 2, 407–440.

[E-D] A. H. Dooley and S. J. Eigen, *A family of generalized Riesz products.* Canad. J. Math. 48 (1996), no. 2, 302–315.

[EA] E. H. El Abdalaoui, *La singularite mutuelle presque sûre du spectre des transformations d'Ornstein*, Israel J. Math. 112 (1999), 135–155.

[EA-P-P] E. H. El Abdalaoui, F. Parreau and A. A. Prikhod'ko, *A new class of Ornstein transformations with singular spectrum*, Ann. Inst. H. Poincaré, to appear.

[E] P. Erdös, *On trigonometric sums with gaps.* Magyar Tud. Akad. Mat. Kutató Int. Közl 7 1962 37–42.

[F-T] K. Fukuyama and S. Takahashi, *The central limit theorem for lacunary series.* Proc. Amer. Math. Soc. 127 (1999), no. 2, 599–608.

[Fr1] N. A. Friedman, *Introduction to ergodic theory*, Van Nostrand Reinhold, New York,(1970) .

[Fr2] N. A. Friedman, *Replication and stacking in ergodic theory.* Amer. Math. Monthly **99** (1992), 31-34.

[P] B. Petit, *Le théorème limite central pour des sommes de Riesz-Raĭkov.* Probab. Theory Related Fields 93 (1992), no. 4, 407–438.

[Pr] A. A. Prikhod'ko, *Stochastic constructions of flows of rank 1*, Sb. Math. 192 (2001), no. 11-12, 1799-1828.

[K] M. Kac, *Statistical independence in probability, analysis and number theory.* The Carus Mathematical Monographs, No. 12 Published by the Mathematical Association of America. Distributed by John Wiley and Sons, Inc., New York, (1959).

[Ka] J. -P. Kahane, *Lacunary Taylor and Fourier series.* Bull. Amer. Math. Soc. 70 1964 199–213.

[K-S] S. J. Kilmer and S. Saeki, *On Riesz Products Measures, mutual absolute continuity and singularity,* Annales de l'Institut Fourier, Grenoble 38 (1988), 63–93.

[Kl] I. Klemes, *The spectral type of staircase transformations, Thohoku Math. J.,* **48** (1994), 247-258.

[K-R] I. Klemes & K. Reinhold, *Rank one transformations with singular spectre type, Isr. J. Math.,* **98** (1997), 1-14.

[Mc] D. L. McLeish, *Dependent central limit theorems and invariance principles.* Ann. Probability 2 (1974), 620–628.

[Mu] T. Murai, *The central limit theorem for trigonometric series.* Nagoya Math. J. 87 (1982), 79–94.

[N] M. G. Nadkarni, *Spectral theory of dynamical systems*, Birkhäuser, Cambridge, MA, 1998. .

[O] D. S. Ornstein, *On the root problem in ergodic theory, Proc. Sixth Berkeley Symposium in Math. Statistics and Probability,* University of California Press, 1971, 347-356.

[T] S. Takahashi, *Probability limit theorems for trigonometric series.* Limit theorems of probability theory (Colloq., Keszthely, 1974), pp. 381–397. Colloq. Math. Soc. Janos Bolyai, Vol. 11, North-Holland, Amsterdam, 1975.

[Z, p.263-264] A. Zygmund, *Trigonometric series* vol. II, second ed., Cambridge Univ. Press, Cambridge, 1959.