

FINITE REPRESENTATIONS OF THE BRAID GROUP COMMUTATOR SUBGROUP

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ABSTRACT. We study the representations of the commutator subgroup K_n of the braid group B_n into a finite group Σ . This is done through a symbolic dynamical system. Some experimental results enable us to compute the number of subgroups of K_n of a given (finite) index, and, as a by-product, to recover the well known fact that every representation of K_n into S_r , with $n > r$, must be trivial.

1. INTRODUCTION

In [SiWi1], D. Silver and S. Williams exploited the structure of the kernel subgroup K of an epimorphism $\chi : G \rightarrow \mathbb{Z}$, where G is a finitely presented group, to show that the set $Hom(K, \Sigma)$ of representations of K into a finite group Σ has a structure of a subshift of finite type (SFT), a symbolic dynamical system described by a graph Γ ; namely, there is a one to one correspondence between representations $\rho : K \rightarrow \Sigma$ and bi-infinite paths in Γ .

We apply this method to the group B_n of braids with n -strands, with χ being the abelianization homomorphism and Σ the symmetric group S_r of degree r or the special linear group over a finite field $SL_2(F_q)$. The subgroup $K_n = \ker \chi$ is then the commutator subgroup of B_n .

It is a well known fact that for a given group K , there is a finite to one correspondence between its subgroups of index no greater than r and representations $\rho : K \rightarrow S_r$. This correspondence can be described by

$$\rho \longmapsto \{g \in K : \rho(g)(1) = 1\}.$$

The pre-image of a subgroup of index exactly r consists of $(r-1)!$ transitive representations ρ . (ρ is said to be transitive if $\rho(K)$ operates transitively on $\{1, 2, \dots, r\}$). This will allow us to draw some conclusions about the subgroups of finite index of K_n , which motivates the choice of S_r ; we motivate that of $SL_2(F_q)$ by the fact that any representation of K_n into any group Σ has range in the commutator subgroup $[\Sigma, \Sigma]$, for $n \geq 6$.

We give an algorithm to compute $Hom(K_n, \Sigma)$, for $n \geq 5$. Some experimental results enable us to compute the number of subgroups of K_n of a given (finite) index, and, as a by-product, to recover the well known fact that $Hom(K_n, S_r)$ is trivial for $n \geq 5$ and $r < n$. Since every representation in $Hom(B_n, \Sigma)$ restricts to

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an element of $\text{Hom}(K_n, \Sigma)$, we enhance the given algorithm in order to compute $\text{Hom}(B_n, \Sigma)$.

2. GENERALITIES

Let B_n be the braid group given by the presentation:

$$\langle \sigma_1, \dots, \sigma_{n-1} \mid \begin{array}{l} \sigma_i \sigma_j = \sigma_j \sigma_i; \quad |i - j| \geq 2 \\ \sigma_i \sigma_j \sigma_i = \sigma_j \sigma_i \sigma_j; \quad |i - j| = 1 \end{array} \rangle,$$

(see [BuZi] for additional background). Let $\beta \in B_n$ be a braid. Then β can be written as :

$$\beta = \sigma_{i_1}^{\varepsilon_1} \cdots \sigma_{i_k}^{\varepsilon_k},$$

with $i_1, \dots, i_k \in \{1, \dots, n-1\}$ and $\varepsilon_i = \pm 1$. Define the exponent sum of β (in terms of the σ_i 's) denoted by $\text{exp}(\beta)$, as:

$$\text{exp}(\beta) = \varepsilon_1 + \cdots + \varepsilon_k.$$

Then $\text{exp}(\beta)$ is an invariant of the braid group, that is, it doesn't depend on the writing of β . Moreover, $\text{exp}(\beta) : B_n \rightarrow \mathbb{Z}$ is a homomorphism. Let H_n denote its kernel. Then the Reidemeister-Schreier theorem [LySc] enables us to find a presentation for H_n . We choose the set

$$\{\cdots, \sigma_1^{-m}, \sigma_1^{-m+1}, \cdots, \sigma_1^{-1}, 1, \sigma_1, \sigma_1^2, \cdots, \sigma_1^m, \cdots\}$$

as a Schreier system of right coset representatives of H_n in B_n . Putting $z_m = \sigma_1^m (\sigma_2 \sigma_1^{-1}) \sigma_1^{-m}$ for $m \in \mathbb{Z}$, and $x_i = \sigma_i \sigma_1^{-1}$ for $i = 3, \dots, n-1$, we get the following presentation of H_n :

$$H_n = \left\langle \begin{array}{l} z_m, \quad m \in \mathbb{Z} \\ x_i, \quad i = 3, \dots, n-1 \end{array} \mid \begin{array}{l} x_i x_j = x_j x_i, \quad |i - j| \geq 2; \\ x_i x_j x_i = x_j x_i x_j, \quad |i - j| = 1; \\ z_m z_{m+2} = z_{m+1}, \quad m \in \mathbb{Z}; \\ z_m x_3 z_{m+2} = x_3 z_{m+1} x_3, \quad i = 4, \dots, n-1; \\ z_m x_i = x_i z_{m+1}, \quad m \in \mathbb{Z}. \end{array} \right\rangle$$

Example 1. We have:

$$H_3 = \langle z_m \mid z_m z_{m+2} = z_{m+1}; \forall m \in \mathbb{Z} \rangle$$

is a free group on two generators $z_0 = \sigma_2 \sigma_1^{-1}$ and $z_{-1} = \sigma_1^{-1} \sigma_2$, and

$$H_4 = \langle z_m, t \mid \begin{array}{l} z_m z_{m+2} = z_{m+1}, \\ z_m t z_{m+2} = t z_{m+1} t, \end{array} m \in \mathbb{Z} \rangle.$$

Note that $H_2 = \{1\}$, since $B_2 = \langle \sigma_1 \mid - \rangle \cong \mathbb{Z}$.

Now, every commutator in B_n has exponent sum zero. Conversely, every generator of H_n is a product of commutators. Hence, we have $H_n = K_n$, and exp is the abelianization homomorphism.

Each K_n fits into a split exact sequence:

$$1 \rightarrow K_n \rightarrow B_n \rightarrow \mathbb{Z} \rightarrow 0,$$

and there are "natural" inclusions $B_n \subset B_{n+1}$ and $K_n \subset K_{n+1}$.

Remark 1. We have the following consequences of relations in K_n :

- (1) $z_{m+1} = x_j^{-1} z_m x_j$, hence $z_m = x_j^{-m} z_0 x_j^m$; for $j \geq 4$. where $z_0 = \sigma_2 \sigma_1^{-1}$.
- (2) The z_m 's are conjugate in B_n for $n \geq 3$ (by σ_1) and in K_n for $n \geq 5$.
- (3) The restriction of conjugation by σ_1 in B_n to K_n induces an action of \mathbb{Z} on K_n . This presentation of K_n is said to be \mathbb{Z} -dynamical.

3. THE REPRESENTATION SHIFT

This work is essentially experimental. It aims to describe the set of representations of K_n into finite group Σ . We start with $n = 3$ and describe $Hom(K_3, \Sigma)$ by means of a graph Γ that we will construct in a step by step fashion [SiWi2]. A representation $\rho : K_3 \rightarrow \Sigma$ is a function ρ from the set of generators z_m to Σ such that for each $m \in \mathbb{Z}$, the relation:

$$\rho(z_m) \rho(z_{m+2}) = \rho(z_{m+1})$$

holds in Σ . Any such function can be constructed as follows, beginning with step 0 and proceeding to steps $\pm 1, \pm 2, \dots$

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(step -1) Choose $\rho(z_{-1})$ if possible such that $\rho(z_{-1}) \rho(z_1) = \rho(z_0)$.

(step 0) Choose values $\rho(z_0)$ and $\rho(z_1)$.

(step 1) Choose $\rho(z_2)$ if possible such that $\rho(z_0) \rho(z_2) = \rho(z_1)$.

(step 2) Choose $\rho(z_3)$ if possible such that $\rho(z_1) \rho(z_3) = \rho(z_2)$.

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This process leads to a bi-infinite graph whose vertices are the maps $\rho : \{z_0, z_1\} \rightarrow \Sigma$, each of which can be regarded as an ordered pair $(\rho(z_0), \rho(z_1))$. There is a directed edge from ρ to ρ' if and only if $\rho(z_1) = \rho'(z_0)$ and $\rho(z_0) \rho'(z_1) = \rho(z_1)$. In such a case, we can extend $\rho : \{z_0, z_1\} \rightarrow \Sigma$ by defining $\rho(z_2)$ to be equal to $\rho'(z_1)$. Now if there is an edge from ρ' to ρ'' , we can likewise extend ρ by defining $\rho(z_3)$ to be $\rho''(z_1)$. We implement this process by starting with an ordered pair (a_0, a_1) of elements of Σ , and computing at each step a new ordered pair from the old one, so that every edge in the graph looks like:

$$(a_m, a_{m+1}) \rightarrow (a_{m+1}, a_{m+2}),$$

with

$$a_{m+2} = a_m^{-1} a_{m+1}.$$

In our case, since the group Σ is finite, the process must end, and the graph Γ we obtain consists necessarily of disjoint cycles. This gives an algorithm for finding $Hom(K_3, \Sigma)$. Observe that $Hom(K_3, \Sigma)$ is endowed with a shift map

$$\sigma : \rho \mapsto \sigma(\rho)$$

defined by

$$\sigma(\rho) : x \mapsto \rho(\sigma_1 x \sigma_1^{-1}).$$

If we regard ρ as a bi-infinite path in the graph Γ , then σ correspond to the shift map $(a_m, a_{m+1}) \mapsto (a_{m+1}, a_{m+2})$, since $z_{m+1} = \sigma_1 z_m \sigma_1^{-1}$. Any cycle in the graph Γ with length p corresponds to p representations having least period p . These are the iterates of some representation $\rho \in Hom(K_3, \Sigma)$ satisfying $\rho(z_m) = a_m$ and $\sigma^p(\rho) = \rho$, since $a_{m+p} = a_m$.

Example 2. A cycle of length 2 has the form $(a, a^2) \rightleftharpoons (a^2, a)$, with $a^3 = 1$.

Remark 2. Since K_3 is a free group of rank 2, $\text{Hom}(K_3, \Sigma)$ has a simple description; namely, there is a one to one correspondence between $\text{Hom}(K_3, \Sigma)$ and Σ^2 . The interest of our approach, beside the dynamical structure, is that it allows to go further and describe $\text{Hom}(K_n, \Sigma)$, for $n \geq 3$. As a consequence of the dynamical approach, we get a partition of $\text{Hom}(K_3, \Sigma)$ into orbits, hence the identity $|\Sigma|^2 = 1 + \sum_{p \geq 2} p.n_p$, where n_p is the number of orbits of (least) period p .

Proposition 1. If a cycle has length p , then the identity $a_0 a_1 \dots a_{p-1} = 1$ holds.

In order to minimize calculations, we extract some foreseeable behaviour for various choices of the ordered pair (a_0, a_1) in the previous algorithm.

First, the dynamics of ordered pairs (a_0, a_1) such that $a_0 = 1$ or $a_1 = 1$ or $a_0 = a_1$ is entirely known. To be precise, we get a cycle of length 6 unless $a = a^{-1}$, in which case it is of length 3 (or 1 if and only if $a = 1$).

$$\begin{aligned} (1, a) &\rightarrow (a, a) \rightarrow (a, 1) \rightarrow (1, a^{-1}) \\ &\rightarrow (a^{-1}, a^{-1}) \rightarrow (a^{-1}, 1) \rightarrow (1, a). \end{aligned}$$

This sort of dynamics will be generalized later to representations into abelian groups. Second, when we proceed to a new step, we do not need to take a pair we have already got in a previous cycle, since we would get indeed the same cycle. The following dichotomy will prove useful in the sequel:

Definition 1. If a vertex of a cycle in Γ has equal components, then the cycle is said to be of type I. Otherwise, it is of type II.

Note that a cycle is determined by any of its vertices. Furthermore, the type I cycles are determined by elements of Σ .

Now let us proceed to compute $\text{Hom}(K_4, \Sigma)$. Since $K_3 \subset K_4$, every representation $\rho \in \text{Hom}(K_4, \Sigma)$ restricts to a representation $\rho|_{K_3} \in \text{Hom}(K_3, \Sigma)$, the latter being described by a cycle. All we have to do is then to check which representation in $\text{Hom}(K_3, \Sigma)$ does extend to K_4 . To this end, observe that K_4 is gotten from K_3 by adjunction of a generator x_3 subject to the relations

$$z_m x_3 z_{m+2} = x_3 z_{m+1} x_3; m \in \mathbb{Z}.$$

Hence we may proceed as follows. Take a cycle in $\text{Hom}(K_3, \Sigma)$ (by abuse of language, i.e. identify each representation with its orbit, since a representation in $\text{Hom}(K_3, \Sigma)$ extends to K_4 if and only if every element in its orbit does), and choose if possible a value $b_3 \in \Sigma$ for $\rho(x_3)$. This value must satisfy the relations

$$a_m b_3 a_{m+2} = b_3 a_{m+1} b_3;$$

for $m = 0, \dots, p-1$, where p is the cycle's length and the indexation is $\text{mod } p$. Observe that the choice $b_3 = 1$ is convenient, so all cycles extend to K_4 . However, this is the only possibility for type I cycles to extend, for if b_3 commute with some a_m , then $b_3 = 1$.

Proposition 2. Let $\rho \in \text{Hom}(K_4, \Sigma)$ be encoded by a cycle $(a_m)_{m=0, \dots, p-1}$ and an element $b_3 \in \Sigma$. Then $b_3^p = 1$.

As a consequence, the order of b_3 divides p ; hence, if $\gcd(p, |\Sigma|) = 1$, then $b_3 = 1$.

Before giving the general procedure, let us proceed one further step to show that all type I cycles will vanish for $n \geq 5$. Take a cycle in $\text{Hom}(K_3, \Sigma)$, along with a convenient value b_3 of $\rho(x_3)$. We look for an element $b_4 \in \Sigma$ satisfying :

$$\begin{aligned} a_m b_4 &= b_4 a_{m+1}, \quad m = 0, \dots, p-1; \text{ and} \\ b_3 b_4 b_3 &= b_4 b_3 b_4 \end{aligned}$$

Hence, if

$$b_3 b_4 = b_4 b_3,$$

then

$$b_3 = b_4.$$

Using

$$a_m b_3 a_{m+2} = b_3 a_{m+1} b_3,$$

and

$$a_m b_4 = b_4 a_{m+1},$$

we get

$$b_3 = b_4 = 1,$$

and

$$a_m = a_{m+1}, \forall m = 0, \dots, p-1,$$

so that the representation is trivial. So, except for the trivial cycle, only type II cycles with $b_3 \neq 1$ possibly extend to K_5 , with b_4 not commuting with b_3 , in particular, $b_4 \neq 1$. Note that b_3 and b_4 are conjugate.

Algorithm 1. *The general procedure for $\text{Hom}(K_n, \Sigma)$, $n \geq 5$ is to consider only type II cycles along with convenient non trivial values b_3, \dots, b_{n-2} , which correspond to representations in $\text{Hom}(K_{n-1}, \Sigma)$ and find a non trivial element $b_{n-1} \in \Sigma$ such that the following relations are satisfied:*

$$\begin{aligned} a_m b_{n-1} &= b_{n-1} a_{m+1}, & m &= 0, \dots, p-1; \\ b_{n-1} b_i &= b_i b_{n-1} & i &= 3, \dots, n-3; \\ b_{n-1} b_{n-2} b_{n-1} &= b_{n-2} b_{n-1} b_{n-2} \end{aligned}$$

The element b_{n-1} has to be non trivial, otherwise the representation is trivial.

Using the various relations between the a_m 's and the b_i 's, we get the following ones :

Proposition 3. *Let $\rho \in \text{Hom}(K_n, \Sigma)$, $n \geq 5$ be encoded by a type II cycle $(a_m)_{m=0}^{p-1}$ of length p , along with elements $b_3, \dots, b_{n-1} \in \Sigma$ as previously. Then we have the following relations :*

- (1) The b_i 's are non trivial and conjugate for $i = 3, \dots, n-1$ and $[b_i^p, a_m] = 1$, for $m = 0, \dots, p-1$; $i = 4, \dots, n-1$.
- (2) $\begin{cases} [b_i, b_j] = 1; & |i-j| \geq 2; \\ [b_i, b_j] \neq 1; & |i-j| = 1. \end{cases}$
- (3) $b_i \neq b_j$, for $i, j = 3, \dots, n-1$ and $i \neq j$, except for the possibility $b_3 = b_5$, in which case ρ doesn't extend to $\text{Hom}(K_7, \Sigma)$.

As a consequence of the first relation, p divides the order of b_i , $i = 4, \dots, n-1$; hence that of Σ .

4. THE ABELIAN CASE

Starting with $n = 3$, we see that the relation between the a_m 's, written additively, becomes :

$$a_{m+2} = a_{m+1} - a_m.$$

Implementing our algorithm gives a matrix A :

$$(a, b) \xrightarrow{A} (b, b - a),$$

with $A^3 = -I$. All cycles have length dividing 6. More precisely, the length may be 1 ($a = b = 0$) or 2 ($a = -b$, with $3a = 0$) or 3 ($2a = 2b = 0$) or 6 (otherwise). Moving to $n = 4$, we find that all cycles extend to K_4 with $b_3 = 1$ (only). No non trivial cycle extends to K_5 , that is $Hom(K_n, \Sigma) = 0$, for Σ abelian and $n \geq 5$. This latter fact can also be seen from :

$$\frac{K_n}{[K_n, K_n]} = 1; n \geq 5.$$

5. EXTENSION TO THE BRAID GROUP

In this section, we address the question of extending representations

$$\rho \in Hom(K_n, \Sigma)$$

to representations

$$\tilde{\rho} \in Hom(B_n, \Sigma).$$

Applying [SiWil (3.5)], the extension is possible if and only if there is an element $c \in \Sigma$ such that

$$\begin{aligned} a_m c &= c a_{m+1}, & m &= 0, \dots, p-1; \\ c b_i &= b_i c & i &= 3, \dots, n-1; \end{aligned}$$

Observe that such an element c must satisfy $c \neq 1$, unless ρ is trivial; and $[c^p, a_m] = 1$, for $m = 0, \dots, p-1$. In particular, p divides the order of c . Hence, if $p \nmid |\Sigma|$, then ρ doesn't extend to B_n . Observe also that a necessary condition for a representation $\rho \in Hom(K_n, \Sigma)$ to extend to $\tilde{\rho} \in Hom(B_n, \Sigma)$ is that $\rho \in Hom(K_n, [\Sigma, \Sigma])$. A sufficient condition is that ρ be the restriction of some representation $\hat{\rho} \in Hom(K_{n+2}, \Sigma)$ for if this is the case, the choice $c = b_{n+1}$ will do. In this case, since $\hat{\rho}$ maps K_n into $[\Sigma, \Sigma]$; it also maps K_{n+2} into $[\Sigma, \Sigma]$, for the b_i 's are conjugate (if $n \geq 4$) and $[\Sigma, \Sigma]$ is normal in Σ . As a result, we get the following

Proposition 4. *for $n \geq 6$, $Hom(K_n, \Sigma) = Hom(K_n, \Sigma^{(r)})$, where $r \geq 0$ is the smallest integer such that $\Sigma^{(r+1)} = \Sigma^{(r)}$.*

As a consequence, we get the fact that $n \geq 6$, $Hom(K_n, \Sigma)$ is trivial for any solvable group Σ . This generalizes the abelian case. In this case, every $\rho \in Hom(B_n, \Sigma)$ has a cyclic image generated by $\rho(\sigma_1) = \dots = \rho(\sigma_1)$, i.e. $|Hom(B_n, \Sigma)| = |\Sigma|$.

Actually we can enhance our algorithm to one which gives for fixed $n \geq 5$ the sets $Hom(K_n, \Sigma)$ and $Hom(B_n, \Sigma)$.

Step one: find all cycles of both types. This gives $Hom(K_3, \Sigma)$.

Step two: For the trivial cycle, take any $c \in \Sigma$ to be arbitrary. For a type II cycle C (with length p dividing $|\Sigma|$), find $c \neq 1$ such that $a_m c = c a_{m+1}$, for $m = 0, \dots, p-1$. This gives $Hom(B_3, \Sigma)$.

Step three: For a cycle of any type, take $b_3 = 1$. For a type II cycle C (with length p such that $\gcd(p, |\Sigma|) \neq 1$), find $b_3 \neq 1$ such that $a_m b_3 a_{m+2} = b_3 a_{m+1} b_3$, for $m = 0, \dots, p-1$. This gives $Hom(K_4, \Sigma)$.

Step four: Beside the trivial cycle, take a type II cycle C along with a convenient b_3 (with length p dividing $|\Sigma|$). If this cycle occurs in $\text{Hom}(B_3, \Sigma)$ with some convenient c then :

if $cb_3 = b_3c$ (which need not be checked if $b_3 = 1$), then the representation $[C, b_3]$ moves up to a representation $[C, b_3, c]$ in $\text{Hom}(B_4, \Sigma)$;

if $cb_3c = b_3cb_3$ (which is impossible if $b_3 = 1$), then the representation $[C, b_3]$ moves up to a representation $[C, b_3, b_4]$ in $\text{Hom}(K_5, \Sigma)$ by taking $b_4 = c$.

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Step i: take a representation ρ in $\text{Hom}(K_i, \Sigma)$, encoded by a type II cycle C along with convenient values b_3, \dots, b_{i-1} . if $[C, b_3, b_4, \dots, b_{i-2}, c]$ occurs in $\text{Hom}(B_{i-1}, \Sigma)$ with some convenient c then:

if $cb_{i-1} = b_{i-1}c$, then the representation $[C, b_3, b_4, \dots, b_{i-1}]$ moves up to a representation $[C, b_3, b_4, \dots, b_{i-1}, c]$ in $\text{Hom}(B_i, \Sigma)$;

if $cb_{i-1}c = b_{i-1}cb_{i-1}$, then the representation $[C, b_3, b_4, \dots, b_{i-1}]$ moves up to a representation $[C, b_3, b_4, \dots, b_{i-1}, b_i]$ in $\text{Hom}(K_{i+1}, \Sigma)$ by taking $b_i = c$.

6. PERMUTATION REPRESENTATIONS

Our goal in this section is to study representations of K_n into the symmetric group S_r . Note that there is a natural homomorphism:

$$\pi : B_n \rightarrow S_n,$$

for all $n \geq 2$, given by $\sigma_i \mapsto (ii + 1)$. This restricts to K_n to give a non trivial homomorphism:

$$\begin{aligned} x_i &\mapsto (12)(ii + 1) \quad i = 3, \dots, n - 1 \\ z_m &\mapsto \begin{cases} (132), & m \text{ is even} \\ (123), & m \text{ is odd} \end{cases} \end{aligned}$$

We start, as usual, with $n = 3$, and describe $\text{Hom}(K_3, S_r)$. Note that it contains $\pi|_{K_3}$, for $r \geq 3$. Recall that a non trivial element $a \in S_r$ has order two if and only if it is a product of disjoint transpositions. Let n_r be the number of such elements. This gives us a means to compute the number of type I cycles to be $\frac{1}{2}(1 + n_r + r!)$ and of representations coming from them to be $3r! - 2$.

Since $S_2 = \{1, (12)\}$, $\text{Hom}(K_3, S_2)$ consists only of the following type I cycle:

$$(1, (12)) \rightarrow ((12), (12)) \rightarrow ((12), 1) \rightarrow (1, (12)),$$

along with the trivial representation. So

Claim 1. $|\text{Hom}(K_3, S_2)| = 4$.

As for $\text{Hom}(K_3, S_3)$, there are three type I cycles of length 3 corresponding to transpositions and one type I cycle of length 6 corresponding to the 3-cycle (123) (and its inverse). Looking at type II cycles, we find two cycles of length 9 corresponding to the pairs $((23), (12))$ and $((23), (123))$ and one cycle of length 2 corresponding to the pair $((123), (132))$. This last one is exactly the orbit (under the shift map σ) of $\pi|_{K_3}$. All by all, we have

Claim 2. $|\text{Hom}(K_3, S_3)| = 36$.

In the last section, we present among other things, the results of computer calculations of type II cycles in the graph of $\text{Hom}(K_3, S_4)$ using Maple. Moving to $n = 4$, we find that all (type I) cycles in $\text{Hom}(K_3, S_2)$ extend to K_4 with $b_3 = 1$. So

Claim 3. $|\text{Hom}(K_4, S_2)| = 4$.

No cycle in $\text{Hom}(K_3, S_3)$ extends to K_4 with non trivial b_3 :

Claim 4. $|\text{Hom}(K_4, S_3)| = |\text{Hom}(K_3, S_3)| = 36$.

Out of 71 (type II) cycles in $\text{Hom}(K_3, S_4)$ only ten do extend to K_4 , each with three possibilities for b_3 (the same for all; see the last section). So

Claim 5. $|\text{Hom}(K_4, S_4)| = |\text{Hom}(K_3, S_4)| + 30$.

For $n = 5$, we find that no type I cycle and no type II cycle with $b_3 = 1$ extends to K_5 , and that

Claim 6. $|\text{Hom}(K_5, S_2)| = |\text{Hom}(K_5, S_3)| = 1$.

As for type II cycles with $b_3 \neq 1$, none of the thirty cycles extends to K_5 :

Claim 7. $|\text{Hom}(K_5, S_4)| = 1$.

We also find :

Claim 8. $|\text{Hom}(K_6, S_5)| = |\text{Hom}(K_7, S_6)| = 1$.

Experimental results recover the well known fact that that the process given by Algorithm 1 will stop at step $n = r$. That is:

Proposition 5. $\text{Hom}(K_n, S_r)$ is trivial for $r \geq 4$ and $n \geq r + 1$.

Proof. see Lin □

It is obvious that a cycle (of any type) can not extend to K_n if it doesn't extend to K_{n-1} , so Proposition 5 asserts exactly that $\text{Hom}(K_{r+1}, S_r)$ is trivial. Recall that for $n \leq r$, $\text{Hom}(K_n, S_r)$ is not trivial since it contains the homomorphism $\pi|_{K_n} : K_n \rightarrow S_n$.

Proposition 6. for $n \geq 6$, $\text{Hom}(K_n, S_r) = \text{Hom}(K_n, A_r)$.

Proof. Apply Proposition 4 to $\Sigma = S_r$. □

Note that according to Proposition 5, for $n \geq r + 1$, every representation $\tilde{\rho} : B_n \rightarrow S_r$ factorizes through the abelianized group $(B_n)_{ab}$, and has a cyclic image. Hence, there are $r!$ possible choices for $\tilde{\rho}$.

7. CONSEQUENCES

Regarding the correspondence between subgroups of finite index of a group K and its representations into symmetric groups, we investigate the subgroups of index r of K_n for low degrees r . The general principle is to compute the number of transitive representations of K into S_r to deduce the number of subgroups of K with index exactly r . We start with K_3 as usual. Note that since K_3 is freely generated by z_0 and z_{-1} , it maps onto any symmetric group, and hence, has subgroups of every index. Now, if a representation in $\text{Hom}(K_3, S_r)$ is transitive,

then so are the representations in its orbit. Consider a type I cycle in $\text{Hom}(K_3, S_r)$. Then the representations it defines are transitive if and only if the defining element a is (with respect to the action of S_r on $\{1, \dots, r\}$). This exactly means that a is an r -cycle. If $r > 2$ then $a^2 \neq 1$ and the cycle has length 6.

Claim 9. *The number of transitive representations $\rho \in \text{Hom}(K_3, S_r)$, $r \geq 2$ coming from type I cycles is $3(r-1)!$.*

For $r = 2$, there are only type I cycles and there is only one 2-cycle, which has length 3; Hence, The number of transitive representations $\rho \in \text{Hom}(K_3, S_2)$ is 3. The kernels of these representations give rise to subgroups of K_3 with index 2.

Claim 10. *There are three subgroups of K_3 with index 2.*

Now we compute the number of subgroups of K_3 with index 3. Among all representations we have seen in example 2, there are six transitive representations coming from the only type I cycle and all representations coming from type II cycles are transitive. Hence:

Claim 11. *The number of transitive representations in $\text{Hom}(K_3, S_3)$ is 26, consequently there are thirteen subgroups of K_3 with index 3.*

We can proceed in this way for every degree r . To compute the number of transitive representations of K_3 into S_r , we need only consider those coming from type II cycles, since we already know the number of those coming from type I cycles. This can be done using a computer algebra system, by taking any cycle $C = (a_0, \dots, a_{p-1})$ and checking if the subgroup $\langle a_0, \dots, a_{p-1} \rangle$ of S_r acts transitively on $\{1, \dots, r\}$. If so, this gives rise to p transitive representations in $\text{Hom}(K_3, S_3)$. Then we divide the total number by $(r-1)!$ to find the number of subgroups of K_3 of index r .

Now let us consider $\text{Hom}(K_4, S_r)$. For $r = 2$ we have, as previously:

Claim 12. *There are three subgroups of K_4 with index 2.*

As for transitive representations in $\text{Hom}(K_4, S_3)$, since all cycles in $\text{Hom}(K_3, S_3)$ extend to K_4 with only $b_3 = 1$, we have:

Claim 13. *There are twenty six transitive representations in $\text{Hom}(K_4, S_3)$, hence thirteen subgroups of K_3 with index 3.*

For $r \geq 4$, we have $3(r-1)!$ transitive representations coming from type I cycles, and we must check which representation coming from a type II cycle is transitive. For a cycle $C = (a_0, \dots, a_{p-1})$ such that $\langle a_0, \dots, a_{p-1} \rangle$ failed to be transitive, we check if $\langle a_0, \dots, a_{p-1}, b_3 \rangle$ (with b_3 non trivial) is transitive. Indeed, if $\langle a_0, \dots, a_{p-1} \rangle$ is transitive, then so is $\langle a_0, \dots, a_{p-1}, b_3 \rangle$. Finally, we divide the total number by $(r-1)!$ to find the number of subgroups of K_4 of index r .

Now, we consider $n \geq 5$, where we get rid of type I cycles. Suppose we have found the transitive representations in $\text{Hom}(K_{n-1}, S_r)$. We then take, for fixed r , a type II cycle $C = (a_0, \dots, a_{p-1})$ along with values b_3, \dots, b_{n-1} , such that $\langle a_0, \dots, a_{p-1}, b_3, \dots, b_{n-2} \rangle$ failed to be transitive and check if $\langle a_0, \dots, a_{p-1}, b_3, \dots, b_{n-1} \rangle$ is transitive. We may enhance algorithm 1 by checking, each time we get a new type II cycle, if it is transitive, and if not, we re-check at each time the cycle extends from K_i to K_{i+1} , $i = 3, \dots, n-1$, after having augmented it with b_i . Dividing by $(r-1)!$ the number of transitive representations in $\text{Hom}(K_n, S_r)$ we find the

number of subgroups of K_n with index r . As a consequence of Proposition 5, we get the following:

Proposition 7. *For $n \geq 5$ and $2 \leq r \leq n - 1$, there are no subgroups of K_n with index r . Moreover, every nontrivial representation ρ of K_n into S_n is transitive.*

Remark 3. *We can likewise investigate the number of subgroups of B_n with a given index r by looking at transitive representations of B_n into S_r . Namely, according to Proposition 5, there is exactly one subgroup of index r in B_n , for $1 \leq r \leq n - 1$. Moreover, if $\rho : B_n \rightarrow S_n$ is a representation, then $\rho|_{K_n}$ is either trivial or transitive, according to Proposition 7. In the first case, ρ has a cyclic image and we know when it is transitive. In the second case, ρ is transitive.*

8. EXPERIMENTAL FACTS

In what follows, we list the type II cycles of $Hom(K_n, S_r)$ for various (small) n and r . A word about the notation: each cycle $B[a_0, a_1] = [a_2, a_3, \dots, a_{p-1}, a_0, a_1]$ is indexed by its first vertex (a_0, a_1) and is followed by its length p . Elements $\tau \in S_r$ are ordered from 1 to $r!$ with respect to the lexicographic order on the vectors $(\tau(1), \dots, \tau(r))$. It would have taken too much space to list the cycles for $r = 5$. We found that there were no (type II) cycles in $Hom(K_5, S_4)$ nor in $Hom(K_6, S_5)$. Furthermore, $Hom(K_4, S_3)$ contains no type II cycles with non trivial b_3 , as predicted by Proposition 5.

$n = 3; r = 3 :$

B[2, 3] = [5, 6, 2, 5, 3, 6, 5, 2, 3]

9

B[2, 4] = [6, 3, 4, 2, 6, 4, 3, 2, 4]

9

B[4, 5] = [4, 5]

2

$n = 3; r = 4 :$

B[2, 3] = [5, 6, 2, 5, 3, 6, 5, 2, 3]

9

B[2, 4] = [6, 3, 4, 2, 6, 4, 3, 2, 4]

9

B[2, 7] = [8, 2, 7]

3

B[2, 8] = [7, 2, 8]

3

B[2, 9] = [11, 6, 16, 18, 3, 20, 19, 2, 9]

9

B[2, 10] = [12, 3, 23, 21, 6, 14, 13, 2, 10]

9

B[2, 11] = [9, 6, 18, 16, 3, 19, 20, 2, 11]

9

B[2, 12] = [10, 3, 21, 23, 6, 13, 14, 2, 12]

9

B[2, 13] = [19, 22, 4, 23, 15, 12, 11, 2, 13]

9

$B[2, 14] = [20, 15, 18, 5, 22, 10, 9, 2, 14]$
 9
 $B[2, 15] = [21, 22, 2, 21, 15, 22, 21, 2, 15]$
 9
 $B[2, 16] = [22, 15, 16, 2, 22, 16, 15, 2, 16]$
 9
 $B[2, 17] = [23, 7, 24, 23, 2, 17]$
 6
 $B[2, 18] = [24, 7, 18, 17, 2, 18]$
 6
 $B[2, 19] = [13, 22, 23, 4, 15, 11, 12, 2, 19]$
 9
 $B[2, 20] = [14, 15, 5, 18, 22, 9, 10, 2, 20]$
 9
 $B[2, 23] = [17, 7, 23, 24, 2, 23]$
 6
 $B[2, 24] = [18, 7, 17, 18, 2, 24]$
 6
 $B[3, 7] = [13, 15, 3, 13, 7, 15, 13, 3, 7]$
 9
 $B[3, 8] = [14, 22, 17, 14, 3, 8]$
 6
 $B[3, 9] = [15, 7, 9, 3, 15, 9, 7, 3, 9]$
 9
 $B[3, 10] = [16, 7, 11, 5, 15, 23, 20, 3, 10]$
 9
 $B[3, 11] = [17, 22, 11, 8, 3, 11]$
 6
 $B[3, 12] = [18, 15, 4, 14, 7, 21, 19, 3, 12]$
 9
 $B[3, 14] = [8, 22, 14, 17, 3, 14]$
 6
 $B[3, 16] = [10, 7, 5, 11, 15, 20, 23, 3, 16]$
 9
 $B[3, 17] = [11, 22, 8, 11, 3, 17]$
 6
 $B[3, 18] = [12, 15, 14, 4, 7, 19, 21, 3, 18]$
 9
 $B[3, 22] = [24, 3, 22]$
 3
 $B[3, 24] = [22, 3, 24]$
 3
 $B[4, 5] = [4, 5]$
 2
 $B[4, 8] = [20, 16, 17, 4, 13, 8, 16, 20, 17, 13, 4, 8]$
 12
 $B[4, 9] = [21, 20, 4, 9]$
 4

$B[4, 10] = [22, 13, 18, 6, 21, 11, 7, 4, 10]$
 9
 $B[4, 11] = [23, 12, 19, 11, 13, 23, 19, 4, 11]$
 9
 $B[4, 12] = [24, 9, 16, 8, 12, 4, 24, 16, 9, 8, 4, 12]$
 12
 $B[4, 16] = [12, 9, 4, 16]$
 4
 $B[4, 17] = [9, 20, 24, 4, 21, 17, 20, 9, 24, 21, 4, 17]$
 12
 $B[4, 18] = [10, 13, 11, 18, 21, 10, 11, 4, 18]$
 9
 $B[4, 19] = [14, 21, 18, 19, 12, 14, 18, 4, 19]$
 9
 $B[4, 20] = [13, 16, 4, 20]$
 4
 $B[4, 22] = [18, 13, 6, 11, 21, 7, 10, 4, 22]$
 9
 $B[5, 7] = [14, 16, 6, 23, 9, 22, 19, 5, 7]$
 9
 $B[5, 8] = [13, 21, 24, 5, 20, 8, 21, 13, 24, 20, 5, 8]$
 12
 $B[5, 9] = [17, 12, 21, 8, 9, 5, 17, 21, 12, 8, 5, 9]$
 12
 $B[5, 10] = [18, 9, 14, 10, 20, 18, 14, 5, 10]$
 9
 $B[5, 12] = [16, 13, 5, 12]$
 4
 $B[5, 13] = [20, 21, 5, 13]$
 4
 $B[5, 14] = [19, 16, 23, 14, 9, 19, 23, 5, 14]$
 9
 $B[5, 16] = [24, 13, 12, 17, 16, 5, 24, 12, 13, 17, 5, 16]$
 12
 $B[5, 19] = [7, 16, 14, 6, 9, 23, 22, 5, 19]$
 9
 $B[5, 21] = [9, 12, 5, 21]$
 4
 $B[5, 23] = [11, 20, 10, 23, 16, 11, 10, 5, 23]$
 9
 $B[6, 7] = [20, 22, 6, 20, 7, 22, 20, 6, 7]$
 9
 $B[6, 8] = [19, 15, 24, 19, 6, 8]$
 6
 $B[6, 10] = [24, 15, 10, 8, 6, 10]$
 6
 $B[6, 12] = [22, 7, 12, 6, 22, 12, 7, 6, 12]$
 9

$B[6, 15] = [17, 6, 15]$
 3
 $B[6, 17] = [15, 6, 17]$
 3
 $B[6, 19] = [8, 15, 19, 24, 6, 19]$
 6
 $B[6, 24] = [10, 15, 8, 10, 6, 24]$
 6
 $B[8, 17] = [24, 8, 17]$
 3
 $B[8, 18] = [23, 8, 23, 18, 8, 18]$
 6
 $B[8, 24] = [17, 8, 24]$
 3
 $B[9, 13] = [9, 13]$
 2
 $B[9, 18] = [11, 16, 19, 18, 20, 11, 19, 9, 18]$
 9
 $B[10, 14] = [12, 23, 10, 21, 14, 23, 13, 10, 14]$
 9
 $B[10, 17] = [10, 19, 17, 19, 10, 17]$
 6
 $B[11, 14] = [24, 14, 11, 24, 11, 14]$
 6
 $B[12, 20] = [12, 20]$
 2
 $B[16, 21] = [16, 21]$
 2

$n = 4; r = 4 :$

ten cycles of length 2 along with three values $b_3 = 8, 17, 24$.

$[8, [4, 5], [4, 9], [4, 16], [4, 20], [5, 12], [5, 13], [5, 21], [9, 13], [12, 20], [16, 21]]$
 $[17, [4, 5], [4, 9], [4, 16], [4, 20], [5, 12], [5, 13], [5, 21], [9, 13], [12, 20], [16, 21]]$
 $[24, [4, 5], [4, 9], [4, 16], [4, 20], [5, 12], [5, 13], [5, 21], [9, 13], [12, 20], [16, 21]]$

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