Smoothing properties for the higher order nonlinear Schrödinger equation with constant coefficients

Mauricio Sepúlveda * Octavio Vera Villagrán.†

Abstract

We study local and global existence and smoothing properties for the initial value problem associated to a higher order nonlinear Schrödinger equation with constant coefficients which appears as a model for propagation of pulse in optical fiber.

Keywords and phrases: Evolution equations, weighted Sobolev space, gain in regularity.

Mathematics Subject Classification: 35Q53, 47J35

1 Introduction

We consider the initial value problem

$$(P)\begin{cases} i\,u_t + \omega\,u_{xx} + i\,\beta\,u_{xxx} + |u|^2\,u = 0 & x,\,t\in\mathbb{R} \\ u(x,\,0) = u_0(x) \end{cases}$$

where $\omega,\ \beta\in\mathbb{R}$, $\beta\neq 0$ and $u = u(x,t)$ is a complex valued function. The above equation is a particular case of the equation

$$(Q)\begin{cases} i\,u_t + \omega\,u_{xx} + i\,\beta\,u_{xxx} + \gamma\,|u|^2\,u + i\,\delta\,|u|^2\,u_x + i\,\epsilon\,u^2\,\overline{u}_x = 0 & x,\,t\in\mathbb{R} \\ u(x,\,0) = u_0(x) \end{cases}$$

where ω, β, γ, δ are real numbers with $\beta\neq 0$. This equation was first proposed by A. Hasegawa and Y. Kodama [13] as a model for the propagation of a signal in an optic fiber (see also [20]). The equation (Q) can be reduced to other well known equations. For instance, setting $\omega = 1$, $\beta = \delta = \epsilon = 0$ in (Q) we have the semilinear Schrödinger equation, i. e.,

$$i\,u_t + u_{xx} + \gamma\,|u|^2\,u = 0. \qquad (Q_1)$$

If we let $\beta = \gamma = 0$ and $\omega = 1$ in (Q), we obtain the derivative nonlinear Schrödinger equation

$$i\,u_t + u_{xx} + i\,\delta\,|u|^2\,u_x + i\,\epsilon\,u^2\,\overline{u}_x = 0. \qquad (Q_2)$$

Letting $\alpha = \gamma = \epsilon = 0$ in (Q), the equation that arises is the complex modified Korteweg-de Vries equation,

$$i\,u_t + i\,\beta\,u_{xxx} + i\,\delta\,|u|^2\,u_x = 0. \qquad (Q_3)$$

The initial value problem for the equations (Q_1), (Q_2) and (Q_3) has been extensively studied in the last few years. See, for instance, [1, 2, 3, 5, 6, 8, 9, 17, 18, 26, 27] and references therein. In 1992, C. Laurey [22] considered the equation (Q) and proved local well-posedness of the initial value problem associated for data in $H^s(\mathbb{R})$, $s > 3/4$, and global well-posedness in $H^s(\mathbb{R})$, $s\geq 1$. In 1997, G. Staffilani [28] for (Q)

*Departamento de Ingeniería Matemática, Universidad de Concepción, Casilla 160-C, Concepción, Chile. mauricio@ing-mat.udec.cl

†Departamento de Matemática, Universidad del Bío-Bío, Collao 1202, Casilla 5-C, Concepción, Chile. overa@ubiobio.cl

established local well-posedness for data in $H^s(\mathbb{R})$, $s \geq 1/4$ improving Laurey's result. A similar result was given in [5, 6] with $w(t)$, $\beta(t)$ real functions.

Our aim in this paper, is to study gain in regularity for the equation (P). Specifically, we prove conditions on (P) for which initial data u_0 possessing sufficient decay at infinity and minimal amount of regularity will lead to a unique solution $u(t) \in C^\infty(\mathbb{R})$ for $0 < t < T$, where T is the existence time of the solution. We are not considering the equation (Q) because of the technique used here, we shall see that the last two terms in (Q) are not outstanding in the main inequality, indeed the two last terms are observed in the last two terms in the main inequality.

In 1986, N. Hayashi *et al.* [13] showed that for the nonlinear Schrödinger equation (NLS): $i\,u_t + u_{xx} = \lambda\,|u|^{p-1}\,u$, $(x,\,t) \in \mathbb{R}\times\mathbb{R}$ with initial condition $u(x,\,0) = u_0(x)$, $x \in \mathbb{R}$ and a certain assumption on λ and p, all solutions of finite energy are smooth for $t \neq 0$ provided the initial functions in $H^1(\mathbb{R})$(or on $L^2(\mathbb{R})$) decay sufficiently fast as $|x| \to \infty$. The main tool is the operator J defined by $Ju = e^{i\,x^2/4\,t}\,(2\,i\,t)\,\partial_x(e^{-i\,x^2/4\,t}\,u) = (x + 2\,i\,t\,\partial_x)u$ which has the remarkable property that it commutes with the operator L defined by $L = (i\,\partial_t + \partial_x^2)$, namely $LJ - JL = [L,\,J] = 0$.

For the Korteweg-de Vries type equation (KdV), J. C. Saut and M. Temam [26] remarked that a solution u cannot gain or lose regularity. They showed that if $u(x,\,0) = u_0(x) \in H^s(\mathbb{R})$ for $s \geq 2$, then $u(\,\cdot\,,\,t) \in H^s(\mathbb{R})$ for all $t > 0$. For the KdV equation on the line, Kato [17] motivated by work of Cohen [11] showed that if $u(x,\,0) = u_0(x) \in L_b^2 \equiv H^2(\mathbb{R}) \cap L^2(e^{bx}\,dx)(b > 0)$ then the solution $u(x,\,t)$ of the KdV equation becomes C^∞ for all $t > 0$. A main ingredient in the proof was the fact that formally the semi-group $S(t) = e^{-\partial_x^3}$ in $L_b^2(\mathbb{R})$ is equivalent to $S_b(t) = e^{-t\,(\partial_x - b)^3}$ in $L^2(\mathbb{R})$ when $t > 0$. One would be inclined to believe that this was a special property of the KdV equation. However, his is not the case. The effect is due to the dispersive nature of the linear part of the equation. Kruzkov and Faminskii [21] proved that $u(x,\,0) = u_0(x) \in L^2(\mathbb{R})$ such that $x^\alpha\,u_0(x) \in L^2((0,\,+\infty))$, the weak solution of the KdV equation, has l-continuous space derivatives for all $t > 0$ if $l < 2\,\alpha$. The proof of this result is based on the asymptotic behavior of the Airy function and its derivatives, and on the smoothing effect of the KdV equation which was found in [17, 21]. While the proof of Kato appears to depend on special a priori estimates, some of this mystery has been solved by the result of local gain of finite regularity for various others linear and nonlinear dispersive equations due to Ginibre and Velo [12] and others. However, all of them require growth conditions on the nonlinear term.

In 1992, W. Craig, T. Kappeler and W. Strauss [8, 9] proved for the fully nonlinear KdV equation $u_t +\ f(u_{xxx},\,u_{xx},\,u_x,\,u,\,x,\,t) = 0$, $x \in \mathbb{R}$, $t > 0$ and certain additional assumption over f that C^∞ solutions $u(x,\,t)$ are obtained for all $t > 0$ if the initial data $u_0(x)$ decays faster than polynomially on $\mathbb{R}^+ = \{x \in \mathbb{R} :\ x > 0\}$ and has certain initial Sobolev regularity. Following this idea, H. Cai [4] studied the nonlinear equation of KdV-type of the form $u_t + u_{xxx} + a(x,\,,t)\,f(u_{xx},\,u_x,\,u,\,x,\,t) = 0$, where $a(x,\,t)$ is positive and bounded, obtaining the same conclusion. Subsequent works were given by O. Vera [30, 31, 32, 33] for a nonlinear dispersive evolution equation, a KdV-Burgers type equation and for KdV-Kawahara type equation, respectively. In more than one spatial dimension, J. Levandosky [23], proved infinite gain in regularity results for nonlinear third-order equations. While [8] included local smoothing results for some mth-order dispersive equation in n spatial dimension, their results and the techniques are different from those presented by Levandosky. First, they consider equations with only a mild solution and Levandosky considers equations with very general nonlinearities including a fully nonlinear equation of the form

$$
\begin{aligned}
&u_t + f(D^3u,\,D^2u,\,Du,\,u,\,x,\,t) = 0,\\
&u(x,\,y,\,0) = u_0(x,\,y).
\end{aligned}
$$

Secondly, they indicate local gain in finite regularity and Levandosky proved complementary results showing the relationship between the decay at infinity of the initial data and the amount of gain in regularity. More specifically, it is proved a condition under which an equation of the form

$$
\begin{aligned}
&u_t + a\,u_{xxx} + b\,u_{xxy} + c\,u_{xyy} + d\,u_{yyy} + f(D^2u,\,Du,\,u,\,x,\,t) = 0,\\
&u(x,\,y,\,0) = u_0(x,\,y),
\end{aligned}
$$

where a, b, c, d are assumed constant. Indeed, Levandosky proved sufficient conditions on this equation for which a solution u will experience an infinite gain in regularity. Specifically, prove conditions for which initial data $u_0(x,\,y)$ possessing sufficient decay at infinity and a minimal amount of regularity will lead to a unique solution $u(t)\in C^\infty(\mathbb{R}^2)$ for T^* where T^* is the existence time of solutions. According to the characteristics of equations (P) and considering the particular cases (Q_1) and (Q_2) we could hope that the (P) equation have gain in regularity following the steps of N. Hayashi *et al.* [13] or W. Craig *et al.* [8].

In our problem, the initial idea is to apply the technique given by N. Hayashi *et al.* [13, 14] to obtain gain in regularity. Firstly, using straightforward calculus we can see that the equation (P) has conservation of the energy, i. e., $||u||_{L^2(\mathbb{R})}=||u_0||_{L^2(\mathbb{R})}$. On the other hand, we look for estimates for u_x that will help to obtain a priori estimates, basically to obtain estimates in $L^\infty(\mathbb{R})$. Indeed, differentiating in the x-variable the equation (P) we have

$$i\,u_{x\,t}+i\,\beta\,u_{xxxx}+\omega\,u_{xxx}+(|u|^2)_x\,u+|u|^2\,u_x=0, \tag{1.1}$$

and multiplying (1.1) by $\overline{u}_x$

$$\begin{aligned}&i\,\overline{u}_x\,u_{x\,t}+i\,\beta\,\overline{u}_x\,u_{xxxx}+\omega\,\overline{u}_x\,u_{xxx}+\,(|u|^2)_x\,u\,\overline{u}_x+|u|^2\,|u_x|^2=0\\ &-i\,u_x\,\overline{u}_{x\,t}-i\,\beta\,u_x\,\overline{u}_{xxxx}+\omega\,u_x\,\overline{u}_{xxx}+\,(|u|^2)_x\,\overline{u}\,u_x+|u|^2\,|u_x|^2=0.\ \text{(applying conjugate)}\end{aligned}$$

Subtracting and integrating over $x\in\mathbb{R}$, we have

$$\begin{aligned}i\,\frac{d}{dt}\int_{\mathbb{R}}|u_x|^2dx+i\,\beta\int_{\mathbb{R}}\overline{u}_x\,u_{xxxx}dx+i\,\beta\int_{\mathbb{R}}u_x\,\overline{u}_{xxxx}dx\\ +\,2\,i\,\omega\,Im\int_{\mathbb{R}}\overline{u}_x\,u_{xxx}dx+2\,i\,Im\int_{\mathbb{R}}(|u|^2)_x\,u\,\overline{u}_xdx=0.\end{aligned}$$

Performing integration by parts and straightforward calculations we obtain

$$\frac{d}{dt}\int_{\mathbb{R}}|u_x|^2dx+2\,Im\int_{\mathbb{R}}(|u|^2)_x\,u\,\overline{u}_xdx=0 \qquad (E_1)$$

where

$$\frac{d}{dt}\,||u_x||^2_{L^2(\mathbb{R})}+2\,Im\int_{\mathbb{R}}u^2\,\overline{u}_x^2dx=0 \qquad (E_2)$$

or integrating by parts the second term in (E_1) we obtain

$$\frac{d}{dt}\,||u_x||^2_{L^2(\mathbb{R})}-2\,Im\int_{\mathbb{R}}|u|^2\,u\,\overline{u}_{xx}dx=0. \qquad (E_3)$$

Thus it is not possible to estimate in $H^1(\mathbb{R})$, because it appears a second term with two derivatives. The reason of having an estimate in the derivative is related to Sobolev embedding. In one spatial dimension we have the embedding $H^1(\mathbb{R})\hookrightarrow L^\infty(\mathbb{R})$. It seems that the term $i\,\beta\,u_{xxx}$ is crucial. It makes the two "top" terms look like KdV equation; that is, $u_t+u_{xxx}+\ldots$. Of course, the solution is complex, so that the equation is like two coupled real KdV equations.

This was our motivation to obtain gain in regularity using the idea of W. Craig *et al.* [8]. We prove conditions on (P) for which initial data $u_0(x)$ possessing sufficient decay at infinity and a minimal amount of regularity will lead to a unique solution $u(t)\in C^\infty(\mathbb{R})$ for $t>0$. We use a technique of nonlinear multipliers, generalizing Kato's original method, together with ideas of Craig and Goodman [7] All the physically significant dispersive equations and systems known to us have linear parts displaying this local smoothing property. To mention only a few, the KdV, Benjamin-Ono, intermediate long wave, various Boussinesq, and Schrödinger equation are included. This paper is organized as follows: Section 2 outlines briefly the notation and terminology to be used subsequently. In section 3 we prove the main inequality. In section 4 we prove an important a priori estimate. In section 5 we prove a basic-local-in-time existence and uniqueness theorem. In section 6 we prove a basic global existence theorem. In section 7 we develop a series of estimates for solutions of equations (P) in weighted Sobolev norms. These provide a starting point for the a priori gain of regularity. In section 8 we prove the following theorem:

Theorem 1.1(Main Theorem). *Let $|\omega| < 3\,\beta$, $T > 0$ and $u(x,\,t)$ be a solution of (P) in the region $\mathbb{R}\times[0,\,T]$ such that*

$$u \in L^\infty([0,\,T]:\, H^3(W_{0\;L\;0})) \tag{1.2}$$

for some $L \geq 2$. Then

$$u \in L^\infty([0,\,T]:\, H^{3+l}(W_{\sigma,\,L-l,\,l})) \cap L^2([0,\,T]:\, H^{4+l}(W_{\sigma,\,L-l-1,\,l})) \tag{1.3}$$

for all $0 \leq l \leq L-1$ and all $\sigma > 0$.

Remark. We consider the Gauge transformation

$$u(x,\,t) \;\;=\;\; e^{i\,d_2\,x+i\,d_3\,t}\,v\,(x-d_1\,t,\,t) \equiv e^\theta\,v\,(\eta,\,\xi) \tag{1.4}$$

where $\theta = i\,d_2\,x + i\,d_3\,t$, $\eta = x - d_1\,t$ and $\xi = t$. Then

$$u_t = i\,d_3\,e^\theta\,v - d_1\,e^\theta\,v_\eta + e^\theta\,v_\xi \quad : \quad u_x = i\,d_2\,e^\theta\,v + e^\theta\,v_\eta$$
$$u_{xx} = -\,d_2^2\,e^\theta\,v + 2\,i\,d_2\,e^\theta\,v_\eta + e^\theta\,v_{\eta\,\eta} \;:\; u_{xxx} = -\,i\,d_2^3\,e^\theta\,v - 3\,d_2^2\,e^\theta\,v_\eta + 3\,i\,d_2\,e^\theta\,v_{\eta\eta} + e^\theta\,v_{\eta\eta\eta}.$$

Replacing in (Q) we have

$$\begin{aligned}
&-d_3\,e^\theta\,v - i\,d_1\,e^\theta\,v_\eta + i\,e^\theta\,v_\xi - \omega\,d_2^2\,e^\theta\,v + 2\,i\,\omega\,d_2\,e^\theta\,v_\eta + \omega\,e^\theta\,v_{\eta\eta}\\
&\beta\,d_3^3\,e^\theta\,v - 3\,i\,\beta\,d_2^2\,e^\theta\,v_\eta - 3\,\beta\,d_2\,e^\theta\,v_{\eta\eta} + i\,\beta\,e^\theta\,v_{\eta\eta\eta} + \gamma\,|v|^2\,e^\theta\,v\\
&-\delta\,d_2\,|v|^2\,e^\theta\,v + i\,\delta\,|v|^2\,e^\theta\,v_\eta + \epsilon\,d_2\,e^\theta\,v^2\overline{v} + i\,\epsilon\,e^\theta\,v^2\,v_\eta = 0
\end{aligned}$$

where

$$\begin{aligned}
&i\,v_\xi + (\omega - 3\,\beta\,d_2)\,v_{\eta\eta} + i\,\beta\,v_{\eta\eta\eta} + (2\,i\,\omega\,d_2 - 3\,i\,\beta\,d_2^2 - i\,d_1 + i\,\delta\,|v|^2 + i\,\epsilon\,v^2)\,v_\eta\\
&(\beta\,d_2^3 - \omega\,d_2^2 - d_3 + \gamma\,|v|^2 - \delta\,d_2\,|v|^2)\,v + \epsilon\,d_2\,v^2\overline{v} = 0
\end{aligned}$$

then

$$d_1 = \frac{\omega^2}{3\,\beta} \quad : \quad d_2 = \frac{\omega}{3\,\beta} \quad : \quad d_3 = \frac{-\,2\omega^3}{27\,\beta^2}. \tag{1.5}$$

This way in (Q) we obtain

$$i\,v_\xi + i\,\beta\,v_{\eta\eta\eta} + i\,(\delta\,|v|^2 + \epsilon\,v^2)\,v_\eta + \left(\gamma - \frac{\omega\,\delta}{3\,\beta}\right)|v|^2 v + \frac{\epsilon\,\delta}{3\,\beta}\,v^2\overline{v} = 0,$$

but $v^2\,\overline{v} = v\,v\,\overline{v} = |v|^2 v$, then using the Gauge transformation we have the equivalent problem to (Q)

$$(\mathbb{Q})\begin{cases} i\,v_\xi + i\,\beta\,v_{\eta\eta\eta} + i\,\delta\,|v|^2\,v_\eta + i\,\epsilon\,v^2\,v_\eta + \left(\gamma + \frac{\epsilon\,\delta}{3\,\beta} - \frac{\omega\,\delta}{3\,\beta}\right)|v|^2 v = 0 & \eta,\,\xi \in \mathbb{R}\\ v(\eta,\,0) = e^{-\,i\,\frac{\omega}{3\,\beta}\,\eta}u_0(\eta). \end{cases}$$

Here, rescaling the equation, we take $\beta = 1$.

$$(\widetilde{\mathbb{Q}})\begin{cases} i\,v_t + i\,v_{xxx} + i\,\delta\,|v|^2\,v_x + i\,\epsilon\,v^2\,v_x + \left(\gamma + \frac{\epsilon\,\delta}{3} - \frac{\omega\,\delta}{3}\right)|v|^2 v = 0 & x,\,t \in \mathbb{R}\\ v(x,\,0) = e^{-\,i\,\frac{\omega}{3}\,x}u_0(x). \end{cases}$$

The above Gauge transformation is a bicontinuous map from $L^p([0,\,T]:\,H^s(W_{\sigma\,i\,k}))$ to itself, as far as $0 < T < +\infty$ and $p,\,s,\,\sigma,\,i,\,k$ used in this paper. With this, the assumption $|\omega| < 3\,\beta$ imposed in Theorem 1.1 can be removed.

2 Preliminaries

We consider the initial value problem

$$(P)\left\{\begin{array}{ll} i\,u_t+\omega\,u_{xx}+i\,\beta\,u_{xxx}+|u|^2\,u=0, & x,\,t\in\mathbb{R}\\ u(x,\,0)=u_0(x) & \end{array}\right.$$

where $\omega,\,\beta\in\mathbb{R}$, $\beta\neq 0$ and $u=u(x,\,t)$ is a complex valued function.

Notation. We write $\partial=\partial/\partial x$, $\partial_t=\partial/\partial t$ and we abbreviate $u_j=\partial^j u$.

Definition 2.1. A function $\xi=\xi(x,\,t)$ belongs to the weight class $W_{\sigma\;i\;k}$ if it is a positive C^∞ function on $\mathbb{R}\times[0,\,T]$, $\partial\xi>0$ and there are constant c_j, $0\leq j\leq 5$ such that

$$0<c_1\leq t^{-\,k}\,e^{-\,\sigma\,x}\,\xi(x,\,t)\leq c_2\qquad\forall\,x<-1,\quad 0<t<T. \tag{2.1}$$

$$0<c_3\leq t^{-\,k}\,x^{-\,i}\,\xi(x,\,t)\leq c_4\qquad\forall\,x>1,\quad 0<t<T. \tag{2.2}$$

$$\big(t\mid\partial_t\xi\mid+\mid\partial^j\xi\mid\big)\,/\xi\leq c_5\quad\forall\,(x,\,t)\in\mathbb{R}\times[0,\,T],\;\forall\,j\in\mathbb{N}. \tag{2.3}$$

Remark. We shall always take $\sigma\geq 0$, $i\geq 1$ and $k\geq 0$.

Example. Let

$$\xi(x)=\left\{\begin{array}{ll} 1+e^{-1/x} & \text{for } x>0\\ 1 & \text{for } x\leq 0\end{array}\right.$$

then $\xi\in W_{0\;i\;0}$.

Definition 2.2. Let N be a positive integer. By $H^N(W_{\sigma\;i\;k})$ we denote the Sobolev space on $\mathbb{R}$ with a weight; that is, with the norm

$$||v||^2_{H^N(W_{\sigma\;i\;k})}=\sum_{j=0}^{N}\int_{\mathbb{R}}|\partial^j v(x)|^2\,\xi(x,\,t)\,dx<+\infty$$

for any $\xi\in W_{\sigma\;i\;k}$ and $0<t<T$., Even though the norm depends on ξ, all such choices leads to equivalent norms.

Remark. $H^s(W_{\sigma\;i\;k})$ depends on t (because $\xi=\xi(x,\,t)$).

Lemma 2.1. (See [4]) For $\xi\in W_{\sigma\;i\;0}$ and $\sigma\geq 0$, $i\geq 0$, there exists a constant $c>0$ such that, for $u\in H^1(W_{\sigma\;i\;0})$,

$$\sup_{x\in\mathbb{R}}||\xi\,u^2||\leq c\int_{\mathbb{R}}\big(\,|u|^2+|\partial u|^2\,\big)\;\xi\,dx$$

Lemma 2.2(The Gagliardo-Nirenberg inequality). Let q, r be any real numbers satisfying $1\leq q,\,r\leq\infty$ and let j and m be nonnegative integers such that $j\leq m$. Then

$$||\partial^j u||_{L^p(\mathbb{R})}\leq c\,||\partial^m u||^a_{L^r(\mathbb{R})}\,||u||^{1-a}_{L^q(\mathbb{R})}$$

where $\frac{1}{p}=j+a\left(\frac{1}{r}-m\right)+\frac{(1-a)}{q}$ for all a in the interval $\frac{j}{m}\leq a\leq 1$, and M is a positive constant depending only on m, j, q, r and a.

Definition 2.3. By $L^2([0,\,T]\,:\,H^N(W_{\sigma\;i\;k}))$ we denote the space of functions $v(x,\,t)$ with the norm (N integer positive)

$$||v||^2_{L^2([0,\,T]:\,H^N(W_{\sigma\;i\;k}))}=\int_0^T||v(x,\,t)||^2_{H^N(W_{\sigma\;i\;k})}dt<+\infty$$

Remark. The usual Sobolev space is $H^N(\mathbb{R}) = H^N(W_{0\,0\,0})$ without a weight.

Remark. We shall derive the a priori estimates assuming that the solution is C^∞, bounded as $x \to -\infty$, and rapidly decreasing as $x \to +\infty$, together with all of its derivatives.

Considering the above notation, the higher order nonlinear Schrödinger equation can be written as

$$i\,u_t + i\,\beta\,u_3 + \omega\,u_2 + |u|^2\,u = 0, \quad x,\,t \in \mathbb{R} \tag{2.4}$$

where $\omega,\,\beta \in \mathbb{R}$, $\beta \neq 0$ and $u = u(x,\,t)$ is a complex valued function.

Throughout this paper c is a generic constant, not necessarily the same at each occasion(it will change from line to line), which depends in an increasing way on the indicated quantities. In this part, we only consider the case $t > 0$. The case $t < 0$ can be treated analogously.

3 Main Inequality

Lemma 3.1. *Let* $|\omega| < 3\,\beta$. *Let* u *be a solution of* (2.4) *with enough Sobolev regularity (for instance,* $u \in H^N(\mathbb{R})$, $N \geq \alpha + 3$*), then*

$$\partial_t \int_{\mathbb{R}} \xi\,|u_\alpha|^2 dx + \int_{\mathbb{R}} \eta\,|u_{\alpha+1}|^2 dx + \int_{\mathbb{R}} \theta\,|u_\alpha|^2 dx + \int_{\mathbb{R}} R_\alpha dx \leq 0 \tag{3.1}$$

where

$$\begin{aligned} \eta &= (3\,\beta - |\omega|)\,\partial\xi \qquad for \qquad |\omega| < 3\,\beta \\ \theta &= -\,[\,\partial_t\xi + \beta\,\partial^3\xi + |\omega|\,\partial\xi + c_0\,\xi\,] \quad where \quad c_0 = ||u||^2_{L^\infty(\mathbb{R})} \end{aligned}$$

and $R_\alpha = R_\alpha(|u_\alpha|,\,|u_{\alpha-1}|,\,\ldots)$.

Proof. Differentiating (2.4) α-times (for $\alpha \geq 0$) over $x \in \mathbb{R}$ leads to

$$i\,u_{\alpha\,t} + i\,\beta\,u_{\alpha+3} + \omega\,u_{\alpha+2} + (|u|^2)_\alpha\,u + \sum_{m=1}^{\alpha-1} \binom{\alpha}{m} (|u|^2)_{\alpha-m}\,u_m + |u|^2\,u_\alpha = 0. \tag{3.2}$$

Let $\xi = \xi(x,\,t)$, then multiplying (3.2) by $\xi\,\overline{u}_\alpha$ we have

$$\begin{aligned} & i\,\xi\,\overline{u}_\alpha\,u_{\alpha\,t} + i\,\beta\,\xi\,\overline{u}_\alpha\,u_{\alpha+3} + \omega\,\xi\,\overline{u}_\alpha\,u_{\alpha+2} + (|u|^2)_\alpha\,\xi\,u\,\overline{u}_\alpha \\ & \quad + \sum_{m=1}^{\alpha-1} \binom{\alpha}{m} (|u|^2)_{\alpha-m}\,\xi\,u_m\,\overline{u}_\alpha + \xi\,|u|^2\,|u_\alpha|^2 = 0 \end{aligned}$$

$$\begin{aligned} & -\,i\,\xi\,u_\alpha\,\overline{u}_{\alpha\,t} - i\,\beta\,\xi\,u_\alpha\,\overline{u}_{\alpha+3} + \omega\,\xi\,u_\alpha\,\overline{u}_{\alpha+2} + (|u|^2)_\alpha\,\xi\,\overline{u}\,u_\alpha \\ & \quad + \sum_{m=1}^{\alpha-1} \binom{\alpha}{m} (|u|^2)_{\alpha-m}\,\xi\,\overline{u}_m\,u_\alpha + \xi\,|u|^2\,|u_\alpha|^2 = 0. \qquad \text{(applying conjugate)} \end{aligned}$$

Subtracting and integrating over $x \in \mathbb{R}$ we have

$$\begin{aligned} & i\,\partial_t \int_{\mathbb{R}} \xi\,|u_\alpha|^2 dx + i\,\beta \int_{\mathbb{R}} \xi\,\overline{u}_\alpha\,u_{\alpha+3} dx + i\,\beta \int_{\mathbb{R}} \xi\,u_\alpha\,\overline{u}_{\alpha+3} dx - i \int_{\mathbb{R}} \xi_t\,|u_\alpha|^2 dx \\ & \quad + \omega \int_{\mathbb{R}} \xi\,\overline{u}_\alpha\,u_{\alpha+2} dx - \omega \int_{\mathbb{R}} \xi\,u_\alpha\,\overline{u}_{\alpha+2} dx + 2\,i\,Im \int_{\mathbb{R}} \xi\,(|u|^2)_\alpha\,u\,\overline{u}_\alpha dx \\ & \quad + 2\,i \sum_{m=1}^{\alpha-1} \binom{\alpha}{m} Im \int_{\mathbb{R}} \xi\,(|u|^2)_{\alpha-m}\,u_m\,\overline{u}_\alpha dx = 0. \end{aligned} \tag{3.3}$$

We estimate the second term integrating by parts

$$\int_{\mathbb{R}} \xi\, \overline{u}_\alpha\, u_{\alpha+3} dx = \int_{\mathbb{R}} \partial^2 \xi\, \overline{u}_\alpha\, u_{\alpha+1} dx + 2 \int_{\mathbb{R}} \partial \xi\, |u_{\alpha+1}|^2 dx + \int_{\mathbb{R}} \xi\, \overline{u}_{\alpha+2}\, u_{\alpha+1} dx.$$

The other terms are calculated in a similar way. Hence, replacing in (3.3) and performing straightforward calculations we obtain

$$\begin{aligned} & i\, \partial_t \int_{\mathbb{R}} \xi\, |u_\alpha|^2 dx + i\, \beta \int_{\mathbb{R}} \partial^2 \xi\, \overline{u}_\alpha\, u_{\alpha+1} dx + 2\, i\, \beta \int_{\mathbb{R}} \partial \xi\, |u_{\alpha+1}|^2 dx \\ & \quad + i\, \beta \int_{\mathbb{R}} \xi\, \overline{u}_{\alpha+2}\, u_{\alpha+1} dx + i\, \beta \int_{\mathbb{R}} \partial^2 \xi\, u_\alpha\, \overline{u}_{\alpha+1} dx + i\, \beta \int_{\mathbb{R}} \partial \xi\, |u_{\alpha+1}|^2 dx \\ & \quad - i\, \beta \int_{\mathbb{R}} \xi\, u_{\alpha+1}\, \overline{u}_{\alpha+2} dx - \omega \int_{\mathbb{R}} \partial \xi\, \overline{u}_\alpha\, u_{\alpha+1} dx - \omega \int_{\mathbb{R}} \xi\, |u_{\alpha+1}|^2 dx \\ & \quad + \omega \int_{\mathbb{R}} \partial \xi\, u_\alpha\, \overline{u}_{\alpha+1} dx + \omega \int_{\mathbb{R}} \xi\, |u_{\alpha+1}|^2 dx - i \int_{\mathbb{R}} \partial_t \xi\, |u_\alpha|^2 dx \\ & \quad + 2\, i\, Im \int_{\mathbb{R}} \xi\, (|u|^2)_\alpha\, u\, \overline{u}_\alpha dx + 2\, i \sum_{m=1}^{\alpha-1} \binom{\alpha}{m} Im \int_{\mathbb{R}} \xi\, (|u|^2)_{\alpha-m}\, u_m\, \overline{u}_\alpha dx = 0 \end{aligned}$$

then

$$\begin{aligned} & \partial_t \int_{\mathbb{R}} \xi\, |u_\alpha|^2 dx - \beta \int_{\mathbb{R}} \partial^3 \xi\, |u_\alpha|^2 dx + 3\, \beta \int_{\mathbb{R}} \partial \xi\, |u_{\alpha+1}|^2 dx - 2\, \omega\, Im \int_{\mathbb{R}} \partial \xi\, \overline{u}_\alpha\, u_{\alpha+1} dx \\ & \quad - \int_{\mathbb{R}} \partial_t \xi\, |u_\alpha|^2 dx + 2\, Im \int_{\mathbb{R}} \xi\, (|u|^2)_\alpha\, u\, \overline{u}_\alpha dx + 2 \sum_{m=1}^{\alpha-1} \binom{\alpha}{m} Im \int_{\mathbb{R}} \xi\, (|u|^2)_{\alpha-m}\, u_m\, \overline{u}_\alpha dx = 0 \end{aligned}$$

hence

$$\begin{aligned} & \partial_t \int_{\mathbb{R}} \xi\, |u_\alpha|^2 dx - \beta \int_{\mathbb{R}} \partial^3 \xi\, |u_\alpha|^2 dx + 3\, \beta \int_{\mathbb{R}} \partial \xi\, |u_{\alpha+1}|^2 dx + 2\, Im \int_{\mathbb{R}} (|u|^2)_\alpha\, \xi\, u\, \overline{u}_\alpha dx \\ & \quad - \int_{\mathbb{R}} \partial_t \xi\, |u_\alpha|^2 dx + 2 \sum_{m=1}^{\alpha-1} \binom{\alpha}{m} Im \int_{\mathbb{R}} \xi\, (|u|^2)_{\alpha-m}\, u_m\, \overline{u}_\alpha dx = 2\, \omega\, Im \int_{\mathbb{R}} \partial \xi\, \overline{u}_\alpha\, u_{\alpha+1} dx \\ & \leq |\omega| \int_{\mathbb{R}} \partial \xi\, |u_\alpha|^2 dx + |\omega| \int_{\mathbb{R}} \partial \xi\, |u_{\alpha+1}|^2 dx \end{aligned}$$

therefore

$$\begin{aligned} & \partial_t \int_{\mathbb{R}} \xi\, |u_\alpha|^2 dx + \int_{\mathbb{R}} [\, 3\, \beta - |\omega|\,]\, \partial \xi\, |u_{\alpha+1}|^2 dx - \int_{\mathbb{R}} [\, \partial_t \xi + \beta\, \partial^3 \xi + |\omega|\, \partial \xi\,]\, |u_\alpha|^2 dx \\ & \quad + 2\, Im \int_{\mathbb{R}} (|u|^2)_\alpha\, \xi\, u\, \overline{u}_\alpha dx + 2 \sum_{m=1}^{\alpha-1} \binom{\alpha}{m} Im \int_{\mathbb{R}} \xi\, (|u|^2)_{\alpha-m}\, u_m\, \overline{u}_\alpha dx \leq 0. \end{aligned} \tag{3.4}$$

But

$$(|u|^2)_\alpha = (\, u\, \overline{u}\,)_\alpha = \sum_{k=0}^{\alpha} \binom{\alpha}{k} u_{\alpha-k}\, \overline{u}_k = \overline{u}\, u_\alpha + \sum_{k=1}^{\alpha-1} \binom{\alpha}{k} u_{\alpha-k}\, \overline{u}_k + u\, \overline{u}_\alpha$$

then

$$(|u|^2)_\alpha\, u\, \overline{u}_\alpha = |u|^2 |u_\alpha|^2 + \sum_{k=1}^{\alpha-1} \binom{\alpha}{k} u_{\alpha-k}\, \overline{u}_k\, u\, \overline{u}_\alpha + u^2\, \overline{u}_\alpha^2$$

thus,

$$
\begin{aligned}
2\,Im\int_{\mathbb{R}}(\,|u|^2)_\alpha\,\xi\,u\,\overline{u}_\alpha dx &= 2\sum_{k=1}^{\alpha-1}\binom{\alpha}{k}Im\int_{\mathbb{R}}\xi\,u_{\alpha-k}\,\overline{u}_k\,u\,\overline{u}_\alpha dx + 2\,Im\int_{\mathbb{R}}\xi\,u^2\,\overline{u}_\alpha^2 dx\\
&\le\; 2\sum_{k=1}^{\alpha-1}\binom{\alpha}{k}\int_{\mathbb{R}}\xi\,|u_{\alpha-k}|\,|u_k|\,|u|\,|u_\alpha|dx + 2\int_{\mathbb{R}}\xi\,|u|^2\,|u_\alpha|^2dx\\
&\le\; 2\sum_{k=1}^{\alpha-1}\binom{\alpha}{k}\int_{\mathbb{R}}\xi\,|u_{\alpha-k}|\,|u_k|\,|u|\,|u_\alpha|dx + 2\,||u||^2_{L^\infty(\mathbb{R})}\int_{\mathbb{R}}\xi\,|u_\alpha|^2dx\\
&\le\; 2\,||u||_{L^\infty(\mathbb{R})}\sum_{k=1}^{\alpha-1}\binom{\alpha}{k}\int_{\mathbb{R}}\xi\,|u_{\alpha-k}|\,|u_k|\,|u_\alpha|dx + 2\,||u||^2_{L^\infty(\mathbb{R})}\int_{\mathbb{R}}\xi\,|u_\alpha|^2dx \qquad (3.5)
\end{aligned}
$$

hence, in (3.4) we have

$$
\begin{aligned}
&\partial_t\int_{\mathbb{R}}\xi\,|u_\alpha|^2dx + \int_{\mathbb{R}}[3\,\beta-|\omega|\,]\,\partial\xi\,|u_{\alpha+1}|^2dx - \int_{\mathbb{R}}[\partial_t\xi+\beta\,\partial^3\xi+|\omega|\,\partial\xi+c_0\,\xi\,]\,|u_\alpha|^2dx\\
&\quad -2\,c\sum_{k=1}^{\alpha-1}\binom{\alpha}{k}\int_{\mathbb{R}}\xi\,|u_{\alpha-k}|\,|u_k|\,|u_\alpha|dx - 2\sum_{m=1}^{\alpha-1}\binom{\alpha}{m}\int_{\mathbb{R}}\xi\,|(|u|^2)_{\alpha-m}|\,|u_m|\,|u_\alpha|dx \le 0.
\end{aligned}
$$

Therefore, using straightforward calculations we obtain the *main inequality*

$$
\partial_t\int_{\mathbb{R}}\xi\,|u_\alpha|^2dx + \int_{\mathbb{R}}\eta\,|u_{\alpha+1}|^2dx + \int_{\mathbb{R}}\theta\,|u_\alpha|^2dx + \int_{\mathbb{R}}R_\alpha dx \le 0
$$

where

$$
\begin{aligned}
\eta &= (3\,\beta-|\omega|\,)\,\partial\xi \qquad \text{for} \qquad |\omega|<3\,\beta\\
\theta &= -\,[\,\partial_t\xi+\beta\,\partial^3\xi+|\omega|\,\partial\xi+c_0\,\xi\,] \quad \text{where} \quad c_0=||u||^2_{L^\infty(\mathbb{R})}
\end{aligned}
$$

and $R_\alpha = R_\alpha(|u_\alpha|,\,|u_{\alpha-1}|,\,\ldots)$.

Remark. In (3.4) using Young's estimate and assuming that $\beta>0$ we have

$$
2\,\omega\,Im\int_{\mathbb{R}}\overline{u_\alpha}\,u_{\alpha+1}\,dx \le \frac{|\omega|^2}{2\,\beta}\int_{\mathbb{R}}|u_\alpha|^2\,dx + 2\,\beta\int_{\mathbb{R}}|u_{\alpha+1}|^2\,dx.
$$

Then, in (3.4) we obtain

$$
\begin{aligned}
&\partial_t\int_{\mathbb{R}}\xi\,|u_\alpha|^2dx - \beta\int_{\mathbb{R}}\partial^3\xi\,|u_\alpha|^2dx + \beta\int_{\mathbb{R}}\partial\xi\,|u_{\alpha+1}|^2dx + 2\,Im\int_{\mathbb{R}}(|u|^2)_\alpha\,\xi\,u\,\overline{u}_\alpha dx\\
&\quad -\int_{\mathbb{R}}\partial_t\xi\,|u_\alpha|^2dx + 2\sum_{m=1}^{\alpha-1}\binom{\alpha}{m}Im\int_{\mathbb{R}}\xi\,(|u|^2)_{\alpha-m}\,u_m\,\overline{u}_\alpha dx = 2\,\omega\,Im\int_{\mathbb{R}}\partial\xi\,\overline{u}_\alpha\,u_{\alpha+1}dx\\
&\le \frac{|\omega|^2}{2\,\beta}\int_{\mathbb{R}}|u_\alpha|^2\,dx
\end{aligned}
$$

and the assumption that $|\omega|<3\,\beta$ can be removed.

Lemma 3.2. *For $\eta\in W_{\sigma\;i\;k}$ an arbitrary weight function and $|\omega|<3\,\beta$, there exists $\xi\in W_{\sigma,\,i+1,\,k}$ that satisfies*

$$
\eta = (3\,\beta-|\omega|)\,\partial\xi \qquad for \qquad |\omega|<3\,\beta. \qquad (3.6)
$$

Indeed, we have

$$
\xi = \frac{1}{(3\,\beta-|\omega|)}\int_{-\infty}^{x}\eta(y,\,t)\,dy. \qquad (3.7)
$$

Lemma 3.3. *The expression R_α in the inequality of Lemma 3.1 is a sum of terms of the form*

$$\xi\, u_{\nu_1}\, \overline{u}_{\nu_2}\, \overline{u}_\alpha \tag{3.8}$$

where $1 \leq \nu_1 \leq \nu_2 \leq \alpha$ and

$$\nu_1 + \nu_2 = \alpha \tag{3.9}$$

Proof. It follows from (3.5).

4 An a priori estimate

We show now a fundamental a priori estimate used for a basic local-in-time existence theorem. We construct a mapping $\mathcal{Z} : L^\infty([0,\, T] : H^s(\mathbb{R})) \longmapsto L^\infty([0,\, T] : H^s(\mathbb{R}))$ with the property:
Given $u^{(n)} = \mathcal{Z}(u^{(n-1)})$ and $essup_{t\in[0,T]}||u^{(n-1)}||_s \leq c_0$ then $essup_{t\in[0,T]}||u^{(n)}||_s \leq c_0$, where s and $c_0 > 0$ are constants. This property tells us that $\mathcal{Z} : \mathbb{B}_{c_0}(0) \longmapsto \mathbb{B}_{c_0}(0)$ where $\mathbb{B}_{c_0}(0) = \{v(x,\, t) : ||v(\,\cdot\,,\, t)||_s \leq c_0\}$ is a ball in $L^\infty([0,\, T] : H^s(\mathbb{R}))$. To guarantee this property, we will appeal to an a priori estimate which is the main object of this section.

Differentiating (2.4) two times leads to

$$i\,\partial_t u_2 + i\,\beta\, u_5 + \omega\, u_4 + (|u|^2)_2\, u + 2\,(|u|^2)_1\, u_1 + |u|^2\, u_2 = 0. \tag{4.1}$$

Let $u = \wedge v$ where $\wedge = (I - \partial^2)^{-1}$. Hence $u = (I - \partial^2)^{-1} v$ then $u - u_2 = v$ where $\partial_t u_2 = -\, v_t + u_t$.

Replacing in (4.1) we have

$$\begin{aligned} -\,i\,v_t + i\,\beta\, \wedge v_5 + \omega\, \wedge v_4 + (|\wedge v|^2)_2 \wedge v + 2\,(|\wedge v|^2)_1 \wedge v_1 \\ + |\wedge v|^2\, \wedge v_2 - (i\,\beta\, \wedge v_3 + \omega\, \wedge v_2 + |\wedge v|^2\, \wedge v) = 0. \end{aligned} \tag{4.2}$$

The (4.2) equation is linearized by substituting a new variable z in each coefficient:

$$\begin{aligned} -\,i\,v_t + i\,\beta\, \wedge v_5 + \omega\, \wedge v_4 + (|\wedge z|^2)_2 \wedge v + 2\,(|\wedge z|^2)_1 \wedge v_1 \\ + |\wedge z|^2\, \wedge v_2 - (i\,\beta\, \wedge v_3 + \omega\, \wedge v_2 + |\wedge z|^2\, \wedge v) = 0. \end{aligned} \tag{4.3}$$

The linear equation which is to be solved at each iteration is of the form

$$i\,\partial_t v = i\,\beta\, \wedge v_5^{(n)} + \omega\, \wedge v_4^{(n)} - i\,\beta\, \wedge v_3^{(n)} - \omega\, \wedge v_2^{(n)} + b^{(1)} \tag{4.4}$$

where $b^{(1)} = (|\wedge z|^2)_2 \wedge v + 2\,(|\wedge z|^2)_1 \wedge v_1 + |\wedge z|^2 \wedge v_2 - |\wedge z|^2 \wedge v$. Equation (4.4) is a linear equation at each iteration which can be solved in any interval of time in which the coefficient is defined.

We consider the following lemma that will help us setting up the iteration scheme.

Lemma 4.1. *Let $|\omega| < 3\,\beta$. Given initial data $u_0(x) \in H^\infty(\mathbb{R}) = \bigcap_{N\geq 0} H^N(\mathbb{R})$ there exists a unique solution of (4.4) where $b^{(1)}$ is a smooth bounded coefficient with $z \in H^\infty(\mathbb{R})$. The solution is defined in any time interval in which the coefficient is defined.*

Proof. Let $T > 0$ be arbitrary and $M > 0$ a constant. Let

$$\Gamma = \xi\,(\, i\,\partial_t - i\,\beta\, \wedge \partial^5 - \omega\, \wedge \partial^4 + i\,\beta\, \wedge \partial^3 + \omega\, \wedge \partial^2\,)$$

then in (4.4) we have $\Gamma u = \xi\, b^{(1)}$. We consider the bilinear form $\mathcal{B} : \mathcal{D} \times \mathcal{D} \longmapsto \mathbb{R}$,

$$\mathcal{B}(u,\, v) =< u,\, v >= Im \int_0^T \int_{\mathbb{R}} e^{-Mt}\, u\, \overline{v}\, dx\, dt$$

where $\mathcal{D} = \{u \in C_0^\infty(\mathbb{R} \times [0,\, T]) :\ u(x,\, 0) = 0\,\}$. We have

$$\begin{aligned} \Gamma u \cdot \overline{u} &= i\,\xi\,\overline{u}\,u_t - i\,\beta\,\xi\,\overline{u}\,\wedge u_5 - \omega\,\xi\,\overline{u}\,\wedge u_4 + i\,\beta\,\xi\,\overline{u}\,\wedge u_3 + \omega\,\xi\,\overline{u}\,\wedge u_2 \\ \overline{\Gamma u \cdot \overline{u}} &= -\,i\,\xi\,u\,\overline{u}_t + i\,\beta\,\xi\,u\,\wedge \overline{u}_5 - \omega\,\xi\,u\,\wedge \overline{u}_4 - i\,\beta\,\xi\,u\,\wedge \overline{u}_3 + \omega\,\xi\,u\,\wedge \overline{u}_2. \text{ (applying conjugate)} \end{aligned}$$

Subtracting and integrating over $x \in \mathbb{R}$ we have

$$\begin{aligned} 2\,i\,Im \int_{\mathbb{R}} \Gamma u \cdot \overline{u} dx &= i\,\partial_t \int_{\mathbb{R}} \xi\,|u|^2 dx - i \int_{\mathbb{R}} \partial_t \xi\,|u|^2 dx - i\,\beta \int_{\mathbb{R}} \xi\,\overline{u}\,\wedge u_5 dx - i\,\beta \int_{\mathbb{R}} \xi\,u\,\wedge \overline{u}_5 dx \\ &\quad -\omega \int_{\mathbb{R}} \xi\,\overline{u}\,\wedge u_4 dx + \omega \int_{\mathbb{R}} \xi\,u\,\wedge \overline{u}_4 dx + i\,\beta \int_{\mathbb{R}} \xi\,\overline{u}\,\wedge u_3 dx + i\,\beta \int_{\mathbb{R}} \xi\,u\,\wedge \overline{u}_3 dx \\ &\quad +\omega \int_{\mathbb{R}} \xi\,\overline{u}\,\wedge u_2 dx - \omega \int_{\mathbb{R}} \xi\,u\,\wedge \overline{u}_2 dx. \end{aligned}$$

Each term is treated separately, integrating by parts

$$\begin{aligned} \int_{\mathbb{R}} \xi\,\overline{u}\,\wedge u_5 dx &= \int_{\mathbb{R}} \xi\,\wedge (I - \partial^2)\overline{u}\,\wedge u_5 dx = \int_{\mathbb{R}} \xi\,\wedge \overline{u}\,\wedge u_5 dx - \int_{\mathbb{R}} \xi\,\wedge \overline{u}_2\,\wedge u_5 dx \\ &= \int_{\mathbb{R}} \partial^4 \xi\,\wedge \overline{u}\,\wedge u_1 dx + \int_{\mathbb{R}} \partial^3 \xi\,|\wedge u_1|^2 dx - 3 \int_{\mathbb{R}} \partial^2 \xi\,\wedge \overline{u}_1\,\wedge u_2 dx - 2 \int_{\mathbb{R}} \partial \xi\,|\wedge u_2|^2 dx \\ &\quad + \int_{\mathbb{R}} \xi\,\wedge \overline{u}_2\,\wedge u_3 dx - \int_{\mathbb{R}} \partial^2 \xi\,\wedge \overline{u}_2\,\wedge u_3 dx - \int_{\mathbb{R}} \partial \xi\,|\wedge u_3|^2 dx + \int_{\mathbb{R}} \xi\,\wedge \overline{u}_3\,\wedge u_4 dx. \end{aligned}$$

The other terms are calculates in a similar way. Then

$$
\begin{aligned}
&2\,i\,Im\int_{\mathbb{R}}\Gamma u\cdot\overline{u}dx\\
=\;& i\,\partial_t\int_{\mathbb{R}}\xi\,|u|^2dx-i\int_{\mathbb{R}}\partial_t\xi\,|u|^2dx-i\,\beta\int_{\mathbb{R}}\partial^4\xi\;\wedge\overline{u}\;\wedge u_1dx-i\,\beta\int_{\mathbb{R}}\partial^3\xi\,|\wedge u_1|^2dx\\
&+3\,i\,\beta\int_{\mathbb{R}}\partial^2\xi\;\wedge\overline{u}_1\;\wedge u_2dx+2\,i\,\beta\int_{\mathbb{R}}\partial\xi\,|\wedge u_2|^2dx-i\,\beta\int_{\mathbb{R}}\xi\;\wedge\overline{u}_2\;\wedge u_3dx\\
&+i\,\beta\int_{\mathbb{R}}\partial^2\xi\;\wedge\overline{u}_2\;\wedge u_3dx+i\,\beta\int_{\mathbb{R}}\partial\xi\,|\wedge u_3|^2dx-i\,\beta\int_{\mathbb{R}}\xi\;\wedge\overline{u}_3\;\wedge u_4dx\\
&-i\,\beta\int_{\mathbb{R}}\partial^4\xi\;\wedge u\;\wedge\overline{u}_1dx-i\,\beta\int_{\mathbb{R}}\partial^3\xi\,|\wedge u_1|^2dx+3\,i\,\beta\int_{\mathbb{R}}\partial^2\xi\;\wedge u_1\;\wedge\overline{u}_2dx\\
&+2\,i\,\beta\int_{\mathbb{R}}\partial\xi\,|\wedge u_2|^2dx-i\,\beta\int_{\mathbb{R}}\xi\;\wedge u_2\;\wedge\overline{u}_3dx+i\,\beta\int_{\mathbb{R}}\partial^2\xi\;\wedge u_2\;\wedge\overline{u}_3dx\\
&+2\,i\,\beta\int_{\mathbb{R}}\partial\xi\,|\wedge u_3|^2dx+i\,\beta\int_{\mathbb{R}}\xi\;\wedge\overline{u}_3\;\wedge u_4dx+\omega\int_{\mathbb{R}}\partial^3\xi\;\wedge\overline{u}\;\wedge u_1dx\\
&+\omega\int_{\mathbb{R}}\partial^2\xi\,|\wedge u_1|^2dx-2\,\omega\int_{\mathbb{R}}\partial\xi\;\wedge\overline{u}_1\;\wedge u_2dx-\omega\int_{\mathbb{R}}\xi\,|\wedge u_2|^2dx\\
&-\omega\int_{\mathbb{R}}\partial\xi\;\wedge\overline{u}_2\;\wedge u_3dx-\omega\int_{\mathbb{R}}\xi\,|\wedge u_3|^2dx-\omega\int_{\mathbb{R}}\partial^3\xi\;\wedge u\;\wedge\overline{u}_1dx\\
&-\omega\int_{\mathbb{R}}\partial^2\xi\,|\wedge u_1|^2dx+2\,\omega\int_{\mathbb{R}}\partial\xi\;\wedge u_1\;\wedge\overline{u}_2dx+\omega\int_{\mathbb{R}}\xi\,|\wedge u_2|^2dx\\
&+\omega\int_{\mathbb{R}}\partial\xi\;\wedge u_2\;\wedge\overline{u}_3dx+\omega\int_{\mathbb{R}}\xi\,|\wedge u_3|^2dx+i\,\beta\int_{\mathbb{R}}\partial^2\xi\;\wedge\overline{u}\;\wedge u_1dx\\
&+i\,\beta\int_{\mathbb{R}}\partial\xi\,|\wedge u_1|^2dx-i\,\beta\int_{\mathbb{R}}\xi\;\wedge\overline{u}_1\;\wedge u_2dx-i\,\beta\int_{\mathbb{R}}\xi\;\wedge\overline{u}_2\;\wedge u_3dx\\
&+i\,\beta\int_{\mathbb{R}}\partial^2\xi\;\wedge u\;\wedge\overline{u}_1dx+i\,\beta\int_{\mathbb{R}}\partial\xi\,|\wedge u_1|^2dx-i\,\beta\int_{\mathbb{R}}\xi\;\wedge u_1\;\wedge\overline{u}_2dx\\
&-i\,\beta\int_{\mathbb{R}}\xi\;\wedge u_2\;\wedge\overline{u}_3dx-\omega\int_{\mathbb{R}}\partial\xi\;\wedge\overline{u}\;\wedge u_1dx-\omega\int_{\mathbb{R}}\xi\,|\wedge u_1|^2dx-\omega\int_{\mathbb{R}}\xi\,|\wedge u_2|^2dx\\
&+\omega\int_{\mathbb{R}}\partial\xi\;\wedge u\;\wedge\overline{u}_1dx+\omega\int_{\mathbb{R}}\xi\,|\wedge u_1|^2dx+\omega\int_{\mathbb{R}}\xi\,|\wedge u_2|^2dx
\end{aligned}
$$

hence

$$
\begin{aligned}
2\,i\,Im\int_{\mathbb{R}}\Gamma u\cdot\overline{u}dx&=i\,\partial_t\int_{\mathbb{R}}\xi\,|u|^2dx-i\int_{\mathbb{R}}\partial_t\xi\,|u|^2dx-i\,\beta\int_{\mathbb{R}}\partial^4\xi\,(|\wedge u|^2)_1dx\\
&-2\,i\,\beta\int_{\mathbb{R}}\partial^3\xi\,|\wedge u_1|^2dx+3\,i\,\beta\int_{\mathbb{R}}\partial^2\xi\,(|\wedge u_1|^2)_1dx+4\,i\,\beta\int_{\mathbb{R}}\partial\xi\,|\wedge u_2|^2dx\\
&-i\,\beta\int_{\mathbb{R}}\xi\,(|\wedge u_2|^2)_1dx+i\,\beta\int_{\mathbb{R}}\partial^2\xi\,(|\wedge u_2|^2)_1dx+3\,i\,\beta\int_{\mathbb{R}}\partial\xi\,|\wedge u_3|^2dx\\
&+2\,i\,\omega\,Im\int_{\mathbb{R}}\partial^3\xi\;\wedge\overline{u}\;\wedge u_1dx-4\,i\,\omega\,Im\int_{\mathbb{R}}\partial\xi\;\wedge\overline{u}_1\;\wedge u_2dx\\
&-2\,i\,\omega\,Im\int_{\mathbb{R}}\partial\xi\;\wedge\overline{u}_2\;\wedge u_3dx+i\,\beta\int_{\mathbb{R}}\partial^2\xi\,(|\wedge u|^2)_1dx+2\,i\,\beta\int_{\mathbb{R}}\partial\xi\,|\wedge u_1|^2dx\\
&-i\,\beta\int_{\mathbb{R}}\xi\,(|\wedge u_1|^2)_1dx-i\,\beta\int_{\mathbb{R}}\xi\,(|\wedge u_2|^2)_1dx-2\,\omega\,Im\int_{\mathbb{R}}\partial\xi\;\wedge\overline{u}\;\wedge u_1dx
\end{aligned}
$$

then, adding similar terms and cutting the letter i we obtain

$$
\begin{aligned}
2\,Im \int_{\mathbb{R}} \Gamma u \cdot \overline{u}\, dx &= \partial_t \int_{\mathbb{R}} \xi\, |u|^2 dx - \int_{\mathbb{R}} \partial_t \xi\, |u|^2 dx + \beta \int_{\mathbb{R}} \partial^5 \xi\, | \wedge u|^2 dx - 5\,\beta \int_{\mathbb{R}} \partial^3 \xi\, | \wedge u_1|^2 dx \\
&+ 6\,\beta \int_{\mathbb{R}} \partial \xi\, | \wedge u_2|^2 dx - \beta \int_{\mathbb{R}} \partial^3 \xi\, | \wedge u_2|^2 dx + 3\,\beta \int_{\mathbb{R}} \partial \xi\, | \wedge u_3|^2 dx \\
&+ 2\,\omega\, Im \int_{\mathbb{R}} \partial^3 \xi\, \wedge \overline{u}\, \wedge u_1 dx - 4\,\omega\, Im \int_{\mathbb{R}} \partial \xi\, \wedge \overline{u}_1\, \wedge u_2 dx - 2\,\omega\, Im \int_{\mathbb{R}} \partial \xi\, \wedge \overline{u}_2\, \wedge u_3 dx \\
&- \beta \int_{\mathbb{R}} \partial^3 \xi\, | \wedge u|^2 dx + 3\,\beta \int_{\mathbb{R}} \partial \xi\, | \wedge u_1|^2 dx - 2\,\omega\, Im \int_{\mathbb{R}} \partial \xi\, \wedge \overline{u}\, \wedge u_1 dx
\end{aligned}
$$

then

$$
\begin{aligned}
&|\omega| \int_{\mathbb{R}} \partial \xi\, | \wedge u_3|^2 dx + |\omega| \int_{\mathbb{R}} \partial \xi\, | \wedge u_2|^2 dx + 2\,|\omega| \int_{\mathbb{R}} \partial \xi\, | \wedge u_1|^2 dx + 2\,|\omega| \int_{\mathbb{R}} \partial \xi\, | \wedge u_2|^2\, dx \\
&\quad + |\omega| \int_{\mathbb{R}} \partial \xi\, | \wedge u|^2 dx + |\omega| \int_{\mathbb{R}} \partial \xi\, | \wedge u_1|^2 dx + |\omega| \int_{\mathbb{R}} |\partial^3 \xi|\, | \wedge u|^2 dx \\
&\quad + |\omega| \int_{\mathbb{R}} |\partial^3 \xi|\, | \wedge u_1|^2 dx + \int_{\mathbb{R}} \partial_t \xi\, |u|^2 dx + 2\, Im \int_{\mathbb{R}} \Gamma u \cdot \overline{u} dx \\
\geq\;& \partial_t \int_{\mathbb{R}} \xi\, |u|^2 dx + 3\,\beta \int_{\mathbb{R}} \partial \xi\, | \wedge u_3|^2 dx - \beta \int_{\mathbb{R}} \partial^3 \xi\, | \wedge u_2|^2 dx + 6\,\beta \int_{\mathbb{R}} \partial \xi\, | \wedge u_2|^2 dx \\
&\quad - 5\,\beta \int_{\mathbb{R}} \partial^3 \xi\, | \wedge u_1|^2 dx + 3\,\beta \int_{\mathbb{R}} \partial \xi\, | \wedge u_1|^2 dx + \beta \int_{\mathbb{R}} \partial^5 \xi\, | \wedge u|^2 dx - \beta \int_{\mathbb{R}} \partial^3 \xi\, | \wedge u|^2 dx
\end{aligned}
$$

where

$$
\begin{aligned}
&3\,|\omega| \int_{\mathbb{R}} \partial \xi\, | \wedge u_2|^2 dx + |\omega| \int_{\mathbb{R}} [|\partial^3 \xi| + 3\, \partial \xi]\, | \wedge u_1|^2 dx \\
&\quad + |\omega| \int_{\mathbb{R}} [|\partial^3 \xi| + \partial \xi + \partial_t \xi]\, | \wedge u|^2 dx + 2\, Im \int_{\mathbb{R}} \Gamma u \cdot \overline{u} dx \\
\geq\;& \partial_t \int_{\mathbb{R}} \xi\, |u|^2 dx + \int_{\mathbb{R}} [3\,\beta - |\omega|]\, \partial \xi\, | \wedge u_3|^2 dx - \beta \int_{\mathbb{R}} \partial^3 \xi\, | \wedge u_2|^2 dx \\
&\quad + 6\,\beta \int_{\mathbb{R}} \partial \xi\, | \wedge u_2|^2 dx - 5\,\beta \int_{\mathbb{R}} \partial^3 \xi\, | \wedge u_1|^2 dx + 3\,\beta \int_{\mathbb{R}} \partial \xi\, | \wedge u_1|^2 dx \\
&\quad + \beta \int_{\mathbb{R}} \partial^5 \xi\, | \wedge u|^2 dx - \beta \int_{\mathbb{R}} \partial^3 \xi\, | \wedge u|^2 dx \\
\geq\;& \partial_t \int_{\mathbb{R}} \xi\, |u|^2 dx + \beta \int_{\mathbb{R}} [-\partial^3 \xi + 5 \partial \xi]\, | \wedge u_2|^2 dx \\
&\quad + \beta \int_{\mathbb{R}} [-5\, \partial^3 \xi + 3 \partial \xi]\, | \wedge u_1|^2 dx + \beta \int_{\mathbb{R}} [\partial^3 \xi - \partial^3 \xi]\, | \wedge u|^2 dx
\end{aligned}
$$

using (2.3), $\wedge u_n = (I - (I - \partial^2)) \wedge u_{n-2} = \wedge u_{n-2} - u_{n-2}$ for n a positive integer and standard estimates we obtain

$$
Im \int_{\mathbb{R}} \Gamma u \cdot \overline{u}\, dx \geq \partial_t \int_{\mathbb{R}} \xi\, |u|^2\, dx - c \int_{\mathbb{R}} \xi\, |u|^2\, dx.
$$

Multiply this equation by e^{-Mt}, and integrate with respect to t for $t \in [0,\, T]$ and $u \in \mathcal{D}$

$$
\begin{aligned}
Im \int_0^T \int_{\mathbb{R}} e^{-Mt}\, \Gamma u \cdot \overline{u}\, dx\, dt &\geq \int_0^T e^{-Mt} \left(\partial_t \int_{\mathbb{R}} \xi\, |u|^2 dx \right) dt - c \int_0^T \int_{\mathbb{R}} \xi\, e^{-Mt}\, |u|^2 dx\, dt \\
&= e^{-Mt} \int_{\mathbb{R}} \xi\, |u|^2 dx\, \Big/_0^T + M \int_0^T \int_{\mathbb{R}} \xi\, e^{-Mt}\, |u|^2 dx\, dt - c \int_0^T \int_{\mathbb{R}} \xi\, e^{-Mt}\, |u|^2 dx\, dt \\
&= e^{-Mt} \int_{\mathbb{R}} \xi(x,\, T)\, |u(x,\, T)|^2 dx + M \int_0^T \int_{\mathbb{R}} \xi\, e^{-Mt}\, |u|^2 dx\, dt - c \int_0^T \int_{\mathbb{R}} \xi\, e^{-Mt}\, |u|^2 dx\, dt.
\end{aligned}
$$

Thus

$$
\begin{aligned}
<\Gamma u,\, u> &= Im \int_0^T \int_{\mathbb{R}} e^{-Mt}\, \Gamma u \cdot \overline{u}\, dx\, dt \\
&\geq \; e^{-Mt} \int_{\mathbb{R}} \xi(x,\, T)\, |u(x,\, T)|^2 dx + (M - c) \int_0^T \int_{\mathbb{R}} \xi\, e^{-Mt}\, |u|^2 dx\, dt \\
&\geq \; \int_0^T \int_{\mathbb{R}} \xi\, e^{-Mt}\, |u|^2 dx\, dt
\end{aligned}
$$

provided that M is chosen large enough. Then $< \Gamma u,\, u > \geq < u,\, u >$, for all $u \in \mathcal{D}$. Let Γ^* be the formal adjoint of Γ defined by $\Gamma^* = \xi(-i\,\partial_t - i\,\beta \wedge \partial^5 - \omega \wedge \partial^4 + i\,\beta \wedge \partial^3 + \omega \wedge \partial^2)$. Let $\mathcal{D}^* = \{w \in C_0^\infty(\mathbb{R} \times [0,\, T]) :\; w(x,\, T) = 0\,\}$. In a similar way we prove that

$$
< \Gamma^* w,\, w > \;\geq\; < w,\, w >, \quad \forall\, w \in \mathcal{D}^*.
$$

¿From this equation, we have that Γ^* is one-one. Therefore, $< \Gamma^* w,\, \Gamma^* v >$ is an inner product on $\mathcal{D}^*$. We denote by X the completion of $\mathcal{D}^*$ with respect to this inner product. By Riesz's Representation Theorem, there exists a unique solution $V \in X$, such that for any $w \in \mathcal{D}^*$, $< \xi b^{(1)},\, w > = < \Gamma^* V,\, \Gamma^* w >$ where we use that $\xi\, b^{(1)} \in X$. Then if $v = \Gamma^* V$ we have $< v,\, \Gamma^* w > = < \xi b^{(1)},\, w >$ or $< \Gamma^* w,\, v > = < w,\, \xi b^{(1)} >$. Hence, $v = \Gamma^* V$ is a weak solution of $\Gamma v = \xi b^{(1)}$ with $v \in L^2(\mathbb{R} \times [0,\, T]) \simeq L^2([0,\, T] :\, L^2(\mathbb{R}))$.

Remark. To obtain higher regularity of the solution, we repeat the proof with higher derivatives. It is a standard approximation procedure to obtain a result for general initial data.

The next step is to estimate the corresponding solutions $v = v(x,\, t)$ of the equation (4.3) via the coefficients of that equation.

The following estimate is related to the existence of solutions theorem.

Lemma 4.2. *Let $|\omega| < 3\,\beta$ and $0 < \gamma_1 \leq \xi \leq \gamma_2$, with γ_2, γ_2 real constants. Let $v,\, z \in C^k([0,\, +\infty) : H^N(\mathbb{R}))$ for all k, N which satisfy (4.3). For each integer α there exist positive nondecreasing functions G and F such that for all $t \geq 0$*

$$
\partial_t \int_{\mathbb{R}} \xi\, |v_\alpha|^2 dx \leq G(||z||_\lambda)\, ||v||_\alpha^2 + F(||z||_\alpha) \tag{4.5}
$$

where $||\,\cdot\,||_\alpha$ is the norm in $H^\alpha(\mathbb{R})$ and $\lambda = \max\{1,\, \alpha\}$.

Proof. Differentiating α-times the equation (4.3), for some $\alpha \geq 0$ we have

$$
-\, i\, \partial_t v_\alpha + i\,\beta \wedge v_{\alpha+5} + \omega \wedge v_{\alpha+4} - i\,\beta \wedge v_{\alpha+3} + \sum_{j=3}^{\alpha+2} h^{(j)} \wedge v_j + (|z|^2)_{\alpha+2} \wedge v + p(\wedge z_{\alpha+1},\, \ldots) = 0 \tag{4.6}
$$

where $h^{(j)}$ is a smooth function depending on $|\wedge z|^2, \ldots$ with $i = 2 + \alpha - j$. For $\alpha \geq 2$, $p(\wedge z_{\alpha+1},\, \ldots)$ depends at most linearly on $\wedge z_{\alpha+1}$, while for $\alpha = 2$, $p(\wedge z_{\alpha+1},\, \ldots)$ depends at most quadratically on $\wedge z_{\alpha+1}$.

We multiply equation (4.6) by $\xi\, \overline{v}_\alpha$ and integrate over $x \in \mathbb{R}$

$$
\begin{aligned}
-\, i \int_{\mathbb{R}} \xi\, \overline{v}_\alpha\, \partial_t v_\alpha dx + i\,\beta \int_{\mathbb{R}} \xi\, \overline{v}_\alpha \wedge v_{\alpha+5} dx + \omega \int_{\mathbb{R}} \xi\, \overline{v}_\alpha \wedge v_{\alpha+4} dx - i\,\beta \int_{\mathbb{R}} \xi\, \overline{v}_\alpha \wedge v_{\alpha+3} dx \\
+ \sum_{j=3}^{\alpha+2} h^{(j)} \int_{\mathbb{R}} \xi\, \overline{v}_\alpha \wedge v_j dx + \int_{\mathbb{R}} \xi\, (|z|^2)_{\alpha+2} \overline{v}_\alpha \wedge v dx + \int_{\mathbb{R}} \xi\, \overline{v}_\alpha p(\wedge z_{\alpha+1},\, \ldots) dx = 0
\end{aligned}
$$

and applying conjugate

$$i\int_{\mathbb{R}}\xi\, v_\alpha\,\partial_t\overline{v}_\alpha dx - i\,\beta\int_{\mathbb{R}}\xi\, v_\alpha\wedge\overline{v}_{\alpha+5}dx + \omega\int_{\mathbb{R}}\xi\, v_\alpha\wedge\overline{v}_{\alpha+4}dx + i\,\beta\int_{\mathbb{R}}\xi\, v_\alpha\wedge\overline{v}_{\alpha+3}dx$$
$$+\sum_{j=3}^{\alpha+2}h^{(j)}\int_{\mathbb{R}}\xi\, v_\alpha\wedge\overline{v}_j dx + \int_{\mathbb{R}}\xi\,(|z|^2)_{\alpha+2}v_\alpha\wedge\overline{v}dx + \int_{\mathbb{R}}\xi\,\overline{v}_\alpha p(\wedge z_{\alpha+1},\,\ldots)dx = 0.$$

Subtracting, it follows that

$$\begin{aligned}
&-i\,\partial_t\int_{\mathbb{R}}\xi\,|v_\alpha|^2dx + i\int_{\mathbb{R}}\partial_t\xi\,|v_\alpha|^2dx + i\,\beta\int_{\mathbb{R}}\xi\,\overline{v}_\alpha\wedge v_{\alpha+5}dx + i\,\beta\int_{\mathbb{R}}\xi\, v_\alpha\wedge\overline{v}_{\alpha+5}dx\\
&+\omega\int_{\mathbb{R}}\xi\,\overline{v}_\alpha\wedge v_{\alpha+4}dx - \omega\int_{\mathbb{R}}\xi\, v_\alpha\wedge\overline{v}_{\alpha+4}dx - i\,\beta\int_{\mathbb{R}}\xi\,\overline{v}_\alpha\wedge v_{\alpha+3}dx - i\,\beta\int_{\mathbb{R}}\xi\, v_\alpha\wedge\overline{v}_{\alpha+3}dx\\
&+\sum_{j=3}^{\alpha+2}h^{(j)}\int_{\mathbb{R}}\xi\,\overline{v}_\alpha\wedge v_j dx - \sum_{j=3}^{\alpha+2}h^{(j)}\int_{\mathbb{R}}\xi\, v_\alpha\wedge\overline{v}_j dx + \int_{\mathbb{R}}\xi\,(|z|^2)_{\alpha+2}v_\alpha\wedge\overline{v}dx \qquad (4.7)\\
&-\int_{\mathbb{R}}\xi\,(|z|^2)_{\alpha+2}\overline{v}_\alpha\wedge v\, dx + \int_{\mathbb{R}}\xi\,\overline{v}_\alpha p(\wedge z_{\alpha+1},\,\ldots)\, dx - \int_{\mathbb{R}}\xi\, v_\alpha\, p(\wedge z_{\alpha+1},\,\ldots)dx = 0.
\end{aligned}$$

Each term is treated separately, integrating by parts

$$\begin{aligned}
&\int_{\mathbb{R}}\xi\,\overline{v}_\alpha\wedge v_{\alpha+5}dx = \int_{\mathbb{R}}\xi\wedge(I-\partial^2)\overline{v}_\alpha\wedge v_{\alpha+5}dx\\
&= \int_{\mathbb{R}}\xi\,\wedge\overline{v}_\alpha\wedge v_{\alpha+5}dx - \int_{\mathbb{R}}\xi\wedge\overline{v}_{\alpha+2}\wedge v_{\alpha+5}dx\\
&= \int_{\mathbb{R}}\partial^4\xi\wedge\overline{v}_\alpha\;\wedge v_{\alpha+1}dx + \int_{\mathbb{R}}\partial^3\xi\,|\wedge v_{\alpha+1}|^2dx - 3\int_{\mathbb{R}}\partial^2\xi\wedge\overline{v}_{\alpha+1}\wedge v_{\alpha+2}dx\\
&\quad -2\int_{\mathbb{R}}\partial\xi\,|\wedge\overline{v}_{\alpha+2}|^2dx + \int_{\mathbb{R}}\xi\wedge\overline{v}_{\alpha+2}\;\wedge v_{\alpha+3}dx - \int_{\mathbb{R}}\partial^2\xi\wedge\overline{v}_{\alpha+2}\wedge v_{\alpha+3}dx\\
&\quad -2\int_{\mathbb{R}}\partial\xi\,|\wedge v_{\alpha+3}|^2dx - \int_{\mathbb{R}}\xi\;\wedge\overline{v}_{\alpha+4}\wedge v_{\alpha+3}dx.
\end{aligned}$$

The other terms are calculated in a similar way. Hence in (4.7) we have performing straightforward calculations as above

$$\begin{aligned}
&-\partial_t\int_{\mathbb{R}}\xi\,|v_\alpha|^2dx + \int_{\mathbb{R}}\partial_t\xi\,|v_\alpha|^2dx - \beta\int_{\mathbb{R}}\partial^5\xi\,|\wedge v_\alpha|^2dx + 2\,\beta\int_{\mathbb{R}}\partial^3\xi\,|\wedge v_{\alpha+1}|^2dx\\
&+3\,\beta\int_{\mathbb{R}}\partial^3\xi\,|\wedge v_{\alpha+1}|^2dx - 4\,\beta\int_{\mathbb{R}}\partial\xi\,|\wedge v_{\alpha+2}|^2dx - \beta\int_{\mathbb{R}}\partial\xi\,|\wedge v_{\alpha+2}|^2dx\\
&+\beta\int_{\mathbb{R}}\partial^2\xi\,|\wedge v_{\alpha+2}|^2dx - 3\,\beta\int_{\mathbb{R}}\partial\xi\,|\wedge v_{\alpha+3}|^2\,dx - 2\,\omega\, Im\int_{\mathbb{R}}\partial^3\xi\;\wedge\overline{v}_\alpha\wedge v_{\alpha+1}dx\\
&+4\,\omega\, Im\int_{\mathbb{R}}\partial\xi\wedge\overline{v}_{\alpha+1}\wedge v_{\alpha+2}dx + 2\,\omega\, Im\int_{\mathbb{R}}\partial\xi\wedge\overline{v}_{\alpha+2}\wedge v_{\alpha+3}dx\\
&+2\,\beta\, Im\int_{\mathbb{R}}\partial\xi\wedge\overline{v}_\alpha\wedge v_{\alpha+2}dx + 2\,\beta\, Im\int_{\mathbb{R}}\xi\wedge\overline{v}_{\alpha+1}\wedge v_{\alpha+2}dx\\
&-\beta\int_{\mathbb{R}}\partial\xi\,|\wedge\overline{v}_{\alpha+2}|^2dx + 2\sum_{j=3}^{\alpha+2}h^{(j)}\, Im\int_{\mathbb{R}}\xi\,\overline{v}_\alpha\wedge v_j dx\\
&+2\, Im\int_{\mathbb{R}}\xi\,(|z|^2)_{\alpha+2}\, v_\alpha\wedge\overline{v}dx + 2\, Im\int_{\mathbb{R}}\xi\,\overline{v}_\alpha p(\wedge z_{\alpha+1},\,\ldots)\,dx = 0
\end{aligned}$$

then

$$
\begin{aligned}
&- \partial_t \int_{\mathbb{R}} \xi\, |v_\alpha|^2 dx + \int_{\mathbb{R}} \partial_t \xi\, |v_\alpha|^2 dx - 3\,\beta \int_{\mathbb{R}} \partial\xi\, | \wedge v_{\alpha+3}|^2 dx + \beta \int_{\mathbb{R}} \partial^2\xi\, | \wedge v_{\alpha+2}|^2 dx \\
&\quad - 6\,\beta \int_{\mathbb{R}} \partial\xi\, | \wedge \overline{v}_{\alpha+2}|^2 dx + 5\,\beta \int_{\mathbb{R}} \partial^3\xi\, | \wedge v_{\alpha+1}|^2 dx - \beta \int_{\mathbb{R}} \partial^5\xi\, | \wedge v_\alpha|^2 dx \\
&= \; - 2\,\omega\, Im \int_{\mathbb{R}} \partial\xi\, \wedge \overline{v}_{\alpha+2} \wedge v_{\alpha+3} dx - 4\,\omega\, Im \int_{\mathbb{R}} \partial\xi \wedge \overline{v}_{\alpha+1} \wedge v_{\alpha+2} dx \\
&\quad - 2\,\beta\, Im \int_{\mathbb{R}} \xi \wedge \overline{v}_{\alpha+1} \wedge v_{\alpha+2} dx - 2\,\beta\, Im \int_{\mathbb{R}} \partial\xi \wedge \overline{v}_\alpha \wedge v_{\alpha+2} dx \\
&\quad + 2\,\omega\, Im \int_{\mathbb{R}} \partial^3\xi \wedge \overline{v}_\alpha \wedge v_{\alpha+1} dx - 2 \sum_{j=3}^{\alpha+2} h^{(j)}\, Im \int_{\mathbb{R}} \xi\, \overline{v}_\alpha \wedge v_j dx \\
&\quad - 2\, Im \int_{\mathbb{R}} \xi\, (|z|^2)_{\alpha+2} v_\alpha \wedge \overline{v} dx - 2\, Im \int_{\mathbb{R}} \xi\, \overline{v}_\alpha p(\wedge z_{\alpha+1}, \ldots)\, dx
\end{aligned}
$$

hence,

$$
\begin{aligned}
&\partial_t \int_{\mathbb{R}} \xi\, |v_\alpha|^2 dx - \int_{\mathbb{R}} \partial_t \xi\, |v_\alpha|^2 dx + 3\,\beta \int_{\mathbb{R}} \partial\xi\, | \wedge v_{\alpha+3}|^2 dx - \beta \int_{\mathbb{R}} \partial^2\xi\, | \wedge v_{\alpha+2}|^2 dx \\
&\quad + 6\,\beta \int_{\mathbb{R}} \partial\xi\, | \wedge \overline{v}_{\alpha+2}|^2 dx - 5\,\beta \int_{\mathbb{R}} \partial^3\xi\, | \wedge v_{\alpha+1}|^2 dx + \beta \int_{\mathbb{R}} \partial^5\xi\, | \wedge v_\alpha|^2 dx \\
&= \; 2\,\omega\, Im \int_{\mathbb{R}} \partial\xi\, \wedge \overline{v}_{\alpha+2} \wedge v_{\alpha+3} dx + 4\,\omega\, Im \int_{\mathbb{R}} \partial\xi \wedge \overline{v}_{\alpha+1} \wedge v_{\alpha+2} dx \\
&\quad + 2\,\beta\, Im \int_{\mathbb{R}} \xi \wedge \overline{v}_{\alpha+1} \wedge v_{\alpha+2} dx + 2\,\beta\, Im \int_{\mathbb{R}} \partial\xi \wedge \overline{v}_\alpha \wedge v_{\alpha+2} dx \\
&\quad - 2\,\omega\, Im \int_{\mathbb{R}} \partial^3\xi \wedge \overline{v}_\alpha \wedge v_{\alpha+1} dx + 2 \sum_{j=3}^{\alpha+2} h^{(j)}\, Im \int_{\mathbb{R}} \xi\, \overline{v}_\alpha \wedge v_j dx \\
&\quad + 2\, Im \int_{\mathbb{R}} \xi\, (|z|^2)_{\alpha+2} v_\alpha \wedge \overline{v} dx + 2\, Im \int_{\mathbb{R}} \xi\, \overline{v}_\alpha p(\wedge z_{\alpha+1}, \ldots)\, dx \\
&\leq \; |\omega| \int_{\mathbb{R}} \partial\xi\, | \wedge v_{\alpha+2}|^2\, dx + |\omega| \int_{\mathbb{R}} \partial\xi | \wedge v_{\alpha+3}|^2 dx + 2\,|\omega| \int_{\mathbb{R}} \partial\xi\, | \wedge v_{\alpha+1}|^2 dx \\
&\quad + 2\,|\omega| \int_{\mathbb{R}} \partial\xi | \wedge v_{\alpha+2}|^2 dx + |\beta| \int_{\mathbb{R}} \xi\, | \wedge v_{\alpha+1}|^2 dx + |\beta| \int_{\mathbb{R}} \xi\, | \wedge v_{\alpha+2}|^2 dx \\
&\quad + |\beta| \int_{\mathbb{R}} \partial\xi | \wedge v_\alpha|^2 dx + |\beta| \int_{\mathbb{R}} \partial\xi | \wedge v_{\alpha+2}|^2 dx + |\omega| \int_{\mathbb{R}} \partial^3\xi | \wedge v_\alpha|^2 dx \\
&\quad + |\omega| \int_{\mathbb{R}} \partial^3\xi | \wedge v_{\alpha+1}|^2 dx + 2 \left| \sum_{j=3}^{\alpha+2} h^{(j)} \int_{\mathbb{R}} \xi\, \overline{v}_\alpha \wedge v_j dx \right| + 2 \left| \int_{\mathbb{R}} \xi\, (|z|^2)_{\alpha+2}\, v_\alpha \wedge \overline{v} dx \right| \\
&\quad + 2 \left| \int_{\mathbb{R}} \xi\, \overline{v}_\alpha p(\wedge z_{\alpha+1}, \ldots)\, dx \right|
\end{aligned}
$$

where

$$
\begin{aligned}
&\partial_t \int_{\mathbb{R}} \xi\, |v_\alpha|^2 dx \\
&\leq \; - \int_{\mathbb{R}} (3\,\beta - |\omega|) \partial\xi\, | \wedge v_{\alpha+3}|^2 dx + \int_{\mathbb{R}} [\beta\, \partial^2\xi - 6\,\beta\, \partial\xi + 3\,|\omega|\, \partial\xi + |\beta|\, \partial\xi + |\beta|\, \xi]\, | \wedge v_{\alpha+2}|^2 dx \\
&\quad + \int_{\mathbb{R}} [5\beta \partial^3\xi + |\omega| \partial^3\xi + 2\,|\omega| \partial\xi + |\beta|\, \xi]\, | \wedge v_{\alpha+1}|^2 dx + \int_{\mathbb{R}} [\partial_t \xi + \beta\, \partial^5\xi + |\omega|\, \partial^3\xi + |\beta|\, \partial\xi]\, | \wedge v_\alpha|^2 dx \\
&\quad + 2 \left| \sum_{j=3}^{\alpha+2} h^{(j)} \int_{\mathbb{R}} \xi\, \overline{v}_\alpha \wedge v_j dx \right| + 2 \left| \int_{\mathbb{R}} \xi\, (|z|^2)_{\alpha+2}\, v_\alpha \wedge \overline{v} dx \right| + 2 \left| \int_{\mathbb{R}} \xi\, \overline{v}_\alpha p(\wedge z_{\alpha+1}, \ldots)\, dx \right|.
\end{aligned}
$$

using that $|\omega| < 3\,\beta$ we have that the first term in the right hand side of the above expression is not positive. Hence,

$$
\begin{aligned}
&\partial_t \int_{\mathbb{R}} \xi\,|v_\alpha|^2 dx \\
&\quad\le \int_{\mathbb{R}} [\beta\,\partial^2\xi - 6\,\beta\,\partial\xi + 3\,|\omega|\,\partial\xi + |\beta|\,\partial\xi + |\beta|\,\xi]\,|\wedge v_{\alpha+2}|^2 dx \\
&\qquad + \int_{\mathbb{R}} [5\,\beta\,\partial^3\xi + |\omega|\,\partial^3\xi + 2\,|\omega|\,\partial\xi + |\beta|\,\xi]\,|\wedge v_{\alpha+1}|^2 dx + \int_{\mathbb{R}} [\partial_t\xi + \beta\,\partial^5\xi + |\omega|\,\partial^3\xi + |\beta|\,\partial\xi]\,|\wedge v_\alpha|^2 dx \\
&\qquad + 2\left|\sum_{j=3}^{\alpha+2} h^{(j)} \int_{\mathbb{R}} \xi\,\overline{v}_\alpha \wedge v_j dx\right| + 2\left|\int_{\mathbb{R}} \xi\,(|z|^2)_{\alpha+2}\,v_\alpha \wedge \overline{v}\,dx\right| + 2\left|\int_{\mathbb{R}} \xi\,\overline{v}_\alpha p(\wedge z_{\alpha+1},\,\ldots)\,dx\right|.
\end{aligned}
$$

Using that $\wedge v_n = \wedge v_{n-2} - v_{n-2}$ and a standard estimate, the lemma follows.

5 Uniqueness and Existence of a Local Solution

In this section, we study the uniqueness and the existence of local strong solutions in the Sobolev space $H^N(\mathbb{R})$ for $N \ge 3$ for the problem (2.4). To establish the existence of strong solutions for (2.4) we use the a priori estimate together with an approximation procedure.

Theorem 5.1(Uniqueness). *Let $|\omega| < 3\,\beta$, $u_0(x) \in H^N(\mathbb{R})$ with $N \ge 3$ and $0 < T < +\infty$. Then there is at most one strong solution $u \in L^\infty([0,\,T] : H^N(\mathbb{R}))$ of (2.4) with initial data $u(x,\,0) = u_0(x)$.*

Proof. Assume that $u,\,v \in L^\infty([0,\,T] : H^N(\mathbb{R}))$ are two solutions of (2.4) with $u_t,\,v_t \in L^\infty([0,\,T] : H^{N-3}(\mathbb{R}))$, and with the same initial data. Then

$$
i\,(u-v)_t + i\,\beta\,(u-v)_3 + \omega\,(u-v)_2 + |u|^2\,u - |v|^2\,v = 0 \tag{5.1}
$$

with $(u-v)(x,\,0) = 0$. By (5.1)

$$
i\,(u-v)_t + i\,\beta\,(u-v)_3 + \omega\,(u-v)_2 + |u|^2\,(u-v) + (|u|^2 - |v|^2)\,v = 0
$$

or

$$
i\,(u-v)_t + i\,\beta\,(u-v)_3 + \omega\,(u-v)_2 + |u|^2\,(u-v) + (|u| - |v|)\,(|u| + |v|)\,v = 0. \tag{5.2}
$$

Multiplying (5.2) by $\xi\overline{(u-v)}$ we have

$$
\begin{aligned}
&i\,\xi\,\overline{(u-v)}\,(u-v)_t + i\,\beta\,\xi\,\overline{(u-v)}\,(u-v)_3 + \alpha\,\xi\,\overline{(u-v)}\,(u-v)_2 \\
&\qquad + |u|^2\,|u-v|^2 + \xi\,\overline{(u-v)}\,(|u| - |v|)\,(|u| + |v|)\,v = 0.
\end{aligned}
$$

$$
\begin{aligned}
&-i\,\xi\,(u-v)\,\overline{(u-v)}_t - i\,\beta\,\xi\,(u-v)\,\overline{(u-v)}_3 + \alpha\,\xi\,(u-v)\,\overline{(u-v)}_2 \\
&\qquad + |u|^2\,|u-v|^2 + \xi\,(u-v)\,(|u| - |v|)\,(|u| + |v|)\,\overline{v} = 0. \quad \text{(applying conjugate)}
\end{aligned}
$$

Subtracting and integrating over $x \in \mathbb{R}$ we obtain

$$
\begin{aligned}
&i\,\partial_t \int_{\mathbb{R}} \xi\,|u-v|^2 dx - i \int_{\mathbb{R}} \partial_t\xi\,|u-v|^2 dx + i\,\beta \int_{\mathbb{R}} \xi\,\overline{(u-v)}\,(u-v)_3 dx \\
&\quad + i\,\beta \int_{\mathbb{R}} \xi\,(u-v)\,\overline{(u-v)}_3 dx + \omega \int_{\mathbb{R}} \xi\,\overline{(u-v)}\,(u-v)_2 dx \\
&\quad - \omega \int_{\mathbb{R}} \xi\,(u-v)\,\overline{(u-v)}_2 dx + 2\,i\,Im \int_{\mathbb{R}} \xi\,\overline{(u-v)}\,(|u| - |v|)\,(|u| + |v|)\,v\,dx = 0
\end{aligned} \tag{5.3}
$$

Each term is treated separately, integrating by parts

$$\begin{aligned}&\int_{\mathbb{R}} \xi\, \overline{(u-v)}\, (u-v)_3 dx\\ &= \int_{\mathbb{R}} \partial^2\xi\, \overline{(u-v)}\, (u-v)_1 dx + 2\int_{\mathbb{R}} \partial\xi\, |(u-v)_1|^2 dx + \int_{\mathbb{R}} \xi\, (u-v)_1\, \overline{(u-v)}_2 dx.\end{aligned}$$

The other terms are calculated in a similar way. Hence in (5.3) we have

$$\begin{aligned}&i\,\partial_t \int_{\mathbb{R}} \xi\, |u-v|^2 dx - i\int_{\mathbb{R}} \partial_t\xi\, |u-v|^2 dx + i\,\beta\int_{\mathbb{R}} \partial^2\xi\, \overline{(u-v)}\, (u-v)_1 dx\\ &+ 2\,i\,\beta\int_{\mathbb{R}} \partial\xi\, |(u-v)_1|^2 dx + i\,\beta\int_{\mathbb{R}} \xi\, (u-v)_1\, \overline{(u-v)}_2 dx + i\,\beta\int_{\mathbb{R}} \partial^2\xi\, (u-v)\, \overline{(u-v)}_1 dx\\ &+ i\,\beta\int_{\mathbb{R}} \partial\xi\, |(u-v)_1|^2 dx - i\,\beta\int_{\mathbb{R}} \xi\, (u-v)_1\, \overline{(u-v)}_2 dx - \omega\int_{\mathbb{R}} \partial\xi\, \overline{(u-v)}\, (u-v)_1 dx\\ &- \omega\int_{\mathbb{R}} \xi\, |(u-v)_1|^2 dx + \omega\int_{\mathbb{R}} \partial\xi\, (u-v)\, \overline{(u-v)}_1 dx + \omega\int_{\mathbb{R}} \xi\, |(u-v)_1|^2 dx\\ &+ 2\,i\,Im\int_{\mathbb{R}} \xi\, \overline{(u-v)}\, (|u|-|v|)\, (|u|+|v|)\, v\, dx = 0\end{aligned}$$

then

$$\begin{aligned}&i\,\partial_t \int_{\mathbb{R}} \xi\, |u-v|^2 dx - i\int_{\mathbb{R}} \partial_t\xi\, |u-v|^2 dx + i\,\beta\int_{\mathbb{R}} \partial^2\xi\, (|u-v|^2)_1 dx + 3\,i\,\beta\int_{\mathbb{R}} \partial\xi\, |(u-v)_1|^2 dx\\ &- 2\,i\,\omega\, Im\int_{\mathbb{R}} \partial\xi\, \overline{(u-v)}\, (u-v)_1 dx + 2\,i\,Im\int_{\mathbb{R}} \xi\, \overline{(u-v)}\, (|u|-|v|)\, (|u|+|v|)\, v\, dx = 0\end{aligned}$$

if and only if

$$\begin{aligned}&\partial_t \int_{\mathbb{R}} \xi\, |u-v|^2 dx - \int_{\mathbb{R}} \partial_t\xi\, |u-v|^2\, dx + \beta\int_{\mathbb{R}} \partial^2\xi\, (|u-v|^2)_1 dx + 3\,\beta\int_{\mathbb{R}} \partial\xi\, |(u-v)_1|^2 dx\\ &= 2\,\omega\, Im\int_{\mathbb{R}} \partial\xi\, \overline{(u-v)}\, (u-v)_1 dx - 2\,Im\int_{\mathbb{R}} \xi\, \overline{(u-v)}\, (|u|-|v|)\, (|u|+|v|)\, v\, dx\\ &\leq |\omega|\int_{\mathbb{R}} \partial\xi\, |u-v|^2 dx + |\omega|\int_{\mathbb{R}} \partial\xi\, |(u-v)_1|^2 dx + 2\int_{\mathbb{R}} \xi\, |u-v|\, |\, |u|-|v|\, |\, (|u|+|v|)\, |v|\, dx.\end{aligned}$$

Using that $|\, |u|-|v|\, | \leq |u-v|$, (2.3) and standard estimates, we have

$$\partial_t \int_{\mathbb{R}} \xi\, |u-v|^2 dx + \int_{\mathbb{R}} [3\,\beta - |\omega|\,]\, \partial\xi\, |(u-v)_1|^2 dx \leq c\int_{\mathbb{R}} \xi\, |u-v|^2 dx.$$

Integrating in $t \in [0,\, T]$, using the fact that $(u-v)$ vanishes at $t=0$ and Gronwall's inequality it follows that $u=v$. This proves the uniqueness of the solution.

We construct the mapping $\mathcal{Z} : L^\infty([0,\, T] : H^s(\mathbb{R})) \longmapsto L^\infty([0,\, T] : H^s(\mathbb{R}))$ where the initial condition is given by $u^{(n)}(x,\, 0) = u_0(x)$ and the first approximation is given by

$$\begin{aligned}u^{(0)} &= u_0(x)\\ u^{(n)} &= \mathcal{Z}(u^{(n-1)}) \qquad n \geq 1,\end{aligned}$$

where $u^{(n-1)}$ is in place of z in equation (4.3) and $u^{(n)}$ is in place of v which is the solution of equation (4.3). That is

$$\begin{aligned}&-i\, u_t^{(n)} + i\,\beta\, \wedge u_5^{(n)} + \omega\, \wedge u_4^{(n)} + (|\wedge u^{(n-1)}|^2)_2 \wedge u^{(n)} + 2\,(|\wedge u^{(n-1)}|^2)_1 \wedge u_1^{(n)}\\ &+ |\wedge u^{(n-1)}|^2\, \wedge u_2^{(n)} - (i\,\beta\, \wedge u_3^{(n)} + \omega\, \wedge u_2^{(n)} + |\wedge u^{(n-1)}|^2\, \wedge u^{(n)}) = 0.\end{aligned}$$

By Lemma 4.1, $u^{(n)}$ exists and is unique in $C((0, +\infty) : H^N(\mathbb{R}))$. A choice of c_0 and the use of the a priori estimate in Section 4 shows that $\mathcal{Z} : \mathbb{B}_{c_0}(0) \longmapsto \mathbb{B}_{c_0}(0)$ where $\mathbb{B}_{c_0}(0)$ is a bounded ball in $L^\infty([0, T] : H^s(\mathbb{R}))$.

Theorem 5.2(Local solution). *Let $|\omega| < 3\,\beta$ and N an integer ≥ 3. If $u_0(x) \in H^N(\mathbb{R})$, then there is $T > 0$ and u such that u is a strong solution of (2.4), $u \in L^\infty([0, T] : H^N(\mathbb{R}))$ and $u(x, 0) = u_0(x)$.*

Proof. We prove that for $u_0(x) \in H^\infty(\mathbb{R}) = \bigcap_{k\geq 0} H^k(\mathbb{R})$ there exists a solution $u \in L^\infty([0, T] : H^N(\mathbb{R}))$ with initial data $u(x, 0) = u_0(x)$ where the time of existence $T > 0$ only depends on the norm of $u_0(x)$. We define a sequence of approximations to equation (4.3) as

$$\begin{aligned} i\, v_t^{(n)} &= i\,\beta \wedge v_5^{(n)} + \omega \wedge v_4^{(n)} - i\,\beta \wedge v_3^{(n)} - \omega \wedge v_2^{(n)} + |\wedge v^{(n-1)}|^2 \wedge v_2^{(n)} \\ &\quad + O[\,(|\wedge v^{(n-1)}|^2)_2,\, (|\wedge v^{(n-1)}|^2)_1,\, \ldots)\,] \end{aligned} \tag{5.4}$$

where the initial condition is $v^{(n)}(x, 0) = u_0(x) - \partial^2 u_0(x)$. The first approximation is given by $v^{(0)}(x, 0) = u_0(x) - \partial^2 u_0(x)$. Equation (5.4) is a linear equation at each iteration which can be solved in any interval of time in which the coefficients are defined. This is shown in Lemma 4.1. By Lemma 4.2, it follows that

$$\partial_t \int_{\mathbb{R}} \xi\, |v_\alpha^{(n)}|^2 dx \leq G(||v^{(n-1)}||_\lambda)\, ||v^{(n)}||_\alpha^2 + F(||v^{(n-1)}||_\alpha). \tag{5.5}$$

Choose $\alpha = 1$ and let $c \geq ||u_0 - \partial^2 u_0||_1 \geq ||u_0||_3$. For each iterate n, $||v^{(n)}(\,\cdot\,, t)||$ is continuous in $t \in [0, T]$ and $||v^{(n)}(\,\cdot\,, 0)|| < c$. Define $c_0 = \frac{\gamma_2}{2\,\gamma_1}\, c^2 + 1$. Let $T_0^{(n)}$ be the maximum time such that $||v^{(k)}(\,\cdot\,, t)||_1 \leq c_3$ for $0 \leq t \leq T_0^{(n)}$, $0 \leq k \leq n$. Integrating (5.5) over $[0, t]$ we have that for $0 \leq t \leq T_0^{(n)}$ and $j = 0, 1$

$$\int_0^t \left(\partial_s \int_{\mathbb{R}} \xi\, |v_j^{(n)}|^2 dx \right) ds \leq \int_0^t G\left(||v^{(n-1)}||_1\right) ||v^{(n)}||_j^2 ds + \int_0^t F\left(||v^{(n-1)}||_j\right) ds.$$

It follows that

$$\begin{aligned} \int_{\mathbb{R}} \xi(x, t) |v_j^{(n)}(x, t)|^2 dx &\leq \int_{\mathbb{R}} \xi(x, 0) |v_j^{(n)}(x, 0)|^2 dx + \int_0^t G\left(||v^{(n-1)}||_1\right) ||v^{(n)}||_j^2 ds \\ &\quad + \int_0^t F\left(||v^{(n-1)}||_j\right) ds \end{aligned}$$

hence

$$\begin{aligned} \gamma_1 \int_{\mathbb{R}} |v_j^{(n)}(x, t)|^2 dx &\leq \int_{\mathbb{R}} \xi(x, t) |v_j^{(n)}(x, t)|^2 dx \\ &\leq \int_{\mathbb{R}} \xi(x, 0) |v_j^{(n)}(x, 0)|^2 dx + \int_0^t G\left(||v^{(n-1)}||_1\right) ||v^{(n)}||_j^2 ds \\ &\quad + \int_0^t F\left(||v^{(n-1)}||_j\right) ds \end{aligned}$$

and

$$\int_{\mathbb{R}} |v_j^{(n)}|^2 dx \leq \frac{\gamma_2}{\gamma_1} \int_{\mathbb{R}} |v_j^{(n)}(x, 0)|^2 dx + \frac{G(c_3)}{\gamma_1}\, c_3^2\, t + \frac{F(c_3)}{\gamma_1}\, t$$

and we obtain for $j = 0, 1$ that

$$||v^{(n)}||_1 \leq \frac{\gamma_2}{\gamma_1}\, c^2 + \frac{G(c_0)}{\gamma_1}\, c_0^2\, t + \frac{F(c_0)}{\gamma_1}\, t.$$

Claim. $T_0^{(n)}$ does not approach to 0.
On the contrary, assume that $T_0^{(n)} \to 0$. Since $||v^{(n)}(\,\cdot\,, t)||$ is continuous for $t \geq 0$, there exists $\tau \in [0, T]$ such that $||v^{(k)}(\,\cdot\,, t)||_1 = c_0$ for $0 \leq \tau \leq T_0^{(n)}$, $0 \leq k \leq n$. Then

$$c_0^2 \leq \frac{\gamma_2}{\gamma_1}\, c^2 + \frac{G(c_0)}{\gamma_1}\, c_0^2\, T_0^{(n)} + \frac{F(c_0)}{\gamma_1}\, T_0^{(n)}$$

as $n \to \infty$, we have

$$\left(\frac{\gamma_2}{2\,\gamma_1}\,c^2+1\right)^2 \le \frac{\gamma_2}{\gamma_1}\,c^2 \quad \text{then} \quad \frac{\gamma_2^2}{4\,\gamma_1^2}\,c^4+1\le 0$$

which is a contradiction. Consequently $T_0^{(n)} \not\to 0$. Choosing $T = T(c)$ sufficiently small, and T not depending on n, one concludes that

$$||v^{(n)}||_1 \le C \tag{5.6}$$

for $0 \le t \le T$. This shows that $T_0^{(n)} \ge T$. Hence, from (5.6) we imply that there exists a subsequence $v^{(n_j)} \equiv v^{(n)}$ such that

$$v^{(n)} \overset{*}{\rightharpoonup} v \quad \text{weakly on} \quad L^\infty([0,\,T]:\,H^1(\mathbb{R})). \tag{5.7}$$

Claim. $u = \wedge v$ is a solution.

In the linearized equation (5.4) we have

$$\wedge v_5^{(n)} = \wedge(I-(I-\partial^2))v_3^{(n)} = \wedge v_3^{(n)} - v_3^{(n)} = \partial^2(\underbrace{\wedge v_1^{(n)}}_{\in L^2(\mathbb{R})}) - \underbrace{\partial^2(v_1^{(n)})}_{\in H^{-2}(\mathbb{R})} \in H^{-2}(\mathbb{R}).$$

Since $\wedge = (I-\partial^2)^{-1}$ is bounded in $H^1(\mathbb{R})$, $\wedge v_5^{(n)}$ belongs to $H^{-2}(\mathbb{R})$. $v^{(n)}$ is still bounded in $L^\infty([0,\,T]:\,H^1(\mathbb{R})) \hookrightarrow L^2([0,\,T]:\,H^1(\mathbb{R}))$ and since $\wedge : L^2(\mathbb{R}) \to H^2(\mathbb{R})$ is a bounded operator,

$$||\wedge v_1^{(n)}||_{H^2(\mathbb{R})} \le c\,||v_1^{(n)}||_{L^2(\mathbb{R})} \le c\,||v_1^{(n)}||_{H^1(\mathbb{R})}.$$

Consequently, $\wedge v_1^{(n)}$ is bounded in $L^2([0,\,T]:\,H^2(\mathbb{R})) \hookrightarrow L^2([0,\,T]:\,L^2(\mathbb{R}))$. It follows that $\partial^2(\wedge v_1^{(n)})$ is bounded in $L^2([0,\,T]:\,H^{-2}(\mathbb{R}))$, and

$$\wedge\, v_5^{(n)} \quad \text{is bounded in} \quad L^2([0,\,T]:\,H^{-2}(\mathbb{R})). \tag{5.8}$$

Similarly, the other terms are bounded. By (5.4), $v_t^{(n)}$ is a sum of terms each of which is the product of a coefficient, uniformly bounded on n and a function in $L^2([0,\,T]:\,H^{-2}(\mathbb{R}))$ uniformly bounded on n such that $v_t^{(n)}$ is bounded in $L^2([0,\,T]:\,H^{-2}(\mathbb{R}))$. On the other hand, $H^1_{loc}(\mathbb{R}) \overset{c}{\hookrightarrow} H^{1/2}_{loc}(\mathbb{R}) \hookrightarrow H^{-4}(\mathbb{R})$. By Lions-Aubin's compactness Theorem [24] there is a subsequence $v^{(n_j)} \equiv v^{(n)}$ such that $v^{(n)} \to v$ strongly on $L^2([0,\,T]:\,H^{1/2}_{loc}(\mathbb{R}))$. Hence, for a subsequence $v^{(n_j)} \equiv v^{(n)}$, we have $v^{(n)} \to v$ a. e. in $L^2([0,\,T]:\,H^{1/2}_{loc}(\mathbb{R}))$. Moreover, from (5.8), $\wedge v_5^{(n)} \rightharpoonup \wedge v_5$ weakly in $L^2([0,\,T]:\,H^{-2}(\mathbb{R}))$. Similarly, $\wedge v_2^{(n)} \rightharpoonup \wedge v_2$ weakly in $L^2([0,\,T]:\,H^{-2}(\mathbb{R}))$. Since $||\wedge v^{(n)}||_{H^2(\mathbb{R})} \le c\,||v^{(n)}||_{L^2(\mathbb{R})} \le c\,||v^{(n)}||_{H^1(\mathbb{R})} \le c\,||v^{(n)}||_{H^{1/2}(\mathbb{R})}$ and $v^{(n)} \to v$ strongly on $L^2([0,\,T]:\,H^{1/2}_{loc}(\mathbb{R}))$ then $\wedge v^{(n)} \to \wedge v$ strongly in $L^2([0,\,T]:\,H^2_{loc}(\mathbb{R}))$. Thus, the fifth term on the right hand side of (5.4), $|\wedge v^{(n-1)}|^2 \wedge v_2^{(n)} \rightharpoonup |\wedge v|^2 \wedge v_2$ weakly in $L^2([0,\,T]:\,L^1_{loc}(\mathbb{R}))$ as $\wedge v_2^{(n)} \rightharpoonup \wedge v_2$ weakly in $L^2([0,\,T]:\,H^{-2}(\mathbb{R}))$ and $|\wedge v^{(n-1)}|^2 \to |\wedge v|^2$ strongly on $L^2([0,\,T]:\,H^2_{loc}(\mathbb{R}))$. Similarly, the other terms in (5.4) converge to their limits, implying $v_t^{(n)} \rightharpoonup v_t$ weakly in $L^2([0,\,T]:\,L^1_{loc}(\mathbb{R}))$. Passing to the limit

$$\begin{aligned} i\,v_t &= \partial^2(i\,\beta\,\wedge v_3 + \omega\,\wedge v_2 + |\wedge v|^2\,\wedge v) - (i\,\beta\,\wedge v_3 + \omega\,\wedge v_2 + |\wedge v|^2\,\wedge v) \\ &= -(I-\partial^2)(i\,\beta\,\wedge v_3 + \omega\,\wedge v_2 + |\wedge v|^2\,\wedge v). \end{aligned}$$

Thus $i\,v_t + (I-\partial^2)(i\,\beta\,\wedge v_3 + \omega\,\wedge v_2 + |\wedge v|^2\,\wedge v) = 0$. This way, we have (2.4) for $u = \wedge v$.

Now, we prove that there exists a solution of (2.4) with $u \in L^\infty([0,\,T]:\,H^N(\mathbb{R}))$ and $N \ge 4$, where T depends only on the norm of u_0 in $H^3(\mathbb{R})$. We already know that there is a solution $u \in L^\infty([0,\,T]:\,H^3(\mathbb{R}))$. It is suffices to show that the approximating sequence $v^{(n)}$ is bounded in $L^\infty([0,\,T]:\,H^{N-2}(\mathbb{R}))$. Taking

$\alpha = N-2$ and considering (5.5) for $\alpha \geq 2$, we define $c_{N-2} = \frac{\gamma_2}{2\,\gamma_1}\,||u_0(\cdot)||_N + 1$. Let $T^{(n)}_{N-3}$ be the largest time such that $||v^{(k)}(\,\cdot\,,\,t)||_\alpha \leq c_{N-3}$ for $0 \leq t \leq T^{(n)}_{N-3}$, $0 \leq k \leq n$. Integrating (5.5) over $[0,\,t]$, for $0 \leq t \leq T^{(n)}_{N-3}$, we have

$$\int_0^t \left(\partial_s \int_{\mathbb{R}} \xi\,|v^{(n)}_\alpha|^2 dx\right) ds \leq \int_0^t G\left(||v^{(n-1)}||_\alpha\right) ||v^{(n)}||^2_\alpha ds + \int_0^t F\left(||v^{(n-1)}||_\alpha\right) ds.$$

It follows that

$$\begin{aligned}\int_{\mathbb{R}} \xi(x,\,t)\,|v^{(n)}_\alpha|^2 dx \quad &\leq \quad \int_{\mathbb{R}} \xi(x,\,0)\,|v^{(n)}_\alpha(x,\,0)|^2 dx + \int_0^t G\left(||v^{(n-1)}||_\alpha\right) ||v^{(n)}||^2_\alpha ds \\ &\quad + \int_0^t F\left(||v^{(n-1)}||_\alpha\right) ds\end{aligned}$$

hence

$$\begin{aligned}\gamma_1 \int_{\mathbb{R}} |v^{(n)}_\alpha|^2 dx \leq \int_{\mathbb{R}} \xi\,|v^{(n)}_\alpha|^2 dx \quad &\leq \quad \int_{\mathbb{R}} \xi(x,\,0)\,|v^{(n)}_\alpha(x,\,0)|^2 dx + \int_0^t G\left(||v^{(n-1)}||_\alpha\right) ||v^{(n)}||^2_\alpha ds \\ &\quad + \int_0^t F\left(||v^{(n-1)}||_\alpha\right) ds\end{aligned}$$

then

$$\begin{aligned}\int_{\mathbb{R}} |v^{(n)}_\alpha|^2 dx &\leq \frac{\gamma_2}{\gamma_1} \int_{\mathbb{R}} |v^{(n)}_\alpha(x,\,0)|^2 dx + \frac{G(c_{N-3})}{\gamma_1}\, c^2_{N-3}\, t + \frac{F(c_{N-3})}{\gamma_1}\, t \\ &\leq \quad \frac{\gamma_2}{\gamma_1}\,||v^{(n)}_\alpha(x,\,0)||^2_\alpha + \frac{G(c_{N-3})}{\gamma_1}\, c^2_{N-3}\, t + \frac{F(c_{N-3})}{\gamma_1}\, t \\ &\leq \quad \frac{\gamma_2}{\gamma_1}\,||u(x,\,0)||^2_N + \frac{G(c_{N-3})}{\gamma_1}\, c^2_{N-3}\, t + \frac{F(c_{N-3})}{\gamma_1}\, t\end{aligned}$$

and we obtain

$$||v^{(n)}_\alpha(\,\cdot\,,\,t)||^2_\alpha dx \leq \frac{\gamma_2}{\gamma_1}\,||u(x,\,0)||^2_N + \frac{G(c_{N-3})}{\gamma_1}\, c^2_{N-3}\, t + \frac{F(c_{N-3})}{\gamma_1}\, t$$

Claim. $T^{(n)}_{N-3}$ does not approach to 0.
On the contrary, assume that $T^{(n)}_{N-3} \to 0$. Since $||v^{(n)}(\,\cdot\,,\,t)||$ is continuous for $t \geq 0$, there exists $\tau \in [0,\,T_{N-3}]$ such that $||v^{(k)}(\,\cdot\,,\,\tau)||_\alpha = c_{N-3}$ for $0 \leq \tau \leq T^{(n)}$, $0 \leq k \leq n$. Then

$$c^2_{N-3} \leq \frac{\gamma_2}{\gamma_1}\,||u(x,\,0)||^2_N + \frac{G(c_{N-3})}{\gamma_1}\, c^2_{N-3}\, T^{(n)}_{N-3} + \frac{F(c_{N-3})}{\gamma_1}\, T^{(n)}_{N-3}$$

as $n \to +\infty$, and we have

$$\left(\frac{\gamma_2}{2\,\gamma_1}\,||u(x,\,0)||^2_N + 1\right)^2 \leq \frac{\gamma_2}{\gamma_1}\,||u(x,\,0)||^2_N \quad \text{then} \quad \frac{\gamma_2^2}{4\,\gamma_1^2}\,||u(x,\,0)||^4_N + 1 \leq 0$$

which is a contradiction. Then $T^{(n)}_{N-3} \not\to 0$. By choosing $T_{N-3} = T_{N-3}(||u(x,\,0)||^2_N)$ sufficiently small, and T_{N-3} not depending on n, we conclude that

$$||v^{(n)}(\,\cdot\,,\,t)||^2_\alpha \leq c^2_{N-3} \quad \text{for all} \quad 0 \leq t \leq T_{N-3}. \tag{5.9}$$

This shows that $T^{(n)}_{N-3} \geq T_{N-3}$. Thus,

$$v \in L^\infty([0,\,T_{N-3}] :\ H^\alpha(\mathbb{R})) \equiv L^\infty([0,\,T_{N-3}] :\ H^{N-2}(\mathbb{R})).$$

Now, denote by $0 \leq T^*_{N-3} \leq +\infty$ the maximal number such that for all $0 < t \leq T^*_{N-3}$, $u = \wedge v \in L^\infty([0,\, t] :\, H^N(\mathbb{R}))$. In particular, $T_{N-3} \leq T^*_{N-3}$ for all $N \geq 4$. Thus, T can be chosen depending only on the norm of u_0 in $H^3(\mathbb{R})$. Approximating u_0 by $\{u_0^{(j)}\} \in C_0^\infty(\mathbb{R})$ such that $||u_0 - u_0^{(j)}||_{H^N(\mathbb{R})} \to 0$ as $j \to +\infty$. Let u^j be a solution of (2.4) with $u^{(j)}(x,\, 0) = u_0^{(j)}$. According to the above argument, there exists T which is independent on n but depending only on $\sup_j ||u_0^{(j)}||$ such that $u^{(j)}$ there exists on $[0,\, T]$ and a subsequence $u^{(j)} \overset{j\to+\infty}{\longrightarrow} u$ in $L^\infty([0,\, T] :\, H^N(\mathbb{R}))$.

As a consequence of Theorem 5.1 and 5.2 and its proof, one obtains the following result.

Corollary 5.3. *Let $|\omega| < 3\,\beta$ and let $u_0 \in H^N(\mathbb{R})$ with $N \geq 3$ such that $u_0^{(j)} \to u_0$ in $H^N(\mathbb{R})$. Let u and $u^{(j)}$ be the corresponding unique solutions given by Theorems 5.1 and 5.2 in $L^\infty([0,\, T] :\, H^N(\mathbb{R}))$ with T depending only on $\sup_j ||u_0^{(j)}||_{H^3(\mathbb{R})}$ such that*

$$\begin{aligned} u^{(j)} &\overset{*}{\rightharpoonup} u \quad \text{weakly on} \quad L^\infty([0,\, T] :\, H^N(\mathbb{R})), \\ u^{(j)} &\to u \quad \text{strongly on} \quad L^2([0,\, T] :\, H^{N+1}(\mathbb{R})). \end{aligned}$$

6 Existence of Global Solutions

Here, we will try to extend the local solution $u \in L^\infty([0,\, T] :\, H^N(W_{0\,i\,0}))$ of (2.4) obtained in Theorem 5.2 to $t \geq 0$. A standard way to obtain these extensions consists into deducing global estimations for the $H^N(W_{0\,i\,0})$-norm of u in terms of the $H^N(W_{0\,i\,0})$-norm of $u(x,\, 0) = u_0(x)$. These estimations are frequently based on conservation laws which contain the L^2-norm of the solution and their spatial derivatives. It is not possible to do the same to give a solution of the problem of global existence because the difficulty here is that the weight depends on the x and t variables. To solve our problem we follow a different method using Leibniz's rule like in the proof of Theorem 3.1 of Bona and Saut [3].

Theorem 6.1. *For $|\omega| < 3\,\beta$ there exists a global solution to (2.4) in the space $H^s(\mathbb{R}) \cap H^N(W_{0\,i\,0})$ with N integer ≥ 3 and $s \geq 2$.*

Proof. The first part was proved in [3]. Differentiating (2.4) α-times (for $\alpha \geq 0$) over $x \in \mathbb{R}$ leads to

$$i\,u_{\alpha\,t} + i\,\beta\,u_{\alpha+3} + \omega\,u_{\alpha+2} + (|u|^2)_\alpha\,u + \sum_{m=1}^{\alpha-1}\binom{\alpha}{m}(|u|^2)_{\alpha-m}\,u_m + |u|^2\,u_\alpha = 0. \tag{6.1}$$

Let $\xi = \xi(x,\, t)$, then multiplying (6.1) by $\xi\,\overline{u}_\alpha$ we have

$$\begin{aligned} &i\,\xi\,\overline{u}_\alpha\,u_{\alpha\,t} + i\,\beta\,\xi\,\overline{u}_\alpha\,u_{\alpha+3} + \omega\,\xi\,\overline{u}_\alpha\,u_{\alpha+2} + (|u|^2)_\alpha\,\xi\,u\,\overline{u}_\alpha \\ &\quad + \sum_{m=1}^{\alpha-1}\binom{\alpha}{m}(|u|^2)_{\alpha-m}\,\xi\,u_m\,\overline{u}_\alpha + \xi\,|u|^2\,|u_\alpha|^2 = 0 \end{aligned}$$

and

$$\begin{aligned} &-i\,\xi\,u_\alpha\,\overline{u}_{\alpha\,t} - i\,\beta\,\xi\,u_\alpha\,\overline{u}_{\alpha+3} + \omega\,\xi\,u_\alpha\,\overline{u}_{\alpha+2} + (|u|^2)_\alpha\,\xi\,\overline{u}\,u_\alpha \\ &\quad + \sum_{m=1}^{\alpha-1}\binom{\alpha}{m}(|u|^2)_{\alpha-m}\,\xi\,\overline{u}_m\,u_\alpha + \xi\,|u|^2\,|u_\alpha|^2 = 0. \qquad \text{(applying conjugate)} \end{aligned}$$

Subtracting and integrating over $x \in \mathbb{R}$ we have

$$
\begin{aligned}
i\,\partial_t \int_{\mathbb{R}} \xi\,|u_\alpha|^2 dx + i\,\beta \int_{\mathbb{R}} \xi\,\overline{u}_\alpha\, u_{\alpha+3} dx + i\,\beta \int_{\mathbb{R}} \xi\, u_\alpha\, \overline{u}_{\alpha+3} dx + \omega \int_{\mathbb{R}} \xi\,\overline{u}_\alpha\, u_{\alpha+2} dx \qquad (6.2)\\
-\,\omega \int_{\mathbb{R}} \xi\, u_\alpha\,\overline{u}_{\alpha+2} dx + 2\,i\,Im \int_{\mathbb{R}} \xi\,(|u|^2)_\alpha\, u\,\overline{u}_\alpha dx + 2\,i \sum_{m=1}^{\alpha-1} \binom{\alpha}{m} Im \int_{\mathbb{R}} \xi\,(|u|^2)_{\alpha-m}\, u_m\,\overline{u}_\alpha dx = 0.
\end{aligned}
$$

Each term is calculated separately, integrating by parts in the second term we have

$$
\int_{\mathbb{R}} \xi\,\overline{u}_\alpha\, u_{\alpha+3} dx = \int_{\mathbb{R}} \partial^2\xi\,\overline{u}_\alpha\, u_{\alpha+1} dx + 2 \int_{\mathbb{R}} \partial\xi\,|u_{\alpha+1}|^2 dx + \int_{\mathbb{R}} \xi\,\overline{u}_{\alpha+2}\, u_{\alpha+1} dx.
$$

The other terms are calculated in a similar way. Hence in (6.2)

$$
\begin{aligned}
\partial_t \int_{\mathbb{R}} \xi\,|u_\alpha|^2 dx - \beta \int_{\mathbb{R}} \partial^3\xi\,|u_\alpha|^2 dx + 3\,\beta \int_{\mathbb{R}} \partial\xi\,|u_{\alpha+1}|^2 dx - 2\,\omega\, Im \int_{\mathbb{R}} \partial\xi\,\overline{u}_\alpha\, u_{\alpha+1} dx\\
- \int_{\mathbb{R}} \partial_t\xi\,|u_\alpha|^2 dx + 2\,Im \int_{\mathbb{R}} \xi\,(|u|^2)_\alpha\, u\,\overline{u}_\alpha dx + 2 \sum_{m=1}^{\alpha-1} \binom{\alpha}{m} Im \int_{\mathbb{R}} \xi\,(|u|^2)_{\alpha-m}\, u_m\,\overline{u}_\alpha dx = 0
\end{aligned}
$$

such that

$$
\begin{aligned}
\partial_t \int_{\mathbb{R}} \xi\,|u_\alpha|^2 dx - \beta \int_{\mathbb{R}} \partial^3\xi\,|u_\alpha|^2 dx + 3\,\beta \int_{\mathbb{R}} \partial\xi\,|u_{\alpha+1}|^2 dx + 2\,Im \int_{\mathbb{R}} (|u|^2)_\alpha\,\xi\, u\,\overline{u}_\alpha dx\\
- \int_{\mathbb{R}} \partial_t\xi\,|u_\alpha|^2 dx + 2 \sum_{m=1}^{\alpha-1} \binom{\alpha}{m} Im \int_{\mathbb{R}} \xi\,(|u|^2)_{\alpha-m}\, u_m\,\overline{u}_\alpha dx\\
= \;\; 2\,\alpha\, Im \int_{\mathbb{R}} \partial\xi\,\overline{u}_\alpha\, u_{\alpha+1} dx \le |\omega| \int_{\mathbb{R}} \partial\xi\,|u_\alpha|^2 dx + |\omega| \int_{\mathbb{R}} \partial\xi\,|u_{\alpha+1}|^2 dx.
\end{aligned}
$$

Hence

$$
\begin{aligned}
\partial_t \int_{\mathbb{R}} \xi\,|u_\alpha|^2 dx + \int_{\mathbb{R}} [\,3\,\beta - |\omega|\,]\,\partial\xi\;|u_{\alpha+1}|^2 dx - \int_{\mathbb{R}} [\,\partial_t\xi + \beta\,\partial^3\xi + |\omega|\,\partial\xi\,]\,|u_\alpha|^2 dx\\
+ 2\,Im \int_{\mathbb{R}} (|u|^2)_\alpha\,\xi\, u\,\overline{u}_\alpha dx + 2 \sum_{m=1}^{\alpha-1} \binom{\alpha}{m} Im \int_{\mathbb{R}} \xi\,(|u|^2)_{\alpha-m} u_m\,\overline{u}_\alpha dx \le 0. \qquad (6.3)
\end{aligned}
$$

But

$$
(|u|^2)_\alpha \;\; = \;\; (\,u\,\overline{u}\,)_\alpha = \sum_{k=0}^{\alpha} \binom{\alpha}{k} u_{\alpha-k}\,\overline{u}_k = \overline{u}\, u_\alpha + \sum_{k=1}^{\alpha-1} \binom{\alpha}{k} u_{\alpha-k}\,\overline{u}_k + u\,\overline{u}_\alpha
$$

then

$$
(|u|^2)_\alpha\, u\,\overline{u}_\alpha = |u|^2\,|u_\alpha|^2 + \sum_{k=1}^{\alpha-1} \binom{\alpha}{k} u_{\alpha-k}\,\overline{u}_k\, u\,\overline{u}_\alpha + u^2\,\overline{u}_\alpha^2
$$

hence

$$
\begin{aligned}
2\;Im \int_{\mathbb{R}} (\,|u|^2\,)_\alpha\,\xi\, u\,\overline{u}_\alpha dx = 2 \sum_{k=1}^{\alpha-1} \binom{\alpha}{k} Im \int_{\mathbb{R}} \xi\, u_{\alpha-k}\,\overline{u}_k\, u\,\overline{u}_\alpha dx + 2\;Im \int_{\mathbb{R}} \xi\, u^2\,\overline{u}_\alpha^2 dx\\
\le \;\; 2 \sum_{k=1}^{\alpha-1} \binom{\alpha}{k} \int_{\mathbb{R}} \xi\,|u_{\alpha-k}|\,|u_k|\,|u|\,|u_\alpha| dx + 2 \int_{\mathbb{R}} \xi\,|u|^2\,|u_\alpha|^2 dx\\
\le \;\; 2 \sum_{k=1}^{\alpha-1} \binom{\alpha}{k} \int_{\mathbb{R}} \xi\,|u_{\alpha-k}|\,|u_k|\,|u|\,|u_\alpha| dx + 2\,||u||^2_{L^\infty(\mathbb{R})} \int_{\mathbb{R}} \xi\,|u_\alpha|^2 dx\\
\le \;\; 2\,||u||_{L^\infty(\mathbb{R})} \sum_{k=1}^{\alpha-1} \binom{\alpha}{k} \int_{\mathbb{R}} \xi\,|u_{\alpha-k}|\,|u_k|\,|u_\alpha| dx + 2\,||u||^2_{L^\infty(\mathbb{R})} \int_{\mathbb{R}} \xi\,|u_\alpha|^2 dx \qquad (6.4)
\end{aligned}
$$

hence in (6.3) we have

$$\partial_t \int_{\mathbb{R}} \xi\, |u_\alpha|^2 dx + \int_{\mathbb{R}} [3\,\beta - |\omega|\,]\, \partial\xi\, |u_{\alpha+1}|^2 dx \le \int_{\mathbb{R}} [\partial_t \xi + \beta\, \partial^3 \xi + |\omega|\, \partial\xi + c\, \xi\,]\, |u_\alpha|^2 dx$$
$$+\, 2\, c \sum_{k=1}^{\alpha-1} \binom{\alpha}{k} \int_{\mathbb{R}} \xi\, |u_{\alpha-k}|\, |u_k|\, |u|\, |u_\alpha| dx - 2 \sum_{m=1}^{\alpha-1} \binom{\alpha}{m} Im \int_{\mathbb{R}} \xi\, (|u|^2)_{\alpha-m}\, u_m\, \overline{u}_\alpha dx.$$

Using (2.3), Gagliardo-Nirenberg's inequality and standard estimates we get

$$\partial_t \int_{\mathbb{R}} \xi\, |u_\alpha|^2 dx + [3\,\beta - |\omega|\,] \int_{\mathbb{R}} \partial\xi\, |u_{\alpha+1}|^2\, dx \le c \int_{\mathbb{R}} \xi\, |u_\alpha|^2 dx. \tag{6.5}$$

Integrating (6.5) in $t \in [0,\, T_{max} = T]$ we obtain

$$\int_{\mathbb{R}} \xi\, |u_\alpha|^2 dx + [3\,\beta - |\omega|\,] \int_0^t \int_{\mathbb{R}} \partial\xi\, |u_{\alpha+1}|^2 dx\, ds \le ||u_0(x)||_\alpha^2 + \int_0^t \left(c \int_{\mathbb{R}} \xi\, |u_\alpha|^2 dx \right) ds,$$

where

$$\int_{\mathbb{R}} \xi\, |u_\alpha|^2 dx \le ||u_0(x)||_\alpha^2 + \int_0^t \left(c \int_{\mathbb{R}} \xi\, |u_\alpha|^2 dx \right) ds.$$

Using Gronwall's inequality

$$\int_{\mathbb{R}} \xi\, |u_\alpha|^2 dx \le ||u_0(x)||_\alpha^2\, e^{c\,t} \le ||u_0(x)||_\alpha^2\, e^{c\,T}$$

it follows that

$$\int_{\mathbb{R}} \xi\, |u_\alpha|^2 dx \le c = c(T,\, ||u_0(x)||_\alpha^2).$$

Then for any $T = T_{max} > 0$ there exists $c = c(T,\, ||u_0(x)||_\alpha^2)$ such that

$$||u||_\alpha^2 + [3\,\beta - |\omega|\,] \int_0^t \int_{\mathbb{R}} \partial\xi\, |u_{\alpha+1}|^2 dx\, ds \le c.$$

This concludes the proof.

7 Persistence Theorem

As a starting point for the a priori gain of regularity results that will be discussed in the next section, we need to develop some estimates for solutions of the equation (2.4) in weighted Sobolev norms. The existence of these weighted estimates is often called the persistence of a property of the initial data u_0. We show that if $u_0 \in H^3(\mathbb{R}) \cap H^L(W_{0\,i\,0})$ for $L \ge 0$, $i \ge 1$, then the solution $u(\,\cdot\,,\, t)$ evolves in $H^L(W_{0\,i\,0})$ for $t \in [0,\, T]$. The time interval of that persistence is at least as long as the interval guaranteed by the existence Theorem 5.2.

Theorem 7.1 (Persistence). *Let $|\omega| < 3\,\beta$ and let $i \ge 1$ and $L \ge 0$ be non-negative integers, $0 < T < +\infty$. Assume that u is the solution to (2.4) in $L^\infty([0,\, T] :\, H^3(\mathbb{R}))$ with initial data $u_0(x) = u(x,\, 0) \in H^3(\mathbb{R})$. If $u_0(x) \in H^L(W_{0\,i\,0})$ then*

$$u \in L^\infty\left([0,\, T] :\, H^3(\mathbb{R}) \cap H^L(W_{0\,i\,0})\right) \tag{7.1}$$

$$\int_0^T \int_{\mathbb{R}} |\partial^{L+1} u(x,\, t)|^2\, \eta\, dx\, dt < +\infty \tag{7.2}$$

where σ is arbitrary, $\eta \in W_{\sigma\, i\, 0}$ for $i \geq 1$.

Proof. We use induction on α. Let

$$u \in L^{\infty}\left([0,\,T]:\,H^{3}(\mathbb{R}) \cap H^{\alpha}(W_{0\,i\,0})\right) \quad \text{for} \quad 0 \leq \alpha \leq L.$$

We derive formally some a priori estimate for the solution where the bound, involves only the norms of u in $L^{\infty}([0,\,T]:\,H^{3}(\mathbb{R}))$ and the norms of u_0 in $H^{3}(W_{0\,i\,0})$. We do this by approximating $u(x,\,t)$ through smooth solutions and the weight functions by smooth bounded functions. By Theorem 5.2, we have

$$u(x,\,t) \in L^{\infty}([0,\,T]:\,H^{N}(\mathbb{R})) \quad \text{with} \quad N = \max\{L,\,3\}.$$

In particular, $u_j(x,\,t) \in L^{\infty}([0,\,T] \times \mathbb{R})$ for $0 \leq j \leq N-1$. To obtain (7.1) and (7.2) there are two ways of approximation. We approximate general solutions by smooth solutions, and we approximate general weight functions by bounded weight functions. The first of these procedure has already been discussed, so we shall concentrate on the second.
Given a smooth weight function $\eta(x) \in W_{\sigma,\,i-1,\,0}$ with $\sigma > 0$, we take a sequence $\eta^{\nu}(x)$ of smooth bounded weight functions approximating $\eta(x)$ from below, uniformly on any half line $(-\infty,\,c)$. Define the weight functions for the α-th induction step as

$$\xi_{\nu} = \frac{1}{(3\,\beta - |\omega|)} \int_{-\infty}^{x} \eta^{\nu}(y,\,t)\,dy$$

then the ξ_{ν} are bounded weight functions which approximate a desired weight function $\xi \in W_{0\,i\,0}$ from below, uniformly on a compact set. For $\alpha = 0$, multiplying (2.4) by $\xi_{\nu}\,\overline{u}$, we have

$$\begin{aligned}&i\,\xi_{\nu}\,\overline{u}\,u_t + i\,\beta\,\xi_{\nu}\,\overline{u}\,u_3 + \omega\,\xi_{\nu}\,\overline{u}\,u_2 + \xi_{\nu}\,|u|^4 = 0\\ &-i\,\xi_{\nu}\,u\,\overline{u}_t - i\,\beta\,\xi_{\nu}\,u\,\overline{u}_3 + \omega\,\xi_{\nu}\,u\,\overline{u}_2 + \xi_{\nu}\,|u|^4 = 0. \quad \text{(applying conjugate)}\end{aligned}$$

Subtracting and integrating over $x \in \mathbb{R}$ we have

$$\begin{aligned}&i\,\partial_t \int_{\mathbb{R}} \xi_{\nu}\,|u|^2 dx - i \int_{\mathbb{R}} \partial_t\xi_{\nu}\,|u|^2 dx + i\,\beta \int_{\mathbb{R}} \xi_{\nu}\,\overline{u}\,u_3 dx + i\,\beta \int_{\mathbb{R}} \xi_{\nu}\,u\,\overline{u}_3 dx\\ &+\omega \int_{\mathbb{R}} \xi_{\nu}\,\overline{u}\,u_2 dx - \omega \int_{\mathbb{R}} \xi_{\nu}\,u\,\overline{u}_2 dx = 0. \qquad (7.3)\end{aligned}$$

Each term is treated separately, integrating by parts in the third term we have

$$\int_{\mathbb{R}} \xi_{\nu}\,\overline{u}\,u_3 dx \;\;=\;\; \int_{\mathbb{R}} \partial^2\xi_{\nu}\,\overline{u}\,u_1 dx + 2 \int_{\mathbb{R}} \partial\xi_{\nu}\,|u_1|^2 dx + \int_{\mathbb{R}} \xi_{\nu}\,\overline{u}_2\,u_1 dx.$$

The other terms are calculated in a similar way. Hence in (7.3) we have

$$\begin{aligned}&\partial_t \int_{\mathbb{R}} \xi_{\nu}\,|u|^2 dx - \int_{\mathbb{R}} \partial_t\xi_{\nu}\,|u|^2 dx - \beta \int_{\mathbb{R}} \partial^3\xi_{\nu}\,|u|^2 dx + 3\,\beta \int_{\mathbb{R}} \partial\xi_{\nu}\,|u_1|^2 dx\\ &= \;2\,\omega\,Im \int_{\mathbb{R}} \partial\xi_{\nu}\,\overline{u}\,u_1 dx \leq |\omega| \int_{\mathbb{R}} \partial\xi_{\nu}\,|u|^2 dx + |\omega| \int_{\mathbb{R}} \partial\xi_{\nu}\,|u_1|^2 dx.\end{aligned}$$

Then, using (2.3) we obtain

$$\begin{aligned}&\partial_t \int_{\mathbb{R}} \xi_{\nu}\,|u|^2 dx + \int_{\mathbb{R}} [3\,\beta - |\omega|]\,\partial\xi_{\nu}\,|u_1|^2 dx\\ &\leq \;\int_{\mathbb{R}} [\partial_t\xi_{\nu} + \beta\,\partial^3\xi_{\nu} + |\omega|\,\partial\xi_{\nu}]\,|u|^2 dx \leq c \int_{\mathbb{R}} \xi_{\nu}\,|u|^2 dx\end{aligned}$$

thus

$$\partial_t \int_{\mathbb{R}} \xi_{\nu}\,|u|^2 dx \leq c \int_{\mathbb{R}} \xi_{\nu}\,|u|^2 dx.$$

We apply Gronwall's Lemma to conclude that

$$\partial_t \int_{\mathbb{R}} \xi_\nu \, |u|^2 dx \le c(T, \, ||u_0||). \tag{7.4}$$

for $0 \le t \le T$, and c not depending on $\beta > 0$, the weighted estimate remains true for $\beta \to 0$.
Now, we assume that the result is true for $(\alpha - 1)$ and we prove that it is true for α. To prove this, we start from the main inequality (3.1) with ξ and η given by ξ_ν and η_ν respectively.

$$\partial_t \int_{\mathbb{R}} \xi_\nu \, |u_\alpha|^2 dx + \int_{\mathbb{R}} \eta_\nu \, |u_{\alpha+1}|^2 dx + \int_{\mathbb{R}} \theta_\nu \, |u_\alpha|^2 dx + \int_{\mathbb{R}} R_\alpha dx \le 0$$

where

$$\begin{array}{rcl} \eta_\nu & = & (3\beta - |\omega|\,)\,\partial\xi_\nu \qquad \text{for} \qquad |\omega| < 3\,\beta \\ \theta_\nu & = & -\,[\,\partial_t \xi_\nu + \beta\,\partial^3 \xi_\nu + |\omega|\,\partial\xi_\nu + c_0\,\xi_\nu\,] \qquad \text{where} \quad c_0 = ||u||^2_{L^\infty(\mathbb{R})} \\ R_\alpha & = & R_\alpha(|u_\alpha|, \, |u_{\alpha-1}|, \, \dots) \end{array}$$

then

$$\begin{aligned} \partial_t \int_{\mathbb{R}} \xi_\nu \, |u_\alpha|^2 dx + \int_{\mathbb{R}} \eta_\nu \, |u_{\alpha+1}|^2 dx &\le - \int_{\mathbb{R}} \theta_\nu \, |u_\alpha|^2 dx - \int_{\mathbb{R}} R_\alpha dx \\ \le \quad \left| - \int_{\mathbb{R}} \theta_\nu \, |u_\alpha|^2 dx - \int_{\mathbb{R}} R_\alpha dx \right| &\le \int_{\mathbb{R}} |\theta_\nu| \, |u_\alpha|^2 dx + \int_{\mathbb{R}} |R_\alpha| dx. \end{aligned}$$

Using (2.3) in the first part of the right hand side we obtain

$$\int_{\mathbb{R}} \theta_\nu \, |u_\alpha|^2 dx \le c \int_{\mathbb{R}} \xi_\nu \, |u_\alpha|^2 dx$$

thus

$$\partial_t \int_{\mathbb{R}} \xi_\nu \, |u_\alpha|^2 dx + \int_{\mathbb{R}} \eta_\nu \, |u_{\alpha+1}|^2 dx \le c \int_{\mathbb{R}} \xi_\nu \, |u_\alpha|^2 dx + \int_{\mathbb{R}} |R_\alpha| dx. \tag{7.5}$$

According to (3.8), $\int_{\mathbb{R}} R_\alpha \, dx$ contains a term of the form

$$\int_{\mathbb{R}} \xi_\nu \, u_{\nu_1} \, \overline{u}_{\nu_2} \, \overline{u}_\alpha dx. \tag{7.6}$$

We estimate the term

$$\int_{\mathbb{R}} \xi_\nu \, u_{\nu_1} \, \overline{u}_{\nu_2} \, \overline{u}_\alpha \, dx \quad \text{for} \quad \nu_1 + \nu_2 = \alpha. \tag{7.7}$$

Let $\nu_2 \le \alpha - 2$. Integrating by parts one time in (7.7) we have

$$\begin{aligned} \int_{\mathbb{R}} \xi_\nu \, u_{\nu_1} \, \overline{u}_{\nu_2} \, \overline{u}_\alpha \, dx \quad &= \quad - \int_{\mathbb{R}} \partial\xi_\nu \, u_{\nu_1} \, \overline{u}_{\nu_2} \, \overline{u}_{\alpha-1} \, dx - \int_{\mathbb{R}} \xi_\nu \, u_{\nu_1+1} \, \overline{u}_{\nu_2} \, \overline{u}_{\alpha-1} \, dx \\ &\quad - \int_{\mathbb{R}} \xi_\nu \, u_{\nu_1} \, \overline{u}_{\nu_2+1} \, \overline{u}_{\alpha-1} \, dx. \end{aligned}$$

We estimates the first term in the right hand side in (7.7). Using Holder's inequality and standard estimates we obtain

$$c \left[\left(\int_{\mathbb{R}} \xi_\nu \, |u_{\nu_2+1}|^2 dx \right)^{1/2} + \left(\int_{\mathbb{R}} \xi_\nu \, |u_{\nu_2}|^2 dx \right)^{1/2} \right] \left(\int_{\mathbb{R}} \xi_\nu \, |u_{\alpha-1}|^2 dx \right)^{1/2} \tag{7.8}$$

where (7.8) is bounded by hypothesis. The other terms are estimates in a similar way. Now suppose that $\nu_1 = \nu_2 = \alpha - 1$, then in (7.7) we have

$$\int_{\mathbb{R}} \xi_\nu \, u_{\alpha-1} \, \overline{u}_{\alpha-1} \, \overline{u}_\alpha dx,$$

hence

$$\left| \int_{\mathbb{R}} \xi_\nu \, |u_{\alpha-1}|^2 \, \overline{u}_\alpha dx \right| \leq ||u_{\alpha-1}||_{L^\infty(\mathbb{R})} \left(\int_{\mathbb{R}} \xi_\nu \, |u_{\alpha-1}|^2 dx \right)^{1/2} \left(\int_{\mathbb{R}} \xi_\nu \, |u_\alpha|^2 dx \right)^{1/2}$$

where $||u_{\alpha-1}||_{L^\infty(\mathbb{R})}$ is bounded by hypothesis, and the estimate is complete. In a similar way we estimate all the other terms of R_α. Using these estimates in (7.5) and applying Gronwall's argument, we obtain for $0 \leq t \leq T$

$$\partial_t \int_{\mathbb{R}} \xi_\nu \, |u_\alpha|^2 dx + \int_{\mathbb{R}} \eta_\nu \, |u_{\alpha+1}|^2 dx \leq c_0 \, e^{c_1 \, t} \left(\int_{\mathbb{R}} \xi_\nu \, |\partial^\alpha u_0(x)|^2 dx + 1 \right)$$

where c_0 and c_1 are independent of ν and such that letting the parameter $\nu \to 0$ the desired estimate (7.2) is obtained.

8 Main Theorem

In this section we state and prove our main theorem, which states that if the initial data $u(x, 0)$ decays faster than polynomially on $\mathbb{R}^+ = \{x \in \mathbb{R} : \; x > 0\}$ and possesses certain initial Sobolev regularity, then the solution $u(x, t) \in C^\infty$ for all $t > 0$.

If η is an arbitrary weight function in $W_{\sigma \; i \; k}$, then by Lemma 3.2, there exists $\xi \in W_{\sigma, \; i+1, \; k}$ which satisfies (3.1). For the main theorem, we take $4 \leq \alpha \leq L + 2$. For $\alpha \leq L + 4$, we take

$$\eta \in W_{\sigma, \, L-\alpha-2, \, \alpha-3} \implies \xi \in W_{\sigma, \, L-\alpha-3, \, \alpha-3}. \tag{8.1}$$

Lemma 8.1(Estimate of error terms). *Let $4 \leq \alpha \leq L+2$ and the weight functions be chosen as in (8.1), then*

$$\left| \int_0^T \int_{\mathbb{R}} (\theta \, |u_\alpha|^2 + R_\alpha) dx \, dt \right| \leq c, \tag{8.2}$$

where c depends only on the norms of u in

$$L^\infty([0, T] : \; H^\beta(W_{\sigma, \, L-\beta+3, \, \beta-3})) \cap L^2([0, T] : \; H^{\beta+1}(W_{\sigma, \, L-\beta+2, \, \beta-3}))$$

for $3 \leq \beta \leq \alpha - 1$, and the norms of u in $L^\infty([0, T] : \; H^3(W_{0 \; L \; 0}))$.

Proof. We must estimate both R_α and θ. We begin with a term in R_α of the form

$$\xi \, |u_{\nu_1}| \; |u_{\nu_2}| \; |u_\alpha| \tag{8.3}$$

assuming that $\nu_1 \leq \alpha - 2$.

By the induction hypothesis, u is bounded in $L^\infty([0, T] : \; H^\beta(W_{\sigma, \, L-(\beta-3)^+, \, (\beta-3)^+}))$ for $0 \leq \beta \leq \alpha - 1$. By Lemma 2.1,

$$\sup_{t \geq 0} \sup_{x \in \mathbb{R}} \zeta \, |u_\beta|^2 < +\infty \tag{8.4}$$

for $0 \leq \beta \leq \alpha - 2$ and $\zeta \in W_{\sigma, \, L-(\beta-2)^+, \, (\beta-2)^+}$. We estimate $|u_{\nu_1}|$ using (8.4). We estimate $|u_{\nu_2}|$ and $|u_\alpha|$ using the weighted L^2 bounds

$$\int_0^T \int_{\mathbb{R}} \zeta \, |u_{\nu_2}|^2 dx \, dt < +\infty \quad \text{for} \quad \zeta \in W_{\sigma, \, L-(\nu_2-3)^+, \, (\nu_2-4)^+} \tag{8.5}$$

and the same with ν_2 replaced by α. It suffices to check the powers to t, the powers of x as $x \to +\infty$ and the exponential of x as $x \to -\infty$.

For $x > 1$. In the (8.3) term, the factor ξ contributed according to (8.1)

$$\xi(x,\, t) = t^{\alpha-3}\, x^{(L-\alpha+3)}\, t^{-(\alpha-3)}\, x^{-(L-\alpha+3)} \xi(x,\, t) \leq c_2\, t^{\alpha-3}\, x^{(L-\alpha+3)} \quad (\text{using}(2.3))$$

then $\xi\, |u_{\nu_1}|\, |u_{\nu_2}|\, |u_\alpha| \leq c_2\, t^{\alpha-3}\, x^{(L-\alpha+3)} |u_{\nu_1}|\, |u_{\nu_2}|\, |u_\alpha|$. Moreover

$$\begin{aligned} |u_{\nu_1}|\, |u_{\nu_2}|\, |u_\alpha| \;\; = \;\; & t^{\frac{(\nu_1-2)^+}{2}}\, x^{\frac{L-(\nu_1-2)^+}{2}}\, t^{\frac{-(\nu_1-2)^+}{2}}\, x^{\frac{(L-(\nu_1-2)^+)}{2}}\, |u_{\nu_1}| \times \\ & t^{\frac{(\nu_2-4)^+}{2}}\, x^{\frac{L-(\nu_2-3)^+}{2}}\, t^{\frac{-(\nu_2-4)^+}{2}}\, x^{\frac{(L-(\nu_2-3)^+)}{2}}\, |u_{\nu_2}| \times \\ & t^{\frac{(\alpha-4)^+}{2}}\, x^{\frac{L-(\alpha-3)^+}{2}}\, t^{\frac{-(\alpha-4)^+}{2}}\, x^{\frac{(L-(\alpha-3)^+)}{2}}\, |u_\alpha|. \end{aligned}$$

tt follows that

$$\begin{aligned} & \xi\, |u_{\nu_1}|\, |u_{\nu_2}|\, |u_\alpha| \\ & \quad \leq c_2\, t^M\, x^T\, t^{\frac{(\nu_1-2)^+}{2}}\, x^{\frac{L-(\nu_1-2)^+}{2}}\, |u_{\nu_1}|\, t^{\frac{(\nu_2-4)^+}{2}}\, x^{\frac{L-(\nu_2-3)^+}{2}}\, |u_{\nu_2}|\, t^{\frac{(\alpha-4)^+}{2}}\, x^{\frac{L-(\alpha-3)^+}{2}}\, |u_\alpha| \end{aligned} \tag{8.6}$$

where

$$M = \alpha - 3 - \frac{1}{2}(\nu_1 - 2)^+ - \frac{1}{2}(\nu_2 - 4)^+ - \frac{1}{2}(\alpha - 4)^+$$

and

$$T = (L - \alpha + 3) - \frac{1}{2}(L - (\alpha - 3)^+) - \frac{1}{2}(L - (\nu_2 - 3)^+) - \frac{1}{2}(L - \nu_1 - 2)^+).$$

Claim. $M \geq 0$ is large enough, that the extra power of t can be omitted

$$\begin{aligned} 2\, M \;\; & = \;\; 2\,\alpha - 6 - (\nu_1 - 2)^+ - (\nu_2 - 4)^+ - (\alpha - 4)^+ \\ & = \;\; \alpha - 2 - (\nu_1 - 2)^+ - (\nu_2 - 4)^+ \\ & = \;\; \alpha - 2 - \nu_1 + 2 - \nu_2 + 4 = \alpha + 4 - (\nu_1 + \nu_2) \\ & = \;\; \alpha + 4 - \alpha = 4 \geq 0. \end{aligned}$$

Claim. $T \leq 0$ is such that the extra power of t can be omitted.

$$\begin{aligned} 2\, T \;\; & = \;\; 2\, L - 2\,\alpha + 6 - L + (\alpha - 3)^+ - L + (\nu_2 - 3)^+ - L + (\nu_1 - 2)^+ \\ & = \;\; -\, L - \alpha + \nu_1 + \nu_2 - 2 = -\, L - \alpha + \alpha - 2 \\ & = \;\; -(L + 2) \leq 0. \end{aligned}$$

Now, we study the behavior as $x \to -\infty$. Since each factor $u_{\nu_j} (j = 1,\, 2)$ must grow slower that an exponential $e^{\sigma'\,|x|}$ and ξ decays as an exponential $e^{-\sigma\,|x|}$, we simply need to choose the appropriate relationship σ and σ' at each induction step. The analysis will be completed with the case where $\nu_1 \geq \alpha - 1$. Then, in (3.9), if $2(\alpha - 1) \leq \alpha$, but $\alpha \geq 3$. So this possibility is impossible. For $x < 1$ the estimate is similar, except for an exponential weight. The analysis of all terms of R_α is estimated in a similar form. This completes the estimate of R_α.

Now, we estimate the term $\theta\, |u_\alpha|^2$ where θ is given in (3.1). We have that θ involves derivatives of u only up to order one, and hence, $\theta\, |u_\alpha|^2$ is a sum of terms of the same type which we have already encountered in R_α. So, its integral can be bounded in the same type. Indeed, (3.1) shows that θ depends on ξ_t, $\partial^3 \xi$ and derivatives of lower order. By using (3.6) we have the claim.

Theorem 8.2(Main Theorem). *Let* $|\omega| < 3\,\beta$, $T > 0$ *and* $u(x,\, t)$ *be a solution of* (2.4) *in the region* $\mathbb{R} \times [0,\, T]$ *such that*

$$u \in L^\infty([0,\, T] :\, H^3(W_{0\; L\; 0})) \tag{8.7}$$

for some $L \geq 2$. *Then*

$$u \in L^\infty([0,\,T]:\,H^{3+l}(W_{\sigma,\,L-l,\,l})) \cap L^2([0,\,T]:\,H^{4+l}(W_{\sigma,\,L-l-1,\,l})) \tag{8.8}$$

for all $0 \leq l \leq L-1$ *and all* $\sigma > 0$.

Remark. If the assumption (8.7) holds for all $L \geq 2$, the solution is infinitely differentiable in the x-variable. ¿From (2.4) we have that the solution is C^∞ in both variables. We are also quantifying the gain of each derivative by the degree of vanishing of the initial data at infinity.

Proof. We use induction on α. For $\alpha = 3$, let u be a solution of (2.4) satisfying (8.7). Therefore, $u_t \in L^\infty([0,\,T]:\,L^2(W_{0\,L\,0}))$ where $u \in L^\infty([0,\,T]:\,H^3(W_{0\,L\,0}))$ and $u_t \in L^\infty([0,\,T]:\,L^2(W_{0\,L\,0}))$. Then $u \in C([0,\,T]:\,L^2(W_{0\,L\,0})) \cap C_w([0,\,T]:\,H^3(W_{0\,L\,0}))$. Hence, $u:[0,\,T] \longmapsto H^3(W_{0\,L\,0})$ is a weakly continuous function. In particular, $u(\,\cdot\,,\,t) \in H^3(W_{0\,L\,0})$ for all t. Let $t_0 \in (0,\,T)$ and $u(\,\cdot\,,\,t_0) \in H^3(W_{0\,L\,0})$, then there are $\{u_0^{(n)}\} \subseteq C_0^\infty(\mathbb{R})$ such that $u_0^{(n)}(\,\cdot\,) \to u(\,\cdot\,,\,t_0)$ in $H^3(W_{0\,L\,0})$. Let $u^{(n)}(x,\,t)$ be a unique solution of (2.4) with $u^{(n)}(x,\,t_0) = u_0^{(n)}$. Then by Theorem 5.1 and 5.2, there exists u in a time interval $[t_0,\,t_0+\delta]$ where $\delta > 0$ does not depend on n and u is a unique solution of (2.4), $u^{(n)} \in L^\infty([t_0,\,t_0+\delta]:\,H^3(W_{0\,L\,0}))$ with $u^{(n)}(x,\,t_0) \equiv u_0^{(n)}(x) \to u(x,\,t_0) \equiv u_0(x)$ in $H^3(W_{0\,L\,0})$. Now, by Theorem 7.1, we have

$$u^{(n)} \in L^\infty([t_0,\,t_0+\delta]:\,H^3(W_{0\,L\,0})) \cap L^2([t_0,\,t_0+\delta]:\,H^4(W_{\sigma,\,L-1,\,0}))$$

with a bound that depends only on the norm of $u_0^{(n)}$ in $H^3(W_{0\,L\,0})$. Furthermore, Theorem 7.1 guarantees the non-uniform bounds

$$\sup_{[t_0,\,t_0+\delta]}\ \sup_x\,(1+|x_+|)^k\,|\,\partial^\alpha u^{(n)}(x,\,t)\,| < +\infty$$

for each n, k and α. The main inequality (3.1) and the estimate (8.2) are therefore valid for each $u^{(n)}$ in the interval $[t_0,\,t_0+\delta]$. η may be chosen arbitrarily in its weight class (8.1) and then ξ is defined by (3.7) and the constant c_1, c_2, c_3, c_4 are independent of n. From (3.1) and (8.1) we have

$$\sup_{[t_0,\,t_0+\delta]}\int_{\mathbb{R}}\xi\,|u_\alpha^{(n)}|^2dx + \int_{t_0}^{t_0+\delta}\int_{\mathbb{R}}\eta\,|u_{\alpha+1}^{(n)}|^2dx \leq c \tag{8.9}$$

where by (8.2), c is independent of n. The estimate (8.9) is proved by induction for $\alpha = 3,\,4,\,5,\ldots$ Thus $u^{(n)}$ is also bounded in

$$L^\infty([t_0,\,t_0+\delta]:\,H^\alpha(W_{\sigma,\,L-\alpha+3,\,\alpha-3})) \cap L^2([t_0,\,t_0+\delta]:\,H^{\alpha+1}(W_{\sigma,\,L-\alpha+2,\,\alpha-3})) \tag{8.10}$$

for $\alpha \geq 3$. Since $u^{(n)} \to u$ in $L^\infty([t_0,\,t_0+\delta]:\,H^3(W_{0\,L\,0}))$. By Corollary 5.3 it follows that u belongs to the space (8.10). Since δ is fixed, this result is valid over the whole interval $[0,\,T]$.

References

[1] E. Bisognin, V. Visognin and O. Vera. *Stabilization of solutions for the higher order nonlinear Schrödinger equation with localized damping.* EJDE. Vol. 2007, 6(2007) 1-18.

[2] V. Bisognin and O. Vera. *On the unique continuation property for the higher order nonlinear Schrödinger equation with constant coefficients,* Turk. J. Math. 30, 2006, 1-38.

[3] J. Bona and J.C. Saut. *Dispersive blow-up of solutions of generalized Korteweg - de Vries equation,* Journal of Diff. equations, 103, 1993, 3-57.

[4] H. Cai. *Dispersive smoothing effect for generalized and high order KdV type equations,* Journal of Diff. Eqns., 136, 1997, 191-221.

[5] X. Carvajal. *Local well-posedness for a higher order nonlinear Schrödinger equation in Sobolev space of negative indices,* EJDE, 13, 2004, 1-10.

[6] X. Carvajal and F. Linares. *A higher order nonlinear Schrödinger equation with variable coefficients,* Differential and Integral Equations, 16, 2003, 1111 - 1130.

[7] W. Craig and J. Goodman. Linear dispersive equations ofAiry Type, J. Diff. equations., Vol. 87(1990) 38-61.

[8] W. Craig and J. Goodman. *Linear dispersive equations of Airy Type,* J. Diff. equations, 87, 1990, 38-61.

[9] W. Craig, T. Kappeler and W. Strauss. *Infinite gain of regularity for dispersive evolution equations,* Microlocal Analysis and Nonlinear waves, I.M.A., 30, 1991, 47-50.

[10] W. Craig, T. Kappeler and W. Strauss. *Gain of regularity for equations of Korteweg - de Vries type,* Ann. Inst. Henri Poincaré, 2, 1992, 147-186.

[11] P. Constantin and J.C. Saut. *Local smoothing properties of dispersive equations,* Journal A.M.S., 1, 1988, 413-439.

[12] J. Ginibre and G. Velo. *Conmutator expansions and smoothing properties of Poincaré* Vol. 51, 2, 1989, 221-229.

[13] A. Hasegawa and Y. Kodama. *Nonlinear pulse propagation in a monomode dielectric guide,* IEEE. J. Quant. Elect., 23, 1987, 510-524.

[14] N. Hayashi, K. Nakamitsu and M. Tsutsumi. *On solutions on the initial value problem for the nonlinear Schrödinger equations in One Space Dimension,* Math. Z., 192, 1986, 637-650.

[15] N. Hayashi, K. Nakamitsu and M. Tsutsumi. *On solutions of the initial value problem for nonlinear Schrödinger equations,* J. of Funct. Anal., 71, 1987, 218-245.

[16] N. Hayashi and T. Ozawa. *Smoothing effect for some Schrödinger equations,* J. of Funct. Anal., 85, 1989, 307-348.

[17] T. Kato. *On the Cauchy problem for the (generalized) Korteweg - de Vries equations,* Adv. in Math. Suppl. Studies, Studies in Appl. Math., 8, 1983, 93-128.

[18] C. Kenig, G. Ponce and L. Vega. *On the (generalized) Korteweg - de Vries equation,* Duke Math. J., Vol. 59, 3, 1989, 585-610.

[19] C. Kenig, G. Ponce and L. Vega. *Oscillatory integrals and regularity equations,* Indiana Univ. Math. J., 40, 1991, 33-69.

[20] Y. Kodama. *Optical solitons in a monomode fiber,* J. Phys. Stat., 39, 1985, 597-614.

[21] S. N. Kruzhkov and A.V. Faminskii. *Generalized solutions to the Cauchy problem for the Korteweg - de Vries equations,* Math. U.S.S.R. Sbornik, 48, 1984, 93-138.

[22] C. Laurey. *Le problème de Cauchy pour une équation de Schrödinger non-linéaire de ordre 3,* C. R. Acad. Sci. Paris, 315, 1992, 165-168.

[23] J. Levandosky. *Smoothing properties of nonlinear dispersion equation in two spatial dimensions,* Journal of Differential Equations, 175, 2001, 275-352.

[24] J. L. Lions. *Quelques méhodes de résolution des problemes aux limites non linéaires,* Gauthiers-Villars.

[25] G. Ponce. *Regularity of solutions to nonlinear dispersive equations,* J. Diff. Eq., 78, 1989, 122-135.

[26] J.C. Saut and R. Temam. *Remark on the Korteweg - de Vries equation,* Israel J. Math., 24, 1976, 78-87.

[27] P. Sjolin. *Regularity of solutions to the Schrödinger equation,* Duke Math. J., 55, 1987, 699-715.

[28] G. Staffilani. *On the generalized Korteweg - de Vries type equation,* Diff, and Int, Eqns., Vol. 10, 4, 1997, 777-796.

[29] R. Temam. *Sur un probleme non-lineaire,* J. Math. Pures Appl., 48(1969) 159-172.

[30] O. Vera. *Gain of regularity for an nonlinear dispersive equation Korteweg-de Vries-Burgers type,* Proyecciones, Vol. 19, 3, 2000, 207-226.

[31] O. Vera. *Existence and uniqueness for a nonlinear dispersive equation,* Divulgaciones Científicas, Vol 10, 2, 2002, 131-148.

[32] O. Vera. *Gain of regularity for a nonlinear dispersive equation,* Series A: Mathematical Sciences, 9, 2003, 53-62.

[33] O. Vera. *Gain of regularity for a Korteweg-de Vries-Kawahara type equation,* EJDE, Vol 2004, 71, 2004, 1-24.