H(CURL) ELEMENTS ON HEXAHEDRAL AND VECTOR ABCS FOR UNBOUNDED MICROWAVE PROBLEMS

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H(curl) Elements on Hexahedral and Vector A.B.C.'s for Unbounded Microwave Problems

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Abstract **-In this paper we present a 3D Engquist-Majda type Absorbing Boundary Conditions version that can preserve the symmetry of the finite element matrix. This vector ABC's is coupled with the vector wave equation. The formulation obtained uses conforming finite elements in *H*(curl) on hexahedral elements. The accuracy of this alternative vector ABC's is evaluated with the radiation from an infinitesimal current dipole**

I. INTRODUCTION

The finite element (F.E.) method is an efficient way of solving Maxwell's equations. However, by nature of this method, the analysis region is bounded. Absorbing boundary conditions (A.B.C.'s) are added on the outer boundary to represent the outside of the finite element domain [1-2]. Two three-dimensional (3D) vector A.B.C.'s symmetric versions were presented in [3-4]. However these Bayliss-Turkel A.B.C.'s work on spherical boundary which generates large amounts of free-space and many unknowns, especially in 3D. Vector A.B.C.'s which can be used on rectangular exterior boundary were developed to reduce the studied region [5]. But this Engquist-Majda (EM) type A.B.C. version is nonsymmetric. An alternative symmetric version proposed for a two-dimensional vector formulation [6] is extended to three-dimensional analysis in this paper.

Mixed finite elements conforming in $\boldsymbol{H}$(curl) of first order, R1 and P1 on Hexahedral [7-8], are used to satisfy continuity conditions at interfaces of materials.

II. THE F.E. FORMULATION

In this section we consider the problem of finding an electromagnetic field **E**, **H** in inhomogeneous medium with perfect electric conductors (pec), Ω_{pec}, obeying the time-harmonic Maxwell equations. To formulate the problem via the FEM, it is necessary to enclose the objects with an artificial outer boundary Γ on which the Silver-Müller radiation condition at infinity is enforced via an outer boundary operator $\boldsymbol{T}$. The problem in the interior domain, Ω, can be mathematically formulated in terms of **H**, for example, as

$$\nabla\times\mathbf{H} = j\omega\varepsilon_0\varepsilon_r\mathbf{E} \qquad \text{in } \Omega/\Omega_{pec}, \qquad (1)$$

$$\nabla\times\varepsilon_r^{-1}\nabla\times\mathbf{H} - k_0^2\mu_r\mathbf{H} = -j\omega\varepsilon_0\mathbf{Jm}, \quad \text{in } \Omega/\Omega_{pec}, \qquad (2)$$

$$\mathbf{n}\times\nabla\times\mathbf{H} = \boldsymbol{T}(\mathbf{H}) \qquad \text{on } \Gamma, \qquad (3)$$

where **n** is the unit outward normal to the contour Γ.

The Galerkin form obtained by weighting the vector wave equation with a right hand side source term (2) is given by:

$$\int_\Omega\left[\varepsilon_r^{-1}(\nabla\times\mathbf{H})\cdot(\nabla\times\mathbf{W}) - k_0^2\mu_r\mathbf{H}\cdot\mathbf{W}\right]d\Omega + \oint_\Gamma \boldsymbol{T}(\mathbf{H})\cdot\mathbf{W}\,d\Gamma = -j\omega\varepsilon_0\int_\Omega \mathbf{Jm}\cdot\mathbf{W}d\Omega \qquad (4)$$

where the magnetic field **H** is taken as unknown and **W** is an arbitrary vector weighting function generated by the basis vectors. The source term **Jm** represents magnetic polarization induced source or equivalent distribution currents.

III. DERIVATION OF 3D VECTOR ENGQUIST-MAJDA ABC'S

We shall use, on the boundary Γ, local cartesian coordinates (n, τ, ν) and associate with them unit vectors ($\mathbf{n}$, τ, ν)

The tangential part $\mathbf{A_t}$ of a vector field **A** will be defined as

$$\mathbf{A_t} = A_\tau\tau + A_\nu\nu \qquad (5)$$

A. Deriving unsymmetric ABC

The three dimensional scalar Engquist-Majda condition is a natural extension of the two-dimensional one [5]. This condition may more generally be written in local cartesian coordinates (n, τ, ν) as

$$-\frac{\partial\psi}{\partial n} = \alpha\psi + \beta\nabla_t^2\psi \qquad (6)$$

which is formally the one obtained in a two-dimensional case [2]. The operator ∇_t^2 is the tangential laplacian with respect to local cartesian coordinates (τ, ν) tangential to ΓThe coefficients α and β are given by

$$\begin{cases}\alpha = jkp_0\\ \beta = -j\dfrac{p_2}{k}\end{cases} \qquad (7)$$

When one knows *a priori* or *a posteriori* propagation directions of wave, it is possible to make the choice of coefficients p_0 and p_2 optimal for two specific directions. The choice $p_0 = 1$ and $p_2 = -1/2$, correspond to the second order Engquist-Majda condition and it is optimal for waves impacting the boundary Γ at near normal incidence.

For solutions of vector wave equation (2), the Silver-Müller radiation condition at infinity is equivalent to the one of Sommerfeld [9]. Then the derivation of vector ABC's for vector field **H**, solution of equation (2), can easily be achieved from scalar ones. As each rectangular component of **H** is solution of scalar Helmholtz equation and satisfies

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Sommerfeld condition, it follows that **H** obeys (6). Thus, tangential parts of **H** satisfies in particular,

$$-\frac{\partial \mathbf{H}_t}{\partial n} = \alpha \mathbf{H}_t + \beta \nabla_t^2 \mathbf{H}_t \tag{8}$$

Using the tangential parts of the vector identity

$$\nabla(\mathbf{n}\cdot\mathbf{A}) = (\mathbf{n}\cdot\nabla)\mathbf{A} + \mathbf{n}\times\nabla\times\mathbf{A} \tag{9}$$

and considering equation (9), we obtain the vector EM condition

$$\mathbf{n}\times\nabla\times\mathbf{H} = \alpha\mathbf{H}_t + \beta\nabla_t^2\mathbf{H}_t + \nabla_t(\mathbf{n}\cdot\mathbf{H}) \tag{10}$$

which is also formally the one obtained in a two-dimensional case [6].
Using the vector identity

$$\nabla_t^2\mathbf{A}_t = \nabla_t(\nabla\cdot\mathbf{A}_t) - \nabla\times(\nabla_t\times\mathbf{A}_t)\,, \tag{11}$$

equation (11) may be written as:

$$\mathbf{n}\times\nabla\times\mathbf{H} = \alpha\mathbf{H}_t + \beta\nabla_t(\nabla\cdot\mathbf{H}_t) - \beta\nabla\times(\nabla_t\times\mathbf{H}_t)$$
$$+\nabla_t(\mathbf{n}\cdot\mathbf{H})\,, \tag{12}$$

which is similar to the one of 3D vector Bayliss-Turkel ABC's kind [3-4].

B. Coupling with FEM formulation

The vector Engquist-Majda (10) is used to approximate the outer boundary operator ***T***. After integration by parts, the surface integral term in (4) is then expanded as :

$$\oint_\Gamma \boldsymbol{T}(\mathbf{H})\cdot\mathbf{W}\,d\Gamma = \alpha\oint_\Gamma \mathbf{H}_t\cdot\mathbf{W}_t\,d\Gamma$$
$$-\beta\oint_\Gamma\left(\frac{\partial\mathbf{H}_t}{\partial\tau}\cdot\frac{\partial\mathbf{W}_t}{\partial\tau} + \frac{\partial\mathbf{H}_t}{\partial\nu}\cdot\frac{\partial\mathbf{W}_t}{\partial\nu}\right)d\Gamma + \beta\sum_{i=1}^{6}\oint_{\partial\Gamma_i}\frac{\partial\mathbf{H}_t}{\partial m}\cdot\mathbf{W}_t\,dl$$
$$+\oint_\Gamma\nabla_t(\mathbf{n}\cdot\mathbf{H})\cdot\mathbf{W}_t\,d\Gamma \tag{13}$$

where Γ is the outer rectangular surface boundary of the FE domain Γ. Γ_i is one of the six faces of Γ. The contour line $\partial\Gamma_i$ of the face Γ_i has unit outward normal vector **m** (see Fig.1.).

The line and the last surface integral terms make the matrix resulting from the discretization of (13) non-symmetric. As it is more convenient to stock and to solve a sparse symmetric system, an attempt is made below to derive a symmetric vector Engquist-Majda ABC.

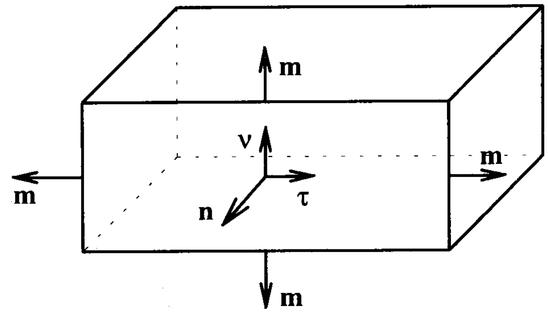

Fig. 1 Definition of the outer rectangular surface.

C. Deriving symmetric ABC

A first method suggested in [4], leads to the following approximation of the first-order derivative term in (10) :

$$\nabla_t(\mathbf{n}\cdot\mathbf{H}) \approx -\frac{\alpha}{k^2}\nabla_t(\nabla\cdot\mathbf{H}_t) \tag{14}$$

Another approach to derive an approximation of $\nabla_t(\mathbf{n}\cdot\mathbf{H})$ is presented here:

- As $(\mathbf{n}\cdot\mathbf{H})$ is a scalar radiation field [8], it obeys (6), and since $\nabla\cdot\mathbf{H} = 0$, we have

$$\nabla\cdot\mathbf{H}_t = \alpha(\mathbf{n}\cdot\mathbf{H}) + \beta\nabla_t^2(\mathbf{n}\cdot\mathbf{H}) \tag{15}$$

Taking the tangential gradient of equation (15) and neglecting the higher order derivatives in the resulting expression since the condition (10) involves at most second order derivatives, we get the approximation

$$\nabla_t(\nabla\cdot\mathbf{H}_t) \approx \alpha\nabla_t(\mathbf{n}\cdot\mathbf{H}) \tag{16}$$

Equations (14) and (16) are compatible for $\alpha = jk$. In this sense the two approachs are complementary and reduce to impose first-order EM condition on normal component of vector field **H**, H_n. Then each of the approximations (14) or (16) affects essentially the component H_n. The residual error of (16) is not available since we have not an explicit expansion of H_n. But it is expected to produce acceptable accuracy as in a two-dimensional case [11].
Multiplying the first member of (17) with a vector testing function **W** and integrating on boundary Γ, we obtain

$$\oint_\Gamma\nabla_t(\nabla\cdot\mathbf{H}_t)\cdot\mathbf{W}\,d\Gamma = -\oint_\Gamma(\nabla\cdot\mathbf{H}_t)(\nabla\cdot\mathbf{W}_t)\,d\Gamma$$
$$+\sum_{i=1}^{6}\oint_{\partial\Gamma_i}(\nabla\cdot\mathbf{H}_t)(\mathbf{W}_t\cdot\mathbf{m})\,dl \tag{17}$$

If we suppose that Γ is smooth, free to round geometrical singularities (edges and corners), then Γ has no boundaries and the line integrals in (13) and (17) vanishe. Thus the "symmetric" 3D vector Engquist-Majda condition is

$$\mathbf{n}\times\nabla\times\mathbf{H} = \alpha\mathbf{H}_t + \beta\nabla_t^2\mathbf{H}_t - 2\beta\nabla_t(\nabla\cdot\mathbf{H}_t) \tag{18}$$

or

$$\mathbf{n}\times\nabla\times\mathbf{H}=\alpha\mathbf{H}_t+\beta\nabla\times(\nabla_t\times\mathbf{H}_t)+\beta\nabla_t(\nabla\cdot\mathbf{H}_t) \quad (19)$$

where $\alpha = jk$ and $\beta = \frac{j}{2k}$.

D. Treatment of domains with geometrical singularities

It is interesting to work with exterior rectangular surface having edges and corners but the line integrals in (13) and (17) do not vanish and adequate ABC's must be prescribed on geometrical singularities. We develop in [11] the appropriate vector conditions which lead a symmetric outer matrix operator. A study on effects of these conditions on the solutions will be discussed in a future paper.

In this paper we limit ourself to use symmetric vector Engquist-Majda ABC's on the rectangular outer surface without corners and edges conditions. We make the approximation that parasite waves generated by geometrical singularities are essentially local and a small amount propagates toward the interior of the domain.

IV. Numerical Approximation

To be solved numerically the integral formulations (6) and (13) are discretized with finite elements methods using mixed elements conforming in space *H*(curl). In particular, the mixed elements of first order , R1 and P1 on hexahedral have been implemented. In [6-7] these finite elements are definied by a list of space of interpolation and corresponding degrees of freedom.

The unknown vector field $\mathbf{H}$ is expanded as

$$\mathbf{H}=\sum_{i=1}^{N}H_i\mathbf{W}_i\ ,\ H_i=\sigma_i(\mathbf{H}) \quad (20)$$

where σ_i is a degree of freedom associated with vector shape function $\mathbf{W}_i$ and N is the total number of degrees of freedom associated with the mesh.

We show how we derive an explicit expression of each vector basis functions and associated degree of freedom in [11]. Here, we rewrite only the final results.

A. Mixed elements R1

- Vector basis functions $\mathbf{W}_{ij}$, associated with edge $C = \{\ a_i\ ,\ a_j\ \}$:

$$\mathbf{W}_{ij}=N_i\nabla L_j-N_j\nabla L_i\ , \quad (21)$$

have degree of freedom σ_{ij} ,

$$\sigma_{ij}(\mathbf{p})=\int_C \mathbf{p}\cdot\tau_{ij}\ ds, \quad (22)$$

where

- L_i and L_j, are first-order node-based functions associated with the two vertices $\{\ a_i\ \}$ and $\{\ a_j\ \}$ in variable u_m (edge C parallel to axis Ou_m in local coordinates),
- N_i , $1\le i\le 8$, are Lagrange shape functions of fisrt order within hexahedral associated with vertices $\{\ a_i\ \}$, $1\le i\le 8$,
- τ_{ij} is a unit tangent vector to edge C and orientates it,
- $\mathbf{p}$ is an arbitrary vector defined inside the hexahedral.

One could shake that these elements correspond to those obtained in [10]. In the form given in (21-22), they appear clearly to be an extension of Whitney edge-elements in hexahedral.

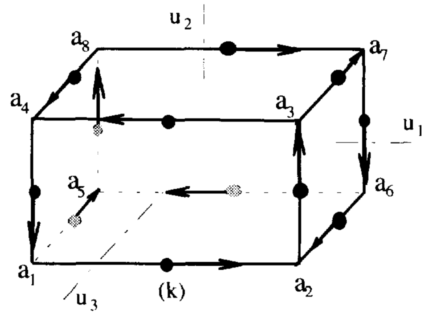

Fig. 2. First-order Hexahedral *H*(curl) element; for mixed elements R1 k=1 and for mixed elements P1 k=2.

B. Mixed elements P1

- Vector basis functions $\mathbf{W}_{ij}^m$, $1\le m\le 2$, associated with edge $C = \{\ a_i\ ,\ a_j\ \}$:

$$\mathbf{W}_{ij}^1=N_i\nabla L_j-N_j\nabla L_i \quad \text{and} \quad \mathbf{W}_{ij}^2=3(N_i\nabla L_j+N_j\nabla L_i)$$

have degrees of freedom σ_{ij}^m , $1\le m\le 2$,

$$\sigma_{ij}^1(\mathbf{p})=\int_C \mathbf{p}\cdot\tau_{ij}\ ds \quad \text{and} \quad \sigma_{ij}^2(\mathbf{p})=\int_C \mathbf{p}\cdot\tau_{ij}(L_i-L_j)\ ds$$

It appears that elements R1 are a subset of elements P1. So, elements P1 are expected to lead more accurate solution [7].

V. Modeling an Infinitesimal Dipole

This test case has already been used in a previous paper [9] to validate a nodal-based F.E. formulation. The infinitesimal dipole, with a constant current, is centered on the origin and directed along the z axis (Fig.3.).

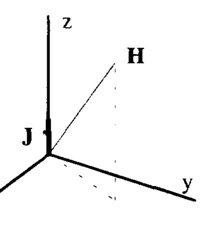

Fig. 3. Definition of the dipole.

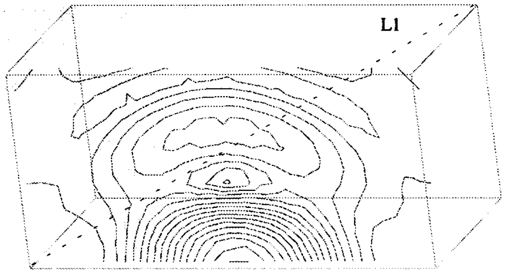

Fig. 4. Near magnetic field in yz plane at x = 0.1λ for the dipole computed by FE with symmetric ABC's.

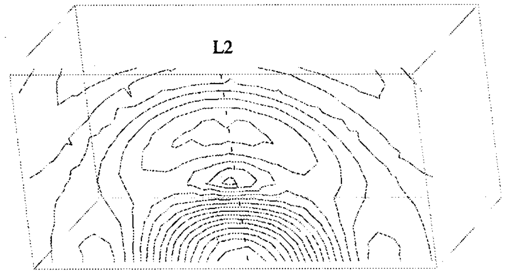

Fig. 5. Near magnetic field in yz plane at x = 0.1λ for the dipole computed with analytical solution.

The problem has been modeled with two symmetries (xz and yz planes). The size of the computational domain is 1λ x 1λ x (2.1)λ . Each finite element is 0.1λ x 0.1λ x 0.1λ brick. The frequency is 3 Ghz.

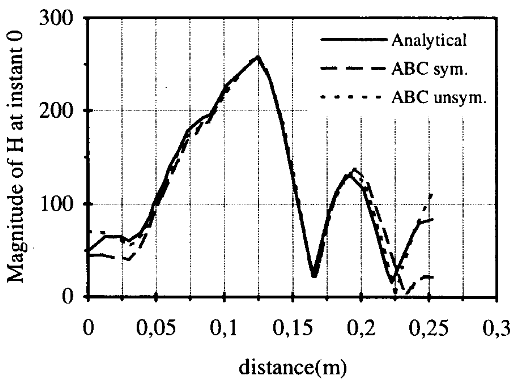

Fig.6: Magnetic field for the dipole, along the diagonal line of the rectangular box (line L1 in Fig.4).

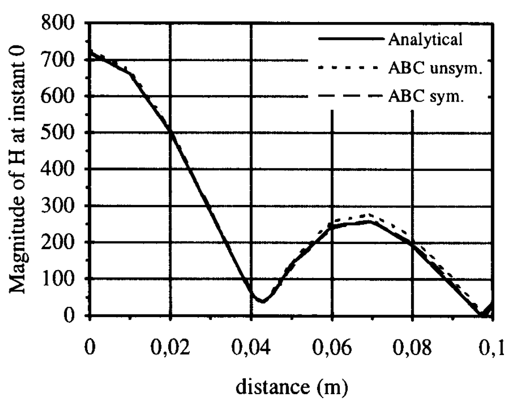

Fig.7: Magnetic field for the dipole, along the line starting at (0.01, 0., 0.) and ending at (0.01, 0.1, 0.),(line L2 in Fig.5).

Fig. 4 and Fig. 5 show that the computed solution removes from the analytical one when we are near the edges and corners of the domain. In the interior of the domain, the two solutions are quite close.

This is confirmed by Fig. 6 and Fig. 7. These figures show also that the accuracy of symmetric Engquist-Majda condition is approximately the same of the unsymmetric one.

Since the source is linear, a constant current line, the elements P1 did not provide improvement in accuracy over elements R1.

CONCLUSION

In this paper, we have presented a symmetric three dimensional vector Engquist-Majda ABC's. These ABC's, when coupled with three-dimensional finite element method using fisrt-order *H*(curl) basis vector functions, produce good accuracy. But when applied on rectangular outer surface which is not smooth, these ABC's generate parasite reflections which are essentially local. Study is under way to quantify the amount of corners and edges waves that propagate toward the interior of the domain.

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