New blow-up rates for fast controls of Schrödinger and heat equations

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Abstract: We consider the null-controllability problem for the Schrödinger and heat equations with boundary control. We concentrate on short-time, or fast, controls. We improve recent estimates (see Miller [14], [15],[16] [17]) on the norm of the operator associating to any initial state the minimal norm control driving the system to zero. Our main results concern the Schrödinger and heat equations in one space dimension. They yield new estimates concerning window problems for series of exponentials as described in Seidman, Avdonin and Ivanov [22]. These results are used, following [17], to deal with the case of several space dimensions.

Keywords: null-controllability, Schrödinger equation, heat equation, series of exponentials.

AMS subject classifications : 93C25, 93B07, 93C20, 11N36.

1 Introduction

In this work we consider the boundary control of systems governed by the Schrödinger or by the heat equation. These systems can be written as an abstract infinite-dimensional linear control system described by the equations

$$\dot{w} = Aw + Bu, \qquad w(0) = \psi, \tag{1.1}$$

where w denotes the state. Here, a dot denotes differentiation with respect to the time t, A is the generator of a strongly continuous operator semigroup on the state space X, B is an admissible control operator for this semigroup (the notion of admissible control operator will be recalled in Section 2) and $\psi \in X$ is the initial state of the system. The system receives the input function (also called control function) u.

Assume the linear system (1.1) is null-controllable in arbitrarily small time, i.e., for every $T > 0$ and every initial state ψ, the set $\mathcal{U}_{T,\psi}$, composed of all controls in $L^2([0,T])$ such that the corresponding state trajectory satisfies $w(T) = 0$, is not empty. Then, as shown in Section 2, $\mathcal{U}_{T,\psi}$ contains a unique minimal norm element, which we denote by $u(T,\psi)$. The *null-controllability operator in time* T, denoted by F_T, is defined by $F_T\psi = u(T,\psi)$. It is clear that the norm of F_T (sometimes called *the controllability cost*, as in Zuazua [28] and

Miller [14], [15]) must increase unboundedly when the available time decreases to zero. We make the terminological choice of calling *control cost* the norm of the null-controllability operator. Thus, we write

$$C_T := \|F_T\| \tag{1.2}$$

and consider the natural question of studying the blow up of C_T as the control time T tends to zero. In the case of finite dimensional systems, this question has been investigated by Seidman [21] and Seidman–Yong [23], who showed that, as T tends to zero, C_T behaves like $1/T^{k+1/2}$, for suitable $k \in \mathbb{N}$. In the infinite dimensional case, a similar analysis has to be limited to systems which are null-controllable in arbitrarily small time, such as systems governed by the Schrödinger or by the heat equations—clearly, delay systems or systems governed by hyperbolic partial differential equations cannot be considered from the above perspective. In the case of the boundary control for the one dimensional heat equation with constant coefficients on the space interval $[0, 1]$, it has been shown by Güichal [10] that

$$\alpha_* := \liminf_{T\to 0} T \ln C_T > 0.$$

This result has been extended and made more precise in [14] and [16], where it is shown that, for the constant coefficients Schrödinger and heat equations on the interval $[0, a]$, we have

$$\alpha_* \geqslant \tfrac{1}{4}a^2. \tag{1.3}$$

On the other hand, Seidman showed in [20] that

$$\alpha^* := \limsup_{T\to 0} T \ln C_T < \infty.$$

More recently (see, for instance, Seidman, Avdonin and Ivanov [22] and Miller [14], [15], [16]) the above estimate on α^* has been extended to the Schrödinger and heat equations with variable coefficients and effective upper bounds have been provided. To our knowledge, the best upper bound for α^* in the case of the one dimensional Schrödinger equation has been obtained in [15] and can be stated as

$$\alpha^* \leqslant 4\left(\tfrac{36}{37}\right)^2 \mu, \tag{1.4}$$

where μ is a constant depending only on the space interval in which the Schrödinger equation holds and on its coefficients: in the case of constant coefficients, μ reduces to the square of the length of the interval.

For systems governed by a variable coefficients heat equation with boundary control, the upper bound in (1.4) becomes (see [14])

$$\alpha^* \leqslant 2\left(\tfrac{36}{37}\right)^2 \mu. \tag{1.5}$$

Although originally dealing with partial differential equations in space dimension one, the above mentioned results have been used in [28], [14], [15] and [16] to derive similar estimates for the Schrödinger and heat equations in several space dimensions.

Our main results provide new upper bounds for the control cost in the case of systems governed by the Schrödinger or the heat equation. Precise statements require some preliminaries, so they are postponed to Section 3. However, we can state at the outset that our upper bounds for $C_T = \|F_T\|$ are valid for every $T > 0$ and imply that

$$\alpha^* \leqslant \tfrac{3}{2}\mu, \tag{1.6}$$

for the Schrödinger equation in one space dimension, and

$$\alpha^* \leqslant \tfrac{3}{4}\mu, \tag{1.7}$$

for the heat equation, thereby improving upon (1.4) and (1.5).

The results described above yield new estimates, of independent interest, on "window problems" for series of exponentials as described in Seidman, Avdonin and Ivanov [22]. They also imply new upper bounds for the control costs of the Schrödinger and heat equations in *several space dimensions*. Another contribution brought in by our work consists in giving a new proof of the lower bound in (1.3).

The plan of this paper is as follows. In Section 2, we give some background on infinite-dimensional systems with emphasis on the null-controllability property. Section 3 is essentially dedicated to the statement of the main results. In Section 4, we establish two lemmas which are essential for the proofs of our theorems, given in Section 5. In Section 6, we apply the earlier obtained results to control problems in several space dimensions. Finally, in Section 7, we provide a simple proof of (1.3).

2 Notation and preliminaries

2.1 Notation

In the sequel, we freely use, according to display convenience, Landau's O-symbol or Vinogradov's $\ll$-notation. Thus $f(x) \ll g(x)$ $(x \in X)$ indicates that, for all x in the set X, the inequality $|f(x)| \leqslant C|g(x)|$ holds for some suitable constant $C > 0$ which may depend on some implicit parameters. In the latter case, dependence may be indicated by inserting appropriate subscripts. We write

$$f(x) \asymp g(x)$$

to indicate that both relations $f(x) \ll g(x)$ and $g(x) \ll f(x)$ hold simultaneously.

Throughout this section, U, Y and X are complex Hilbert spaces, identified with their duals. The inner product and the norm in X are denoted by $\langle \cdot, \cdot \rangle$ and $\|\cdot\|$, respectively. If $P \in \mathscr{L}(X; Y)$ then the *null-space* and the *range* of P are the subspaces of X and Y respectively defined by

$$\text{Ker } P = \{x \in X \ : \ Px = 0\}, \qquad \text{Ran } P = \{Px \ : \ x \in X\}.$$

We denote by $\mathbb{W} = (\mathbb{W}_t)_{t \geqslant 0}$ a strongly continuous semigroup on X generated by an operator $A : \mathscr{D}(A) \to X$ with resolvent set $\varrho(A)$. The notation X_1 stands for $\mathscr{D}(A)$ equipped with the norm $\|z\|_1 := \|(\beta I - A)z\|$, where $\beta \in \varrho(A)$ is fixed, while X_{-1} is the completion of X with respect to the norm $\|z\|_{-1} := \|(\beta I - A)^{-1}z\|$. We use the notation A and $\mathbb{W}_t$ also for the extensions of the original generator to X and of the original semigroup to X_{-1}. Recall that X_{-1} is the dual of $\mathscr{D}(A^*)$ with respect to the pivot space X. For $B \in \mathscr{L}(U; X_{-1})$ and $T > 0$ we define $\Phi_T \in \mathscr{L}(L^2([0,T], U); X_{-1})$ by

$$\Phi_T u = \int_0^T \mathbb{W}_{T-\sigma} B u(\sigma) \mathrm{d}\sigma. \tag{2.8}$$

2.2 Some background on null-controllability

Definition 2.1. With the above notation, the operator $B \in \mathscr{L}(U; X_{-1})$ is called an *admissible control operator* for $\mathbb{W}$ if Ran $\Phi_\tau \subset X$ for some $\tau > 0$.

It is known (see Weiss [25]) that, if B is an admissible control operator for $\mathbb{W}$, if $T > 0$, if $u \in L^2([0,T], U)$ and if $\psi \in X$, then the solution of the initial value problem (1.1), viz.

$$w(t) = \mathbb{W}_t \psi + \Phi_t u, \tag{2.1}$$

satisfies $w \in \mathscr{C}([0,T], X)$ and we have

$$\Phi_T \in \mathscr{L}(L^2([0,T], U); X).$$

Definition 2.2. Given $T > 0$ and $B \in \mathscr{L}(U; X_{-1})$, an admissible control operator for $\mathbb{W}$, the pair (A, B) is said *null-controllable in time* T if, for any $\psi \in X$, there exists a u in $L^2([0,T]; U)$ such that the solution w of (1.1) satisfies $w(T) = 0$.

It is easy to see that the null-controllability of the pair (A, B) in time T is equivalent to the property Ran $\Phi_T \supset$ Ran $\mathbb{W}_T$.

Definition 2.3. For $T > 0$ and $B \in \mathscr{L}(U; X_{-1})$, an admissible control operator for $\mathbb{W}$, the pair (A, B) is said *exactly controllable in time* T if Ran $\Phi_T = X$.

It is clear that if (A, B) is exactly controllable in time T then (A, B) is null-controllable in time T. The converse is false in the general case but holds if A generates a strongly continuous group on X— this last condition being satisfied for systems governed by a Schrödinger equation. Therefore, in all the statements below concerning Schrödinger type equations, one can replace the term null-controllability by exact controllability.

The next proposition is essential for defining the null-controllability operator correctly. Since we did not find in the literature the required version valid for unbounded input operators (see, for instance, [27] for the bounded case), we provide below a precise statement and a short proof, with no claim of originality.

Proposition 2.4. *Suppose that* (A, B) *is null-controllable in time* T. *Then there exists an operator* $F_T \in \mathscr{L}(X; L^2([0,T], U))$ *such that*

1. $\mathbb{W}_T + \Phi_T F_T = 0.$
2. *If* $u \in L^2([0,T], U)$ *is a control driving the solution* (2.1) *of* (1.1) *to rest in time* T, *then*
$$\|u\|_{L^2([0,T],U)} \geqslant \|F_T \psi\|_{L^2([0,T],U)}.$$

Proof. Let $\psi \in X$. Then we have $-\mathbb{W}_T\psi \in$ Ran $\mathbb{W}_T \subset$ Ran Φ_T, so there exists a unique $y \in (\text{Ker } \Phi_T)^\perp$ such that $\Phi_T y = -\mathbb{W}_T \psi$. By setting $F_T \psi = y$, we have that $\mathbb{W}_T\psi + \Phi_T F_T \psi = 0$. We still have to prove that F_T is bounded from X to $L^2([0,T], U)$. Since F_T is defined on all of X, it suffices to show that F_T has a closed graph. Let (ψ_n, y_n) be a sequence in the graph of F_T such that $\lim(\psi_n, y_n) = (\psi, y)$ in $X \times L^2([0,\infty), U)$, then $\lim \mathbb{W}_T \psi_n = \mathbb{W}_T \psi$ and $\lim \Phi_T y_n = \Phi_T y$. Thus, $\mathbb{W}_T \psi = \Phi_T y$ and, since $(\text{Ker } \Phi_T)^\perp$ is closed, $y \in (\text{Ker } \Phi_T)^\perp$, so $F_T \psi = y$.

It remains to show that the minimality property in the second assertion of the proposition also holds. If $u \in L^2([0,T],U)$ is a control driving the solution w of (1.1) to rest in time T, then

$$\mathbb{W}_T\psi + \Phi_T u = 0. \tag{2.2}$$

Let $u = u_1 + u_2$ be the orthogonal decomposition of u with $u_1 \in \operatorname{Ker}\,\Phi_T$ and $u_2 \in (\operatorname{Ker}\,\Phi_T)^\perp$. From (2.2) and the definition of F_T, we deduce that $u_2 = F_T\psi$, hence

$$\|u\|^2_{L^2([0,T],U)} = \|u_1\|^2_{L^2([0,T],U)} + \|F_T\psi\|^2_{L^2([0,T],U)} \geqslant \|F_T\psi\|^2_{L^2([0,T],U)},$$

so F_T does satisfy the second required condition. □

Proposition 2.4 says that, for any $\psi \in X$, $F_T\psi$ is the control of minimal norm driving the system (1.1) to rest in time T. We refer to F_T as the *null-controllability operator in time* T.

The admissibility and null-controllability properties of a control operator are respectively dual to the admissibility and final state observability properties of an observation operator. We now recall the definitions of the latter concepts.

Definition 2.5. The operator $C \in \mathscr{L}(X_1;Y)$ is called an *admissible observation operator* for $\mathbb{W}$ if, for some $T > 0$, there exists a constant $K_T > 0$ such that

$$\int_0^T \|C\mathbb{W}_t\psi\|^2_U \mathrm{d}t \leqslant K_T^2\|\psi\|^2_X \qquad (\psi \in \mathscr{D}(A)). \tag{2.3}$$

Definition 2.6. Let $T > 0$ and let $C \in \mathscr{L}(X_1;Y)$ be an admissible observation operator for $\mathbb{W}$. The pair (A,C) is *final state observable in time* T if there exists $k_T > 0$ such that

$$\|\mathbb{W}_T\psi\|^2 \leqslant k_T^2 \int_0^T \|C\mathbb{W}_t\psi\|^2 \mathrm{d}t \qquad (\psi \in \mathscr{D}(A)).$$

The duality mentioned above is made precise in the following result, essentially due to Dolecki and Russell [7].

Proposition 2.7. *Suppose that $B \in \mathscr{L}(U;X_{-1})$. Then B is an admissible control operator for $\mathbb{W}$ if, and only if, B^* is an admissible observation operator for $\mathbb{W}^*$. The pair (A,B) is null-controllable in time T if, and only if, the pair (A^*,B^*) is final state observable in time T. Moreover, if (A,B) is null-controllable in time T, then the norm of the associated null-controllability operator coincides with the greatest lower bound of the set of those numbers C_T satisfying*

$$\|\mathbb{W}^*_T\psi\|^2 \leqslant C_T^2 \int_0^T \|B^*\mathbb{W}^*_t\psi\|^2 \mathrm{d}t \qquad (\psi \in \mathscr{D}(A^*)).$$

2.3 Systems with self-adjoint or skew-adjoint generator and one dimensional input

Here, we specialize the notions and results of the two previous subsections to the case of systems with self-adjoint or skew-adjoint generator A and with one dimensional input space U—i.e. we take $U = \mathbb{C}$.

Let $A_0 : \mathscr{D}(A_0) \to X$ be a strictly negative self-adjoint operator, with non-empty resolvent set $\varrho(A_0)$ and with compact resolvents. We denote by $(\varphi_k)_{k\in\mathbb{N}^*}$ an orthonormal basis of X consisting of eigenvectors of A_0. For every $k \in \mathbb{N}^*$, we denote by $-\lambda_k$ the eigenvalue associated to the eigenvector φ_k. Since A_0 is self-adjoint, λ_k is real for all $k \in \mathbb{N}^*$. We assume that $\lambda_k > 0$ for every $k \in \mathbb{N}$, so that A_0 is a strictly negative operator. According to the Lummer-Phillips theorem, A_0 generates a $\mathscr{C}^0$ contraction semigroup in X. This semigroup, denoted by $\mathbb{S} = (\mathbb{S}_t)_{t\geqslant 0}$, acts on X according to the formula

$$\mathbb{S}_t\psi = \sum_{k\in\mathbb{N}^*} \langle \psi, \varphi_k\rangle \mathrm{e}^{-\lambda_k t}\varphi_k \qquad (t \geqslant 0, \quad \psi \in X). \tag{2.4}$$

On the other hand, the operator iA_0 is skew-adjoint in X so, according to Stone's theorem, it generates a strongly continuous group of linear isometries in X. The action of this group, denoted $\mathbb{U} = (\mathbb{U}_t)_{t\in\mathbb{R}}$, is described by the formula

$$\mathbb{U}_t\psi = \sum_{k\in\mathbb{N}^*} \langle \psi, \varphi_k\rangle \mathrm{e}^{-i\lambda_k t}\varphi_k \qquad (t \in \mathbb{R}, \quad \psi \in X). \tag{2.5}$$

We introduce the scale of Hilbert spaces X_α, $\alpha \in \mathbb{R}$, as follows: for every $\alpha \geqslant 0$, we set $X_\alpha := \mathscr{D}((-A_0)^\alpha)$, equipped with the norm

$$\|\psi\|_\alpha^2 := \sum_{k\in\mathbb{N}^*} \lambda_k^{2\alpha} \left|\langle \psi, \varphi_k\rangle\right|^2 .$$

For $\alpha > 0$, the space $X_{-\alpha}$ is defined as the dual space of X_α with respect to the pivot space X. Equivalently, $X_{-\alpha}$ is the completion of X for the norm

$$\|\psi\|_{-\alpha}^2 = \sum_{k\in\mathbb{N}^*} \lambda_k^{-2\alpha} \left|\langle \psi, \varphi_k\rangle\right|^2 .$$

The operator A_0 and the semigroups $\mathbb{S}$ and $\mathbb{U}$ can be extended (or restricted) to each X_α, in such a way that A_0 becomes a bounded operator

$$A_0 : X_\alpha \to X_{\alpha-1} \qquad (\alpha \in \mathbb{R}),$$

and $\mathbb{U}$ (respectively $\mathbb{S}$) becomes a $\mathscr{C}^0$ group of isometries (respectively a $\mathscr{C}^0$ contraction semigroup) on $X_{\alpha-1}$ with generator iA_0 (respectively A_0).

Assume that the control space U is one dimensional (i.e. that $U = \mathbb{C}$) and that the control operator $B \in \mathscr{L}(U; X_{-1})$ is given by

$$Bu = ub \qquad (u \in \mathbb{C}), \tag{2.6}$$

where b is a fixed element of X_{-1}. For b as above and $\psi \in \mathscr{D}(A_0)$, the notation $\langle b, \psi\rangle$ stands for the duality product of b and ψ. For every $k \in \mathbb{N}^*$ we put

$$b_k := \langle b, \varphi_k\rangle. \tag{2.7}$$

Sufficient conditions for the admissibility of a control of the form (2.6) are given in the result below, which is a particular case of the admissibility conditions given in Ho and Russell [11] and Weiss [24].

Proposition 2.8. *With the above notation, assume that* $\sup_{k\in\mathbb{N}^*}|b_k|<\infty$ *and that the sequence* $\Lambda=(\lambda_n)_{n\in\mathbb{N}}$ *is* regular*, i.e., that*

$$\gamma=\gamma(\Lambda):=\inf_{\substack{m,n\in\mathbb{N}^*\\ m\neq n}}|\lambda_m-\lambda_n|>0. \tag{2.8}$$

Then B *defined by* (2.6) *is an admissible control operator for* $\mathbb{S}$ *and for* $\mathbb{U}$.

The control cost can be interpreted in terms of a window problem for a sequence of complex exponentials. More precisely we may derive from Proposition 2.7 the following statement where, for $\mathrm{a}:=(a_n)_{n\in\mathbb{N}}\in\ell^2(\mathbb{C})$, we denote by $f_{\mathrm{a},\Lambda}$ and $g_{\mathrm{a},\Lambda}$ the elements of $L^2_{\mathrm{loc}}(\mathbb{R})$ defined almost everywhere by

$$\begin{aligned} f_{\mathrm{a},\Lambda}(t)&:=\sum_{n\in\mathbb{N}}a_n\mathrm{e}^{i\lambda_n t},\\ g_{\mathrm{a},\Lambda}(t)&:=\sum_{n\in\mathbb{N}}a_n\mathrm{e}^{\lambda_n(T-t)}, \end{aligned}\qquad (t\in\mathbb{R}). \tag{2.9}$$

Proposition 2.9. *Let* Λ *be a regular sequence of real numbers, let* B *be given by* (2.6), *and assume that* $b_k\asymp 1$ $(k\geqslant 1)$.

For the pair (iA_0,B), *we have*

$$\|F_T\|\asymp\sup_{\mathrm{a}\in\ell^2(\mathbb{C})\setminus\{0\}}\|\mathrm{a}\|_{\ell^2(\mathbb{C})}\big/\|f_{\mathrm{a},\Lambda}\|_{L^2([-T/2,T/2],\mathbb{C})}\qquad (T>0).$$

For the pair (A_0,B), *we have*

$$\|F_T\|\asymp\sup_{\mathrm{a}\in\ell^2(\mathbb{C})\setminus\{0\}}\|\mathrm{a}\|_{\ell^2(\mathbb{C})}\big/\|g_{\mathrm{a},\Lambda}\|_{L^2([-T/2,T/2],\mathbb{C})}\qquad (T>0).$$

In both cases, the implicit constants depend only on $\inf|b_k|$ *and* $\sup|b_k|$.

Proof. From (2.4) and (2.5) it follows that

$$(\mathbb{S}_t)^*\psi=\sum_{k\in\mathbb{N}^*}\langle\psi,\varphi_k\rangle\mathrm{e}^{-\lambda_k t}\varphi_k\qquad (t\geqslant 0,\ \psi\in X).$$

$$(\mathbb{U}_t)^*\psi=\sum_{k\in\mathbb{N}^*}\langle\psi,\varphi_k\rangle\mathrm{e}^{i\lambda_k t}\varphi_k\qquad (t\in\mathbb{R},\ \psi\in X).$$

The above relation and the fact that

$$B^*\psi=\sum_{k\in\mathbb{N}^*}b_k\langle\psi,\varphi_k\rangle\qquad (\psi\in X_1),$$

imply that

$$B^*(\mathbb{S}_t)^*\psi=\sum_{k\in\mathbb{N}^*}b_k\langle\psi,\varphi_k\rangle\mathrm{e}^{-\lambda_k t}\qquad (t\geqslant 0,\ \psi\in X),$$

and

$$B^*(\mathbb{U}_t)^*\psi=\sum_{k\in\mathbb{N}^*}b_k\langle\psi,\varphi_k\rangle\mathrm{e}^{i\lambda_k t}\qquad (t\in\mathbb{R},\ \psi\in X).$$

Since $\psi\mapsto\big(b_k\langle\psi,\varphi_k\rangle\big)_{k\in\mathbb{N}^*}$ maps X onto $\ell^2(\mathbb{N}^*,\mathbb{C})$, the desired conclusions follow from the last two formulas and Proposition 2.7. □

3 Statement of the main results

3.1 Results on Schrödinger type equations and complex exponentials

In this subsection and in the following one we use the notation introduced in the previous sections. Recall, in particular, that X is a Hilbert space, $A_0 : \mathscr{D}(A_0) \to X$ is a self-adjoint strictly negative operator with compact resolvents and with eigenvalues $(-\lambda_k)_{k\geqslant 1}$, $b \in X_{-1}$, the sequence (b_k) is given by (2.7) and that $\mathbb{S}$ is the semigroup generated by A_0.

Our first result gives an estimate (with explicit constants) for the norm of the control operator in the case of a system governed by an abstract Schrödinger equation.

Theorem 3.1. *Assume that $|b_k| \asymp 1$ $(k \geqslant 1)$, that the sequence $\Lambda := (\lambda_n)_{n\in\mathbb{N}}$ is regular and it satisfies*

$$|\lambda_n - rn^2| \leqslant Cn \qquad (n \geqslant 1). \tag{3.1}$$

for some $r > 0$, $C \geqslant 0$. Then the pair (iA_0, B), with B given by (2.6), is null-controllable in time T and, for every $\kappa > \frac{3}{2}\pi^2$, the control cost $C_T = \|F_T\|$ satisfies the estimate

$$C_T \ll \mathrm{e}^{\kappa/(rT)} \qquad (T > 0), \tag{3.2}$$

where the implicit constant depends only on κ and Λ.

The result above can be applied for the control of the one-dimensional Schrödinger equation with variable coefficients and with various boundary conditions. In the particular case of Dirichlet boundary control we obtain:

Corollary 3.2. *Let $a > 0$, $p \in \mathscr{C}^2([0,a],\mathbb{R})$, $q \in \mathscr{C}([0,a],\mathbb{R})$. Assume that $p(x) > 0$ for all $x \in [0,a]$ and write $\mu := \left(\int_0^a \sqrt{p(x)}\,\mathrm{d}x\right)^2$. Let $\alpha > \frac{3}{2}$. Then, for every $\psi \in H^{-1}(]0,a[)$ and every $T > 0$, there exists $u \in L^2[0,T]$ verifying*

$$\|u\|_{L^2[0,T]} \ll \mathrm{e}^{\alpha\mu/T} \|\psi\|_{H^{-1}(]0,a[)}, \tag{3.3}$$

(the implicit constant being independent of T) such that the solution w of

$$\begin{cases} -i\dfrac{\partial w}{\partial t}(x,t) = \dfrac{\partial}{\partial x}\left(p(x)\dfrac{\partial w}{\partial x}(x,t)\right) + q(x)w(x,t) & (0 < x < a,\ 0 < t < T) \\ w(0,t) = u(t) & (0 < t < T) \\ w(a,t) = 0 & (0 < t < T) \\ w(x,0) = \psi(x) & (0 < x < a), \end{cases} \tag{3.4}$$

satisfies $w(x,T) = 0$ for all $x \in]0,a[$.

Therefore, the system (3.4) has control cost $\ll \mathrm{e}^{\alpha\mu/T}$ for every $\alpha > \frac{3}{2}$, which implies (1.6). This improves upon Theorem 4.1 in [15], where a similar assertion is established under the stronger condition $\alpha > 4\left(\frac{36}{37}\right)^2$.

The duality viewpoint in Theorem 2.7 suggests that Theorem 3.1 can be equivalently stated in terms of a window problem for a sequence of complex exponentials. We actually derive the following statement where, for $\mathrm{a} := (a_n)_{n\in\mathbb{N}} \in \ell^2(\mathbb{C})$, we denote by $f_{\mathrm{a},\Lambda}$ the element of $L^2_{\mathrm{loc}}(\mathbb{R})$ defined almost everywhere by

$$f_{\mathrm{a},\Lambda}(t) := \sum_{n\in\mathbb{N}} a_n \mathrm{e}^{i\lambda_n t} \qquad (t \in \mathbb{R}).$$

Corollary 3.3. *Let $\kappa > \frac{3}{2}\pi^2$. Assume that $\Lambda := (\lambda_n)_{n\in\mathbb{N}}$ is a regular sequence satisfying* (3.1)*. Then, uniformly for $(a_n)_{n\in\mathbb{N}} \in \ell^2(\mathbb{C})$ and $T > 0$, we have*

$$\sum_{n\geqslant 0} |a_n|^2 \ll \mathrm{e}^{2\kappa/(rT)} \int_{-T/2}^{T/2} \Big| \sum_{n\geqslant 0} a_n \mathrm{e}^{i\lambda_n t} \Big|^2 \, \mathrm{d}t.$$

For real sequences $(\lambda_n)_{n\in\mathbb{N}}$ satisfying (3.1), this yields an improvement of the constants in the corresponding estimates of [22].

3.2 Results on heat type equations and real exponentials

The analogue of Theorem 3.1 for abstract heat equations is the following statement.

Theorem 3.4. *Let $\kappa > \frac{3}{4}\pi^2$. Then, under the assumptions in Theorem 3.1, the pair (A_0, B), with B given by* (2.6)*, is null-controllable in any time $T > 0$. Moreover, for every $\psi \in X$ there exists $u \in \mathscr{C}[0, T]$ driving the system* (1.1) *(with $A = A_0$ and $Bu = ub$) to rest in time T and satisfying*

$$\|u\|_{\mathscr{C}([0,T])} \ll \mathrm{e}^{\kappa/(rT)} \|\mathbb{S}_{T/2}\,\psi\| \qquad (T > 0,\ \psi \in X), \tag{3.5}$$

the implicit constant being independent of T. In particular, the control cost of the pair (A_0, b) satisfies

$$C_T \ll \mathrm{e}^{\kappa/(rT)} \qquad (T > 0). \tag{3.6}$$

The above result can be applied to parabolic equations in one space dimension, with various boundary conditions. In the case of Dirichlet boundary control of the one-dimensional heat equation with variable coefficients, it yields the following statement.

Corollary 3.5. *Let $a > 0$, p, q, μ, T be as in Corollary* 3.2*. Let $\alpha > \frac{3}{4}$. For every $\psi \in H^{-1}(]0, a[)$, there exists $u \in \mathscr{C}[0, T]$ verifying*

$$\|u\|_{\mathscr{C}[0,T]} \ll \mathrm{e}^{\alpha\mu/T} \|\psi\|_{H^{-1}(]0,a[)},$$

(the implicit constant being independent of T) and such that the solution w of

$$\begin{cases} \dfrac{\partial w}{\partial t}(x,t) = \dfrac{\partial}{\partial x}\left(p(x)\dfrac{\partial w}{\partial x}(x,t)\right) + q(x)w(x,t) & (0 < x < a,\ 0 < t < T) \\ w(0,t) = u(t) & (0 < t < T) \\ w(a,t) = 0 & (0 < t < T) \\ w(x,0) = \psi(x) & (0 < x < a), \end{cases} \tag{3.7}$$

satisfies $w(x, T) = 0$ for all $x \in]0, a[$.

The above result, implying estimate (1.7), improves Theorem 4.1 in [14], where a similar assertion is shown to hold under the stronger hypothesis $\alpha > 2\left(\frac{36}{37}\right)^2$. Another improvement brought in by our Corollary 3.5 is that our estimate involves $\|u\|_{\mathscr{C}[0,T]}$ and $\|\psi\|_{H^{-1}]0,a[}$ instead of $\|u\|_{L^2[0,T]}$ and $\|\psi\|_{L^2[0,a]}$ employed in [14].

The dual version of Theorem 3.4 may be stated as follows.

Corollary 3.6. *Let $\kappa > \frac{3}{4}\pi^2$. Assume that $\Lambda := (\lambda_n)_{n\in\mathbb{N}}$ is a regular sequence satisfying* (3.1)*. Then, uniformly for $(a_n)_{n\in\mathbb{N}} \in \ell^2(\mathbb{C})$ and $T > 0$, we have*

$$\sum_{n\geqslant 0} |a_n|^2 \mathrm{e}^{-\lambda_n T} \ll \mathrm{e}^{2\kappa/(rT)} \int_0^T \Big|\sum_{n\geqslant 0} a_n \mathrm{e}^{-\lambda_n t}\Big|^2 \,\mathrm{d}t. \tag{3.8}$$

Note that a direct application of Theorem 2.7 would provide only (3.8) with

$$\sum_{n\geqslant 0} |a_n|^2 \mathrm{e}^{-2\lambda_n T}$$

in the left-hand side.

For real sequences $(\lambda_n)_{n\in\mathbb{N}}$ satisfying (3.1), this result improves the constants obtained in [22]—see [14] for detailed comments on this issue.

4 Two lemmas

The proofs of our main results rest upon two lemmas. The first one furnishes sharp estimates for the exponential type, and growth on the real axis, for a sequence of entire functions defined by certain infinite products. Recall that a sequence of real numbers is said to be *regular* if it satisfies (2.8).

Throughout, we denote by $\lfloor x \rfloor$ the integer part of the real number x. Also, we use Kronecker's symbol

$$\delta_{nk} := \begin{cases} 1 & \text{if } n = k, \\ 0 & \text{if } n \neq k. \end{cases}$$

Lemma 4.1. *Let $(\lambda_n)_{n\geqslant 1}$ be a regular sequence of positive real numbers satisfying*

$$|\lambda_n - n^2| \leqslant Cn \qquad (n \geqslant 1), \tag{4.1}$$

for some $C > 0$. Let $(a_n) \in \ell^2(\mathbb{N}^*, \mathbb{C})$. For $n \in \mathbb{N}^*$, define

$$\Phi_n(z) := \prod_{k\neq n} \Big(1 - \frac{z}{\lambda_k - \lambda_n}\Big) \qquad (z \in \mathbb{C}). \tag{4.2}$$

Then, for suitable $B = B(C)$, we have, uniformly with respect to $n \geqslant 1$,

$$\Phi_n(z) \ll_\Lambda \mathrm{e}^{\pi\sqrt{|z|}} (1 + |z|)^B \qquad (z \in \mathbb{C}), \tag{4.3}$$

$$\Phi_n(-ix - \lambda_n) \ll_\Lambda (\lambda_n + |x|)^B \mathrm{e}^{\pi\sqrt{|x|/2}} \qquad (x \in \mathbb{R}). \tag{4.4}$$

The implicit constants depend at most upon C and γ, as defined in (2.8).

Proof. Let $\delta=\delta_n:=\inf_{k\neq n}|\lambda_k-\lambda_n|\geqslant\gamma(\Lambda):=\inf_{k\neq j}|\lambda_k-\lambda_j|$. We have

$$\begin{aligned}\ln|\Phi_n(z)| &\leqslant \sum_{k\geqslant1}\ln\Big(1+\frac{|z|}{|\lambda_k-\lambda_n|}\Big)=\sum_{k\geqslant1}\int_0^{|z|}\frac{\mathrm{d}t}{t+|\lambda_k-\lambda_n|}\\ &=\int_0^{|z|}\sum_{k\geqslant1}\frac{1}{t+|\lambda_k-\lambda_n|}\,\mathrm{d}t=\int_0^{|z|}\sum_{k\geqslant1}\int_{|\lambda_k-\lambda_n|}^\infty\frac{\mathrm{d}s}{(t+s)^2}\,\mathrm{d}t\\ &=\int_0^{|z|}\int_\delta^\infty\frac{L_n(s)}{(t+s)^2}\,\mathrm{d}s\mathrm{d}t\end{aligned}\tag{4.5}$$

with $L_n(s):=\sum_{|\lambda_k-\lambda_n|\leqslant s}1$. From assumption (4.1), we readily get

$$L_n(s)\leqslant\sqrt{\lambda_n+s}-\sqrt{(\lambda_n-s)^+}+O(1),\tag{4.6}$$

where the implicit constant depends on C.

The contribution of the term $\sqrt{\lambda_n+s}-\sqrt{(\lambda_n-s)^+}$ to the right hand side of (4.5) is

$$\begin{aligned}\int_0^{|z|}\int_\delta^\infty\frac{\sqrt{\lambda_n+s}-\sqrt{(\lambda_n-s)^+}}{(t+s)^2}\,\mathrm{d}s\,\mathrm{d}t &=|z|\int_\delta^\infty\frac{\sqrt{\lambda_n+s}-\sqrt{(\lambda_n-s)^+}}{s(s+|z|)}\,\mathrm{d}s\\ &\leqslant\frac{|z|}{\sqrt{\lambda_n}}\Big\{U\Big(\frac{|z|}{\lambda_n}\Big)+V\Big(\frac{|z|}{\lambda_n}\Big)\Big\},\end{aligned}$$

with

$$\begin{aligned}U(x)&:=\int_0^1\frac{\sqrt{1+v}-\sqrt{1-v}}{v(v+x)}\,\mathrm{d}v=\int_0^1\frac{2\,\mathrm{d}v}{(v+x)\{\sqrt{1+v}+\sqrt{1-v}\}},\\ V(x)&:=\int_1^\infty\frac{\sqrt{v+1}}{v(v+x)}\,\mathrm{d}v.\end{aligned}$$

Since the global contribution of the term $O(1)$ from the right hand side of (4.6) to the right hand side of (4.5) is

$$\ll\ln\big(1+|z|/\delta\big),$$

the conclusion of our lemma follows provided that we show the inequality

$$\sqrt{x}\{U(x)+V(x)\}\leqslant\pi\qquad(x\geqslant0).\tag{4.7}$$

Since (4.7) can be easily verified numerically for $x\leqslant3$, we assume $x>3$ henceforth. Denote

$$a:=\int_0^1\frac{2\,\mathrm{d}v}{\sqrt{1+v}+\sqrt{1-v}}\qquad b:=\int_1^\infty\frac{\mathrm{d}v}{v\{\sqrt{v}+\sqrt{v+1}\}}\cdot$$

We notice that

$$U(x)\leqslant\frac{a}{x}$$

and

$$\begin{aligned}V(x)&=\int_1^\infty\frac{\sqrt{v}}{v(v+x)}\,\mathrm{d}v+\int_1^\infty\frac{\mathrm{d}v}{v(v+x)\{\sqrt{v}+\sqrt{v+1}\}}\\ &=\frac{1}{\sqrt{x}}\int_{1/\sqrt{x}}^\infty\frac{2\,\mathrm{d}t}{1+t^2}+\int_1^\infty\frac{\mathrm{d}v}{v(v+x)\{\sqrt{v}+\sqrt{v+1}\}}\\ &\leqslant\frac{\pi}{\sqrt{x}}-\frac{2}{\sqrt{x}}\arctan\Big(\frac{1}{\sqrt{x}}\Big)+\frac{b}{x+1}\\ &\leqslant\frac{\pi}{\sqrt{x}}-\frac{2}{x}+\frac{2}{3x^2}+\frac{b}{x}-\frac{b}{x(x+1)}.\end{aligned}$$

Therefore

$$\sqrt{x}\{U(x)+V(x)\} \leqslant \pi + \frac{a+b-2}{\sqrt{x}} + \frac{2}{3x^{3/2}} - \frac{b}{\sqrt{x}(x+1)}. \tag{4.8}$$

We shall see that

$$a = 2\sqrt{2} - 2\ln\big(1+\sqrt{2}\big) \approx 1.0656, \qquad b = 2 + 2\ln\big(1+\sqrt{2}\big) - 2\sqrt{2} \approx 0.9343.$$

Thus $a+b=2$ and the sum of the last two terms in (4.8) is negative for $x>3$, which yields estimate (4.7).

It remains to establish the above formulae for a and b. From the successive changes of variables $v = 1-2(\sin\vartheta)^2$ and $t = \tan(\vartheta/2)$, we obtain

$$a = 4\sqrt{2}\int_0^{\pi/4} \frac{\sin\vartheta\cos\vartheta}{\sin\vartheta+\cos\vartheta}\,\mathrm{d}\vartheta = 8\sqrt{2}\int_0^1 \frac{(1-t^2)t}{(1+t^2)^2(1+2t-t^2)}\,\mathrm{d}t.$$

This furnishes the announced value for a after routine calculations.

Similarly, writing $v = (\tan\vartheta)^2$ and then $t = \tan(\vartheta/2)$ yields

$$b = \int_{\pi/4}^{\pi/2} \frac{2\,\mathrm{d}\vartheta}{(1+\sin\vartheta)\sin\vartheta} = 2\int_{\sqrt{2}-1}^{1} \frac{1+t^2}{t(1+t)^2}\,\mathrm{d}t,$$

from which the stated formula for b stems by standard calculus.

The proof of (4.4) is similar but easier. We have

$$\Phi_n(-ix-\lambda_n) := \prod_{k\neq n}\Big(\frac{1+ix/\lambda_k}{1-\lambda_n/\lambda_k}\Big).$$

Now, similarly to (4.5), writing $\lambda_0 := \min_{k\geqslant 1}\lambda_k$, we have

$$\sum_{k\geqslant 1}\ln\big(1+x^2/\lambda_k^2\big) = \int_0^{|z|^2/\lambda_0^2} \frac{M(t)}{1+t}\,\mathrm{d}t$$

with

$$M(t) := \sum_{\lambda_k\leqslant |z|/\sqrt{t}} 1 \leqslant \sqrt{|x|}t^{-1/4} + O(1).$$

The bounded remainder term contributes $\ll \ln(1+|x|)$ and the main term does not exceed

$$\sqrt{|x|}\int_0^\infty \frac{\mathrm{d}t}{(t+1)t^{1/4}} = \pi\sqrt{2|x|},$$

since

$$\int_0^\infty \frac{\mathrm{d}t}{(t+1)t^{1/4}} = \int_{-\infty}^\infty \frac{2w^2\,\mathrm{d}w}{1+w^4} = 2\pi i\big\{\operatorname{Res}(J;\mathrm{e}^{i\pi/4}) + \operatorname{Res}(J;\mathrm{e}^{3\pi i/4})\big\} = \pi\sqrt{2},$$

with $J(w) := 2w^2/(1+w^4)$.

Thus, we have shown so far that, for a suitable constant D, we have

$$|\Phi_n(-ix-\lambda_n)| \ll B_n(1+|x|)^D\mathrm{e}^{\pi\sqrt{|x|/2}}$$

with

$$B_n := \prod_{k\neq n}\big(1-\lambda_n/\lambda_k\big)^{-1}.$$

It remains to bound $|B_n|$ from above. We may plainly assume that n is sufficiently large.

Put $m := \lfloor C \rfloor + 1$, so that $(k-m)^2 \leqslant \lambda_k \leqslant (k+m)^2$ for all $k > m$. For $n > 3m$, we have from Euler's product formula for $\sin(\pi z)$ (see, for instance, Ahlfors [1, p.195])

$$\begin{aligned}
|B_n| &\leqslant \prod_{k<n-2m} \Big(\frac{\lambda_n}{(k+m)^2} - 1\Big)^{-1} \prod_{n-2m\leqslant k\leqslant n+2m} \Big(\frac{\lambda_n}{\lambda_n - \lambda_k}\Big) \prod_{k>n+2m} \Big(1 - \frac{\lambda_n}{(k-m)^2}\Big)^{-1} \\
&\leqslant (\lambda_n/\gamma)^{4m+1} \prod_{1\leqslant k\leqslant m} \Big|1 - \frac{\lambda_n}{k^2}\Big| \prod_{n-m\leqslant k\leqslant n+m} \Big|1 - \frac{\lambda_n}{k^2}\Big| \prod_{k\geqslant 1} \Big|1 - \frac{\lambda_n}{k^2}\Big|^{-1} \\
&\ll_\Lambda \lambda_n^{7m+1} \frac{\big|1 - \lambda_n/\lfloor\sqrt{\lambda_n}\rfloor^2\big|\sqrt{\lambda_n}}{\sin \pi\sqrt{\lambda_n}} \ll_\Lambda \lambda_n^{7m+2}.
\end{aligned}$$

□

Remark 4.2. Euler's product formula for $\sin(\pi z)$ shows the optimality of the exponent π in (4.3).

In our second lemma, we construct an entire function with fast decay on the real line. This will be essentially obtained as the Fourier transform of the $\mathscr{C}^\infty$ function defined by

$$\sigma_\nu(t) := \begin{cases} \exp\Big\{-\dfrac{\nu}{1-t^2}\Big\} & \text{if } |t| < 1, \\ 0 & \text{if } |t| \geqslant 1, \end{cases} \tag{4.9}$$

where ν is a positive constant. We note straightaway that, for every $\eta \in]0,1[$, we have

$$\int_{-1}^{1} \sigma_\nu(t)\mathrm{d}t \geqslant 2\eta \exp\Big\{-\frac{\nu}{1-\eta^2}\Big\}.$$

Selecting $\eta := 1/\sqrt{\nu+1}$ readily yields

$$\frac{2\mathrm{e}^{-\nu-1}}{\sqrt{\nu+1}} \leqslant \int_{-1}^{1} \sigma_\nu(t)\mathrm{d}t \leqslant 2\mathrm{e}^{-\nu}. \tag{4.10}$$

The following result furnishes the required fast decay property—see Bombieri, Friedlander and Iwaniec [3] and Jaffard and Micu [12] for related estimates.

Lemma 4.3. *Let $\beta > 0, \delta > 0$, and set $\nu := (\pi+\delta)^2/\beta$. The function σ_ν being defined as in (4.9), put $C_\nu := 1/\|\sigma_\nu\|_1$ and denote by H_β the entire function defined by*

$$H_\beta(z) := C_\nu \int_{-1}^{1} \sigma_\nu(t)\mathrm{e}^{-i\beta tz}\,\mathrm{d}t. \tag{4.11}$$

Then we have

$$\begin{cases}
H_\beta(0) = 1, & (4.12) \\
H_\beta(ix) \geqslant \dfrac{\mathrm{e}^{\beta|x|/(2\sqrt{\nu+1})}}{11\sqrt{\nu+1}} \quad (x \in \mathbb{R}) & (4.13) \\
|H_\beta(z)| \leqslant \mathrm{e}^{\beta|y|} \quad (z = x+iy, \quad x,y \in \mathbb{R}), & (4.14) \\
|H_\beta(x)| \ll \sqrt{\nu+1}\,\mathrm{e}^{3\nu/4-(\pi+\delta/2)\sqrt{|x|}} \qquad (x \in \mathbb{R}). & (4.15)
\end{cases}$$

Proof. Conditions (4.12) and (4.14) immediately follow from the definition of C_ν.

To show (4.13), we may assume $x \geqslant 0$. We first note that, from(4.10), we have

$$\tfrac{1}{2}\mathrm{e}^{\nu} \leqslant C_\nu \leqslant \tfrac{3}{2}\sqrt{\nu+1}\,\mathrm{e}^{\nu}. \tag{4.16}$$

Then, since $\sigma_\nu(t) \geqslant \mathrm{e}^{-\nu-1}$ for $\frac{1}{2}\eta \leqslant t \leqslant \eta$ with $\eta := 1/\sqrt{\nu+1}$, we may write

$$H_\beta(ix) \geqslant \tfrac{1}{2}C_\nu \eta \mathrm{e}^{-\nu-1+\beta x\eta/2} \geqslant \tfrac{1}{11}\eta \mathrm{e}^{\beta\eta x/2},$$

as required.

Thus, it only remains to establish condition (4.15). Since H_β is even, we restrict to the case $x > 0$. Since $\sigma_\nu \in \mathscr{C}^\infty(\mathbb{R})$, $\sigma_\nu(-1) = \sigma_\nu(1) = 0$, we obtain by partial integration

$$|H_\beta(x)| \leqslant \frac{C_\nu \|\sigma_\nu^{(j)}\|_\infty}{(\beta x)^j} \qquad (x > 0, \quad j \in \mathbb{N}). \tag{4.17}$$

For $t \in]-1, 1[$ we set $\varrho = 1 - t$ and $z = t + \varrho\mathrm{e}^{i\vartheta}$, with $\vartheta \in]-\pi, \pi]$. We have

$$\Re\frac{2}{1-z^2} = \Re\frac{1}{1-z} + \Re\frac{1}{1+z} = \frac{1}{2\varrho} + \frac{1-\varrho(\sin\vartheta/2)^2}{2-2\varrho(2-\varrho)(\sin\vartheta/2)^2}.$$

Since the last term is an increasing function of $(\sin\vartheta/2)^2$, we obtain

$$\Re\frac{2}{1-z^2} \geqslant \frac{1}{2\varrho} + \frac{1}{2} \qquad (|z-t| = \varrho).$$

Therefore

$$|\sigma_\nu(z)| \leqslant \mathrm{e}^{-\nu/4\varrho - \nu/4} \qquad (|z-t| = \varrho). \tag{4.18}$$

Applying Cauchy's integral formula, we obtain that

$$|\sigma_\nu^{(j)}(t)| \leqslant \mathrm{e}^{-\nu/4} \sup_{\varrho>0} \frac{j!\mathrm{e}^{-\nu/4\varrho}}{\varrho^j} \qquad (j \in \mathbb{N}, \quad t \in [-1, 1]),$$

which, in view of the elementary inequality $j! > j^j\mathrm{e}^{-j}$ $(j \geqslant 1)$, yields

$$|\sigma_\nu^{(j)}(t)| \leqslant \mathrm{e}^{-\nu/4}\frac{\left(2^j j!\right)^2}{\nu^j} \qquad (j \in \mathbb{N}, \quad t \in [-1, 1]). \tag{4.19}$$

From this, (4.16), (4.17) and the fact that H_β is even, we get that

$$|H_\beta(x)| \leqslant \tfrac{3}{2}\sqrt{\nu+1}\,\mathrm{e}^{3\nu/4}\frac{\left(2^j j!\right)^2}{(\beta\nu x)^j} \qquad (x > 0, \; j \in \mathbb{N}).$$

Selecting $j := 0$ when $0 \leqslant x \leqslant 1$ and $j := \left\lfloor \frac{1}{2}\sqrt{\beta\nu x}\right\rfloor$ otherwise, we readily check that (4.15) holds as required. Indeed, we deduce from the above that, for $x > 1$,

$$\begin{aligned} \frac{|H_\beta(x)|}{\sqrt{\nu+1}\,\mathrm{e}^{3\nu/4}} &\ll \frac{\left(2^j j!\right)^2}{(2j)^{2j}} \ll \mathrm{e}^{-2j} j \\ &\ll \mathrm{e}^{-(\pi+\delta)\sqrt{x}}\sqrt{x} \ll \mathrm{e}^{-(\pi+\delta/2)\sqrt{x}}. \end{aligned}$$

This concludes the proof. □

5 Proofs of the main results

Proof of Theorem 3.1. A simple change of variables shows that it suffices to prove the result for any given special value of r. For simplicity, we choose $r = 1$. The proof, following the strategy in Fattorini and Russell [8], is divided into two steps.

First step: construction of a family bi-orthogonal to $(\mathrm{e}^{i\lambda_n t})_{n\geqslant 1}$. For $n \in \mathbb{N}^*$, we define

$$\Psi_n(z) := \Phi_n(z-\lambda_n) = \prod_{k\neq n}\Big(1-\frac{z-\lambda_n}{\lambda_k-\lambda_n}\Big) = \prod_{k\neq n}\Big(\frac{1-z/\lambda_k}{1-\lambda_n/\lambda_k}\Big),$$

where $(\Phi_n)_{n\in\mathbb{N}^*}$ is the sequence of entire functions constructed in Lemma 4.1. By (4.3), we have

$$\Psi_n(z) \ll \mathrm{e}^{\pi\sqrt{|z-\lambda_n|}}\big\{1+|z-\lambda_n|\big\}^B.$$

Let $T>0$, $\kappa > \frac{3}{2}\pi^2$, $3\pi^2T/(4\kappa) < \beta < T/2$ and select $\delta > 0$ so small that

$$\nu := (\pi+\delta)^2/\beta \leqslant (4-\delta)\kappa/(3T). \tag{5.1}$$

We next define the functions

$$g_n(z) := \Psi_n(-z)H_\beta(z+\lambda_n),$$

where H_β is the entire function constructed in Lemma 4.3. Since $\Psi_n(\lambda_k) = \delta_{kn}$ and $H_\beta(0) = 1$, we have

$$g_n(-\lambda_k) = \delta_{kn} \qquad (k,\, n\in\mathbb{N}^*). \tag{5.2}$$

Moreover, it follows from (4.3), (4.15) and (5.1) that

$$|g_n(x)| \ll_\Lambda K_T\frac{(1+|x+\lambda_n|)^B}{\mathrm{e}^{(\delta/2)\sqrt{|x+\lambda_n|}}} \ll \frac{K_T}{1+|x+\lambda_n|^2} \qquad (x\in\mathbb{R}), \tag{5.3}$$

with

$$K_T := \sqrt{\nu+1}\,\mathrm{e}^{3\nu/4} \ll \mathrm{e}^{\kappa/T}. \tag{5.4}$$

Finally, since $\beta < T/2$, we infer from (4.3) and (4.14) that

$$g_n(z) \ll_n \mathrm{e}^{T|z|/2} \qquad (z\in\mathbb{C}). \tag{5.5}$$

By the Paley–Wiener theorem (see, for instance, Rudin [19, p.375]), g_n is, for every $n\in\mathbb{N}^*$, the Fourier transform of a function $f_n \in L^2(\mathbb{R})$ with support included in $\big[-\frac{1}{2}T, \frac{1}{2}T\big]$, i.e.

$$g_n(z) = \int_{-T/2}^{T/2} f_n(t)\mathrm{e}^{-itz}\mathrm{d}t \qquad (z\in\mathbb{C}). \tag{5.6}$$

Since (5.2) and (5.6) imply

$$\int_{-T/2}^{T/2} f_n(t)\mathrm{e}^{i\lambda_k t}\mathrm{d}t = \delta_{kn} \qquad (k,n\in\mathbb{N}^*), \tag{5.7}$$

we see that the sequence $(g_n)_{n\geqslant 1}$ is, in $L^2[-T/2,T/2]$, biorthogonal to the the family $(\mathrm{e}^{i\lambda_n t})_{n\geqslant 1}$.

Second step: construction of the control. Given $\psi \in X$, we define $u \in L^2[0,T]$ by

$$u(t) = -\sum_{k\in\mathbb{N}^*} a_k \mathrm{e}^{-iT\lambda_k/2} f_k\,(t - T/2) \qquad (0 \leqslant t \leqslant T), \tag{5.8}$$

where $a_k := \langle \psi, \varphi_k\rangle / b_k$ $(k \in \mathbb{N})$. We deduce from (5.3) and (5.4) that

$$\int_0^T |u(t)|^2 \mathrm{d}t \ll \mathrm{e}^{2\kappa/T} \sum_{m\in\mathbb{N}^*}\sum_{n\in\mathbb{N}^*} |a_m a_n| \int_{\mathbb{R}} \frac{\mathrm{d}x}{(1+|x+\lambda_m|^2)(1+|x+\lambda_n|^2)}.$$

Inserting the elementary inequality

$$\int_{\mathbb{R}} \frac{\mathrm{d}x}{(1+|x+\lambda_m|^2)(1+|x+\lambda_n|^2)} \leqslant \frac{4\pi}{1+|\lambda_m-\lambda_n|^2} \qquad (m,\, n \in \mathbb{N}^*),$$

yields

$$\int_0^T |u(t)|^2 \mathrm{d}t \ll \mathrm{e}^{2\kappa/T} \sum_{m\in\mathbb{N}^*}\sum_{n\in\mathbb{N}^*} \frac{|a_m a_n|}{1+(\lambda_m-\lambda_n)^2}.$$

We plainly have from (2.8), that $|\lambda_m - \lambda_n| \geqslant \gamma |m-n|$ for all positive integers m, n. So we derive from the above estimate that

$$\|u\|_{L^2[0,T]} \ll \mathrm{e}^{\kappa/T} \|\psi\| \qquad (T > 0). \tag{5.9}$$

Now, (2.8) and (2.1) (with $\mathbb{W} = \mathbb{U}$), together with (2.5) and (5.8), imply that the solution w of (1.1) (with $A = iA_0$ and $Bu = ub$) satisfies

$$w(T) = \sum_{k\in\mathbb{N}^*} \left[\langle \psi, \varphi_k \rangle + b_k \int_0^T u(s) \mathrm{e}^{i\lambda_k s} \mathrm{d}s \right] \mathrm{e}^{-i\lambda_k T} \varphi_k. \tag{5.10}$$

We deduce from (5.7) and (5.8) that, for every $k \in \mathbb{N}^*$, we have

$$\int_0^T u(s) \mathrm{e}^{-i\lambda_k s} \mathrm{d}s \;\; = -\sum_{m\in\mathbb{N}^*} a_m \mathrm{e}^{-iT\lambda_m/2} \int_0^T f_m\,(s - T/2)\, \mathrm{e}^{i\lambda_k s} \mathrm{d}s = -a_k.$$

In view of (5.10), this yields that $w(T) = 0$. By (5.9), this implies in turn that the pair (iA_0, b) is null controllable in time T and that the control cost satisfies (3.2). □

Proof of Corollary 3.2. It is easily checked that, without loss of generality, we may assume $q \leqslant 0$—see, for instance [15]. In order to apply Theorem 3.1, we write $X := H^{-1}(]0,a[)$ and we consider the linear operator $A_0 : X_1 \to X$ defined by

$$X_1 = H_0^1(]0,a[), \quad A_0\varphi = \frac{\mathrm{d}}{\mathrm{d}x}\left(p \frac{\mathrm{d}\varphi}{\mathrm{d}x} \right) + q\varphi \qquad (\varphi \in X_1). \tag{5.11}$$

That A_0 is self-adjoint and strictly negative readily follows from our assumptions on p and q. Let $\mathbb{U}$ stand for the group of isometries on X generated by iA_0. We select $U = \mathbb{C}$ as input space and we consider $b \in X_{-1}$ defined by

$$b = \delta_1', \tag{5.12}$$

where δ_1 is the Dirac distribution supported at $x = 1$. The input operator $B \in \mathscr{L}(\mathbb{C}; X_{-1})$ is defined by $Bu = ub$ for all $u \in \mathbb{C}$. It is known (see, for instance, Curtain and Pritchard [5] and Curtain and Weiss [6]) that the system (3.4) can be written in the form (1.1) with X, A_0 and B chosen as above and $A := iA_0$. Therefore, in order to prove the theorem, it suffices to show that X, A and B satisfy the assumptions in Theorem 3.1 with a suitable constant r.

According to classical estimates on Sturm-Liouville operators (see, for instance, Courant and Hilbert [4, p.415]), the sequence $(\lambda_k)_{k\in\mathbb{N}^*}$ formed by the eigenvalues of $-A_0$, is regular and satisfies

$$\lambda_n - n^2\pi^2/\mu \ll 1 \qquad (n \geqslant 1). \tag{5.13}$$

Moreover, let $(\varphi_k)_{k\in\mathbb{N}^*}$ be an orthonormal basis in $X = H^{-1}(\Omega)$ consisting of eigenvectors of A_0. Using standard estimates on the eigenvectors of Sturm-Liouville operators (see, for instance, [4, Section V.5]), we can check that $b_k = \langle b, \varphi_k\rangle = -\varphi_k'(1) \asymp 1$. Therefore, we can apply Theorem 3.1 (with $r = \pi^2/\mu$) to conclude that the system (3.4) is null-controllable in any time $T > 0$ and that its null-controllability operator satisfies $\|F_T\| \ll \mathrm{e}^{\alpha\mu/T}$ for every $\alpha > \frac{3}{2}$. □

Proof of Corollary 3.3. Put $X := \ell^2(\mathbb{N}, \mathbb{C})$ and let $A_0 : \mathscr{D}(A_0) \to X$ be the diagonal operator defined by

$$\mathscr{D}(A_0) = \Big\{\psi \in \ell^2(\mathbb{N}, \mathbb{C}) : \textstyle\sum_{k\in\mathbb{N}}(1 + |\lambda_k|^2)|\psi_k|^2 < \infty\Big\}, \qquad (A_0\psi)_k = -\lambda_k\psi_k \quad (k \in \mathbb{N}).$$

Note that the canonical basis $(e_k)_{k\in\mathbb{N}}$ of $\ell^2(\mathbb{N}, \mathbb{C})$ is the sequence of eigenvectors of A_0, with corresponding sequence of eigenvalues $(-\lambda_k)$. The operator $A := iA_0$ generates a (diagonal) group of isometries defined by

$$(\mathbb{U}_t\psi)_k = \mathrm{e}^{-i\lambda_k t}\psi_k \qquad (k \in \mathbb{N},\ t \in \mathbb{R}). \tag{5.14}$$

Let $b \in X_{-1}$ be defined by

$$b_k = 1 \qquad (k \in \mathbb{N}).$$

The operator $B \in \mathscr{L}(\mathbb{C}; X_{-1})$ corresponding by (2.6) to the above choice of b is defined by

$$(Bu)_k = u \qquad (u \in \mathbb{C},\ k \in \mathbb{N}),$$

and its adjoint $B^* \in \mathscr{L}(X_1; \mathbb{C})$ is given by

$$B^*\psi = \sum_{k\in\mathbb{N}^*} \psi_k \qquad (\psi \in \mathscr{D}(A)). \tag{5.15}$$

The operators A and B clearly satisfy the assumptions in Theorem 3.1, so the pair (A, B) is null-controllable in any time $T > 0$ and the control cost $C_T = \|F_T\|$ satisfies (3.2) for every $\kappa > \frac{3}{4}\pi$. Combined to (5.14), (5.15) and to Proposition 2.7, this implies the required conclusion. □

Proof of Theorem 3.4. Arguing in a similar way than for Theorem 3.1, we first choose $r = 1$ and then proceed in two steps.
First step: construction of a family of functions bi-orthogonal to $(\mathrm{e}^{\lambda_n t})_{n\geqslant 1}$. Let $T > 0$ and $\kappa > \frac{3}{4}\pi^2$. We choose $\beta \in]3\pi^2 T/(4\kappa), T[$ and $\delta > 0$ so small that $\nu := (\pi + \delta)^2/\beta$ satisfies

$$\nu < (4 - \delta)\kappa/(3T). \tag{5.16}$$

For $n \in \mathbb{N}^*$ and $z \in \mathbb{C}$, we set

$$G_n(z) := \Phi_n(-iz - \lambda_n)\frac{H_\beta(z/2)}{H_\beta(i\lambda_n/2)}, \tag{5.17}$$

where $(\Phi_n)_{n\in\mathbb{N}^*}$ is as defined in Lemma 4.1 and H_β is the entire function constructed in Lemma 4.3. Clearly, G_n is, for each positive integer n, an entire function. It immediately follows from (4.2), (4.3), (4.12), (4.13) and (4.14) that

$$G_n \in L^1(\mathbb{R}) \cap L^2(\mathbb{R}) \qquad (n \in \mathbb{N}^*); \tag{5.18}$$

$$G_n(z) \ll_n \mathrm{e}^{T|z|/2} \qquad (z \in \mathbb{C}); \tag{5.19}$$

$$G_n(i\lambda_k) = \delta_{nk} \qquad (n \in \mathbb{N}^*). \tag{5.20}$$

Moreover, (4.4), (4.13) and (4.15) readily yield that

$$G_n(x) \ll (\nu+1)(\lambda_n + |x|)^B\, \mathrm{e}^{3\nu/4 - \beta\lambda_n/(4\sqrt{\nu+1}) - (\delta/2)\sqrt{|x|/2}} \qquad (x \in \mathbb{R}),$$

so that, in view of (5.16), we plainly have

$$\|G_n\|_{L^1(\mathbb{R})} \ll \lambda_n^{-1}\mathrm{e}^{\kappa/T}. \tag{5.21}$$

By the Paley–Wiener theorem, G_n is, for each $n \in \mathbb{N}^*$, the Fourier transform of a function $F_n \in L^2(\mathbb{R})$ with support included in $\left[-\frac{1}{2}T, \frac{1}{2}T\right]$, i.e.

$$G_n(z) = \int_{-T/2}^{T/2} F_n(t)\mathrm{e}^{-itz}\mathrm{d}t \qquad (z \in \mathbb{C}).$$

By (5.20), this implies that, for all $n, k \in \mathbb{N}^*$, we have

$$\int_{-T/2}^{T/2} F_n(t)\mathrm{e}^{\lambda_k t}\,\mathrm{d}t = \int_{-T/2}^{T/2} F_n(t)\mathrm{e}^{-i(i\lambda_k)t}\,\mathrm{d}t = \delta_{nk}. \tag{5.22}$$

The family $(F_n)_{n\geqslant 1}$ is therefore bi-orthogonal, in $L^2[-T/2, T/2]$, to $(\mathrm{e}^{\lambda_n t})_{n\geqslant 1}$.

Second step: construction of the control. Given arbitrary $\psi \in X$, we define $u \in L^2[0,T]$ by

$$u(t) := -\sum_{k\in\mathbb{N}^*} a_k \mathrm{e}^{-T\lambda_k/2} F_k\left(t - T/2\right), \tag{5.23}$$

with $a_k := \langle \psi, \varphi_k\rangle / b_k$ $(k \geqslant 1)$.

We claim that u satisfies (3.5). Indeed, from (5.23) and (5.21), we have

$$\|u\|_{\mathscr{C}[0,T]} \ll \mathrm{e}^{\kappa/T} \sum_{k\in\mathbb{N}^*} |a_k| \mathrm{e}^{-T\lambda_k/2}\lambda_k^{-1},$$

whence, by the Cauchy-Schwarz inequality,

$$\|u\|^2_{\mathscr{C}[0,T]} \ll \mathrm{e}^{2\kappa/T} \sum_{k\in\mathbb{N}^*} |a_k|^2 \mathrm{e}^{-T\lambda_k} \sum_{m\in\mathbb{N}^*} \lambda_m^{-2} \ll \mathrm{e}^{2\kappa/T} \sum_{k\in\mathbb{N}^*} |a_k|^2\mathrm{e}^{-T\lambda_k}. \tag{5.24}$$

By the choice of a_k and the estimate $|b_k| \asymp 1$, this implies that

$$\|u\|^2_{\mathscr{C}[0,T]} \ll \mathrm{e}^{2\kappa/T} \sum_{k\geqslant 1} |\langle \mathbb{S}_{T/2}\psi, \varphi_k\rangle|^2 = \mathrm{e}^{2\kappa/T}\|\mathbb{S}_{T/2}\psi\|^2.$$

Therefore, u satisfies (3.5).

It remains to show that u as defined in (5.23) drives the solution w of (1.1) (with $A = A_0$ and $Bu = ub$) to rest in time T. From (2.8), (2.1), (2.4) and (5.23), we have

$$w(T) = \sum_{k\in\mathbb{N}^*} \left[\langle\psi,\varphi_k\rangle + b_k \int_0^T u(s)\mathrm{e}^{\lambda_k s}\mathrm{d}s\right] \mathrm{e}^{-\lambda_k T}\varphi_k. \tag{5.25}$$

From (5.22) and (5.23) it follows that

$$\int_0^T u(s)\mathrm{e}^{\lambda_k s}\mathrm{d}s = \quad -\sum_{m\in\mathbb{N}^*} a_m \mathrm{e}^{-T\lambda_m/2}\int_0^T F_m\left(s - T/2\right)\mathrm{e}^{\lambda_k s}\mathrm{d}s = -a_k.$$

Substituting in (5.25) yields $w(T) = 0$. Hence the control u drives the system w to rest in time T. This implies the required conclusion. □

Corollary 3.5 may be derived from Theorem 3.4 by following step by step the proof of Corollary 3.2. We omit the details.

Proof of Corollary 3.6. Consider the operators A_0 and B introduced in the proof of Corollary 3.3. According to Theorem 3.4 the pair (A_0, B) is null-controllable in any time $T > 0$. Let $u \in L^2[0,T]$ be the control given by (5.23) with $b_k = 1$ and $-\overline{a_k}$ in place of a_k. Writing

$$h(t) := u(T-t) = \sum_{k\in\mathbb{N}^*} \overline{a_k}\mathrm{e}^{-T\lambda_k/2} F_k(T/2 - t) \quad (0 \leqslant t \leqslant T),$$

it follows from (3.5) that

$$\int_0^T |h(t)|^2\,\mathrm{d}t \ll \mathrm{e}^{2\kappa/t}\sum_{k\geqslant 1}|a_k|^2\mathrm{e}^{-\lambda_k T}. \tag{5.26}$$

Put

$$f(t) := \sum_{n\geqslant 1} a_n \mathrm{e}^{-\lambda_n t} \qquad (t \geqslant 0).$$

We have

$$\begin{aligned}\int_0^T h(t)f(t)\,\mathrm{d}t &= \sum_{n,\,k\geqslant 1} \overline{a_k}a_n\,\mathrm{e}^{-T(\lambda_n+\lambda_k)/2}\int_0^T F_k(T/2-t)\mathrm{e}^{\lambda_n(T/2-t)}\mathrm{d}t \\ &= \sum_{k\geqslant 1}|a_k|^2\mathrm{e}^{-\lambda_k T},\end{aligned}$$

where the last equality follows from (5.22). In view of (5.26), this yields the required conclusion by the Cauchy-Schwarz inequality. □

6 Results in several space dimensions

As noted in the introduction, the results of the previous sections have consequences on null-controllability problems in several space dimensions. The passage from one dimensional results (as Corollaries 3.4 and 3.7) to several space dimensions estimates has been studied in [14], [15], [16] by the *control transmutation method.*

Let $\Omega \subset \mathbb{R}^n$ be an open bounded set and let Γ be an open non empty subset of $\partial\Omega$. We consider the initial and boundary value problem:

$$\text{(6.1)}\qquad \begin{cases} i\dot{w} + \Delta w = 0 & (x \in \Omega,\ t \geqslant 0),\\ w = u & (x \in \Gamma,\ t \geqslant 0),\\ w = 0 & (x \in \partial\Omega \setminus \overline{\Gamma},\ t \geqslant 0),\\ w(x,0) = \psi(x) & (x \in \Omega). \end{cases}$$

We also introduce a corresponding initial and boundary value problem for the heat equation

$$\text{(6.2)}\qquad \begin{cases} \dot{w} - \Delta w = 0 & (x \in \Omega,\ t \geqslant 0),\\ w = u & (x \in \Gamma,\ t \geqslant 0),\\ w = 0 & (x \in \partial\Omega \setminus \overline{\Gamma},\ t \geqslant 0),\\ w(x,0) = \psi(x) & (x \in \Omega). \end{cases}$$

It is classical knowledge that, under some regularity assumptions on Ω and Γ, each of the systems (6.1) and (6.2) determines a well-posed linear system with input space $L^2(\Gamma)$ and state space $H^{-1}(\Omega)$. A sufficient condition for the null-controllability of these systems is that Γ satisfies the generalized geodesics condition of Bardos, Lebeau and Rauch [2]. In our case, this means, roughly speaking, that any light ray travelling in Ω and reflected according to geometrical optic laws when it hits $\partial\Omega$, will intersect Γ (see [2] or [14] for more details on this condition).

Proposition 6.1. *Assume that $\Omega \subset \mathbb{R}^n$ is a bounded domain with smooth frontier and that $\Gamma \subset \partial\Omega$ satisfies the generalized geodesics condition. Then, the system determined by* (6.1) *is null-controllable in any time $T > 0$ and the control cost satisfies*

$$\text{(6.3)}\qquad \limsup_{T\to 0} T \ln C_T \leqslant \tfrac{3}{2} L_\Gamma^2\,,$$

where L_Γ is the length of the longest generalized geodesic in $\overline{\Omega}$ not intersecting Γ.

Proof. Consider the system with state space $L^2(\Omega) \times H^{-1}(\Omega)$ and input space $L^2(\Gamma)$ determined by the equations

$$\text{(6.4)}\qquad \begin{cases} \ddot{w} - \Delta w = 0 & (x \in \Omega,\ t \geqslant 0),\\ w = u & (x \in \Gamma,\ t \geqslant 0),\\ w = 0 & (x \in \partial\Omega \setminus \overline{\Gamma},\ t \geqslant 0),\\ w(x,0) = \psi_0(x),\ \dot{w}(x,0) = \psi_1(x) & (x \in \Omega). \end{cases}$$

According to [2], our assumptions imply that there exists $T_0 > 0$ such that the system (6.4) is exactly controllable in time T_0. This fact, combined to Theorem 3.1 of [16] and to our Corollary 3.2, implies the required result. □

If Ω is the rectangle $]0,a[\times]0,b[$, with $a,\, b > 0$, a sufficient condition ensuring that Γ satisfies the generalized geodesics condition is that $\Gamma \supset ([0,a] \times \{0\}) \cup (\{0\} \times [0,b])$. The result in Proposition 6.1, although not directly applicable to Ω and Γ as above (since Ω does not have a smooth boundary), suggests that the control cost satisfies

$$\limsup_{T\to 0} T \ln C_T \leqslant \tfrac{3}{2}(a^2+b^2).$$

The result below improves the above estimate inasmuch the constant a^2+b^2 is replaced by a smaller one and we obtain an effective estimate valid for every $T>0$.

Proposition 6.2. *Let Ω be the rectangle $]0,a[\times]0,b[$, with $a,\, b>0$ and assume that $\Gamma\supset([0,a]\times\{0\})\cup(\{0\}\times[0,b])$. Denote $\mu := \max(a^2,b^2)$ and let $\alpha>\frac{3}{2}$. Then the system* (6.1) *is null-controllable in any time $T>0$ and the control cost satisfies the bound*

$$C_T \ll \mathrm{e}^{\alpha\mu/T} \qquad (T>0).$$

Proof. Appealing to Proposition 2.7, it can be checked (see, for instance, [13]) that $\|F_T\| = C_T$ is the smallest quantity satisfying

$$\|\nabla\psi\|^2_{L^2(\Omega)} \leqslant C_T^2 \int_0^T \int_\Gamma \left|\frac{\partial\varphi}{\partial\nu}\right|^2 \mathrm{d}\Gamma \mathrm{d}t \qquad \big(\psi\in H^2(\Omega)\cap H_0^1(\Omega)\big), \tag{6.5}$$

where φ is the solution of the initial and boundary value problem

$$\begin{cases} \dot\varphi + i\Delta\varphi = 0 & (x\in\Omega,\, t\geqslant 0),\\ \varphi = 0 & (x\in\partial\Omega,\, t\geqslant 0),\\ \varphi(x,0)=\psi(x), & (x\in\Omega). \end{cases} \tag{6.6}$$

Define

$$\varphi_{mn}(x,y) = \frac{2\sqrt{ab}}{\pi\sqrt{b^2m^2+a^2n^2}} \sin\Big(\frac{m\pi x}{a}\Big)\sin\Big(\frac{n\pi y}{b}\Big) \qquad (m,n\in\mathbb{N}^*).$$

The above family forms an orthogonal basis in $L^2(\Omega)$ and the family $(\nabla\varphi_{mn})$ is orthonormal in $L^2(\Omega)$. Let (c_{mn}) be the components of ψ with respect to this basis so that

$$\psi = \sum_{m,n\in\mathbb{N}} c_{mn}\varphi_{mn},$$

with $(c_{mn})\in\ell^2(\mathbb{C})$. It follows that

$$\begin{aligned}\int_0^T\int_\Gamma\left|\frac{\partial\varphi}{\partial\nu}\right|^2\mathrm{d}\Gamma\mathrm{d}t \geqslant \frac{4a}{b}\int_0^T\int_0^a\Big|\sum_{m,n\geqslant 1}\frac{nc_{mn}\mathrm{e}^{i\pi^2(m^2/a^2+n^2/b^2)\,t}}{\sqrt{b^2m^2+a^2n^2}}\sin\Big(\frac{m\pi x}{a}\Big)\Big|^2\mathrm{d}x\mathrm{d}t\\ +\frac{4b}{a}\int_0^T\int_0^b\Big|\sum_{m,n\geqslant 1}\frac{mc_{mn}\mathrm{e}^{i\pi^2(m^2/a^2+n^2/b^2)\,t}}{\sqrt{b^2m^2+a^2n^2}}\sin\Big(\frac{n\pi y}{a}\Big)\Big|^2\mathrm{d}y\mathrm{d}t.\end{aligned}$$

The above formula, combined to the orthogonality of the family $(\sin(m\pi x/a))_{m\geqslant 1}$ (respectively $(\sin(n\pi y/b))_{n\geqslant 1}$) in $L^2[0,a]$ (respectively in $L^2[0,b]$), implies that

$$\begin{aligned}&\int_0^T\int_\Gamma\left|\frac{\partial\varphi}{\partial\nu}\right|^2\mathrm{d}\Gamma\mathrm{d}t\\ &\geqslant \frac{a}{b}\sum_{m\geqslant 1}\int_0^T\Big|\sum_{n\geqslant 1}\frac{nac_{mn}\mathrm{e}^{i\pi^2n^2t/b^2}}{\sqrt{b^2m^2+a^2n^2}}\Big|^2\mathrm{d}t + \frac{b}{a}\sum_{n\geqslant 1}\int_0^T\Big|\sum_{m\geqslant 1}\frac{mbc_{mn}\mathrm{e}^{i\pi^2m^2t/a^2}}{\sqrt{b^2m^2+a^2n^2}}\Big|^2\mathrm{d}t.\end{aligned}$$

By applying Corollary 3.3 it follows that there exists $C_T > 0$ such that $C_T \ll \mathrm{e}^{\alpha\mu/T}$ for every $\alpha > \frac{3}{2}$ and

$$C_T^2 \int_0^T \int_\Gamma \left|\frac{\partial \varphi}{\partial \nu}\right|^2 \mathrm{d}\Gamma \mathrm{d}t \geqslant \sum_{m,n \geqslant 1} |c_{mn}|^2,$$

which is exactly (6.5). This ends up our proof. □

Remark 6.3. It has been recently shown in Ramdani, Takahashi, Tenenbaum and Tucsnak [18] that if Ω is a square in $\mathbb{R}^2$ then the system determined by (6.1) is null-controllable even when the controlled part Γ of the boundary is arbitrarily small. It would be interesting to prove that this property holds in arbitrarily small time and to estimate the corresponding control cost.

The result in Proposition 6.1 has the following counterpart for the heat equation.

Proposition 6.4. *Assume that $\Omega \subset \mathbb{R}^n$ is a smooth bounded domain and that Γ satisfies the generalized geodesics condition. Then, the system determined by* (6.2) *is null-controllable in any time $T > 0$ and the control cost satisfies*

$$\limsup_{T \to 0} T \ln C_T \leqslant \tfrac{3}{4} L_\Gamma^2\,, \tag{6.7}$$

where L_Γ is the length of the longest generalized geodesic in $\overline{\Omega}$ not intersecting Γ.

Proof. The result follows directly from Corollary 7.6 and Theorem 6.2 in [17]. □

Proposition 6.2 also has a counterpart for the heat equation. Since the proof is identical, we omit it.

Proposition 6.5. *Let Ω, a, $b > 0$, Γ and μ be as in Proposition 6.2 and let $\alpha > \frac{3}{4}$. Then then the system* (6.2) *is null-controllable in any time $T > 0$ and the control cost C_T satisfies*

$$C_T \ll \mathrm{e}^{\alpha\mu/T} \qquad (T > 0).$$

7 Lower bounds

The question of giving *lower bounds* of the control cost for the Schrödinger and the heat equations has also been investigated in the literature. The first result in this direction, due to Güichal [10], concerns the heat equation. It asserts that, for $p \equiv 1$ and $q \equiv 0$, the control cost for the parabolic system (3.7) satisfies the condition

$$\liminf_{T \to 0} T \ln C_T > 0.$$

This is extended to the heat equation in several space dimensions and with internal control in [28]—see also Fernandez-Cara and Zuazua [9]. These results are improved in [14], where it is shown, in particular, that for $p \equiv 1$ and $q \equiv 0$, the control cost for the system (3.7) satisfies the condition

$$\liminf_{T \to 0} T \ln C_T \geqslant \tfrac{1}{4} a^2. \tag{7.1}$$

As far as we know, the only available lower bound for the control cost of the Schrödinger equation appears in [15], where the inequality (7.1) is proved to hold also for the system (3.4), with $p \equiv 1$ and $q \equiv 0$. Moreover, in the case of Schrödinger and heat equations in several space dimensions (with internal control), lower bounds for the control cost are provided in [28], [14] and [15].

In this section, we give, for both the Schrödinger and heat equations in one space dimension, a simple proof of the estimate (7.1). Moreover, we use the same method to establish a lower bound for the Schrödinger equation in a rectangular domain in $\mathbb{R}^2$ (this estimate is slightly different of that obtained in the general case). Our approach is based on classical properties of the Jacobi theta function, while the arguments of [14] and [15] rest upon a deep formula of Varadhan.

Proposition 7.1. *Let $r > 0$ and let $\Lambda := (\lambda_n)$ be the sequence defined by $\lambda_n = rn^2$ for all $n \in \mathbb{N}$. Assume that, for every $T \in]0,1]$, there is a real number K_T such that the inequality*

$$\sum_{n\geqslant 0} |a_n|^2 \leqslant K_T^2 \int_{-T/2}^{T/2} \Big|\sum_{n\geqslant 0} a_n \mathrm{e}^{i\lambda_n t}\Big|^2 \mathrm{d}t, \tag{7.2}$$

holds uniformly for $(a_n) \in \ell^2(\mathbb{C})$. Then

$$K_T \gg_r T^{1/4} \mathrm{e}^{\pi^2/(4rT)}.$$

Proof. We may plainly restrict to proving the result for any given special value of r. Let us select $r = \pi$. Let $\xi > 0$ be a parameter which will be specified later. We choose

$$a_0 = \tfrac{1}{2}, \quad a_n := (-1)^n \mathrm{e}^{-\pi n^2 \xi} \quad (n \geqslant 1). \tag{7.3}$$

Denoting $z := \xi - it$, we can write

$$f(t) = \sum_{n\geqslant 0} a_n \mathrm{e}^{-2\pi n^2 t i} = \tfrac{1}{2} \sum_{n\in\mathbb{Z}} (-1)^n \mathrm{e}^{-\pi n^2 z}.$$

We next introduce the Jacobi theta function defined by

$$\vartheta(z) := \sum_{n\in\mathbb{Z}} \mathrm{e}^{-\pi n^2 z} \qquad (\Re z > 0). \tag{7.4}$$

It is known (see, for instance, [26], §21.51) that ϑ satisfies the functional equation

$$\vartheta(z) = \frac{1}{\sqrt{z}} \vartheta\Big(\frac{1}{z}\Big),$$

where the square root is chosen to be positive for real positive z. Therefore

$$\begin{aligned} f(t) &= \tfrac{1}{2}\Big\{\sum_{n\in\mathbb{Z}} \mathrm{e}^{-4\pi n^2 z} - \Big(\sum_{n\in\mathbb{Z}} \mathrm{e}^{-\pi n^2 z} - \sum_{n\in\mathbb{Z}} \mathrm{e}^{-4\pi n^2 z}\Big)\Big\} \\ &= \tfrac{1}{2}\big\{2\vartheta(4z) - \vartheta(z)\big\} = \frac{1}{2\sqrt{z}}\Big\{\vartheta\Big(\frac{1}{4z}\Big) - \vartheta\Big(\frac{1}{z}\Big)\Big\}. \end{aligned} \tag{7.5}$$

It follows that for T, $\xi\in]0,1[$, $|t|\leqslant \frac12 T$, we have

$$f(t)\ll \frac{\mathrm{e}^{-\pi\xi/(4\xi^2+T^2)}}{\xi+T}$$

so that

$$\int_I |f(t)|^2\,\mathrm{d}t \ll \frac{T\mathrm{e}^{-2\pi\xi/(4\xi^2+T^2)}}{\xi^2+T^2}.$$

Since

$$\sum_{n\geqslant 0}|a_n|^2=\tfrac12\vartheta(2\xi)\asymp\frac{1}{\sqrt{\xi}},$$

we see that by selecting $\xi=T/2$ that if (7.2) holds then

$$\frac{1}{\sqrt{T}}\ll K_T^2\frac{\mathrm{e}^{-\pi/2T}}{T}.$$

The required estimate follows immediately. □

The above result yields, at least for the case $\lambda_n=rn^2$, a lower bound for the control cost of the system in Theorem 3.1. More precisely, we obtain the following statement as a consequence of Propositions 2.7 and 7.1.

Corollary 7.2. *Let $r>0$. Put $X=\ell^2(\mathbb{N},\mathbb{C})$ and let $A:\mathscr{D}(A)\to X$ be the diagonal operator defined by*

$$\mathscr{D}(A)=\left\{\psi\in\ell^2(\mathbb{N},\mathbb{C}):\textstyle\sum_{k\in\mathbb{N}}(1+k^2)|\psi_k|^2<\infty\right\},\qquad (A\psi)_k=irk^2\psi_k\quad(k\in\mathbb{N}).$$

Let $B\in\mathscr{L}(\mathbb{C};X_{-1})$ be the operator defined by

$$(Bu)_k=u\qquad(u\in\mathbb{C},\ k\in\mathbb{N}).$$

Then the control cost for the pair (A,B) satisfies

$$C_T\gg T^{1/4}\mathrm{e}^{\pi^2/(4rT)}\qquad(0<T\leqslant 1).$$

Applying the above to the system (3.4), with $p\equiv 1$ and $q\equiv 0$ we obtain the following lower bound for the constant α appearing in (3.3).

Corollary 7.3. *Let $a>0$, $T\in]0,1[$ and assume that, for every $\psi\in H^{-1}(]0,a[)$, there exists $u\in L^2[0,T]$ verifying*

$$\|u\|_{L^2[0,T]}\ll \mathrm{e}^{\alpha a^2/T}\|\psi\|_{H^{-1}(]0,a[)},$$

(the implicit constant being independent of T) and such that the solution w of

$$\begin{cases} -i\dfrac{\partial w}{\partial t}(x,t)=\dfrac{\partial^2 w}{\partial x^2}(x,t) & (0<x<a,\ 0<t<T)\\ w(0,t)=u(t) & (0<t<T)\\ w(a,t)=0 & (0<t<T)\\ w(x,0)=\psi(x) & (0<x<a),\end{cases}\tag{7.6}$$

satisfies $w(x,T)=0$ for all $x\in]0,a[$. Then $\alpha\geqslant\frac14 a^2$.

In order to obtain a corresponding lower bound for the control cost of the heat equation, we need the following counterpart for real exponentials of the estimate in Proposition 7.1.

Proposition 7.4. *Let $r > 0$ and set $\lambda_n = rn^2$ for all $n \in \mathbb{N}$. Assume that, for each $T \in]0,1]$, there is a number K_T such that the inequality*

$$\sum_{n\geqslant 0} |a_n|^2 \mathrm{e}^{-\lambda_n T} \leqslant K_T^2 \int_0^T \Big| \sum_{n\geqslant 0} a_n \mathrm{e}^{-\lambda_n t} \Big|^2 \mathrm{d}t, \tag{7.7}$$

holds uniformly for $(a_n)_{n\geqslant 0} \in \ell^2(\mathbb{C})$. Then

$$K_T \gg_r T^{1/4} \mathrm{e}^{\pi^2/(4rT)}.$$

Proof. As before, we may select any convenient value for r and we pick up $r = \pi$: the general result follows by a change of variables. Let $\xi > 0$ to be specified later. Consider the the sequence $(a_n)_{n\geqslant 0}$ with $a_n = (2 - \delta_{0n})(-1)^n \mathrm{e}^{-\pi n^2 \xi}$, so that

$$\begin{aligned} f(t) := \sum_{n\geqslant 0} a_n \mathrm{e}^{-\pi n^2 t} &= \sum_{n\in\mathbb{Z}} (-1)^n \mathrm{e}^{-\pi n^2(\xi+t)} \\ &= 2\vartheta(4\xi + 4t) - \vartheta(\xi + t) \\ &= \frac{1}{\sqrt{\xi + t}} \Big\{ \vartheta\Big(\frac{1}{4\xi + 4t}\Big) - \vartheta\Big(\frac{1}{\xi + t}\Big) \Big\}. \end{aligned} \tag{7.8}$$

It follows that

$$f(t) \ll \frac{\mathrm{e}^{-\pi/(4\xi+4t)}}{\sqrt{\xi + t}},$$

and hence

$$\int_0^T |f(t)|^2 \,\mathrm{d}t \ll \frac{\mathrm{e}^{-\pi/(2\xi+2T)}}{T + \xi}.$$

Since

$$\sum_{n\geqslant 0} |a_n|^2 \mathrm{e}^{-\pi n^2 T} = 2\vartheta(2\xi + T) - 1 \asymp \frac{1}{\sqrt{\xi + T}},$$

we see, by selecting for instance $\xi = T^2$, that

$$\frac{1}{\sqrt{T}} \ll K_T^2 \frac{\mathrm{e}^{-\pi/\{2T(1+T)\}}}{T} \ll K_T^2 \frac{\mathrm{e}^{-\pi/(2T)}}{T},$$

which implies the stated inequality. □

Proposition 7.4 implies two corollaries, which are the counterparts for the heat equation of the results in corollaries 7.2 and 7.3. More precisely, we have the following statements.

Corollary 7.5. *Put $X = \ell^2(\mathbb{N}, \mathbb{C})$ and let $A : \mathscr{D}(A) \to X$ be the diagonal operator defined by*

$$(A\psi)_k = rk^2 \psi_k, \qquad \mathscr{D}(A) = \Big\{ \psi \in \ell^2(\mathbb{N}, \mathbb{C}) : \textstyle\sum_{k\in\mathbb{N}} (1 + k^2)|\psi_k|^2 < \infty \Big\}.$$

Let $B \in \mathscr{L}(\mathbb{C}; X_{-1})$ be the operator defined by

$$(Bu)_k = u \qquad (u \in \mathbb{C},\ k \in \mathbb{N}),$$

Then the control cost for the pair (A, B) satisfies the condition

$$C_T \gg T^{1/4} \mathrm{e}^{\pi^2/(4rT)} \qquad (0 < T \leqslant 1).$$

Corollary 7.6. *Let $a > 0$, $T \in]0,1[$ and assume that, for every $\psi \in H^{-1}(]0,a[)$, there exists $u \in L^2[0,T]$ verifying*

$$\|u\|_{L^2[0,T]} \ll \mathrm{e}^{\alpha a^2/T} \|\psi\|_{H^{-1}(]0,a[)},$$

(the implicit constant being independent of T) and such that the solution w of

$$\begin{cases} \dfrac{\partial w}{\partial t}(x,t) = \dfrac{\partial^2 w}{\partial x^2}(x,t) & (0 < x < a,\ 0 < t < T) \\ w(0,t) = u(t) & (0 < t < T) \\ w(a,t) = 0 & (0 < t < T) \\ w(x,0) = \psi(x) & (0 < x < a), \end{cases} \tag{7.9}$$

satisfies $w(x,T) = 0$ for all $x \in]0,a[$. Then $\alpha \geqslant \frac{1}{4}a^2$.

We omit the proofs of the two above corollaries, since they are almost identical to those of Corollaries 7.2 and 7.3.

We end up with the remark that the estimate

$$\sum_{n \in \mathbb{Z}} (-1)^n \mathrm{e}^{-\pi z n^2 + i\pi n x} \ll \frac{\mathrm{e}^{-\pi(1-|x|)^2 \Re z/\{4|z|^2\}}}{\sqrt{|z|}} \qquad (|x| \leqslant \tfrac{1}{2},\ \Re z > 0), \tag{7.10}$$

easily established by Poisson's summation formula, readily provides a lower bound for the control cost of a system governed by a Schrödinger or a heat equation in a rectangular domain. We omit the proof, which is very similar to that of Proposition 7.4.

Proposition 7.7. *Let Ω be the rectangle $]0,a[\times]0,b[$, with a, $b > 0$ and assume that $\Gamma = ([0,a] \times \{0\}) \cup (\{0\} \times [0,b])$. Let $\mu = \min(a^2, b^2)$. Then the control costs of the systems* (6.1) *and* (6.2) *satisfy*

$$C_T \gg T^{1/4} \mathrm{e}^{\mu\pi^2/(4T)} \qquad (0 < T \leqslant 1). \tag{7.11}$$

References

[1] L. V. Ahlfors, *Complex analysis*, McGraw-Hill Book Co., New York, third ed., 1978. An introduction to the theory of analytic functions of one complex variable, International Series in Pure and Applied Mathematics.

[2] C. Bardos, G. Lebeau, and J. Rauch, *Sharp sufficient conditions for the observation, control and stabilization of waves from the boundary*, SIAM J. Control. and Optim., 30 (1992), pp. 1024–1065.

[3] E. Bombieri, J. B. Friedlander, and H. Iwaniec, *Primes in arithmetic progressions to large moduli. II*, Math. Ann., 277 (1987), pp. 361–393.

[4] R. Courant and D. Hilbert, *Methods of Mathematical Physics*, Interscience, New York, London, 1953.

[5] R. F. Curtain and A. J. Pritchard, *Infinite dimensional linear systems theory*, vol. 8 of Lecture Notes in Control and Information Sciences, Springer-Verlag, Berlin, 1978.

[6] R. F. Curtain and G. Weiss, *Well posedness of triples of operators (in the sense of linear systems theory)*, in Control and estimation of distributed parameter systems (Vorau, 1988), vol. 91 of Internat. Ser. Numer. Math., Birkhäuser, Basel, 1989, pp. 41–59.

[7] S. Dolecki and D. L. Russell, *A general theory of observation and control.*, SIAM J. Control Optimization, 15 (1977), pp. 185–220.

[8] H. O. Fattorini and D. L. Russell, *Exact controllability theorems for linear parabolic equations in one space dimension*, Arch. Rational Mech. Anal., 43 (1971), pp. 272–292.

[9] E. Fernández-Cara and E. Zuazua, *The cost of approximate controllability for heat equations: the linear case*, Adv. Differential Equations, 5 (2000), pp. 465–514.

[10] E. N. Güichal, *A lower bound of the norm of the control operator for the heat equation*, J. Math. Anal. Appl., 110 (1985), pp. 519–527.

[11] L. F. Ho and D. L. Russell, *Admissible input elements for systems in Hilbert space and a Carleson measure criterion*, SIAM J. Control Optim., 21 (1983), pp. 614–640.

[12] S. Jaffard and S. Micu, *Estimates of the constants in generalized Ingham's inequality and applications to the control of the wave equation*, Asymptot. Anal., 28 (2001), pp. 181–214.

[13] G. Lebeau, *Contrôle de l'équation de Schrödinger*, J. Math. Pures Appl. (9), 71 (1992), pp. 267–291.

[14] L. Miller, *Geometric bounds on the growth rate of null-controllability cost for the heat equation in small time*, J. Differential Equations, 204 (2004), pp. 202–226.

[15] ——, *How violent are fast controls for Schrödinger and plate vibrations?*, Arch. Ration. Mech. Anal., 172 (2004), pp. 429–456.

[16] ——, *Controllability cost of conservative systems: resolvent condition and transmutation*, J. Funct. Anal., 218 (2005), pp. 425–444.

[17] L. Miller, *The control transmutation method and the cost of fast controls*, SIAM J. Control Optim., 45 (2006), pp. 762–772.

[18] K. Ramdani, T. Takahashi, G. Tenenbaum, and M. Tucsnak, *A spectral approach for the exact observability of infinite-dimensional systems with skew-adjoint generator*, J. Funct. Anal., 226 (2005), pp. 193–229.

[19] W. Rudin, *Real and complex analysis*, McGraw-Hill Book Co., New York, third ed., 1987.

[20] T. I. Seidman, *Two results on exact boundary control of parabolic equations*, Appl. Math. Optim., 11 (1984), pp. 145–152.

[21] ——, *How violent are fast controls?*, Math. Control Signals Systems, 1 (1988), pp. 89–95.

[22] T. I. Seidman, S. A. Avdonin, and S. A. Ivanov, *The "window problem" for series of complex exponentials*, J. Fourier Anal. Appl., 6 (2000), pp. 233–254.

[23] T. I. Seidman and J. Yong, *How violent are fast controls? II*, Math. Control Signals Systems, 9 (1996), pp. 327–340.

[24] G. Weiss, *Admissibility of input elements for diagonal semigroups on l^2*, Systems Control Lett., 10 (1988), pp. 79–82.

[25] ——, *Admissibility of unbounded control operators*, SIAM J. Control Optim., 27 (1989), pp. 527–545.

[26] E. T. Whittaker and G. N. Watson, *A course of modern analysis. An introduction to the general theory of infinite processes and of analytic functions: with an account of the principal transcendental functions*, Fourth edition. Reprinted, Cambridge University Press, New York, 1962.

[27] J. Zabczyk, *Mathematical control theory: an introduction*, Systems & Control: Foundations & Applications, Birkhäuser Boston Inc., Boston, MA, 1992.

[28] E. Zuazua, *Some results and open problems on the controllability of linear and semilinear heat equations*, in Carleman estimates and applications to uniqueness and control theory (Cortona, 1999), vol. 46 of Progr. Nonlinear Differential Equations Appl., Birkhäuser Boston, Boston, MA, 2001, pp. 191–211.