Upwind discretisation of a time-dependent two dimensional grade-two fluid model

Hyam ABBOUD[†,⋆] and Toni SAYAH[⋆]

† Faculté des Sciences et d'Ingénierie
Université Saint-Esprit de Kaslik
B.P. 446 Jounieh, Liban

⋆ Faculté des Sciences
Université Saint-Joseph
B.P. 11-514 Riad El Solh
Beyrouth 1107 2050, Liban

E-mail: hyamabboud@usek.edu.lb, tsayah@fs.usj.edu.lb

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Abstract

In this paper, we propose a finite-element scheme for solving numerically the equations of a transient two dimensional grade-two fluid non-Newtonian Rivlin-Ericksen fluid model. This system of equations is considered an appropriate model for the motion of a water solution of polymers. As expected, the difficulties of this problem arise from the transport equation. As one of our aims is to derive unconditional a priori estimates from the discrete analogue of the transport equation, we stabilize our scheme by adding a consistent stabilizing term. We use the $\mathbb{P}_2 - \mathbb{P}_1$ Taylor-Hood finite-element scheme for the velocity $\mathbf{v}$ and the pressure p, and the discontinuous $\mathbb{P}_1$ finite element for an auxiliary variable $\mathbf{z}$. The error is of the order of $h^{3/2} + k$, considering that the discretization of the transport equation loses inevitably a factor $h^{1/2}$.

Keywords Grade-two fluid, non-linear problem, incompressible flow, time and space discretizations.

1 Introduction

This article is devoted to the numerical solution of the equation of a grade two fluid non-Newtonian Rivlin-Ericksen fluid ([16]) :

$$\frac{\partial}{\partial t}(\mathbf{u} - \alpha\Delta\mathbf{u}) - \nu\Delta\mathbf{u} + \mathbf{curl}(\mathbf{u} - \alpha\Delta\mathbf{u}) \times \mathbf{u} + \nabla p = \mathbf{f} \text{ in }]0, T[\times\Omega, \tag{1}$$

with the incompressibility condition :

$$\operatorname{div} \mathbf{u} = 0 \text{ in }]0, T[\times\Omega, \tag{2}$$

where the velocity $\mathbf{u} = (u_1, u_2, 0)$,

$$\operatorname{div} \mathbf{u} = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2}, \quad \mathbf{curl}\, \mathbf{u} = (0, 0, \operatorname{curl} \mathbf{u}), \quad \operatorname{curl}\ \mathbf{u} = \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2},$$

here $\mathbf{f}$ denotes an external force, ν the viscosity and α is a constant normal stress modulus.

This model is considered an appropriate one for the motion of water solutions of polymers ([7]). The case $\alpha = 0$ represents the transient Navier-Stokes problem. Here, p is not the pressure, but the formula which gives the pressure from $\mathbf{u}$ and p is complex. To simplify, we refer to p as the "pressure" in the sequel. According to Dunn and Fosdick's work [8], in order to be consistent with thermodynamics, a grade-two fluid must satisfy $\alpha \geq 0$ and $\nu \geq 0$. The reader can refer to [7] for a discussion on the sign of α.

The equations of a grade two fluid model have been studied by many authors (Videmann gives in [17] a very extensive list of references), but the best construction of solutions for the problem, with homogeneous Dirichlet boundary conditions and mildly smooth data, is given by Ouazar [15] and by Cioranescu and Ouazar [3], [4]. They prove existence of solutions, with H^3 regularity in space, by looking for a velocity $\mathbf{u}$ such that

$$\mathbf{z} = \mathbf{curl}(\mathbf{u} - \alpha \Delta \mathbf{u})$$

has L^2 regularity in space, introducing $\mathbf{z}$ as an auxiliary variable and discretizing the equations of motion by Galerkin's method in the basis of the eigenfunctions of the operator $\mathbf{curl}\,\mathbf{curl}(\mathbf{u} - \alpha \Delta \mathbf{u})$. This choice allows one to recover estimates from the transport equation in two dimensions

$$\alpha \frac{\partial \mathbf{z}}{\partial t} + \nu \mathbf{z} + \alpha \mathbf{u} \cdot \nabla \mathbf{z} = \nu\, \mathbf{curl}\, \mathbf{u} + \alpha\, \mathbf{curl}\, \mathbf{f}, \tag{3}$$

whenever $\mathbf{curl}\, \mathbf{f}$ belongs to $L^2(\Omega)^3$. In this case, $\mathbf{z} = (0, 0, z)$ with $z = \operatorname{curl}(\mathbf{u} - \alpha \Delta \mathbf{u})$. Hence, $\mathbf{z}$ is necessarily orthogonal to $\mathbf{u}$.

In this article, we propose finite-element schemes for solving numerically the equation of a two dimensional grade-two fluid model. Defining $\mathbf{z}$ as above, the equations of motion becomes :

$$\frac{\partial}{\partial t}(\mathbf{u} - \alpha \Delta \mathbf{u}) - \nu \Delta \mathbf{u} + z \times \mathbf{u} + \nabla p = \mathbf{f}, \tag{4}$$

and

$$\alpha \frac{\partial z}{\partial t} + \nu z + \alpha \mathbf{u} \cdot \nabla z = \nu\ \operatorname{curl} \mathbf{u} + \alpha\ \operatorname{curl} \mathbf{f}, \tag{5}$$

the Dirichlet boundary condition :

$$\mathbf{u} = \mathbf{0} \quad \text{on }]0, T[\times \partial\Omega, \tag{6}$$

and the initial conditions :

$$\mathbf{u}(x, t) = \mathbf{0}, \quad \text{and} \quad z(x, t) = 0. \tag{7}$$

This problem is analyzed by Girault and Saadouni [10]. If we want to derive an unconditional a priori estimate for the discrete analogue of (3), we add to the left-hand side of this last equation a stabilizing, consistent term, so it becomes

$$\alpha\frac{\partial z}{\partial t} + \nu z + \alpha \mathbf{u}\cdot\nabla z + \frac{\alpha}{2}(\operatorname{div}\mathbf{u})z = \nu\operatorname{curl}\mathbf{u} + \alpha\operatorname{curl}\mathbf{f}. \tag{8}$$

In this work, we propose to discretize this last equation, as Girault and Scott did in [11], by an upwind scheme based on the discontinuous Galerkin method of degree one introduced by Lesaint and Raviart in [12]. Let X_h, M_h and Z_h be the discrete spaces for the velocity and the pressure. We approximate the velocity and the pressure by the standard $\mathbb{P}_2 - \mathbb{P}_1$ Taylor-Hood scheme, where $\mathbb{P}_k$ denotes the space of polynomials of degree k in two variables. Also, in each element of the triangulation, z_h^n is a polynomial of degree one, without continuity requirement on interelement boundaries. Our discrete system corresponding to (4) and (8) is :

Find $\mathbf{u}_h^{n+1} \in X_h, p_h^{n+1} \in M_h$ and $z_h^{n+1} \in Z_h$ such that

$$\begin{aligned}\forall \mathbf{v}_h \in X_h, \frac{1}{k}(\mathbf{u}_h^{n+1} - \mathbf{u}_h^n, \mathbf{v}_h) + \frac{\alpha}{k}(\nabla(\mathbf{u}_h^{n+1} - \mathbf{u}_h^n), \nabla\mathbf{v}_h) + \nu(\nabla\mathbf{u}_h^{n+1}, \nabla\mathbf{v}_h)\\ +(z_h^n \times \mathbf{u}_h^{n+1}, \mathbf{v}_h) - (p_h^{n+1}, \operatorname{div}\mathbf{v}_h) = (\mathbf{f}^{n+1}, \mathbf{v}_h),\end{aligned} \tag{9}$$

$$\begin{aligned}\forall \theta_h \in Z_h, \frac{\alpha}{k}(z_h^{n+1} - z_h^n, \theta_h) + \nu(z_h^{n+1}, \theta_h) + c(\mathbf{u}_h^{n+1}; z_h^{n+1}, \theta_h)\\ = \nu(\operatorname{curl}\mathbf{u}_h^{n+1}, \theta_h) + \alpha(\operatorname{curl}\mathbf{f}^{n+1}, \theta_h),\end{aligned} \tag{10}$$

where $c(\mathbf{u}_h^{n+1}; z_h^{n+1}, \theta_h)$ is the discrete non-linear part of the transport equation and the functions of X_h vanish on $\partial\Omega$. This system is linearized in the sense that in (9), knowing z_h^n, we calculate $\mathbf{u}_h^{n+1}$ and p_h^{n+1} with a linear equation. Then, we calculate z_h^{n+1} with the second linear equation (10). For both the velocity and pressure discretizations, the error is of order $h^{3/2}$ and k. This is the best that can be achieved, considering that the discretization of the transport equation loses inevitably a factor $h^{1/2}$. Other finite elements can be used, cf. Crouzeix and Raviart [6], Brezzi and Fortin [2] and Girault and Raviart [9].

Now, we recall some notation and basic functional results. As usual, for handling time-dependent problems, it is convenient to consider functions defined on a time interval $]a, b[$ with values in a functional space, say X (cf. Lions and Magenes [13]). More precisely, let $\| \,.\, \|_X$ denote the norm of X; then for any r, $1 \le r \le \infty$, we define

$$L^r(a, b; X) = \{f \text{ mesurable in }]a, b[; \int_a^b \| f(t) \|_X^r \, dt < \infty\}$$

equipped with the norm

$$\| f \|_{L^r(a,b;X)} = (\int_a^b \| f(t) \|_X^r \, dt)^{1/r},$$

with the usual modifications if $r = \infty$. It is a Banach space if X is a Banach space.
Let (k_1, k_2) denote a pair of non-negative integers, set $|k| = k_1 + k_2$ and define the partial derivative ∂^k by

$$\partial^k v = \frac{\partial^{|k|} v}{\partial \mathbf{x}_1^{k_1} \partial \mathbf{x}_2^{k_2}}.$$

We denote by :

$$W^{m,r}(\Omega) = \{v \in L^r(\Omega); \partial^k v \in L^r(\Omega)\ \forall |k| \le m\},$$

This space is equipped with the seminorm

$$|v|_{W^{m,r}(\Omega)} = [\sum_{|k|=m} \int_\Omega |\partial^k v|^r d\mathbf{x}]^{1/r},$$

and is a Banach space for the norm

$$\| v \|_{W^{m,r}(\Omega)} = [\sum_{0 \le |k| \le m} |v|^r_{W^{k,r}(\Omega)} d\mathbf{x}]^{1/r}.$$

When $r = 2$, this space is the Hilbert space $H^m(\Omega)$. In particular, the scalar product of $L^2(\Omega)$ is denoted by $(.\,,\,.)$.

Similarly, $L^2(a, b; H^m(\Omega))$ is a Hilbert space and in particular $L^2(a, b; L^2(\Omega))$ coincides with $L^2(\Omega\times]a, b[)$. The definitions of these spaces are extended straightforwardly to vectors, with the same notation, but with the following modification for the norms in the non-Hilbert case. Let $\mathbf{u} = (u_1, u_2)$; then we set

$$\| \mathbf{u} \|_{L^r(\Omega)} = [\int_\Omega \| \mathbf{u}(\mathbf{x}) \|^r \, d\mathbf{x}]^{1/r},$$

where $\| \,.\, \|$ denotes the Euclidean vector norm.

For functions that vanish on the boundary, we define for any $r \ge 1$

$$W_0^{1,r}(\Omega) = \{v \in W^{1,r}(\Omega); v|_{\partial\Omega} = 0\}$$

and recall Sobolev's imbeddings in two dimensions: for each $r \in [2, \infty[$, there exits a constant S_r such that

$$\forall \mathbf{v} \in H_0^1(\Omega)\ ,\ \| \mathbf{v} \|_{L^r(\Omega)} \le S_r |\mathbf{v}|_{H^1(\Omega)}, \tag{11}$$

where

$$|\mathbf{v}|_{H^1(\Omega)} = \| \nabla \mathbf{v} \|_{L^2(\Omega)}\ . \tag{12}$$

When $r = 2$, (11) reduces to Poincaré's inequality and S_2 is Poincaré's constant.

The case $r = \infty$ is excluded and is replaced by: for any $r > 2$, there exists a constant M_r such that

$$\forall v \in W_0^{1,r}(\Omega), \quad \| v \|_{L^\infty(\Omega)} \le M_r |v|_{W^{1,r}(\Omega)}. \tag{13}$$

We have also in dimension 2,

$$||g||_{L^4(\Omega)} \le 2^{1/4} \| g \|^{1/2}_{L^2(\Omega)} \| \nabla g \|^{1/2}_{L^2(\Omega)}\ . \tag{14}$$

Owing to Poincaré's inequality, the seminorm $|.|_{H^1(\Omega)}$ is a norm on $H_0^1(\Omega)$ and we use it to define the dual norm:

$$\| f \|_{H^{-1}(\Omega)} = \sup_{v \in H_0^1(\Omega)} \frac{\langle f, v \rangle}{|v|_{H^1(\Omega)}},$$

where $\langle ., . \rangle$ denotes the duality pairing between $H^{-1}(\Omega)$ and $H_0^1(\Omega)$.
Also, we introduce the space:

$$L_0^2(\Omega) = \{q \in L^2(\Omega); \int_\Omega q \, d\mathbf{x} = 0\},$$

2 The exact problem

Let Ω be a bounded polygon in two dimensions with boundary $\partial\Omega$ and let $]0,T[$ be a given time-interval. We want to find a vector velocity $\mathbf{u}$, a scalar pressure p and an auxiliary scalar function z solution of

$$\frac{\partial}{\partial t}(\mathbf{u}-\alpha\Delta\mathbf{u})-\nu\Delta\mathbf{u}+z\times\mathbf{u}+\nabla p=\mathbf{f} \text{ in }]0,T[\times\Omega, \tag{15}$$

$$\alpha\frac{\partial z}{\partial t}+\nu z+\alpha\mathbf{u}\cdot\nabla z=\nu\,\mathrm{curl}\mathbf{u}+\alpha\,\mathrm{curl}\mathbf{f} \text{ in }]0,T[\times\Omega, \tag{16}$$

$$\mathbf{u}=\mathbf{0} \text{ on }]0,T[\times\partial\Omega, \tag{17}$$

$$\mathbf{u}(x,t)=\mathbf{0} \text{ and } z(x,t)=0, \tag{18}$$

where $\mathbf{z}\times\mathbf{u}=(-zu_2,zu_1)$. Here $\nu>0$ and $\alpha>0$ are given constants.

A straightforward formulation of (15)–(18) is :
Find $(\mathbf{u}(t),p(t),z(t))\in L^\infty(0,T;H_0^1(\Omega)^2)\times L^2(0,T;L_0^2(\Omega))\times L^\infty(0,T;L^2(\Omega))$, $\mathbf{u}'\in L^2(0,T;H_0^1(\Omega)^2)$ such that

$$\begin{aligned}\forall\mathbf{v}\in H_0^1(\Omega),\quad &(\mathbf{u}'(t),\mathbf{v})+\alpha(\nabla\mathbf{u}'(t),\nabla\mathbf{v})+\nu(\nabla\mathbf{u}(t),\nabla\mathbf{v})\\ &+(z(t)\times\mathbf{u}(t),\mathbf{v})-(p(t),\mathrm{div}\ \mathbf{v})=(\mathbf{f}(t),\mathbf{v}) \text{ in } \Omega\times]0,T],\end{aligned} \tag{19}$$

$$\forall q\in L_0^2(\Omega),\quad (q(t),\mathrm{div}\,\mathbf{u}(t))=0, \tag{20}$$

$$\alpha\frac{\partial z}{\partial t}+\nu z+\alpha\mathbf{u}\cdot\nabla z=\nu\,\mathrm{curl}\mathbf{u}+\alpha\,\mathrm{curl}\mathbf{f} \text{ in }]0,T[\times\Omega, \tag{21}$$

$$\mathbf{u}(0)=\mathbf{0} \qquad \text{and } z(0)=0 \text{ in } \Omega. \tag{22}$$

The following theorem is established in [10]:

Theorem 2.1. *Let Ω be a lipschitz polygon. For all $\nu>0$ and $\mathbf{f}\in L^2(0,T;L^2(\Omega)^2)$ such that* $\mathrm{curl}\,\mathbf{f}\in L^2(0,T;L^2(\Omega))$, (15)–(18) *has at least one solution $(\mathbf{u},z,p)$ that satisfies the following estimates :*

$$||z||_{L^\infty(0,T;L^2(\Omega))}\le\sqrt{2}\frac{S_2}{\nu}||\mathbf{f}||_{L^2(0,T;L^2(\Omega)^2)}+\frac{|\alpha|}{\nu}||\,\mathrm{curl}\,\mathbf{f}||_{L^2(0,T;L^2(\Omega))},$$

$$||\mathbf{u}||_{L^\infty(0,T;H^1(\Omega)^2)}\le\frac{S_2}{\nu}||\mathbf{f}||_{L^2(0,T;L^2(\Omega)^2)},$$

$$||p||_{L^2(0,T;L^2(\Omega))}\le\frac{1}{\beta}(S_2||\mathbf{f}||_{L^2(0,T;L^2(\Omega)^2)}+S_4^2||\mathbf{u}||_{L^\infty(0,T;H^1(\Omega)^2)}||z||_{L^\infty(0,T;L^2(\Omega))}).$$

3 A discontinuous upwind scheme

Let $h>0$ be a discretization parameter and let $\mathcal{T}_h$ be a regular family of triangulation of $\overline{\Omega}$, consisting of triangles κ with maximum mesh size h: There exists a constant σ_0, independent

of h, such that $\forall \kappa \in \mathcal{T}_h$, $\dfrac{h_\kappa}{\rho_\kappa} \leq \sigma_0$, where h_κ is the diameter and ρ_κ is the diameter of the ball inscribed in κ. We introduce $\rho_{min} = \min_{\kappa} \rho_k$. As usual, the triangulation is such that any two triangles are either disjoint or share a vertex or a complete side.
We first recall how upwinding can be achieved by the discontinuous Galerkin approximation introduced in [12]. Let Z_h be the discontinuous finite-element space :

$$Z_h = \{\theta_h \in L^2(\Omega); \forall \kappa \in \mathcal{T}_h, \theta_h|_\kappa \in \mathbb{P}_1\}.$$

There exists an approximation operator, [5], $R_h \in \mathcal{L}(W^{1,p}(\Omega); Z_h \cap C^0(\overline{\Omega}))$ such that for any $p \geq 1$, for $m = 0, 1$ and $0 \leq l \leq 1$

$$\forall z \in W^{l+1,p}(\Omega), \;\; |R_h(z) - z|_{W^{m,p}(\Omega)} \leq C h^{l+1-m} |z|_{W^{l+1,p}(\Omega)}.$$

Let $\mathbf{u}_h$ be a discrete velocity in $H_0^1(\Omega)^2$, and for each triangle κ, let

$$\partial\kappa_- = \{x \in \partial\kappa; \; \alpha \mathbf{u}_h \cdot \mathbf{n} < 0\},$$

where $\mathbf{n}$ denotes the unit exterior normal to ∂k. Note that, for all triangles κ of $\mathcal{T}_h$, $\partial\kappa_-$ only involves interior segments of $\mathcal{T}_h$ because $\mathbf{u}_h = 0$ on $\partial\Omega$. Then, the non-linear term $\alpha[(\mathbf{u} \cdot \nabla z, \theta) + \dfrac{1}{2}(\operatorname{div} \mathbf{u}\, z, \theta)]$ is approximated by

$$c(\mathbf{u}_h^{n+1}; z_h^{n+1}, \theta_h^{n+1}) = \frac{\alpha}{2} \int_\Omega \operatorname{div} \mathbf{u}_h^{n+1} z_h^{n+1} \theta_h^{n+1} d\mathbf{x} + \sum_{\kappa \in \mathcal{T}_h} \Big(\int_\kappa \alpha (\mathbf{u}_h^{n+1} \cdot \nabla z_h^{n+1}) \theta_h^{n+1} d\mathbf{x}$$
$$+ \int_{\partial\kappa_-} |\alpha \mathbf{u}_h^{n+1} \cdot \mathbf{n}| (z_{h,\text{int}}^{n+1} - z_{h,\text{ext}}^{n+1}) \theta_{h,\text{int}}^{n+1} ds \Big).$$

The subscript int (resp. ext) refers to the trace on the segment $\partial\kappa$ of the function taken inside (resp. outside) κ. Note that in the above sum, the boundary integrations act in fact over complete interior segments.

On the other hand, let us recall the standard Taylor-Hood discretization of the velocity and pressure. The discrete space of the pressure is :

$$M_h = \{q_h \in H^1(\Omega) \cap L_0^2(\Omega); \; \forall \kappa \in \mathcal{T}_h, \; q_h \in \mathbb{P}_1\}.$$

There exists an operator $r_h \in \mathcal{L}(L_0^2(\Omega); M_h)$ such that for $0 \leq l \leq 2$,

$$\forall q \in H^l(\Omega) \cap L_0^2(\Omega), \; ||r_h(q) - q||_{L^2(\Omega)} \leq C h^l ||q||_{H^l(\Omega)}.$$

The discrete velocity space is :

$$X_h = \{\mathbf{v}_h \in C^0(\overline{\Omega}); \; \forall \kappa \in \mathcal{T}_h, \mathbf{v}_h|_\kappa \in \mathbb{P}_2, \mathbf{v}_h|_{\partial\Omega} = \mathbf{0}\},$$

and let

$$V_h = \{\mathbf{v}_h \in X_h; \; (q_h, \operatorname{div} \mathbf{v}_h) = 0 \quad \forall q \in M_h\}.$$

There exists an operator $P_h \in \mathcal{L}(H_0^1(\Omega)^2; X_h)$, such that

$$\begin{cases} \forall \mathbf{v} \in H_0^1(\Omega)^2, \quad \forall \kappa \in \mathcal{T}_h, \quad \forall q_h \in M_h, \quad \displaystyle\int_\kappa q_h \operatorname{div}(P_h(\mathbf{v}) - \mathbf{v}) d\mathbf{x} = 0, \\ \text{for all number } p \geq 2; \quad \forall \mathbf{v} \in H_0^1(\Omega)^2, \quad \|P_h(\mathbf{v}) - \mathbf{v}\|_{L^p(\Omega)} \leq C h^{2/p} |\mathbf{v}|_{H^1(\Omega)}, \\ \text{for all number } p \geq 2, 1 \leq s \leq 3, m = 0 \text{ or } 1, \quad \forall \mathbf{v} \in [W^{s,p}(\Omega) \cap H_0^1(\Omega)]^2, \\ \qquad \|P_h(\mathbf{v}) - \mathbf{v}\|_{W^{m,p}(\Omega)} \leq C h^{s-m} |\mathbf{v}|_{W^{s,p}(\Omega)}. \end{cases} \tag{23}$$

We take $\mathbf{f}^{n+1}(\mathbf{x}) = \dfrac{1}{k} \displaystyle\int_{t^n}^{t^{n+1}} \mathbf{f}(t, \mathbf{x}) dt$. Then the discrete system corresponding to the formulation (19)–(22) is :

Given $(\mathbf{u}_h^0, z_h^0) = (\mathbf{0}, 0)$ and $z_h^n \in Z_h$, find $(\mathbf{u}_h^{n+1}, p_h^{n+1}) \in X_h \times M_h$ such that :

$$\begin{aligned} \forall \mathbf{v}_h \in X_h, \quad \frac{1}{k}(\mathbf{u}_h^{n+1} - \mathbf{u}_h^n, \mathbf{v}_h) \ + \ \frac{\alpha}{k}(\nabla(\mathbf{u}_h^{n+1} - \mathbf{u}_h^n), \nabla \mathbf{v}_h) \ + \ \nu(\nabla \mathbf{u}_h^{n+1}, \nabla \mathbf{v}_h) \\ + (\mathbf{z}_h^n \times \mathbf{u}_h^{n+1}, \mathbf{v}_h) - \ (p_h^{n+1}, \operatorname{div} \mathbf{v}_h) = (\mathbf{f}^{n+1}, \mathbf{v}_h), \end{aligned} \tag{24}$$

$$\forall q_h \in M_h, \qquad (q_h, \operatorname{div} \mathbf{u}_h^{n+1}) = 0. \tag{25}$$

Once we have $\mathbf{u}_h^{n+1}$, we compute z_h^{n+1} by solving the system :

$$\begin{aligned} \forall \theta_h \in Z_h, \quad \frac{1}{k}(z_h^{n+1} - z_h^n, \theta_h) + \nu(z_h^{n+1}, \theta_h) + c(\mathbf{u}_h^{n+1}; z_h^{n+1}, \theta_h) \\ = \nu(\operatorname{curl} \mathbf{u}_h^{n+1}, \theta_h) + \alpha(\operatorname{curl} \mathbf{f}^{n+1}, \theta_h). \end{aligned} \tag{26}$$

In order to prove the existence of solutions of (24)–(26), let us recall the following identity established by Lesaint and Raviart [12] :

Lemma 3.1. *For all $\mathbf{v}_h^n \in X_h$, z_h^n and θ_h^n in Z_h, we have*

$$\begin{aligned} c(\mathbf{v}_h^n; z_h^n, \theta_h^n) = \sum_{\kappa \in \mathcal{T}_h} \Big(- \int_\kappa \alpha(\mathbf{v}_h^n \cdot \nabla \theta_h^n) z_h^n d\mathbf{x} + \int_{\partial \kappa_-} \alpha |\mathbf{v}_h^n \cdot n| z_{h,ext}^n (\theta_{h,ext}^n - \theta_{h,int}^n) ds \Big) \\ - \frac{\alpha}{2} \int_\Omega (\operatorname{div} \mathbf{v}_h^n) \theta_h^n z_h^n d\mathbf{x}. \end{aligned}$$

For $\theta_h^n \in H^1(\Omega)$, we have

$$c(\mathbf{v}_h^n; z_h^n, \theta_h^n) = - \int_\Omega \alpha(\mathbf{v}_h^n \cdot \nabla \theta_h^n) z_h^n d\mathbf{x} - \frac{\alpha}{2} \int_\Omega (\operatorname{div} \mathbf{v}_h^n) \theta_h^n z_h^n d\mathbf{x}.$$

For $\theta_h^n = z_h^n \in Z_h$ we have

$$c(\mathbf{v}_h^n; z_h^n, z_h^n) = \frac{1}{2} \sum_{\kappa \in \mathcal{T}_h} \int_{\partial \kappa_-} |\alpha \mathbf{v}_h^n \cdot n| (z_{h,ext}^n - z_{h,int}^n)^2 ds.$$

Theorem 3.2. *Given $\mathbf{f}^{n+1} \in L^2(\Omega)^2$ with curl $\mathbf{f}^{n+1} \in L^2(\Omega)$, for all $(\mathbf{u}_h^n, z_h^n) \in X_h \times Z_h$, there exists a unique solution $(\mathbf{u}_h^{n+1}, p_h^{n+1}, \mathbf{z}_h^{n+1})$ of problem (24)–(26) that belongs to $X_h \times M_h \times Z_h$.*

Proof. On the one hand, for $\mathbf{z}_h^n \in Z_h$, it is clear that problem (24)–(25) has a unique solution $(\mathbf{u}_h^{n+1}, p_h^{n+1})$ as a consequence of the coerciveness of the corresponding bilinear form on $X_h \times X_h$. On the other hand, the last lemma proves that the bilinear form corresponding to the equation (26) is also coercive on $Z_h \times Z_h$. Then (26) has a unique solution. □

Theorem 3.3. *We assume that* $\mathbf{f} \in L^2(0,T;L^2(\Omega)^2)$ *with* $\textit{curl}\,\mathbf{f} \in L^2(0,T;L^2(\Omega))$. *The solution of the problem* (24)–(26) *satisfies :*

$$||\mathbf{u}_h||_{L^\infty(0,T;H^1(\Omega)^2)} \le C_1||\mathbf{f}||_{L^2(0,T;L^2(\Omega)^2)},$$
$$||z_h||^2_{L^\infty(0,T;L^2(\Omega))} \le C_2||\mathbf{f}||^2_{L^2(0,T;L^2(\Omega)^2)} + C_3||\ \textit{curl}\ \mathbf{f}||^2_{L^2(0,T;L^2(\Omega))},$$
$$||p_h||^2_{L^2(0,T;L^2(\Omega))} \le C_4||\mathbf{f}||^2_{L^2(0,T;L^2(\Omega)^2)} + C_5||\mathbf{u}_h||^2_{L^\infty(0,T;H^1(\Omega)^2)} + C_6||z_h||^2_{L^\infty(0,T;L^2(\Omega))}||\mathbf{u}_h||^2_{L^\infty(0,T;H^1(\Omega)^2)},$$

where $C_i, i = 1, \ldots, 6$ *are positive constants that depend on* Ω *and* T.

Proof. On the one hand, we take $\mathbf{v}_h = \mathbf{u}_h^{n+1}$ in (24) and we obtain :

$$\frac{1}{2}||\mathbf{u}_h^{n+1}||^2_{L^2(\Omega)} - \frac{1}{2}||\mathbf{u}_h^n||^2_{L^2(\Omega)} + \frac{\alpha}{2}|\mathbf{u}_h^{n+1}|^2_{H^1(\Omega)} - \frac{\alpha}{2}|\mathbf{u}_h^n|^2_{H^1(\Omega)} + \nu k|\mathbf{u}_h^{n+1}|^2_{H^1(\Omega)}$$
$$\le \frac{k\varepsilon}{2}||\mathbf{f}^{n+1}||^2_{L^2(\Omega)} + \frac{k\mathcal{S}_2^2}{2\varepsilon}|\mathbf{u}_h^{n+1}|^2_{H^1(\Omega)}.$$

We choose $\varepsilon = \dfrac{\mathcal{S}_2^2}{2\nu}$ and sum over $n = 0, \ldots i$. We obtain :

$$\frac{1}{2}||\mathbf{u}_h^i||^2_{L^2(\Omega)} + \frac{\alpha}{2}|\mathbf{u}_h^i|^2_{H^1(\Omega)} \le \sum_{n=1}^{i} \frac{k\mathcal{S}_2^2}{4\nu}||\mathbf{f}^n||^2_{L^2(\Omega)} \le \sum_{n=1}^{N} \frac{k\mathcal{S}_2^2}{4\nu}||\mathbf{f}^n||^2_{L^2(\Omega)}.$$

This implies the first estimate :

$$||\mathbf{u}_h||^2_{L^\infty(0,T;H^1(\Omega)^2)} = \sup_{0\le i\le N} |\mathbf{u}_h^i|^2_{H^1(\Omega)} \le \frac{\mathcal{S}_2^2}{2\nu\alpha}||\mathbf{f}||^2_{L^2(0,T;L^2(\Omega)^2)}.$$

On the other hand, we choose $\theta_h = z_h^{n+1}$ in (26), use the third relation in Lemma 3.1 and we obtain :

$$\frac{\alpha}{2}||z_h^{n+1}||^2_{L^2(\Omega)} - \frac{\alpha}{2}||z_h^n||^2_{L^2(\Omega)} + \nu k||z_h^{n+1}||^2_{L^2(\Omega)} \le \frac{k\nu}{2\varepsilon_1}||\ \text{curl}\ \mathbf{u}_h^{n+1}||^2_{L^2(\Omega)}$$
$$+\frac{k\nu\varepsilon_1}{2}||z_h^{n+1}||^2_{L^2(\Omega)} + \frac{k\alpha}{2\varepsilon_2}||\ \text{curl}\ \mathbf{f}^{n+1}||^2_{L^2(\Omega)} + \frac{k\alpha\varepsilon_2}{2}||z_h^{n+1}||^2_{L^2(\Omega)},$$

taking $\varepsilon_1 = 1$, $\varepsilon_2 = \dfrac{\nu}{\alpha}$ and summing over $n = 0, \ldots, i$, this becomes :

$$\alpha||z_h^i||^2_{L^2(\Omega)} \le \nu T||\mathbf{u}_h||^2_{L^\infty(0,T;H^1(\Omega)^2)} + \frac{\alpha^2}{\nu}||\ \text{curl}\ \mathbf{f}||^2_{L^2(0,T;L^2(\Omega))}.$$

Then we obtain the second estimate :

$$||z_h||^2_{L^\infty(0,T;L^2(\Omega))} \le \frac{\nu T}{\alpha}||\mathbf{u}_h||^2_{L^\infty(0,T;H^1(\Omega)^2)} + \frac{\alpha}{\nu}||\ \text{curl}\ \mathbf{f}||^2_{L^2(0,T;L^2(\Omega))}.$$

The third estimate is obtained in two steps: First, we take the function test $\mathbf{v}_h^{n+1} = \dfrac{\mathbf{u}_h^{n+1} - \mathbf{u}_h^n}{k}$ in (24). We obtain :

$$\frac{1}{k^2}||\mathbf{u}_h^{n+1} - \mathbf{u}_h^n||^2_{L^2(\Omega)} + \frac{\alpha}{k^2}|\mathbf{u}_h^{n+1} - \mathbf{u}_h^n|^2_{H^1(\Omega)} + \frac{\nu}{2}|\mathbf{u}_h^{n+1}|^2_{H^1(\Omega)} + \frac{\nu}{2k^2}|\mathbf{u}_h^{n+1} - \mathbf{u}_h^n|^2_{H^1(\Omega)}$$
$$\leq \frac{1}{2\varepsilon_1}||\mathbf{f}^{n+1}||^2_{L^2(\Omega)} + \frac{\varepsilon_1}{2k^2}||\mathbf{u}_h^{n+1} - \mathbf{u}_h^n||^2_{L^2(\Omega)} + \frac{1}{2\varepsilon_2}||z_h^n||^2_{L^2(\Omega)}||\mathbf{u}_h^{n+1}||^2_{L^4(\Omega)} + \frac{S_4^2\varepsilon_2}{2k^2}|\mathbf{u}_h^{n+1} - \mathbf{u}_h^n|^2_{H^1(\Omega)}.$$

Then by choosing $\varepsilon_1 = 2$ and $\varepsilon_2 = \dfrac{\alpha}{2S_4^2}$ we obtain :

$$|\frac{\mathbf{u}_h^{n+1} - \mathbf{u}_h^n}{k}|^2_{H^1(\Omega)} \leq \frac{1}{2\alpha}||\mathbf{f}^{n+1}||^2_{L^2(\Omega)} + \frac{2S_4^2}{\alpha^2}||z_h^n||^2_{L^2(\Omega)}||\mathbf{u}_h^{n+1}||^2_{L^4(\Omega)} + \frac{2\nu^2}{\alpha^2}|\mathbf{u}_h^{n+1}|^2_{H^1(\Omega)}.$$

Next, owing that the pair (X_h, M_h) satisfies a uniform discrete inf-sup condition, we associate with $p_h^{n+1} \in M_h$ the function $\mathbf{v}_h \in X_h$ defined by

$$\begin{cases} \forall \mathbf{w}_h \in V_h, \quad (\nabla \mathbf{v}_h, \nabla \mathbf{w}_h) = 0, \\ \forall q_h \in M_h, \quad (\operatorname{div} \mathbf{v}_h, q_h) = (p_h^{n+1}, q_h), \\ |\mathbf{v}_h|_{H^1(\Omega)} \leq \dfrac{1}{\beta}||p_h^{n+1}||_{L^2(\Omega)}, \end{cases} \tag{27}$$

we substitute this $\mathbf{v}_h$ into (24) and we obtain :

$$\begin{aligned} ||p_h^{n+1}||^2_{L^2(\Omega)} \leq & \frac{\mathcal{P}+\alpha}{2\varepsilon_1}|\frac{\mathbf{u}_h^{n+1} - \mathbf{u}_h^n}{k}|^2_{H^1(\Omega)} + \frac{\mathcal{P}+\alpha}{2\beta^2}\varepsilon_1||p_h^{n+1}||^2_{L^2(\Omega))} + \frac{\nu}{2\varepsilon_2}|\mathbf{u}_h^{n+1}|^2_{H^1(\Omega)} \\ & + \frac{\nu\varepsilon_2}{2\beta^2}||p_h^{n+1}||^2_{L^2(\Omega))} + \frac{1}{2\varepsilon_3}||z_h^n||^2_{L^2(\Omega)}||\mathbf{u}_h^{n+1}||^2_{L^4(\Omega)} \\ & + \frac{S_4^2\varepsilon_3}{2\beta^2}||p_h^{n+1}||^2_{L^2(\Omega))} + \frac{1}{2\varepsilon_4}||\mathbf{f}^{n+1}||^2_{L^2(\Omega)} + \frac{\mathcal{P}^2\varepsilon_4}{2\beta^2}||p_h^{n+1}||^2_{L^2(\Omega)}. \end{aligned}$$

By choosing $\varepsilon_1 = \dfrac{\beta^2}{4(\mathcal{P}+\alpha)}$, $\varepsilon_2 = \dfrac{\beta^2}{4\nu}$, $\varepsilon_3 = \dfrac{\beta^2}{4S_4^2}$ and $\varepsilon_4 = \dfrac{\beta^2}{4\mathcal{P}^2}$ and summing over n from 0 to $N-1$, we obtain the third estimate. □

4 Error estimates

Theorem 4.1. *Under the assumptions* $\mathbf{u} \in L^\infty(0,T;W^{1,4}(\Omega)^2) \cap L^2(0,T;H^3(\Omega)^2)$, $\mathbf{u}' \in L^2(0,T;H^3(\Omega)^2)$, $p \in L^2(0,T;H^2(\Omega))$, $z \in L^\infty(0,T;L^2(\Omega))$ *and* $z' \in L^2(0,T;L^2(\Omega))$, *there exist positive constants* C *and* C' *that depend on* $\mathbf{u}, z, \Omega$ *and* T *such that :*

$$\begin{aligned} \frac{1}{2}||\mathbf{u}_h^N - \mathbf{u}(t^N)||^2_{L^2(\Omega)} + \frac{\alpha}{2}||\nabla\mathbf{u}_h^N - \nabla\mathbf{u}(t^N)||^2_{L^2(\Omega)} + \frac{\nu}{2}\sum_{n=0}^{N-1} k|\mathbf{u}_h^{n+1} - \mathbf{u}(t^{n+1})|^2_{H^1(\Omega)} \\ \leq C(h^4 + k^2) + C'\sum_{n=0}^{N-1} k||z_h^{n+1} - z(t^{n+1})||^2_{L^2(\Omega)}. \end{aligned} \tag{28}$$

Proof. We consider (19), choose the function test $\mathbf{v}_h^{n+1} = \mathbf{u}_h^{n+1} - P_h\mathbf{u}(t^{n+1})$, integrate from t^n to t^{n+1} and take the difference between this and (24) multiplied by k. We obtain :

$$\Big((\mathbf{u}_h^{n+1} - \mathbf{u}(t^{n+1})) - (\mathbf{u}_h^n - \mathbf{u}(t^n)), \mathbf{v}_h^{n+1}\Big) + \alpha\Big(\nabla(\mathbf{u}_h^{n+1} - \mathbf{u}(t^{n+1})) - \nabla(\mathbf{u}_h^n - \mathbf{u}(t^n)), \nabla\mathbf{v}_h^{n+1}\Big)$$
$$+\nu\Big(k\nabla\mathbf{u}_h^{n+1} - \int_{t^n}^{t^{n+1}} \nabla\mathbf{u}(t)dt, \nabla\mathbf{v}_h^{n+1}\Big) - \Big(kp_h^{n+1} - \int_{t^n}^{t^{n+1}} p(t)dt, \operatorname{div}\mathbf{v}_h^{n+1}\Big)$$
$$+\Big(kz_h^n \times \mathbf{u}_h^{n+1} - \int_{t^n}^{t^{n+1}} z(t)\times\mathbf{u}(t)dt, \mathbf{v}_h^{n+1}\Big) = 0.$$

Let us treat the terms in the left-hand side of this equation that we denote $(a_i), i = 1, ..., 5$.

For the first term, we insert $P_h\mathbf{u}(t^{n+1})$ and $P_h\mathbf{u}(t^n)$ and we split (a_1) into two terms that we treat separately. The first part is as follows :

$$(a_{1,1}) = \frac{1}{2}||\mathbf{v}_h^{n+1}||^2_{L^2(\Omega)} - \frac{1}{2}||\mathbf{v}_h^n||^2_{L^2(\Omega)} + \frac{1}{2}||\mathbf{v}_h^{n+1} - \mathbf{v}_h^n||^2_{L^2(\Omega)},$$

and the second part is as follows :

$$|(a_{1,2})| = \Big|\Big(\int_{t^n}^{t^{n+1}} (P_h\mathbf{u}'(\tau) - \mathbf{u}'(\tau))d\tau, \mathbf{v}_h^{n+1}\Big)\Big|$$
$$\leq \frac{1}{2\varepsilon_1}Ch^4||\mathbf{u}'||^2_{L^2(t^n,t^{n+1};H^2(\Omega)^2)} + \frac{S_2\varepsilon_1}{2}k|\mathbf{v}_h^{n+1}|^2_{H^1(\Omega)}.$$

We treat the second term (a_2) as the first one and we obtain :

$$(a_{2,1}) = \frac{\alpha}{2}|\mathbf{v}_h^{n+1}|^2_{H^1(\Omega)} - \frac{\alpha}{2}|\mathbf{v}_h^n|^2_{H^1(\Omega)} + \frac{\alpha}{2}|\mathbf{v}_h^{n+1} - \mathbf{v}_h^n|^2_{H^1(\Omega)},$$

and

$$|(a_{2,2})| \leq \frac{C\alpha}{2\varepsilon_2}h^4||\mathbf{u}'||^2_{L^2(t^n,t^{n+1};H^3(\Omega)^2)} + \frac{\varepsilon_2\alpha}{2}k|\mathbf{v}_h^{n+1}|^2_{H^1(\Omega)}.$$

For the third term (a_3), we insert $\nabla P_h\mathbf{u}(t^{n+1})$ and $\nabla P_h\mathbf{u}(t)$ and we split it into three parts that are treated successively as follows :

$$(a_{3,1}) = \nu k|\mathbf{u}_h^{n+1} - P_h\mathbf{u}(t^{n+1})|^2_{H^1(\Omega)},$$

$$|(a_{3,2})| = \nu\Big|\Big(\int_{t^n}^{t^{n+1}} \nabla P_h(\mathbf{u}(t^{n+1}) - \mathbf{u}(t))dt, \nabla\mathbf{v}_h^{n+1}\Big)\Big| = \nu\Big|\Big(\int_{t^n}^{t^{n+1}} \nabla P_h\mathbf{u}'(\tau)(\tau - t^n)d\tau, \nabla\mathbf{v}_h^{n+1}\Big)\Big|$$
$$\leq \frac{\nu\varepsilon_3}{2\sqrt{3}}k|\mathbf{v}_h^{n+1}|^2_{H^1(\Omega)} + \frac{\nu C^2k^2}{2\varepsilon_3\sqrt{3}}||\mathbf{u}'||^2_{L^2(t^n,t^{n+1};H^2(\Omega)^2)},$$

and

$$|(a_{3,3})| = \nu\Big|\Big(\nabla\int_{t^n}^{t^{n+1}} (P_h\mathbf{u}(t) - \mathbf{u}(t))dt, \nabla\mathbf{v}_h^{n+1}\Big)\Big|$$
$$\leq \frac{\nu C_2}{2\varepsilon_4}h^4||\mathbf{u}||^2_{L^2(t^n,t^{n+1},H^3(\Omega)^2)} + \frac{\nu\varepsilon_4}{2}k|\mathbf{v}_h^{n+1}|^2_{H^1(\Omega)}.$$

To study the fourth term, we use the fact that $\displaystyle\int_\Omega p_h^{n+1}\operatorname{div}(P_h\mathbf{u}(t^{n+1}) - \mathbf{u}(t^{n+1})) = 0,$

$\operatorname{div}\mathbf{u}(t^{n+1}) = 0$ and $\displaystyle\int_\Omega p_h^{n+1}\operatorname{div}\mathbf{u}_h^{n+1} = 0$ and we obtain :

$$|(a_4)| = \Big|\Big(\int_{t^n}^{t^{n+1}} (r_h p(t) - p(t))dt, \operatorname{div}\mathbf{v}_h^{n+1}\Big)\Big|$$

$$\leq \frac{C_1}{2\varepsilon_5}\, h^4 ||p||^2_{L^2(t^n,t^{n+1};H^2(\Omega))} + \frac{\varepsilon_5}{2}\, k|\mathbf{v}_h^{n+1}|^2_{H^1(\Omega)}.$$

Finally, for the last term (a_5), we have $(z_h^n \times \mathbf{u}_h^{n+1}, \mathbf{v}_h^{n+1}) = (z_h^n \times P_h\mathbf{u}(t^{n+1}), \mathbf{v}_h^{n+1})$, because $(a\times b, b) = 0$. But

$$z_h^n \times P_h\mathbf{u}(t^{n+1}) - z(t)\times\mathbf{u}(t) = (z_h^n - z(t^n))\times P_h\mathbf{u}(t^{n+1}) + z(t^n)\times P_h(\mathbf{u}(t^{n+1}) - \mathbf{u}(t))$$

$$+z(t^n)\times(P_h\mathbf{u}(t) - \mathbf{u}(t)) + (z(t^n) - z(t))\times(\mathbf{u}(t) - \mathbf{u}(t^n)) + (z(t^n) - z(t))\times\mathbf{u}(t^n),$$

than (a_5) is split into five parts that we treat successively.

The first part is as follows :

$$|(a_{5,1})| = \Big|\int_{t^n}^{t^{n+1}} ((z_h^n - z(t^n))\times P_h\mathbf{u}(t^{n+1}), \mathbf{v}_h^{n+1})dt\Big|$$

$$\leq \frac{S_4^2\varepsilon_6}{2}||P_h\mathbf{u}||^2_{L^\infty(0,T;H^1(\Omega)^2)}\, k||z_h^n - z(t^n)||^2_{L^2} + \frac{S_4^2}{2\varepsilon_6}\, k|\mathbf{v}_h^{n+1}|^2_{H^1(\Omega)}.$$

$$\leq \frac{S_4^2\varepsilon_6}{2}||\mathbf{u}||^2_{L^\infty(0,T;W^{1,4}(\Omega)^2)}\, k||z_h^n - z(t^n)||^2_{L^2} + \frac{S_4^2}{2\varepsilon_6}\, k|\mathbf{v}_h^{n+1}|^2_{H^1(\Omega)}.$$

The second part is as follows :

$$\begin{aligned}|(a_{5,2})| &= \Big|(z(t^n)\times\int_{t^n}^{t^{n+1}}(P_h\mathbf{u}(t^{n+1}) - P_h\mathbf{u}(t))dt, \mathbf{v}_h^{n+1})\Big| \\ &= \Big|(z(t^n)\times\int_{t^n}^{t^{n+1}} P_h\mathbf{u}'(\tau)(\tau - t^n)d\tau, \mathbf{v}_h^{n+1})\Big| \\ &\leq \frac{S_4^2}{2\sqrt{3}}\Big(\frac{C'''}{\varepsilon_7}||z||^2_{L^\infty(0,T;L^2(\Omega))}||\mathbf{u}'||^2_{L^2(t^n,t^{n+1};H^2(\Omega)^2)}k^2 + k\varepsilon_7|\mathbf{v}_h^{n+1}|^2_{H^1(\Omega)}\Big).\end{aligned}$$

For the third part, we have

$$\begin{aligned}|(a_{5,3})| &= \Big|\int_{t^n}^{t^{n+1}}(z(t^n)\times(P_h\mathbf{u}(t) - \mathbf{u}(t)), \mathbf{v}_h^{n+1})dt\Big| \\ &\leq S_4^2||z(t^n)||_{L^2(\Omega)}|\mathbf{v}_h^{n+1}|_{H^1(\Omega)}\int_{t^n}^{t^{n+1}}|P_h\mathbf{u}(t) - \mathbf{u}(t)|_{H^1(\Omega)}dt \\ &\leq \frac{C_2S_4^2}{2\varepsilon_8}||z||^2_{L^\infty(0,T;L^2(\Omega))}||\mathbf{u}||^2_{L^2(t^n,t^{n+1};H^3(\Omega)^2)}h^4 + \frac{C_2S_4^2\varepsilon_8}{2}k|\mathbf{v}_h^{n+1}|^2_{H^1(\Omega)}.\end{aligned}$$

The fourth part is treated as follows :

$$\begin{aligned}|(a_{5,4})| &= \Big|\Big(\int_{t^n}^{t^{n+1}} (z(t^n)-z(t))\times(\mathbf{u}(t)-\mathbf{u}(t^n)),\mathbf{v}_h^{n+1}\Big)dt\Big|\\ &= \Big|\int_{t^n}^{t^{n+1}} \Big(\Big(\int_{t^n}^{t} z'(\tau)d\tau\Big)\times\Big(\int_{t^n}^{t}\mathbf{u}'(\tau)d\tau\Big),\mathbf{v}_h^{n+1}\Big)dt\Big|\\ &\leq \frac{S_4^2\varepsilon_9}{2\sqrt{2}}\; k\;|\mathbf{v}_h^{n+1}|^2_{H^1(\Omega)} + \frac{S_4^2\; k^3}{2\sqrt{2}\varepsilon_9}||\mathbf{u}'||^2_{L^2(0,T;H^1(\Omega)^2)}||z'||^2_{L^2(0,T;L^2(\Omega))}.\end{aligned}$$

Finally, for the last part, we have

$$\begin{aligned}|(a_{5,5})| &= \Big|\Big(\int_{t^n}^{t^{n+1}} (z(t^n)-z(t))\times\mathbf{u}(t^n),\mathbf{v}_h^{n+1}\Big)dt\Big|\\ &= \Big|\Big(\Big(\int_{t^n}^{t^{n+1}} z'(t)(t-t^{n+1})dt\Big)\times\mathbf{u}(t^n),\mathbf{v}_h^{n+1}\Big)\Big|\\ &\leq \frac{S_4^2\varepsilon_{10}}{2\sqrt{3}}k|\mathbf{v}_h^{n+1}|^2_{H^1(\Omega)} + \frac{S_4^2}{2\sqrt{3}\varepsilon_{10}}k^2||z'||^2_{L^2(t^n,t^{n+1};L^2(\Omega))}||\mathbf{u}||^2_{L^\infty(0,T;H^1(\Omega)^2)}\end{aligned}$$

At the end, (28) follows easily after the decomposition

$$(a_{1,1})+(a_{2,1})+(a_{3,1}) \leq |(a_{1,2})|+|(a_{2,2})|+|(a_{3,2})|+|(a_{3,3})|+|(a_4)|+|(a_5)|,$$

the sum over $n=1,\ldots,N-1$, a suitable choice of $\varepsilon_i,\ i=1,\ldots,10$ and by using the properties of P_h in :

$$|\mathbf{u}_h^{n+1}-\mathbf{u}(t^{n+1})|_{H^1(\Omega)} \leq |\mathbf{u}_h^{n+1}-P_h\mathbf{u}(t^{n+1})|_{H^1(\Omega)} + |P_h\mathbf{u}(t^{n+1})-\mathbf{u}(t^{n+1})|_{H^1(\Omega)}.$$

□

We define ρ_h as the L^2 projection of z onto $\mathbb{P}_1$ in each triangle κ : for $z\in L^2(\Omega)$,

$$\forall q\in\mathbb{P}_1,\ \int_\kappa (\rho_h(z)-z)q\,d\mathbf{x}=0.$$

This operator has locally the same accuracy as R_h.

Theorem 4.2. *We suppose that there exists a constant $\gamma>0$ such that $k\leq\gamma h$. In addition to the assumptions of Theorem 4.1, we assume that $\mathbf{u}\in L^\infty(0,T;W^{1,\infty}(\Omega)^2)$, $z\in L^\infty(0,T;W^{1,4}(\Omega))$ and $z'\in L^\infty(0,T;L^4(\Omega))\cap L^2(0,T;W^{1,4}(\Omega))$. Denoting $\theta_h^{n+1}=z_h^{n+1}-\rho_h z(t^{n+1})$ we have :*

$$\begin{aligned}\sum_{n=0}^{N-1}\int_{t^n}^{t^{n+1}} c(\mathbf{u}_h^{n+1};\rho_h z(t^{n+1})-z(t),\theta_h^{n+1})dt &\leq L_1(h^3+k^2)+L_3\sum_{n=0}^{N-1}k||\theta_h^{n+1}||^2_{L^2(\Omega)}\\ +L_2\sum_{n=0}^{N-1}k|\mathbf{u}_h^{n+1}-\mathbf{u}(t^{n+1})|^2_{H^1(\Omega)} &+ \frac{\alpha}{2}\sum_{\kappa\in\mathcal{T}_h}\sum_{n=0}^{N-1}\int_{t^n}^{t^{n+1}}\int_{\partial\kappa_-}|\mathbf{u}_h^{n+1}\cdot\mathbf{n}|(\theta_{h,ext}^{n+1}-\theta_{h,int}^{n+1})^2dsdt\end{aligned} \tag{29}$$

where L_i are constants that only depend on $\mathbf{u},z,\Omega,\ T$ and arbitrary coefficients $\varepsilon_i(i=1,\ldots,8)$

Proof. Owing to Lemma 3.1 and denoting $\xi_h = \rho_h(z(t^{n+1}))$, we have :

$$\int_{t^n}^{t^{n+1}} c(\mathbf{u}_h^{n+1}; \rho_h z(t^{n+1}) - z(t), \theta_h^{n+1})dt = \alpha \sum_{\kappa \in \mathcal{T}_h} \Big[-\int_{t^n}^{t^{n+1}} \int_\kappa (\mathbf{u}_h^{n+1} \cdot \nabla \theta_h^{n+1})(\xi_h - z(t))dxdt$$
$$+ \int_{t^n}^{t^{n+1}} \int_{\partial\kappa_-} |\mathbf{u}_h^{n+1} \cdot \mathbf{n}|(\xi_h - z(t))^{\text{ext}}(\theta_{h,\text{ext}}^{n+1} - \theta_{h,\text{int}}^{n+1})dsdt\Big]$$
$$-\frac{\alpha}{2} \int_{t^n}^{t^{n+1}} \int_\Omega \text{div}(\mathbf{u}_h^{n+1} - \mathbf{u}(t))(\xi_h - z(t))\theta_h^{n+1} dxdt$$

In the left-hand side, we denote the terms by $(d_i), i = 1 \ldots, 3.$ as $\nabla(z_h - \rho_h(z))$ is a constant vector, the first term (d_1), for any constant vector $\mathbf{c}$, can be treated as :

$$\Big|\int_{t^n}^{t^{n+1}} \int_\kappa (\mathbf{u}_h^{n+1} \cdot \nabla \theta_h^{n+1})(\xi_h - z(t))dxdt\Big| \le \alpha \Big| \int_{t^n}^{t^{n+1}} \int_\kappa (\mathbf{u}_h^{n+1} - \mathbf{u}(t^{n+1}))\nabla \theta_h^{n+1}(\xi_h - z(t))dxdt\Big|$$
$$+\alpha\Big| \int_{t^n}^{t^{n+1}} \int_\kappa (\mathbf{u}(t^{n+1}) - \mathbf{c})\nabla \theta_h^{n+1}(\xi_h - z(t^{n+1}))dxdt$$
$$+ \int_{t^n}^{t^{n+1}} \int_\kappa \mathbf{u}(t^{n+1})\nabla \theta_h^{n+1}(z(t^{n+1}) - z(t))dxdt\Big|.$$

With

$$\alpha\Big| \int_{t^n}^{t^{n+1}} \int_\kappa (\mathbf{u}_h^{n+1} - \mathbf{u}(t^{n+1}))\nabla \theta_h^{n+1}(\xi_h - z(t))dxdt\Big|$$
$$\le \frac{\alpha S_4}{\rho_k} |\mathbf{u}_h^{n+1} - \mathbf{u}(t^{n+1})|_{H^1(\kappa)} ||\theta_h^{n+1}||_{L^2(\kappa)} \Big\{ c_1 k ||\xi_h - z(t^{n+1})||_{L^4(\kappa)} + \frac{c_2 k^2}{\sqrt{2}} ||z'||_{L^\infty(0,T;L^4(\kappa))} \Big\}$$
$$\le \alpha c_3 S_4 \big(\sigma_0 ||z||_{L^\infty(0,T;W^{1,4}(\kappa))} + \gamma ||z'||_{L^\infty(0,T;L^4(\kappa))}\big) \Big(\frac{k}{2\varepsilon_1} ||\theta_h^{n+1}||^2_{L^2(\kappa)} + \frac{\varepsilon_1 k}{2} |\mathbf{u}_h^{n+1} - \mathbf{u}(t^{n+1})|^2_{H^1(\kappa)} \Big),$$

and

$$\alpha\Big| \int_{t^n}^{t^{n+1}} \int_\kappa (\mathbf{u}(t^{n+1}) - \mathbf{c})\nabla \theta_h^{n+1}(\xi_h - z(t^{n+1}))dxdt + \int_{t^n}^{t^{n+1}} \int_\kappa \mathbf{u}(t^{n+1})\nabla \theta_h^{n+1}(z(t^{n+1}) - z(t))dxdt\Big|$$
$$= \alpha\Big| \int_{t^n}^{t^{n+1}} \int_\kappa (\mathbf{u}(t^{n+1}) - \mathbf{c})\nabla \theta_h^{n+1}\ (\xi_h - z(t^{n+1}))dxdt - \int_{t^n}^{t^{n+1}} \int_\kappa \theta_h^{n+1}\ \mathbf{u}(t^{n+1})\nabla(z(t^{n+1}) - z(t))dxdt\Big|$$
$$\le ||\theta_h^{n+1}||_{L^2(\kappa)} \Big\{ \frac{\alpha c_3}{\rho_k} k h^{3/2} ||z(t^{n+1})||_{W^{1,4}(\kappa)} ||\mathbf{u}(t^{n+1}) - \mathbf{c}||_{L^\infty(\kappa)} + ||\mathbf{u}||_{L^\infty(0,T;\kappa)} \frac{\alpha k^{3/2}}{\sqrt{3}} ||z'||_{L^2(t^n,t^{n+1};H^1(\kappa))} \Big\}$$
$$\le \alpha c_4 \Big(\sigma_0 k h^{3/2} ||\theta_h^{n+1}||_{L^2(\kappa)} ||\mathbf{u}(t^{n+1})||_{W^{1,\infty}(\kappa)} ||z||_{L^\infty(0,T;W^{1,4}(\kappa))}$$
$$+ \frac{k^{3/2}}{\sqrt{3}} ||\theta_h^{n+1}||_{L^2(\kappa)} ||z'||_{L^2(t^n,t^{n+1};H^1(\kappa))} ||\mathbf{u}||_{L^\infty(0,T;\kappa)} \Big)$$
$$\le \alpha c_5 ||\mathbf{u}||_{L^\infty(0,T;W^{1,\infty}(\kappa)^2)} \Big(\frac{k}{2\varepsilon_2} ||\theta_h^{n+1}||^2_{L^2(\kappa)} + \frac{\varepsilon_2}{2} (k^2 ||z'||^2_{L^2(t^n,t^{n+1};H^1(\kappa))} + k h^3 ||z||^2_{L^\infty(t^n,t^{n+1};W^{1,4}(\kappa))}) \Big).$$

For the second part (d_2), we write :

$$\alpha \int_{t^n}^{t^{n+1}} \int_{\partial\kappa_-} |\mathbf{u}_h^{n+1} \cdot \mathbf{n}| (\xi_h - z(t))^{ext} (\theta_{h,\text{ext}}^{n+1} - \theta_{h,\text{int}}^{n+1}) dsdt$$
$$\leq \frac{\alpha}{2} \int_{t^n}^{t^{n+1}} \int_{\partial\kappa_-} |\mathbf{u}_h^{n+1} \cdot \mathbf{n}| (\theta_{h,\text{ext}}^{n+1} - \theta_{h,\text{int}}^{n+1})^2 dsdt + \frac{\alpha}{2} \int_{t^n}^{t^{n+1}} \int_{\partial\kappa_-} |\mathbf{u}_h^{n+1} \cdot \mathbf{n}| \Big((\xi_h - z(t))^{\text{ext}} \Big)^2 dsdt$$

We keep the first term in the right-hand side of this inequality. The second term can be written as follows :

$$\frac{\alpha}{2} \int_{t^n}^{t^{n+1}} \int_{\partial\kappa_-} |\mathbf{u}_h^{n+1} \cdot \mathbf{n}| \Big((\xi_h - z(t))^{\text{ext}} \Big)^2 dsdt$$
$$\leq \alpha \int_{t^n}^{t^{n+1}} \int_{\partial\kappa_-} |\mathbf{u}_h^{n+1} \cdot \mathbf{n}| \Big((\xi_h - z(t^{n+1}))^{\text{ext}} \Big)^2 dsdt + \alpha \int_{t^n}^{t^{n+1}} \int_{\partial\kappa_-} |\mathbf{u}_h^{n+1} \cdot \mathbf{n}| \Big((z(t^{n+1}) - z(t))^{\text{ext}} \Big)^2 dsdt$$

with

$$\Big| \alpha \int_{t^n}^{t^{n+1}} \int_{\partial\kappa_-} |\mathbf{u}_h^{n+1} \cdot \mathbf{n}| \Big((\xi_h - z(t^{n+1}))^{\text{ext}} \Big)^2 dsdt \Big|$$
$$\leq \alpha c_6 k \Big(||\xi_h - z(t^{n+1})||^2_{L^4(\partial\kappa_-)} ||\mathbf{u}_h^{n+1} - \mathbf{u}(t^{n+1})||_{L^2(\partial\kappa_-)} + ||\xi_h - z(t^{n+1})||^2_{L^2(\partial\kappa_-)} ||\mathbf{u}(t^{n+1})||_{L^\infty(\omega_k)} \Big)$$
$$\leq \alpha c_7 k \Big(h^{3/2} ||z(t^{n+1})||^2_{W^{1,4}(\omega_k)} ||\mathbf{u}_h^{n+1} - \mathbf{u}(t^{n+1})||_{H^1(\omega_k)} + h^3 ||z(t^{n+1})||^2_{W^{1,4}(\omega_k)} ||\mathbf{u}(t^{n+1})||_{L^\infty(\omega_k)} \Big)$$
$$\leq \alpha \tilde{c}_7 h^3 k \big(||z||^2_{L^\infty(0,T;W^{1,4}(\omega_k))} ||\mathbf{u}||_{L^\infty(0,T;\omega_k)} + \frac{1}{2\varepsilon_3} ||z||^2_{L^\infty(0,T;W^{1,4}(\omega_k))} \big) + \frac{\varepsilon_3}{2} k ||\mathbf{u}_h^{n+1} - \mathbf{u}(t^{n+1})||^2_{H^1(\omega_k)},$$

where ω_k denotes the union of triangles adjacent to κ and

$$\Big| \alpha \int_{t^n}^{t^{n+1}} \int_{\partial\kappa_-} |\mathbf{u}_h^{n+1} \cdot \mathbf{n}| \Big((z(t^{n+1}) - z(t))^{\text{ext}} \Big)^2 dsdt \Big|$$
$$\leq \alpha \int_{\partial\kappa_-} |\mathbf{u}_h^{n+1} \cdot \mathbf{n}| \int_{t^n}^{t^{n+1}} \Big[(t^{n+1} - t) \Big(\int_t^{t^{n+1}} |z'(\tau)|^2 d\tau \Big)^{\text{ext}} \Big] dtds$$
$$\leq \alpha \int_{\partial\kappa_-} |\mathbf{u}_h^{n+1} \cdot \mathbf{n}| \Big(\int_{t^n}^{t^{n+1}} |z'(\tau)|^2 d\tau \Big)^{\text{ext}} \int_{t^n}^{t^{n+1}} (t^{n+1} - t) dtds$$
$$\leq \frac{\alpha k^2}{2} \Big\{ \int_{\partial\kappa_-} |(\mathbf{u}_h^{n+1} - \mathbf{u}(t^{n+1})) \cdot n| \Big(\int_{t^n}^{t^{n+1}} |z'(\tau)|^2 d\tau \Big)^{\text{ext}} ds$$
$$+ \int_{\partial\kappa_-} |u(t^{n+1}) \cdot n| \Big(\int_{t^n}^{t^{n+1}} |z'(\tau)|^2 d\tau \Big)^{\text{ext}} ds \Big\}$$
$$\leq \alpha c_8 \Big(\frac{k}{2\varepsilon_4} |\mathbf{u}_h^{n+1} - \mathbf{u}(t^{n+1})|^2_{H^1(\omega_k))} + \frac{\varepsilon_4 k^3}{2} ||z'||^2_{L^2(t^n,t^{n+1};W^{1,4}(\omega_k))}$$
$$+ k^2 ||\mathbf{u}||_{L^\infty(0,T;H^1(\omega_k)^2)} ||z'||^2_{L^2(t^n,t^{n+1};W^{1,4}(\omega_k))} \Big)$$

For the third part (d_3), we insert respectively in the divergence term and the term in z, $\pm\mathbf{u}(t^{n+1})$ and $\pm z(t^{n+1})$. We get four parts that we treat as follow :

$$\begin{aligned} |(d_{3,1})| &= \Big|\frac{\alpha}{2}\int_{t^n}^{t^{n+1}}\int_\Omega \mathrm{div}(\mathbf{u}_h^{n+1}-\mathbf{u}(t^{n+1}))(\xi_h - z(t^{n+1}))\theta_h^{n+1}dxdt\Big| \\ &\le \frac{\alpha}{2}\sigma_0 C_9 ||z||_{L^\infty(0,T;W^{1,4}(\Omega))}\Big(\frac{k}{2\varepsilon_5}|\mathbf{u}_h^{n+1}-\mathbf{u}(t^{n+1})|^2_{H^1(\Omega)} + \frac{\varepsilon_5}{2}k||\theta_h^{n+1}||^2_{L^2(\Omega)}\Big), \end{aligned}$$

$$\begin{aligned} |(d_{3,2})| &= \Big|\frac{\alpha}{2}\int_{t^n}^{t^{n+1}}\int_\Omega \mathrm{div}(\mathbf{u}(t^{n+1})-\mathbf{u}(t))(\xi_h - z(t^{n+1}))\theta_h^{n+1}dxdt\Big| \\ &\le \frac{\alpha}{2}C_{10}\sigma_0 k^{3/2}||z||_{L^\infty(0,T;W^{1,4}(\Omega))}||\theta_h^{n+1}||_{L^2(\Omega)}||\mathbf{u}'||_{L^2(t^n,t^{n+1};H^1(\Omega)^2)} \\ &\le \frac{\alpha}{2}C_{10}\sigma_0 ||z||_{L^\infty(0,T;W^{1,4}(\Omega))}\Big(\frac{k^2}{2\varepsilon_6}||\mathbf{u}'||^2_{L^2(t^n,t^{n+1};H^1(\Omega)^2)} + \frac{\varepsilon_6}{2}k||\theta_h^{n+1}||^2_{L^2(\Omega)}\Big), \\ |(d_{3,3})| &= \Big|\frac{\alpha}{2}\int_{t^n}^{t^{n+1}}\int_\Omega \mathrm{div}(\mathbf{u}(t^{n+1})-\mathbf{u}(t))(z(t^{n+1}) - z(t))\theta_h^{n+1}dxdt\Big| \\ &\le \frac{\alpha}{2}C_{11}\frac{k^2}{\rho_{min}}||z'||_{L^2(t^n,t^{n+1};L^4(\Omega))}||\theta_h^{n+1}||_{L^2(\Omega)}||\mathbf{u}'||_{L^2(t^n,t^{n+1},H^1(\Omega)^2)} \\ &\le \frac{\alpha}{2}C_{11}\sigma_0\gamma||\mathbf{u}'||_{L^\infty(0,T;H^1(\Omega)^2)}\Big(\frac{k^2}{2\varepsilon_7}||z'||^2_{L^2(t^n,t^{n+1};L^4(\Omega))} + \frac{\varepsilon_7}{2}k||\theta_h^{n+1}||^2_{L^2(\Omega)}\Big), \end{aligned}$$

and

$$\begin{aligned} |(d_{3,4})| &= \Big|\frac{\alpha}{2}\int_{t^n}^{t^{n+1}}\int_\Omega \mathrm{div}(\mathbf{u}_h^{n+1}-\mathbf{u}(t^{n+1}))(z(t^{n+1}) - z(t))\theta_h^{n+1}dxdt\Big| \\ &\le \frac{\alpha}{2}C_{13}\frac{k^{3/2}}{\rho_{min}}||\theta_h^{n+1}||_{L^2(\Omega)}|\mathbf{u}_h^{n+1}-\mathbf{u}(t^{n+1})|_{H^1(\Omega)}||z'||_{L^2(t^n,t^{n+1};L^4(\Omega))} \\ &\le \frac{\alpha}{2}C_{13}\gamma||z'||_{L^\infty(0,T;L^4(\Omega))}\Big(\frac{k}{2\varepsilon_8}||\theta_h^{n+1}||^2_{L^2(\Omega)} + \frac{\varepsilon_8}{2}k|\mathbf{u}_h^{n+1}-\mathbf{u}(t^{n+1})|^2_{H^1(\Omega)}\Big). \end{aligned}$$

We deduce the result after summing over $n = 0, \ldots, N-1$

□

Theorem 4.3. *With the same assumptions of Theorem 4.2, we have :*

$$\sum_{n=1}^{N-1} k||z_h^{n+1} - z(t^{n+1})||^2_{L^2(\Omega)} \le F_1(h^3+k^2) + F_2\sum_{n=0}^{N-1} k|\mathbf{u}_h^{n+1}-\mathbf{u}(t^{n+1})|^2_{H^1(\Omega)}, \tag{30}$$

where F_i are constants that only depend on $\mathbf{u}, z, \Omega$ and T.

Proof. We consider (21), take the test function $\theta_h = \theta_h^{n+1} = z_h^{n+1} - \rho_h z(t^{n+1})$, integrate from t^n

to t^{n+1} and subtract (26) multiplied by k. We obtain :

$$\alpha\Big((z_h^{n+1}-z(t^{n+1}))-(z_h^n-z(t^n)),\theta_h^{n+1}\Big)+\nu\Big(\int_{t^n}^{t^{n+1}}(z_h^{n+1}-z(t))dt,\theta_h^{n+1}\Big)$$
$$+\Big\{k\,c(\mathbf{u}_h^{n+1};z_h^{n+1},\theta_h^{n+1})-\alpha\int_{t^n}^{t^{n+1}}\big(\mathbf{u}(t)\nabla z(t)+\frac{1}{2}\operatorname{div}\mathbf{u}(t)z(t),\theta_h^{n+1}\big)dt\Big\} \tag{31}$$
$$=\nu\Big(\int_{t^n}^{t^{n+1}}(\mathrm{curl}\mathbf{u}_h^{n+1}-\mathrm{curl}\mathbf{u}(t))dt,\theta_h^{n+1}\Big)$$

Let us treat each term of this equation that we denote by $(b_i), i=1,...,4$.
For the first term, we follow the same steps as for the term (a_1) in the Theorem 4.1. We obtain :

$$(b_{1,1})=\frac{\alpha}{2}||\theta_h^{n+1}||^2_{L^2(\Omega)}-\frac{\alpha}{2}||\theta_h^n||^2_{L^2(\Omega)}+\frac{\alpha}{2}||\theta_h^{n+1}-\theta_h^n||^2_{L^2(\Omega)},$$

and

$$|(b_{1,2})|=\alpha\Big|\Big(\int_{t^n}^{t^{n+1}}(\rho_h z'(\tau)-z'(\tau))d\tau,\theta_h^{n+1}\Big)\Big|$$
$$\leq\frac{C\alpha h^3}{2\varepsilon_9}||z'||^2_{L^2(t^n,t^{n+1};W^{1,4}(\Omega))}+\frac{\alpha\varepsilon_9}{2}k||\theta_h^{n+1}||^2_{L^2(\Omega)}.$$

For the second term (b_2), we write :

$$z_h^{n+1}-z(t)=z_h^{n+1}-\rho_h z(t^{n+1})+\rho_h z(t^{n+1})-\rho_h z(t)+\rho_h z(t)-z(t),$$

and we obtain three parts that we treat successively.

The first one gives : $(b_{2,1})=\nu k||\theta_h^{n+1}||^2_{L^2(\Omega)}$.
The second part is bounded as follows :

$$|(b_{2,2})|=\nu\Big|(\int_{t^n}^{t^{n+1}}(\rho_h z(t^{n+1})-\rho_h z(t))dt,\theta_h\big)\Big|=\nu\Big|(\int_{t^n}^{t^{n+1}}\rho_h z'(t)(t-t^n)dt,\theta_h^{n+1})\Big|$$
$$\leq\frac{\nu k^2}{2\sqrt{3}\varepsilon_{10}}||z'||^2_{L^2(t^n,t^{n+1};H^1(\Omega))}+\frac{\varepsilon_{10}}{2\sqrt{3}}k||\theta_h^{n+1}||^2_{L^2(\Omega)},$$

and the last part is bounded as follows :

$$|(b_{2,3})|=\nu\Big|(\int_{t^n}^{t^{n+1}}(\rho_h z(t)-z(t))dt,\theta_h^{n+1})\Big|$$
$$\leq\frac{\nu h^3}{2\varepsilon_{11}}||z||^2_{L^2(t^n,t^{n+1};W^{1,4}(\Omega))}+\frac{\nu k\varepsilon_{11}}{2}||\theta_h^{n+1}||^2_{L^2(\Omega)}.$$

The third term can be written as follows :

$$(b_3)=\int_{t^n}^{t^{n+1}}c(\mathbf{u}_h^{n+1};z_h^{n+1},\theta_h^{n+1})dt-\alpha\int_{t^n}^{t^{n+1}}\Big(\mathbf{u}(t)\nabla z(t)+\frac{1}{2}\operatorname{div}\mathbf{u}(t)z(t),\theta_h^{n+1}\Big)dt$$
$$=\int_{t^n}^{t^{n+1}}c(\mathbf{u}_h^{n+1};\theta_h^{n+1},\theta_h^{n+1})dt+\int_{t^n}^{t^{n+1}}c(\mathbf{u}_h^{n+1};\rho_h z(t^{n+1})-z(t),\theta_h^{n+1})dt$$
$$+\int_{t^n}^{t^{n+1}}c(\mathbf{u}_h^{n+1};z(t),\theta_h^{n+1})dt-\alpha\int_{t^n}^{t^{n+1}}\Big(\mathbf{u}(t)\nabla z(t)+\frac{1}{2}\operatorname{div}\mathbf{u}(t)z(t),\theta_h^{n+1}\Big)dt.$$

Owing to Lemma 3.1 and denoting $\xi_h = \rho_h(z(t^{n+1}))$, (b_3) becomes :

$$(b_3) = \frac{\alpha}{2}\sum_{\kappa\in\mathcal{T}_h}\int_{t^n}^{t^{n+1}}\int_{\partial\kappa_-}|\mathbf{u}_h^{n+1}\cdot\mathbf{n}|(\theta_{h,\text{ext}}^{n+1}-\theta_{h,\text{int}}^{n+1})^2 ds dt + \int_{t^n}^{t^{n+1}} c(\mathbf{u}_h^{n+1};\rho_h z(t^{n+1})-z(t),\theta_h^{n+1})dt$$

$$+\alpha\int_{t^n}^{t^{n+1}}\int_\Omega(\mathbf{u}_h^{n+1}-\mathbf{u}(t))\nabla z(t)\theta_h^{n+1}dxdt + \frac{\alpha}{2}\int_{t^n}^{t^{n+1}}\int_\Omega \text{div}(\mathbf{u}_h^{n+1}-\mathbf{u}(t))z(t)\theta_h^{n+1}dxdt.$$

We divide (b_3) into four terms $(b_{3,i}), i = 1,\ldots,4$. We keep the term $(b_{3,1})$ in the left-hand side of (31). The second term $(b_{3,2})$ is bounded as in the previous theorem.

For the third part $(b_{3,3})$, we have :

$$\begin{aligned}|(b_{3,3})| &= \Big|\alpha\int_{t^n}^{t^{n+1}}\int_\Omega(\mathbf{u}_h^{n+1}-\mathbf{u}(t))\nabla z(t)\theta_h^{n+1}dxdt\Big|\\ &\le \alpha C_{14}||z||_{L^\infty(0,T;W^{1,4}(\Omega))}\Big(\frac{k}{2\varepsilon_{12}}|\mathbf{u}_h^{n+1}-\mathbf{u}(t^{n+1})|^2_{H^1(\Omega)} + \frac{\varepsilon_{12}}{2}k||\theta_h^{n+1}||^2_{L^2(\Omega)}\Big)\\ &\quad +\alpha C_{15}||z||_{L^\infty(0,T;W^{1,4}(\Omega))}\Big(\frac{k^2}{2\varepsilon_{13}}||\mathbf{u}'||^2_{L^2(t^n,t^{n+1};H^1(\Omega)^2)} + \frac{\varepsilon_{13}}{2}k||\theta_h^{n+1}||^2_{L^2(\Omega)}\Big),\end{aligned}$$

and the last part of the third term (b_3) is :

$$\begin{aligned}|(b_{3,4})| &= \Big|\frac{\alpha}{2}\int_{t^n}^{t^{n+1}}\int_\Omega \text{div}(\mathbf{u}_h^{n+1}-\mathbf{u}(t))z(t)\theta_h^{n+1}dxdt\Big|\\ &\le \frac{\alpha}{2}C_{16}||z||_{L^\infty(0,T;L^\infty(\Omega))}\Big(\frac{k}{2\varepsilon_{14}}|\mathbf{u}_h^{n+1}-\mathbf{u}(t^{n+1})|^2_{H^1(\Omega)} + \frac{\varepsilon_{14}}{2}k||\theta_h^{n+1}||^2_{L^2(\Omega)}\Big)\\ &\quad +\frac{\alpha}{2}C_{17}||z||_{L^\infty(0,T;L^\infty(\Omega))}\Big(\frac{k^2}{2\varepsilon_{15}}||\mathbf{u}'||_{L^2(t^n,t^{n+1};H^1(\Omega)^2)} + \frac{\varepsilon_{15}}{2}k||\theta_h^{n+1}||^2_{L^2(\Omega)}\Big).\end{aligned}$$

For the last term (b_4), we split it into two parts, as follows : Using $||\text{ curl }\mathbf{u}_h||^2_{L^2(\Omega)} \le 2|\mathbf{u}_h|^2_{H^1(\Omega)}$, we have

$$\begin{aligned}|(b_{4,1})| &= \Big|\nu\Big(\int_{t^n}^{t^{n+1}}(\text{ curl }\mathbf{u}_h^{n+1} - \text{ curl }\mathbf{u}(t^{n+1}))dt,\theta_h^{n+1}\Big)\Big|\\ &\le \frac{\nu k}{\varepsilon_{16}}|\mathbf{u}_h^{n+1}-\mathbf{u}(t^{n+1})|^2_{H^1(\Omega)} + \frac{\nu\varepsilon_{16}k}{2}||z_h^{n+1}-\rho_h z(t^{n+1})||^2_{L^2(\Omega)},\end{aligned}$$

and

$$\begin{aligned}|(b_{4,2})| &= \Big|\nu\Big(\int_{t^n}^{t^{n+1}}(\text{ curl }\mathbf{u}(t^{n+1}) - \text{ curl }\mathbf{u}(t))dt,\theta_h^{n+1}\Big)\Big|\\ &\le \frac{\nu k^2}{\sqrt{3}\varepsilon_{17}}||\mathbf{u}'||^2_{L^2(t^n,t^{n+1};H^1(\Omega)^2)} + \frac{\nu\varepsilon_{17}k}{\sqrt{3}}||z_h^{n+1}-\rho_h z(t^{n+1})||^2_{L^2(\Omega)}.\end{aligned}$$

Collecting all these results, we obtain :

$$(b_{1,1}) + (b_{2,1}) + (b_{3,1}) \quad\le\quad |(b_{1,2})| + |(b_{2,2})| + |(b_{2,3})| + |\sum_{i=2}^{4}(b_{3,i})| + |(b_4)|.$$

Then (30) follows easily after the sum over $n = 1, \ldots, N-1$, a suitable choice of $\varepsilon_i, i = 1, \ldots, 17$ and by applying a triangular inequality to $||z_h^{n+1} - z(t^{n+1})||_{L^2(\Omega)}$:

$$||z_h^{n+1} - z(t^{n+1})||_{L^2(\Omega)} \leq ||z_h^{n+1} - \rho_h z(t^{n+1})||_{L^2(\Omega)} + ||\rho_h z(t^{n+1}) - z_h^{n+1}||_{L^2(\Omega)}$$

and properties of P_h. □

Corollary 4.4. *Under the assumptions of Theorem 4.1 and Theorem 4.3, and for k sufficiently small, there exist constants* C_1, C_2 *and* C_3 *independent of* h *and* k *such that :*

$$\sum_{n=0}^{N-1} k|\mathbf{u}_h^{n+1} - \mathbf{u}(t^{n+1})|^2_{H^1(\Omega)} \leq C_1(h^3 + k^2), \tag{32}$$

$$\sum_{n=0}^{N-1} k||z_h^{n+1} - z(t^{n+1})||^2_{L^2(\Omega)} \leq C_2(h^3 + k^2), \tag{33}$$

and

$$\sup_n |\mathbf{u}_h^n - \mathbf{u}(t^n)|^2_{H^1(\Omega)} \leq C_3(h^3 + k^2). \tag{34}$$

Proof. On one hand, we consider (30). On the other hand, the only difference between this proof and that of Theorem 4.1 is the upper bound of the term $(a_{5,1})$. Here, using the inequality

$$||\mathbf{u}||^2_{L^4(\Omega)} \leq C|\mathbf{u}|_{H^1(\Omega)}||\mathbf{u}||_{L^2(\Omega)},$$

we have :

$$\begin{aligned} |(a_{5,1})| \quad \leq \quad & \frac{S_4\varepsilon_6}{2}||P_h\mathbf{u}||^2_{L^\infty(0,T;H^1(\Omega)^2)}k||z_h^n - z(t^n)||^2_{L^2(\Omega)} + \frac{S_4}{4\tilde{\varepsilon}_6}\, k||\mathbf{u}_h^{n+1} - P_h\mathbf{u}(t^{n+1})||^2_{L^2(\Omega)} \\ + \quad & \frac{S_4}{4\tilde{\varepsilon}_6}k|\mathbf{u}_h^{n+1} - P_h\mathbf{u}(t^{n+1})|^2_{H^1(\Omega)}. \end{aligned}$$

Then, using this result with (30) and after a suitable choice of $\varepsilon_i, i = 1, \ldots, 10$ and $\tilde{\varepsilon}_6$, we obtain :

$$\begin{aligned} &||\mathbf{u}_h^N - P_h\mathbf{u}(t^N)||^2_{L^2(\Omega)} + \sum_{n=0}^{N-1} ||(\mathbf{u}_h^{n+1} - P_h(\mathbf{u}(t^{n+1}))) - (\mathbf{u}_h^n - P_h(\mathbf{u}(t^n)))||^2_{L^2(\Omega)} \\ &+\alpha|\mathbf{u}_h^N - P_h\mathbf{u}(t^N)|^2_{H^1(\Omega)} + \alpha\sum_{n=0}^{N-1} |(\mathbf{u}_h^{n+1} - P_h\mathbf{u}(t^{n+1})) - (\mathbf{u}_h^n - P_h\mathbf{u}(t^n))|^2_{H^1(\Omega)} \\ &+\nu\sum_{n=0}^{N-1} k|\mathbf{u}_h^{n+1} - P_h\mathbf{u}(t^{n+1})|^2_{H^1(\Omega)} \leq C(h^3 + k^2) + C'\sum_{n=0}^{N-1} k||\mathbf{u}_h^{n+1} - P_h\mathbf{u}(t^{n+1})||^2_{L^2(\Omega)}. \end{aligned}$$

Then by applying the discrete Gronwall lemma, we obtain, for k sufficiently small :

$$||\mathbf{u}_h^N - \mathbf{u}(t^N)||^2_{L^2(\Omega)} + \nu\sum_{n=0}^{N-1} k|\mathbf{u}_h^{n+1} - \mathbf{u}(t^{n+1})|^2_{H^1(\Omega)} \leq Ce^{C'kN}(h^3 + k^2),$$

and the results follow easily. □

Remark 4.5. *If we have, for example,*

$$\| \mathbf{v}_h^m \|^2_{L^2(\Omega)} + \sum_{n=0}^{m-1} \| \mathbf{v}_h^{n+1} - \mathbf{v}_h^n \|^2_{L^2(\Omega)} + \nu \sum_{n=0}^{m-1} k |\mathbf{v}_h^{n+1}|^2_{H^1(\Omega)} \leq C_1 + C_2 \sum_{n=0}^{m-1} k \| \mathbf{v}_h^{n+1} \|^2_{L^2(\Omega)},$$

by writing,

$$\| \mathbf{v}_h^m \|_{L^2(\Omega)} \leq \| \mathbf{v}_h^m - \mathbf{v}_h^{m-1} \|_{L^2(\Omega)} + \| \mathbf{v}_h^{m-1} \|_{L^2(\Omega)},$$

we obtain

$$C_2 k \| \mathbf{v}_h^m \|^2_{L^2(\Omega)} \leq 2C_2 k \| \mathbf{v}_h^m - \mathbf{v}_h^{m-1} \|^2_{L^2(\Omega)} + 2C_2 k \| \mathbf{v}_h^{m-1} \|^2_{L^2(\Omega)} .$$

By assuming k sufficiently small such that $2C_2 k \leq 1$, we obtain :

$$\| \mathbf{v}_h^m \|^2_{L^2(\Omega)} + \sum_{n=0}^{m-2} \| \mathbf{v}_h^{n+1} - \mathbf{v}_h^n \|^2_{L^2(\Omega)} + \nu \sum_{n=0}^{m-1} k |\mathbf{v}_h^{n+1}|^2_{H^1(\Omega)} \leq C_1 + 3C_2 \sum_{n=1}^{m-1} k \| \mathbf{v}_h^n \|^2_{L^2(\Omega)},$$

then we can apply the discrete classic Gronwall lemma.

Theorem 4.6. *Under the assumptions of Corollary 4.4, we suppose that $p' \in L^2(0,T;L^2(\Omega))$. Then the pressure satisfies the following estimate :*

$$\begin{aligned} \sum_{n=0}^{N-1} k ||p_h^{n+1} - p(t^{n+1})||^2_{L^2(\Omega)} \leq \frac{1}{\beta^\star} \Big\{ C(h^3 + k^2) \\ + (\alpha + S_2^2) \sum_{n=0}^{N-1} k \Big| \frac{(\mathbf{u}_h^{n+1} - \mathbf{u}(t^{n+1})) - (\mathbf{u}_h^n - \mathbf{u}(t^n))}{k} \Big|^2_{H^1(\Omega)} \Big\}, \end{aligned} \tag{35}$$

where the coefficients C and S_2 are respectively the inf-sup constant and Poincaré's constant and are independent of h and k.

Proof. We consider again the first equation of the proof of Theorem 4.1, insert $\pm k r_h p(t^{n+1})$ in the terms of the pression and we get :

$$\begin{aligned} &\int_{t^n}^{t^{n+1}} \Big(p_h^{n+1} - r_h p(t^{n+1}), \operatorname{div} \mathbf{v}_h^{n+1} \Big) dt = \Big((\mathbf{u}_h^{n+1} - \mathbf{u}(t^{n+1})) - (\mathbf{u}_h^n - \mathbf{u}(t^n)), \mathbf{v}_h^{n+1} \Big) \\ &+ \alpha \Big(\nabla(\mathbf{u}_h^{n+1} - \mathbf{u}(t^{n+1})) - \nabla(\mathbf{u}_h^n - \mathbf{u}(t^n)), \nabla \mathbf{v}_h^{n+1} \Big) + \nu \Big(\int_{t^n}^{t^{n+1}} \nabla(\mathbf{u}_h^{n+1} - \mathbf{u}(t)) dt, \nabla \mathbf{v}_h^{n+1} \Big) \\ &+ \int_{t^n}^{t^{n+1}} \Big((z_h^n \wedge \mathbf{u}_h^{n+1} - z(t) \wedge \mathbf{u}(t)) dt, \mathbf{v}_h^{n+1} \Big) - \int_{t^n}^{t^{n+1}} \Big(r_h p(t^{n+1}) - p(t), \operatorname{div} \mathbf{v}_h^{n+1} \Big) dt. \end{aligned}$$

Owing to the inf-sup condition, $\forall q_h \in M_h$,

$$\exists \mathbf{w}_h \in V_h^\perp;\ (\operatorname{div} \mathbf{w}_h, q_h) = ||q_h||^2_{L^2(\Omega)} \text{ and } ||\nabla \mathbf{w}_h||_{L^2(\Omega)} \leq ||q_h||_{L^2(\Omega)},$$

and summing over $n = 0, \ldots, N-1$, the left-hand side of this equation becomes $\sum_{n=0}^{N-1} k ||p_h^{n+1} - r_h p(t^{n+1})||^2_{L^2(\Omega)}$. Let us treat the terms in the right-hand side of the equation.

For the first term, we have

$$\begin{aligned}
&\Big|\sum_{n=0}^{N-1}\Big((\mathbf{u}_h^{n+1}-\mathbf{u}(t^{n+1}))-(\mathbf{u}_h^{n}-\mathbf{u}(t^{n})),\mathbf{v}_h^{n+1}\Big)\Big| \\
&\qquad\le S_2\Big(\sum_{n=0}^{N-1}k\Big|\Big|\frac{(\mathbf{u}_h^{n+1}-\mathbf{u}_h^{n})-(\mathbf{u}(t^{n+1})-\mathbf{u}(t^{n}))}{k}\Big|\Big|^2_{L^2(\Omega)}\Big)^{1/2}\Big(\sum_{n=0}^{N-1}k|\mathbf{v}_h^{n+1}|^2_{H^1(\Omega)}\Big)^{1/2} \\
&\qquad\le S_2^2\Big(\sum_{n=0}^{N-1}k\Big|\frac{(\mathbf{u}_h^{n+1}-\mathbf{u}_h^{n})-(\mathbf{u}(t^{n+1})-\mathbf{u}(t^{n}))}{k}\Big|^2_{H^1(\Omega)}\Big)^{1/2}\Big(\sum_{n=0}^{N-1}k|\mathbf{v}_h^{n+1}|^2_{H^1(\Omega)}\Big)^{1/2},
\end{aligned}$$

and for the second,

$$\begin{aligned}
&\Big|\alpha\sum_{n=0}^{N-1}\Big(\nabla((\mathbf{u}_h^{n+1}-\mathbf{u}(t^{n+1}))-(\mathbf{u}_h^{n}-\mathbf{u}(t^{n}))),\nabla\mathbf{v}_h^{n+1}\Big)\Big| \\
&\qquad\le \alpha\Big(\sum_{n=0}^{N-1}k\Big|\frac{(\mathbf{u}_h^{n+1}-\mathbf{u}_h^{n})-(\mathbf{u}(t^{n+1})-\mathbf{u}(t^{n}))}{k}\Big|^2_{H^1(\Omega)}\Big)^{1/2}\Big(\sum_{n=0}^{N-1}k|\mathbf{v}_h^{n+1}|^2_{H^1(\Omega)}\Big)^{1/2}.
\end{aligned}$$

For the third term, we have :

$$\begin{aligned}
&\nu\Big|\sum_{n=0}^{N-1}\int_{t^n}^{t^{n+1}}(\nabla(\mathbf{u}_h^{n+1}-\mathbf{u}(t)),\nabla\mathbf{v}_h^{n+1})\,dt\Big| \\
&\quad\le \nu\sum_{n=0}^{N-1}\int_{t^n}^{t^{n+1}}\Big|(\nabla(\mathbf{u}_h^{n+1}-\mathbf{u}(t^{n+1})),\nabla\mathbf{v}_h^{n+1})\Big|dt+\nu\sum_{n=0}^{N-1}\Big|\int_{t^n}^{t^{n+1}}\Big(\int_{t^{n+1}}^{t}\nabla\mathbf{u}'(\tau)d\tau,\nabla\mathbf{v}_h^{n+1}\Big)dt\Big| \\
&\quad\le \nu\Big(C_1(h^3+k^2)^{1/2}+C_2k||\mathbf{u}'||_{L^2(0,T;H^1(\Omega)^2)}\Big)\Big(\sum_{n=0}^{N-1}k|\mathbf{v}_h^{n+1}|^2_{H^1(\Omega)}\Big)^{1/2}.
\end{aligned}$$

The fourth term is treated as the fifth term in the proof of Theorem 4.1 and by using the result of Theorem 4.3, the result is the following :

$$\Big|\sum_{n=0}^{N-1}\int_{t^n}^{t^{n+1}}\Big(\big(z_h^n\wedge\mathbf{u}_h^{n+1}-z(t)\wedge\mathbf{u}(\mathbf{t})\big)dt,\nabla\mathbf{v}_h^{n+1}\Big)\Big|\le C(h^3+k^2)^{1/2}\Big(\sum_{n=0}^{N-1}k|\mathbf{v}_h^{n+1}|^2_{H^1(\Omega)}\Big)^{1/2}.$$

Finally, the last term is treated as follows :

$$\begin{aligned}
&\int_{t^n}^{t^{n+1}}\Big(r_hp(t^{n+1})-p(t),\operatorname{div}\mathbf{v}_h^{n+1}\Big)dt \\
&\le\Big|\sum_{n=0}^{N-1}\int_{t^n}^{t^{n+1}}\Big((r_hp(t^{n+1})-r_hp(t))dt,\operatorname{div}\mathbf{v}_h^{n+1}\Big)\Big|+\Big|\sum_{n=0}^{N-1}\int_{t^n}^{t^{n+1}}\Big((r_hp(t)-p(t))dt,\operatorname{div}\mathbf{v}_h^{n+1}\Big)\Big| \\
&\le C\sum_{n=0}^{N-1}\int_{t^n}^{t^{n+1}}(\tau-t^n)||p'(\tau)||_{L^2(\Omega)}|\mathbf{v}_h^{n+1}|_{H^1(\Omega)}d\tau+C_2\sum_{n=0}^{N-1}h^2k^{1/2}||p||_{L^2(t^n,t^{n+1};H^2(\Omega))}|\mathbf{v}_h^{n+1}|_{H^1(\Omega)} \\
&\le\Big(\frac{C_1k}{\sqrt{3}}||p'||_{L^2(0,T;L^2(\Omega))}+C_2h^2||p||_{L^2(0,T;H^2(\Omega))}\Big)\Big(\sum_{n=0}^{N-1}k|\mathbf{v}_h^{n+1}|^2_{H^1(\Omega)}\Big)^{1/2}.
\end{aligned}$$

Then (35) follows easily from these inequalities. □

We still have to estimate $\Big(\sum_{n=0}^{N-1} k\big|\dfrac{(\mathbf{u}_h^{n+1}-\mathbf{u}(t^{n+1}))-(\mathbf{u}_h^n-\mathbf{u}(t^n))}{k}\big|^2_{H^1(\Omega)}\Big)^{1/2}$.

We introduce a variant of Stokes projection as follows : $\forall(\mathbf{u},p)\in V\times L^2_0(\Omega), S_h\mathbf{u}\in V_h$ is defined by

$$\forall \mathbf{v}_h\in V_h,\quad \nu(\nabla(S_h\mathbf{u}-\mathbf{u}),\nabla\mathbf{v}_h)=-(p,\operatorname{div}\mathbf{v}_h), \tag{36}$$

Lemma 4.7. *Let* $(\mathbf{u},p)\in V\times L^2_0(\Omega)$*. Then* $S_h\mathbf{u}$ *defined by* (36) *satisfies :*

$$|S_h\mathbf{u}-\mathbf{u}|_{H^1(\Omega)}\le 2|P_h\mathbf{u}-\mathbf{u}|_{H^1(\Omega)}+\frac{1}{\nu}||r_hp-p||_{L^2(\Omega)}. \tag{37}$$

If, in addition, Ω *is convex, there exists a constant* C *independent of* h *such that:*

$$||S_h\mathbf{u}-\mathbf{u}||_{L^2(\Omega)}\le Ch\big(|S_h\mathbf{u}-\mathbf{u}|_{H^1(\Omega)}+||r_hp-p||_{L^2(\Omega)}\big). \tag{38}$$

Theorem 4.8. *Under the assumptions of Theorem 4.6 and assuming* $p'\in L^2(0,T;H^2(\Omega))$*, we have :*

$$\sum_{n=0}^{N-1} k\Big|\frac{(\mathbf{u}_h^{n+1}-\mathbf{u}(t^{n+1}))-(\mathbf{u}_h^n-\mathbf{u}(t^n))}{k}\Big|^2_{H^1(\Omega)}\le C(h^3+k^2). \tag{39}$$

Proof. We consider, once more, the first equation in the proof of Theorem 4.6, choose $\mathbf{v}_h^{n+1}\in V_h$, insert $r_hp(s)$ and $S_h\mathbf{u}'=(S_h\mathbf{u})'$ and we set $\mathbf{e}_h^n=\mathbf{u}_h^n-S_h\mathbf{u}(t^n)$. We obtain :

$$\begin{aligned}
&\Big((\mathbf{u}_h^{n+1}-S_h\mathbf{u}(t^{n+1}))-(\mathbf{u}_h^n-S_h\mathbf{u}(t^n)),\mathbf{v}_h^{n+1}\Big)-\int_{t^n}^{t^{n+1}}\Big(\mathbf{u}'(s)-S_h\mathbf{u}'(s),\mathbf{v}_h^{n+1}\Big)ds\\
&+\alpha\Big(\nabla(\mathbf{u}_h^{n+1}-S_h\mathbf{u}(t^{n+1}))-\nabla(\mathbf{u}_h^n-S_h\mathbf{u}(t^n)),\nabla\mathbf{v}_h^{n+1}\Big)-\alpha\int_{t^n}^{t^{n+1}}\Big(\nabla(\mathbf{u}'(s)-S_h\mathbf{u}'(s)),\nabla\mathbf{v}_h^{n+1}\Big)ds\\
&+\nu\Big(\int_{t^n}^{t^{n+1}}\nabla(\mathbf{u}_h^{n+1}-S_h\mathbf{u}(s))ds,\nabla\mathbf{v}_h^{n+1}\Big)+\nu\Big(\int_{t^n}^{t^{n+1}}\nabla(S_h\mathbf{u}(s)-\mathbf{u}(s))ds,\nabla\mathbf{v}_h^{n+1}\Big)\\
&+\int_{t^n}^{t^{n+1}}\Big((z_h^n\wedge\mathbf{u}_h^{n+1}-z(s)\wedge\mathbf{u}(s))ds,\mathbf{v}_h^{n+1}\Big)+\int_{t^n}^{t^{n+1}}\Big(p(s),\operatorname{div}\mathbf{v}_h^{n+1}\Big)ds=0.
\end{aligned}$$

We sum this above equation over $n=0,\dots,N-1$ and we treat the terms denoted again $(a_i), i=1,\dots,6$ in the left-hand side. We take $\mathbf{v}_h^{n+1}=\dfrac{\mathbf{e}_h^{n+1}-\mathbf{e}_h^n}{k}$.

The first term is composed of two parts as follows :

$$(a_1)=\sum_{n=0}^{N-1}(\mathbf{e}_h^{n+1}-\mathbf{e}_h^n,\mathbf{v}_h^{n+1})-\sum_{n=0}^{N-1}\int_{t^n}^{t^{n+1}}\Big(\mathbf{u}'(s)-S_h\mathbf{u}'(s),\mathbf{v}_h^{n+1}\Big)ds=(a_{1,1})+(a_{1,2}),$$

where

$$
\begin{aligned}
(a_{1,1}) &= \sum_{n=0}^{N-1} k||\frac{\mathbf{e}_h^{n+1}-\mathbf{e}_h^n}{k}||^2_{L^2(\Omega)},\\
|(a_{1,2})| &\le \sum_{n=0}^{N-1} ||\mathbf{u}'-S_h\mathbf{u}'||_{L^2(t^n,t^{n+1};L^2(\Omega)^2)}\Big(k||\frac{\mathbf{e}_h^{n+1}-\mathbf{e}_h^n}{k}||^2_{L^2(\Omega)}\Big)^{1/2}\\
&\le \frac{1}{2\varepsilon_1}||\mathbf{u}'-S_h\mathbf{u}'||^2_{L^2(0,T;L^2(\Omega)^2)}+\frac{\varepsilon_1}{2}\sum_{n=0}^{N-1} k||\frac{\mathbf{e}_h^{n+1}-\mathbf{e}_h^n}{k}||^2_{L^2(\Omega)}\\
&\le \frac{C}{2\varepsilon_1}h^4\big(||\mathbf{u}'||^2_{L^2(0,T;H^2(\Omega)^2)}+||p'||^2_{L^2(0,T;H^1(\Omega))}\big)+\frac{\varepsilon_1}{2}\sum_{n=0}^{N-1} k||\frac{\mathbf{e}_h^{n+1}-\mathbf{e}_h^n}{k}||^2_{L^2(\Omega)}.
\end{aligned}
$$

The second term (a_2) is treated as the first one. We have :

$$
(a_2)=\alpha\sum_{n=0}^{N-1}(\nabla\mathbf{e}_h^{n+1}-\nabla\mathbf{e}_h^n,\nabla\mathbf{v}_h^{n+1})-\alpha\sum_{n=0}^{N-1}\int_{t^n}^{t^{n+1}}\Big(\nabla\mathbf{u}'(s)-\nabla S_h\mathbf{u}'(s),\nabla\mathbf{v}_h^{n+1}\Big)ds=(a_{2,1})+(a_{2,2}),
$$

with

$$
\begin{aligned}
(a_{2,1}) &= \alpha\sum_{n=0}^{N-1} k|\frac{\mathbf{e}_h^{n+1}-\mathbf{e}_h^n}{k}|^2_{H^1(\Omega)},\\
|(a_{2,2})| &\le \frac{C\alpha}{2\varepsilon_2}h^4\big(||\mathbf{u}'||^2_{L^2(0,T;H^3(\Omega)^2)}+||p'||^2_{L^2(0,T;H^2(\Omega))}\big)+\frac{\alpha\varepsilon_2}{2}\sum_{n=0}^{N-1} k||\frac{\mathbf{e}_h^{n+1}-\mathbf{e}_h^n}{k}||^2_{H^1(\Omega)}.
\end{aligned}
$$

The third term is treated as :

$$
\begin{aligned}
|(a_3)| &= \nu\Big|\sum_{n=0}^{N-1}\int_{t^n}^{t^{n+1}}(\nabla(\mathbf{u}_h^{n+1}-S_h\mathbf{u}(s)),\nabla\mathbf{v}_h^{n+1})ds\Big|\\
&= \nu\Big|\sum_{n=0}^{N-1}(\nabla\mathbf{e}_h^{n+1},\nabla\mathbf{v}_h^{n+1})+\nu\sum_{n=0}^{N-1}\int_{t^n}^{t^{n+1}}(\nabla S_h(\mathbf{u}(t^{n+1})-\mathbf{u}(s)),\nabla\mathbf{v}_h^{n+1})ds\Big|\\
&= |(a_{3,1})+(a_{3,2})|.
\end{aligned}
$$

Using the relation $(a^{n+1}-a^n,a^{n+1})=\frac{1}{2}||a^{n+1}||^2_{L^2(\Omega)}-\frac{1}{2}||a^n||^2_{L^2(\Omega)}+\frac{1}{2}||a^{n+1}-a^n||^2_{L^2(\Omega)}$ and by defining $S_h\mathbf{u}^0$ with $p=0$, we obtain :

$$
\begin{aligned}
|(a_{3,1})| &= \frac{\nu}{2}\sum_{n=0}^{N-1} k|\frac{\mathbf{e}_h^{n+1}-\mathbf{e}_h^n}{k}|^2_{H^1(\Omega)}+\frac{\nu}{2}\sum_{n=0}^{N-1} k\big(|\frac{\mathbf{e}_h^{n+1}}{k}|^2_{H^1(\Omega)}-|\frac{\mathbf{e}_h^n}{k}|^2_{H^1(\Omega)}\big)\\
&= \frac{\nu}{2}\sum_{n=0}^{N-1} k|\frac{\mathbf{e}_h^{n+1}-\mathbf{e}_h^n}{k}|^2_{H^1(\Omega)}+\frac{\nu}{2}k|\frac{\mathbf{e}_h^N}{k}|^2_{H^1(\Omega)}.
\end{aligned}
$$

and

$$\begin{aligned}
|(a_{3,2})| &= \nu\Big|\sum_{n=0}^{N-1}\int_{t^n}^{t^{n+1}}(\nabla S_h(\mathbf{u}(t^{n+1})-\mathbf{u}(s)),\nabla\mathbf{v}_h^{n+1})ds\Big| \\
&= \nu\Big|\sum_{n=0}^{N-1}\int_{t^n}^{t^{n+1}}(\nabla S_h\mathbf{u}'(t)(t-t^n),\nabla\mathbf{v}_h^{n+1})dt\Big| \\
&\leq \nu\sum_{n=0}^{N-1}|\mathbf{v}_h^{n+1}|_{H^1(\Omega)}\int_{t^n}^{t^{n+1}}(t-t^n)|S_h\mathbf{u}'(t)|_{H^1(\Omega)}dt \\
&\leq \nu\sum_{n=0}^{N-1}k^{3/2}|\mathbf{v}_h^{n+1}|_{H^1(\Omega)}||S_h\mathbf{u}'||_{L^2(t^n,t^{n+1};H^1(\Omega)^2)} \\
&\leq \frac{\nu}{2\varepsilon_3}\sum_{n=0}^{N-1}k|\frac{\mathbf{e}_h^{n+1}-\mathbf{e}_h^n}{k}|^2_{H^1(\Omega)}+\frac{\nu\varepsilon_3}{2}k^2||S_h\mathbf{u}'||^2_{L^2(0,T;H^1(\Omega)^2)}.
\end{aligned}$$

Using the definition of S_h, we have

$$(a_4)+(a_6)=\nu\Big(\int_{t^n}^{t^{n+1}}\nabla(S_h\mathbf{u}(s)-\mathbf{u}(s))ds,\nabla\mathbf{v}_h^{n+1}\Big)+\int_{t^n}^{t^{n+1}}\Big(p(s),\operatorname{div}\mathbf{v}_h^{n+1}\Big)ds=0$$

Finally, the last term (a_5) is bounded as previously in Theorem 4.6 :

$$|(a_5)|\leq\frac{C\varepsilon_4}{2}(h^3+k^2)+\frac{C}{2\varepsilon_4}\sum_{n=0}^{N-1}k|\frac{\mathbf{e}_h^{n+1}-\mathbf{e}_h^n}{k}|^2_{H^1(\Omega)}.$$

Collecting these results, writing

$$(a_{1,1})+(a_{2,1})+(a_{3,1})\leq|(a_{1,2})|+|(a_{2,2})|+|(a_{3,2})|+|(a_5)|,$$

choosing suitably $\varepsilon_i, i=1,\ldots,4$ and by applying the following triangular inequality

$$|\mathbf{u}_h^{n+1}-\mathbf{u}(t^{n+1})|_{H^1(\Omega)}\leq|\mathbf{u}_h^{n+1}-S_h\mathbf{u}(t^{n+1})|_{H^1(\Omega)}+|S_h\mathbf{u}(t^{n+1})-\mathbf{u}(t^{n+1})|_{H^1(\Omega)},$$

(39) follows easily. □

Theorem 4.9. *Under the assumptions of Theorem 4.6, there exists a constant C that does not depend on h and k such that*

$$\sum_{n=0}^{N-1}k||p_h^{n+1}-p(t^{n+1})||^2_{L^2(\Omega)}\quad\leq C(h^3+k^2). \tag{40}$$

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