VERY SINGULAR SOLUTIONS TO A NONLINEAR PARABOLIC EQUATION WITH ABSORPTION. II – UNIQUENESS

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Abstract

We prove the uniqueness of the very singular solution to

$$u_t - \Delta u + |\nabla u|^p = 0 \quad \text{in} \quad (0,+\infty) \times \mathbb{R}^N,$$

when $1 < p < (N+2)/(N+1)$, thus completing the previous result by Qi & Wang (2001) restricted to self-similar solutions.

Key words. viscous Hamilton-Jacobi equation, very singular solution, uniqueness

1 Introduction

The existence of a self-similar *very singular solution* at the origin to the following viscous Hamilton-Jacobi equation

$$u_t - \Delta u + |\nabla u|^p = 0 \quad \text{in} \quad (0,+\infty) \times \mathbb{R}^N, \tag{1.1}$$

has been established in [2] and in [18] by two different methods, when $1 < p < (N+2)/(N+1)$. Recall that a very singular solution at the origin to (1.1) is a nonnegative solution to (1.1) which is smooth in $(0,+\infty) \times \mathbb{R}^N$ and fulfils the following two conditions

$$\lim_{t\to 0} \int_{\{|x|\le r\}} u(t,x)\, dx = +\infty,$$

$$\lim_{t\to 0} \int_{\{|x|\ge r\}} u(t,x)\, dx = 0,$$

for every $r \in (0,+\infty)$. In addition, a very singular solution u is *self-similar* if there is a smooth function $f \in L^1((0,+\infty); r^{N-1}dr)$ such that

$$u(t,x) = t^{-a/2}\, f\left(|x| t^{-1/2}\right), \quad (t,x) \in (0,+\infty) \times \mathbb{R}^N,$$

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where $a = (2-p)/(p-1)$. In [18], Qi & Wang show that there is one and only one self-similar very singular solution to (1.1). The purpose of this paper is to extend this uniqueness result without the self-similarity assumption.

Before describing our results, let us mention that the name very singular solution has been introduced by Brezis, Peletier and Terman [4] who proved the existence and uniqueness of a self-similar very singular solution to

$$u_t - \Delta u + u^p = 0 \quad \text{in} \quad (0,+\infty) \times \mathbb{R}^N, \tag{1.2}$$

when $1 < p < 1 + 2/N$. As self-similar very singular solutions to (1.2) satisfy an ordinary differential equation, the uniqueness proof in [4] relies on ordinary differential equations techniques. The uniqueness of the very singular solution to (1.2) (without the self-similarity assumption) was subsequently obtained by Oswald [14]. Since then, the existence and uniqueness of nonnegative very singular solutions have been studied for other nonlinear parabolic equations with absorption such as

$$u_t - \Delta u^m + u^p = 0 \quad \text{in} \quad (0,+\infty) \times \mathbb{R}^N, \tag{1.3}$$

where $m > (1 - 2/N)^+$, $m \neq 1$ [16, 9, 15, 11, 12], or

$$u_t - \text{div}\left(|\nabla u|^{m-2} \nabla u\right) + u^p = 0 \quad \text{in} \quad (0,+\infty) \times \mathbb{R}^N, \tag{1.4}$$

where $m > 2N/(N+1)$, $m \neq 2$ [17, 6, 8, 7]. Let us mention at this point that the uniqueness results obtained in the above mentioned papers are either restricted to self-similar very singular solutions or use the finite speed of propagation of the support of solutions to (1.3) and (1.4) when $m > 1$ and $m > 2$, respectively.

¿From another viewpoint, let us notice that a very singular solution u formally satisfies $u(0,x) = 0$ if $x \in \mathbb{R}^N \setminus \{0\}$ and $u(0,0) = +\infty$ and thus belongs to the class of solutions having initial data taking infinite values on some subset of $\mathbb{R}^N$. Existence and uniqueness of such solutions have been investigated in [13] for (1.2) and in [5] for (1.1) on a bounded open subset Ω of $\mathbb{R}^N$ with homogeneous Dirichlet boundary conditions. In the latter work [5] the initial data are required to take infinite values on a bounded subset of Ω with non-empty interior and thus do not include very singular solutions.

We now state our main result : we first make more precise the definition of a very singular solution to (1.1) we will use in this paper.

Definition 1.1 *A very singular solution to (1.1) is a function $u \in \mathcal{C}((0,+\infty); L^1(\mathbb{R}^N))$ satisfying for each $t \in (0,+\infty)$ and $\tau \in (0,t)$:*

$$u(t) \geq 0 \quad a.e. \ in \quad \mathbb{R}^N \quad and \quad u \in L^p((\tau,t); W^{1,p}(\mathbb{R}^N)), \tag{1.5}$$

$$u(t) = G(t-\tau)u(\tau) - \int_\tau^t G(t-\sigma) \ \left(|\nabla u(\sigma)|^p\right) \ d\sigma, \tag{1.6}$$

$$\sup_{s \in (t,+\infty)} (s-t)^{N/2} \ \|u(s)\|_{L^\infty} < \infty, \tag{1.7}$$

$$\sup_{s \in (t,+\infty)} (s-t)^{(p(N+1)-N)/2p} \ \left\|\nabla u^{(p-1)/p}(s)\right\|_{L^\infty} < \infty, \tag{1.8}$$

$$\lim_{s\to 0} \int_{\{|x|\le r\}} u(s,x)\ dx = +\infty, \quad r \in (0,+\infty), \tag{1.9}$$

$$\lim_{s\to 0} \int_{\{|x|\ge r\}} u(s,x)\ dx = 0, \quad r \in (0,+\infty). \tag{1.10}$$

Here, $G(t)$ denotes the linear heat semigroup in $\mathbb{R}^N$.

Our result then reads as follows.

Theorem 1.2 *Assume that $1 < p < (N+2)/(N+1)$ and put $a = (2-p)/(p-1)$. There is one and only one very singular solution U to (1.1) in the sense of Definition 1.1. More precisely, there is a nonnegative and non-increasing function*

$$f \in L^1((0,+\infty); r^{N-1}dr) \cap \mathcal{C}^\infty((0,+\infty))$$

such that

$$U(t,x) = t^{-a/2}\ f\left(|x|t^{-1/2}\right), \quad (t,x) \in (0,+\infty)\times\mathbb{R}^N, \tag{1.11}$$

and f is a solution to the ordinary differential equation

$$f''(r) + \left(\frac{N-1}{r} + \frac{r}{2}\right)\ f'(r) + \frac{a}{2}\ f(r) - \left|f'(r)\right|^p = 0, \quad r \in (0,+\infty), \tag{1.12}$$

with the boundary conditions

$$f'(0) = 0 \quad \text{and} \quad \lim_{r\to+\infty} r^a\ f(r) = 0. \tag{1.13}$$

As already mentioned the existence of a very singular solution to (1.1) which has the self-similar form (1.11) and with a profile f satisfying (1.12)-(1.13) has been proved in [2, 18]. The main achievement of the present paper is the uniqueness part of Theorem 1.2 which we prove in the following way : we first proceed as in the proof of [3, Theorem 2] to show that any very singular solution to (1.1) takes on the initial value zero uniformly on compact subsets of $\mathbb{R}^N \setminus \{0\}$. At this point a suitable modification of the proof of [3, Theorem 2] is needed to handle the gradient term. This result then enables us to derive some estimates which are valid for every very singular solution to (1.1) and to prove that the very singular solution to (1.1) we constructed in [2] (denoted by U throughout the paper) is the minimal very singular solution to (1.1). The next section is devoted to the existence of a maximal very singular solution V to (1.1), following the approach of [8]. Both minimal and maximal very singular solutions being self-similar with profiles satisfying (1.12)-(1.13), the conclusion $U = V$ readily follows from [18, Theorem 2.1].

Let us finally mention that the very singular solutions to (1.2) (when they exist) play an important role in the description of the large time behaviour of the solutions to (1.2) (see, e.g., the survey paper [19] and the references therein). A similar result is expected for (1.1) and the above uniqueness result thus opens the path towards the study of the large time behaviour of the solutions to (1.1) when $1 < p < (N+2)/(N+1)$.

2 Preliminaries

We first recall the well-posedness of (1.1) in the space of nonnegative and bounded measures $\mathcal{M}_b^+(\mathbb{R}^N)$ [1, Theorems 1 & 3].

Theorem 2.1 *Consider $p \in (1,(N+2)/(N+1))$ and $u_0 \in \mathcal{M}_b^+(\mathbb{R}^N)$. There is a unique nonnegative function*

$$u \in \mathcal{C}((0,T);L^1(\mathbb{R}^N)) \cap L^p((0,T);W^{1,p}(\mathbb{R}^N)), \quad T \in (0,+\infty),$$

satisfying

$$u(t) = G(t-s)u(s) - \int_s^t G(t-\sigma)\left(|\nabla u(\sigma)|^p\right)\ d\sigma, \quad 0 < s \le t,$$

$$\lim_{t\to 0} \int u(t,x)\ \psi(x)\ dx = \int \psi(x)\ du_0(x), \quad \psi \in \mathcal{BC}(\mathbb{R}^N),$$

and

$$\left\{ \begin{array}{l} \sup\limits_{t\in(0,+\infty)} \ t^{N/2}\ \|u(t)\|_{L^\infty} \le C_H, \\ \sup\limits_{t\in(0,+\infty)} \ t^{(p(N+1)-N)/2p}\ \left\|\nabla u^{(p-1)/p}(t)\right\|_{L^\infty} \le C_H. \end{array} \right. \tag{2.1}$$

Here $\mathcal{BC}(\mathbb{R}^N)$ denotes the space of bounded and continuous functions in $\mathbb{R}^N$ and C_H is a positive real number depending only on N, p and $\|u_0\|_{\mathcal{M}_b}$.

In addition, there holds

$$\sup_{t\in(0,+\infty)} \ t^{1/p}\ \left\|\nabla u^{(p-1)/p}(t)\right\|_{L^\infty} \le (p-1)^{1-1/p}\ p^{-1} := C_{HJ}. \tag{2.2}$$

It follows from Theorem 2.1 that, if u is a very singular solution to (1.1) in the sense of Definition 1.1, the conditions (1.6)-(1.8) imply that $s \mapsto u(s+t)$ is the unique solution to (1.1) with initial datum $u(t)$ given by Theorem 2.1 for each $t \in (0,+\infty)$. Therefore $s \mapsto u(s+t)$ satisfies (2.2), i.e.

$$\sup_{s\in(0,+\infty)} \ s^{1/p}\ \left\|\nabla u^{(p-1)/p}(s+t)\right\|_{L^\infty} \le C_{HJ}.$$

As this is valid for every $t \in (0,+\infty)$ we may let $t \to 0$ in the above inequality and obtain the following result.

Lemma 2.2 *Let u be a very singular solution to (1.1) in the sense of Definition 1.1. Then*

$$\sup_{t\in(0,+\infty)} \ t^{1/p}\ \left\|\nabla u^{(p-1)/p}(t)\right\|_{L^\infty} \le C_{HJ}. \tag{2.3}$$

We now recall additional estimates for solutions to (1.1) satisfying a growth condition for large values of x. For $p \in (1,2)$ we put

$$\Gamma_p(r) = \gamma_p\ r^{-a}, \quad r \in (0,+\infty), \tag{2.4}$$

where

$$\gamma_p = (p-1)^{(p-2)/(p-1)}\ (2-p)^{-1}.$$

Recall that $a = (2-p)/(p-1)$. The following result is proved in [2, Lemma 2.2 & Proposition 2.4].

Lemma 2.3 *Let $p \in (1, (N+2)/(N+1))$ and consider a nonnegative function u_0 in $L^1(\mathbb{R}^N)$. We define*

$$R(u_0) := \inf \{R > 0\ ,\ |x|^a\ u_0(x) \le \gamma_p \ \ \text{a.e. in} \ \ \{|x| \ge R\}\} \in [0, +\infty],$$

and denote by u the nonnegative solution to (1.1) with initial datum u_0 given by Theorem 2.1. If $R(u_0) < +\infty$ and $t \in (0, +\infty)$ there holds

$$0 \le u(t,x) \le \Gamma_p\left(|x| - R(u_0)\right), \quad x \in \mathbb{R}^N, \quad |x| > R(u_0).$$

In addition, there is a positive real number C_1 depending only on N and p such that

$$\begin{aligned} \|u(t)\|_{L^1} &\le C_1\ t^{-((N+2)-p(N+1))/2(p-1)}, \\ \|u(t)\|_{L^\infty} &\le C_1\ t^{-a/2}, \\ \|\nabla u(t)\|_{L^\infty} &\le C_1\ t^{-1/2(p-1)}, \end{aligned}$$

for each $t > \tau(u_0)$, where

$$\tau(u_0) = \left(\frac{(N+2) - p(N+1)}{(N+1)p - N}\right)^{1-p}\ R(u_0)^2.$$

3 Some properties of very singular solutions

In this section we investigate the behaviour of the very singular solutions of (1.1) near $t = 0$ and in $(0, +\infty) \times \mathbb{R}^N$.

Proposition 3.1 *Let u be a very singular solution to (1.1) and $r \in (0, +\infty)$. Putting $\Omega_r := \{x \in \mathbb{R}^N\,;\, |x| > r\}$ we have*

$$u \in L^\infty((0,1)\,; L^1(\Omega_r)), \quad \nabla u \in L^p((0,1) \times \Omega_r). \tag{3.1}$$

The proof of this assertion follows step 2 of [3, Theorem 2].

Proposition 3.2 *Let u be a very singular solution to (1.1). Then*

$$u \in \mathcal{C}^{1,2}_{t,x}([0, +\infty) \times (\mathbb{R}^N \setminus \{0\})), \tag{3.2}$$

and for every compact subset K of $\mathbb{R}^N \setminus \{0\}$ there holds

$$\lim_{t \to 0} \|u(t)\|_{\mathcal{C}(K)} = 0. \tag{3.3}$$

Proof. We adapt step 3 of the proof of [3, Theorem 2]. Let Ω be a bounded open subset of $\mathbb{R}^N$ such that $\overline{\Omega} \subset \mathbb{R}^N \setminus \{0\}$. We define a function v on $(-1,1)\times\Omega$ by

$$v(t,x) = \begin{cases} u(t,x) & \text{if} \quad (t,x) \in (0,1)\times\Omega, \\ 0 & \text{if} \quad (t,x) \in (-1,0)\times\Omega. \end{cases}$$

Then

$$\nabla v(t,x) = \begin{cases} \nabla u(t,x) & \text{if} \quad (t,x) \in (0,1)\times\Omega, \\ 0 & \text{if} \quad (t,x) \in (-1,0)\times\Omega, \end{cases}$$

and (3.1) ensures that

$$v \in L^\infty((-1,1)\ ;\ L^1(\Omega)), \quad \nabla v \in L^p((-1,1)\times\Omega). \tag{3.4}$$

Also, v is a nonnegative function in $(-1,1)\times\Omega$.

We may then proceed as in step 3 of [3, Theorem 2] to show that

$$v_t - \Delta v + |\nabla v|^p = 0 \quad \text{in} \quad \mathcal{D}'((-1,1)\times\Omega). \tag{3.5}$$

Since v satisfies (3.4) and is a subsolution to the heat equation, parabolic regularity theory then entails

$$v \in L^\infty_{\text{loc}}((-1,1)\times\Omega).$$

As Ω is an arbitrary open subset of $\mathbb{R}^N$ with $\overline{\Omega} \subset \mathbb{R}^N \setminus \{0\}$ we have shown that

$$v \in L^\infty_{\text{loc}}((-1,1)\times(\mathbb{R}^N \setminus \{0\})). \tag{3.6}$$

At this point we need to extend the argument of [3] to obtain some more regularity on $|\nabla v|^p$.

Since v satisfies (3.5), (3.6) and is a nonnegative subsolution of the heat equation, classical arguments yield

$$\nabla v \in L^2_{\text{loc}}((-1,1)\times(\mathbb{R}^N \setminus \{0\})). \tag{3.7}$$

On the other hand, we recall a well-known regularity result for a solution to the heat equation.

Lemma 3.3 *[10] Let Ω be a bounded open subset of $\mathbb{R}^N$, $T>0$ and $q \in (1,+\infty)$. Consider $w \in L^q((0,T)\ ;\ W^1_q(\Omega))$ and $f \in L^q((0,T)\times\Omega)$ such that*

$$w_t - \Delta w = f \quad in \quad \mathcal{D}'((0,T)\times\Omega).$$

Then, for each bounded open subset $\mathcal{O}$ of Ω such that $\mathcal{O} \subset \overline{\mathcal{O}} \subset \Omega$ and $\varepsilon \in (0,T/2)$, there holds

$$w\,,\, w_t\,,\, \nabla w\,,\, D^2 w \in L^q((\varepsilon, T-\varepsilon)\times\mathcal{O}).$$

In addition

$$\nabla w \in L^\sigma((\varepsilon,T-\varepsilon)\times\mathcal{O}), \quad where \quad \sigma = \begin{cases} +\infty & if \quad q \ge N+2, \\ \dfrac{q(N+2)}{N+2-q} & if \quad 1 < q < N+2. \end{cases}$$

The last statement of Lemma 3.3 is a consequence of the embedding theorem for anisotropic Sobolev spaces [10, Lemma II.3.3].

We continue with the proof of Proposition 3.2. By (3.6) and (3.7) we have

$$\begin{aligned} v &\in L^{2/p}_{\text{loc}}((-1,1)\times(\mathbb{R}^N\setminus\{0\})),\\ \nabla v &\in L^{2/p}_{\text{loc}}((-1,1)\times(\mathbb{R}^N\setminus\{0\})), \end{aligned}$$

(recall that $1<p<(N+2)/(N+1)$). Thus, v satisfies (3.5). We may then apply Lemma 3.3 and obtain

$$\nabla v\in L^{q_1}_{\text{loc}}((-1,1)\times(\mathbb{R}^N\setminus\{0\})),\quad q_1=\frac{2(N+2)}{p(N+2)-2}. \tag{3.8}$$

Consequently, (3.6) and (3.8) yield

$$\begin{aligned} v &\in L^{q_1/p}_{\text{loc}}((-1,1)\times(\mathbb{R}^N\setminus\{0\})),\\ |\nabla v|^p &\in L^{q_1/p}_{\text{loc}}((-1,1)\times(\mathbb{R}^N\setminus\{0\})). \end{aligned}$$

Noticing that $q_1>q_0=2$ (since $p<(N+2)/(N+1)<(N+4)/(N+2)$) we have indeed a better regularity for $|\nabla v|^p$. Applying again Lemma 3.3 we obtain

$$\nabla v\in L^{q_2}_{\text{loc}}((-1,1)\times(\mathbb{R}^N\setminus\{0\})),\quad q_2=\begin{cases} +\infty & \text{if}\quad q_1\geq p(N+2)\\ \dfrac{q_1(N+2)}{p(N+2)-q_1} & \text{if}\quad q_1<p(N+2). \end{cases}$$

We then define, by induction, a sequence (q_k) by $q_0=2$ and

$$q_{k+1}=\begin{cases} +\infty & \text{if}\quad q_k\geq p(N+2),\\ \dfrac{q_k(N+2)}{p(N+2)-q_k} & \text{if}\quad q_k<p(N+2). \end{cases}$$

We claim that

$$q_k\geq 2,\quad \forall k\geq 0. \tag{3.9}$$

Indeed, we have $q_0=2$. Assume that $q_k\geq 2$ for some k. Then either $q_k\geq p(N+2)$ and $q_{k+1}=+\infty$. Or $q_k<p(N+2)$ and

$$q_{k+1}=q_k\frac{(N+2)}{p(N+2)-q_k}\geq q_k\frac{N+2}{p(N+2)-2}.$$

Since $1<p<(N+2)/(N+1)$, we see that

$$\frac{N+2}{p(N+2)-2}>1, \tag{3.10}$$

whence $q_{k+1}>q_k>2$. Thus (3.9) holds true.

It then follows from (3.9) that

$$q_{k+1} \geq q_k \frac{(N+2)}{p(N+2)-2} > q_k.$$

Therefore the sequence $(q_k)_{k\geq 0}$ is increasing and (3.10) ensures that there is an integer $k_0 > 2$ such that $q_k < p(N+2)$ for $k \in \{1, \ldots, k_0\}$ and $q_k = +\infty$ if $k > k_0$. Proceeding by induction we infer from Lemma 3.3 that

$$\nabla v \in L^\infty_{\text{loc}}((-1,1) \times (\mathbb{R}^N \setminus \{0\})).$$

Classical parabolic regularity results then entail

$$v \in \mathcal{C}^{1,2}_{t,x}((-1,1) \times (\mathbb{R}^N \setminus \{0\})),$$

whence (3.2) and (3.3). □

Thanks to Proposition 3.2, we may now use comparison principle arguments to obtain additional information on the short time behaviour of the very singular solutions to (1.1).

Lemma 3.4 *Let u be a very singular solution to (1.1). Then*

$$0 \leq u(t,x) \leq \Gamma_p(|x|), \quad (t,x) \in (0,+\infty) \times \mathbb{R}^N. \tag{3.11}$$

Proof. Take $r \in (0,+\infty)$ and put $\Omega_r := \{x \in \mathbb{R}^N, |x| > r\}$. Then u and $x \mapsto \Gamma_p(|x|-r)$ are solutions to (1.1) on $(0,+\infty) \times \Omega_r$ with

$$u(0,x) = 0 \leq \Gamma_p(|x|-r) \quad \text{if} \quad x \in \Omega_r,$$
$$u(t,x) \leq +\infty = \Gamma_p(0) \quad \text{if} \quad (t,x) \in (0,+\infty) \times \partial\Omega_r.$$

The comparison principle then entails

$$u(t,x) \leq \Gamma_p(|x|-r), \quad (t,x) \in (0,+\infty) \times \Omega_r.$$

Now fix $x_0 \in \mathbb{R}^N \setminus \{0\}$. For $r \in (0,|x_0|)$, we have $x_0 \in \Omega_r$ and

$$u(t,x_0) \leq \Gamma_p(|x_0|-r), \quad t \in (0,+\infty).$$

We then let $r \to 0$ and obtain (3.11) for $x_0 \in \mathbb{R}^N \setminus \{0\}$. As $\Gamma(0) = +\infty$, (3.11) also holds true for $x = 0$. □

Lemma 3.5 *There is a constant K_1 depending only on p and N such that, if u is a very singular solution to (1.1) and $t \in (0,+\infty)$, there holds*

$$\|u(t)\|_{L^1} \leq K_1 t^{-(N+2-p(N+1))/2(p-1)}, \tag{3.12}$$
$$\|u(t)\|_{L^\infty} \leq K_1 t^{-a/2}, \tag{3.13}$$
$$\|\nabla u(t)\|_{L^\infty} \leq K_1 t^{-1/2(p-1)}, \tag{3.14}$$

with $a = (2-p)/(p-1)$.

Proof. Owing to Lemma 2.2 and Lemma 3.4, the proof of (3.12), (3.13) and (3.14) is similar to that of [2, Proposition 2.4] to which we refer. □

The next lemma follows from [5, p. 186].

Lemma 3.6 *For $y \in \mathbb{R}^N$ and $\rho > 0$, we denote by $\alpha_{y,\rho}$ the solution to*

$$\begin{aligned} -\Delta \alpha_{y,\rho} &= 1 \quad \textit{in} \quad B(y,\rho), \\ \alpha_{y,\rho} &= 0 \quad \textit{on} \quad \partial B(y,\rho). \end{aligned}$$

For every $\lambda \in (0,+\infty)$ there is $C_\lambda \in (0,+\infty)$ such that, if u is a very singular solution to (1.1), $y \in \mathbb{R}^N \setminus \{0\}$ and $\rho \in (0,|y|)$, there holds

$$u(t,x) \le \lambda e^{C_\lambda t} \exp\left(\frac{1}{\alpha_{y,\rho}(x)}\right), \quad (t,x) \in (0,+\infty) \times B(y,\rho).$$

The last lemma of this section will allow us to prove that the very singular solution we constructed in [2] is the minimal very singular solution.

Lemma 3.7 *If u is a very singular solution to (1.1) and $M \in (0,+\infty)$ we have*

$$u_M \le u,$$

where u_M denotes the solution to (1.1) with initial datum $M\delta$.

Proof. As u is a very singular solution to (1.1) we have

$$\lim_{t\to 0} \|u(t)\|_{L^1} = +\infty.$$

By a suitable truncation, it is possible to construct a sequence of nonnegative functions $(u_{0,k})_{k \ge k_M}$ such that

$$\begin{aligned} u_{0,k}(x) &\le u(1/k, x), \quad x \in \mathbb{R}^N, && (3.15) \\ \|u_{0,k}\|_{L^1} &= M, && (3.16) \end{aligned}$$

for $k \ge k_M$, where k_M is a sufficiently large integer. We denote by u_k the unique nonnegative solution to (1.1) with initial datum $u_{0,k}$ given by Theorem 2.1. Since $(u_{0,k})$ is bounded in $L^1(\mathbb{R}^N)$, we may use (2.1) to proceed as in the proof of [1, Theorem 3] and show that there are a subsequence of (u_k) (not relabeled) and a function $\overline{u} \in \mathcal{C}((0,+\infty), L^1(\mathbb{R}^N))$ such that, as $k \to +\infty$:

$$u_k \to \overline{u} \quad \text{in} \quad \mathcal{C}((0,+\infty)\,; L^1(\mathbb{R}^N)) \cap L^p((s,t)\,; W^{1,p}(\mathbb{R}^N)), \qquad (3.17)$$

$$\overline{u}(t) = G(t-s)\overline{u}(s) - \int_s^t G(t-\sigma)(|\nabla \overline{u}(\sigma)|^p) d\sigma,$$

for every $s \in (0,+\infty)$ and $t \in (s,+\infty)$.

It also follows from (3.17) that $\overline{u}$ satisfies (2.1). It remains to identify the initial datum taken by $\overline{u}$. Let $\rho \in \mathcal{D}(\mathbb{R}^N)$, $k \geq k_M$ and $t \in (0,1)$. By (1.1), (2.1) and (3.16) we have

$$\begin{aligned}
& \left| \int u_k(t,x)\rho(x)dx - \int u_{0,k}(x)\rho(x)dx \right| \\
\leq\ & \|\Delta\rho\|_{L^\infty} \int_0^t \int u_k(\sigma,x)dxd\sigma + \|\rho\|_{L^\infty} \int_0^t \int |\nabla u_k(\sigma,x)|^p dxd\sigma \\
\leq\ & Mt\|\Delta\rho\|_{L^\infty} + \left(\frac{p}{p-1}\right)^p \|\rho\|_{L^\infty} \int_0^t \int u_k \left|\nabla u_k^{(p-1)/p}\right|^p dxd\sigma \\
\leq\ & Mt\|\Delta\rho\|_{L^\infty} + C(\rho,p,M,N) \int_0^t \sigma^{(N-p(N+1))/2} d\sigma \\
\leq\ & C(\rho,p,M,N) \left(t + t^{(N+2-p(N+1))/2}\right).
\end{aligned} \tag{3.18}$$

For $r \in (0,+\infty)$, we also have by (3.15)

$$\begin{aligned}
& \left| \int u_{0,k}(x)\rho(x)dx - M\rho(0) \right| \\
\leq\ & 2\|\rho\|_{L^\infty} \int_{\{|x|\geq r\}} u\left(\frac{1}{k},x\right) dx + \left(\int_{\{|x|\leq r\}} u_{0,k}(x)dx \right) \sup_{\{|x|\leq r\}} |\rho(x)-\rho(0)| \\
\leq\ & 2\|\rho\|_{L^\infty} \int_{\{|x|\geq r\}} u\left(\frac{1}{k},x\right) dx + M \sup_{\{|x|\leq r\}} |\rho(x)-\rho(0)|.
\end{aligned}$$

We let $k \to +\infty$ and use Definition 1.1 to obtain that

$$\limsup_{k\to+\infty} \left| \int u_{0,k}(x)\rho(x)dx - M\rho(0) \right| \leq M \sup_{\{|x|\leq r\}} |\rho(x)-\rho(0)|.$$

Passing to the limit as $r \to 0$ then yields

$$\lim_{k\to+\infty} \int u_{0,k}(x)\rho(x)dx = M\rho(0). \tag{3.19}$$

Thanks to (3.17) and (3.19) we may let $k \to +\infty$ in (3.18) and obtain

$$\left| \int \overline{u}(t,x)\rho(x)dx - M\rho(0) \right| \leq C(\rho,p,M,N) \left(t + t^{(N+2-p(N+1))/2}\right).$$

Consequently, for each $\rho \in \mathcal{D}(\mathbb{R}^N)$,

$$\lim_{t\to 0} \int \overline{u}(t,x)\rho(x)dx = M\rho(0)\,. \tag{3.20}$$

As $\overline{u}$ is a subsolution to the heat equation, a comparison argument yields that (3.20) actually holds for every $\rho \in \mathcal{BC}(\mathbb{R}^N)$.

Summarizing, we have proved that $\overline{u}$ is a solution to (1.1) with initial datum $M\delta$ fulfilling all the requirements of Theorem 2.1. Such a solution being unique, we conclude that

$$\overline{u} = u_M. \tag{3.21}$$

To complete the proof we only have to notice that the comparison principle and (3.15) entail that for $k \geq k_M$,

$$u_k(t,x) \leq u(t+1/k,x), \quad (t,x) \in (0,+\infty) \times \mathbb{R}^N.$$

We then use (3.17) and (3.21) to obtain Lemma 3.7. $\square$

We end up this section by recalling the main result of [2].

Theorem 3.8 *The function*

$$U(t,x) = \sup_{\{M>0\}} u_M(t,x), \quad (t,x) \in (0,+\infty) \times \mathbb{R}^N,$$

is a very singular solution to (1.1), where u_M denotes the solution to (1.1) with initial datum $M\delta$. Moreover, there is a nonnegative and non-increasing function

$$f \in L^1((0,+\infty), r^{N-1}dr) \cap \mathcal{C}^\infty((0,+\infty))$$

satisfying (1.12) and (1.13) and such that

$$U(t,x) = t^{-a/2} f\left(|x| t^{-1/2}\right), \quad (t,x) \in (0,+\infty) \times \mathbb{R}^N.$$

As a consequence of Lemma 3.7 and Theorem 3.8, we see that, if u is a very singular solution to (1.1), there holds

$$u \geq U. \tag{3.22}$$

Then U is the minimal very singular solution to (1.1), and it is the unique self-similar very singular solution to (1.1) by [18, Theorem 2.1].

4 Existence of a maximal very singular solution

We denote by $\mathcal{S}$ the set of very singular solutions to (1.1) in the sense of Definition 1.1. Notice that, as the minimal very singular solution U (defined in Theorem 3.8) belongs to $\mathcal{S}$, the set $\mathcal{S}$ is non-empty.

We now proceed as in [8, Theorem 4.1] to prove that $\mathcal{S}$ has a maximal element. More precisely we put

$$V(t,x) = \sup_{u \in \mathcal{S}} u(t,x), \quad (t,x) \in (0,+\infty) \times \mathbb{R}^N. \tag{4.1}$$

Lemma 4.1 *For each $t \in (0,+\infty)$, $V(t)$ is a nonnegative function in $L^1(\mathbb{R}^N) \cap W^{1,\infty}(\mathbb{R}^N)$ which satisfies*

$$t^{a/2} \|V(t)\|_{L^\infty} + t^{1/2(p-1)} \|\nabla V(t)\|_{L^\infty} \leq 2K_1, \tag{4.2}$$

$$t^{1/p} \left\| \nabla V^{(p-1)/p}(t) \right\|_{L^\infty} \leq C_{HJ}, \tag{4.3}$$

$$U(t,x) \leq V(t,x) \leq \Gamma_p(|x|), \quad x \in \mathbb{R}^N, \tag{4.4}$$

where C_{HJ}, Γ_p and K_1 are defined in (2.2), (2.4) and Lemma 3.5, respectively.

Proof. Since $U \in \mathcal{S}$, (4.4) is a straightforward consequence of (3.11) and (4.1). Next, (3.13) and (4.4) entail that

$$0 \le V(t,x) \le \min\left(K_1 t^{-a/2}, \Gamma_p(x)\right).$$

Consequently, for each $t \in (0,+\infty)$, $V(t) \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ and

$$t^{a/2} \|V(t)\|_{L^\infty} \le K_1, \quad t \in (0,+\infty).$$

It next follows from (3.14) that if $u \in \mathcal{S}$ and $(x,y) \in \mathbb{R}^N \times \mathbb{R}^N$

$$\begin{aligned} u(t,x) &\le u(t,y) + K_1 t^{-1/2(p-1)} |x-y| \\ &\le V(t,y) + K_1 t^{-1/2(p-1)} |x-y|. \end{aligned}$$

Therefore,

$$V(t,x) \le V(t,y) + K_1 t^{-1/2(p-1)} |x-y|,$$

and $V(t)$ is Lipschitz continuous with Lipschitz constant $K_1 t^{-1/2(p-1)}$. We have thus shown (4.2). Since $\sigma \mapsto \sigma^{(p-1)/p}$ is non-decreasing on $(0,+\infty)$, a similar argument yields (4.3). □

Lemma 4.2 $V \in \mathcal{S}$.

Proof. Fix $\tau \in (0,+\infty)$. As $V(\tau)$ is a nonnegative function in $L^1(\mathbb{R}^N) \cap W^{1,\infty}(\mathbb{R}^N)$, we denote by v^τ the unique nonnegative solution to (1.1) in $(\tau,+\infty) \times \mathbb{R}^N$ with $v^\tau(\tau) = V(\tau)$ given by Theorem 2.1. For $u \in \mathcal{S}$ it follows from (4.1) that

$$v^\tau(\tau) = V(\tau) \ge u(\tau),$$

and the comparison principle entails

$$v^\tau(t) \ge u(t), \quad t \in [\tau,+\infty).$$

Consequently

$$v^\tau(t) \ge V(t), \quad t \in [\tau,+\infty). \tag{4.5}$$

Next, on the one hand, it follows from (4.4) and the comparison principle that

$$v^\tau(t,x) \le \Gamma_p(|x|), \quad (t,x) \in [\tau,+\infty) \times \mathbb{R}^N. \tag{4.6}$$

On the other hand, by (2.2) we have

$$\sup_{t \in (\tau,+\infty)} (t-\tau)^{1/p} \left\| \nabla (v^\tau)^{(p-1)/p}(t) \right\|_{L^\infty} \le C_{HJ}. \tag{4.7}$$

We may therefore proceed as in [2, Proposition 2.4] to show that there is a positive constant K_2 depending only on p and N such that for each $\tau \in (0,+\infty)$ and $t \in (\tau,+\infty)$ there holds

$$\|v^\tau(t)\|_{L^1} \le K_2 (t-\tau)^{-(N+2-p(N+1))/2(p-1)}, \tag{4.8}$$

$$\|v^\tau(t)\|_{L^\infty} \le K_2 (t-\tau)^{-a/2}, \tag{4.9}$$

$$\|\nabla v^\tau(t)\|_{L^\infty} \le K_2 (t-\tau)^{-1/2(p-1)}. \tag{4.10}$$

We next claim that

$$\tau < \sigma \Rightarrow v^\sigma(t,x) \le v^\tau(t,x), \quad (t,x) \in [\sigma,+\infty) \times \mathbb{R}^N. \tag{4.11}$$

Indeed, by (4.5) we have

$$v^\sigma(\sigma) = V(\sigma) \le v^\tau(\sigma), \quad \sigma \ge \tau,$$

and (4.11) follows from the comparison principle.
We now define

$$W(t,x) = \sup_{\tau\in(0,t/2)} v^\tau(t,x), \quad (t,x) \in (0,+\infty) \times \mathbb{R}^N. \tag{4.12}$$

Thanks to (4.8), (4.11) and the monotone convergence theorem, we realize that, for each $t \in (0,+\infty)$,

$$W(t) \in L^1(\mathbb{R}^N) \quad \text{and} \quad \lim_{\tau\to 0} \|W(t) - v^\tau(t)\|_{L^1} = 0. \tag{4.13}$$

Next, owing to (4.8)-(4.10), we may proceed as in [1, Section 3] to show that, for each $t \in (0,+\infty)$ and $s \in (0,t)$, we have

$$W \in \mathcal{C}((s,t)\,;L^1(\mathbb{R}^N)) \cap L^p((s,t)\,;W^{1,p}(\mathbb{R}^N)),$$

and W satisfies

$$W(t) = G(t-s)W(s) - \int_s^t G(t-\sigma)(|\nabla W(\sigma)|^p)d\sigma. \tag{4.14}$$

Also, if $t_0 \in (0,+\infty)$, it follows from (4.8) that

$$Y(t_0) := \sup_{\tau\in(0,t_0/2)} \|v^\tau(t_0)\|_{L^1} < \infty.$$

We then infer from Theorem 2.1 that

$$(t-t_0)^{N/2}\|v^\tau(t)\|_{L^\infty} + (t-t_0)^{(p(N+1)-N)/2p}\left\|\nabla(v^\tau)^{(p-1)/p}(t)\right\|_{L^\infty} \le K_3(t_0),$$

for $t \in (t_0,+\infty)$ and $\tau \in (0,t_0/2)$, where $K_3(t_0)$ only depends on N, p and $Y(t_0)$. Owing to (4.13), we may let $\tau \to 0$ in the above estimate and see that

$$(t-t_0)^{N/2}\|W(t)\|_{L^\infty} + (t-t_0)^{(p(N+1)-N)/2p}\left\|\nabla W^{(p-1)/p}(t)\right\|_{L^\infty} \le K_3(t_0), \tag{4.15}$$

for $t \in (t_0,+\infty)$. We next infer from (4.4), (4.5), (4.6) and (4.12) that

$$U(t,x) \le V(t,x) \le W(t,x) \le \Gamma_p(|x|), \quad (t,x) \in (0,+\infty) \times \mathbb{R}^N. \tag{4.16}$$

Since U is a very singular solution, we obtain

$$\lim_{t\to 0} \int_{\{|x|\le r\}} W(t,x)dx = +\infty, \quad r \in (0,+\infty). \tag{4.17}$$

Finally, take $r \in (0,+\infty)$ and consider $\rho \in \mathcal{C}^\infty(\mathbb{R}^N)$ such that

$$0 \le \rho \le 1, \quad \rho(x) = 1 \quad \text{if} \quad |x| \ge r \quad \text{and} \quad \rho(x) = 0 \quad \text{if} \quad |x| \le \frac{r}{2}.$$

Since v^τ is a solution to (1.1), we have for $t \in (0,+\infty)$ and $\tau \in (0,t)$

$$\begin{aligned} &\int \rho(x) v^\tau(t,x)dx + \int_\tau^t \int \rho(x)|\nabla v^\tau(\sigma,x)|^p dx d\sigma \\ = &\int \rho(x) v^\tau(\tau,x)dx + \int_\tau^t \int \Delta\rho(x) v^\tau(\sigma,x) d\sigma dx. \end{aligned}$$

Recalling that $v^\tau(\tau,x) = V(\tau,x)$, we find

$$\int \rho(x) v^\tau(t,x)dx \le \int \rho(x) V(\tau,x)dx + \int_\tau^t \int_{\{r/2<|x|<r\}} |\Delta\rho| v^\tau(\sigma,x) d\sigma dx. \tag{4.18}$$

On the one hand, it follows from (4.4) that

$$\begin{aligned} \int_\tau^t \int_{\{r/2<|x|<r\}} |\Delta\rho| v^\tau(\sigma,x) d\sigma dx &\le \|\Delta\rho\|_{L^\infty} \int_\tau^t \int_{\{r/2<|x|<r\}} \Gamma_p(|x|)dx \\ &\le C(p,r,\rho)t. \end{aligned} \tag{4.19}$$

On the other hand, we have by (4.4) that

$$\int \rho(x) V(\tau,x)dx \le \int_{\{r/2<|x|<r\}} V(\tau,x)dx + \int_{\{|x|>R\}} \Gamma_p(|x|)dx, \tag{4.20}$$

for every $R \in [2r,+\infty)$. Since

$$K(r,R) := \left\{ x \in \mathbb{R}^N, r/2 \le |x| \le R \right\}$$

is compact, there is a finite number of points $(y_i)_{1\le i\le k}$ in $K(r,R)$ such that

$$K(r,R) \subset \bigcup_{i=1}^k B\left(y_i, r/8\right).$$

Let $i \ge 1$. Notice that, as $|y_i| \ge r/2$, we have $|y_i| > r/4$. We then infer from Lemma 3.6 that, if u belong to $\mathcal{S}$, there holds

$$u(t,x) \le \lambda e^{C_\lambda t} \exp\left(\frac{1}{\alpha_{y_i,r/4}(x)}\right), \quad (t,x) \in (0,+\infty) \times B\left(y_i, r/4\right)$$

for every $\lambda \in (0,+\infty)$. The above pointwise estimate being true for every u in $\mathcal{S}$, we deduce that

$$V(t,x) \le \lambda e^{C_\lambda t} \exp\left(\frac{1}{\alpha_{y_i,r/4}(x)}\right), \quad (t,x) \in (0,+\infty) \times B\left(y_i, r/4\right).$$

Therefore,

$$\int_{K(r,R)} V(\tau,x)dx \le \lambda e^{C_\lambda \tau} \sum_{i=1}^k \int_{B(y_i,r/8)} \exp\left(\frac{1}{\alpha_{y_i,r/4}(x)}\right) dx,$$

and

$$0 \le \limsup_{\tau\to 0} \int_{K(r,R)} V(\tau,x)dx \le \lambda \sum_{i=1}^{k} \int_{B(y_i,r/8)} \exp\left(\frac{1}{\alpha_{y_i,r/4}(x)}\right) dx.$$

This inequality being valid for every $\lambda \in (0,+\infty)$, we finally obtain

$$\lim_{\tau\to 0} \int_{K(r,R)} V(\tau,x)dx = 0. \tag{4.21}$$

We may then let $\tau \to 0$ in (4.18) and use (4.13), (4.19), (4.20) and (4.21) to obtain

$$\int \rho(x)W(t,x)dx \le C(p,r,\rho)t + \int_{\{|x|>R\}} \Gamma_p(|x|)dx$$

for every $t \in (0,+\infty)$ and $R \ge 2r$. As $\Gamma_p \in L^1(\mathbb{R}^N \setminus B(0,1))$, we may let $R \to +\infty$ in the above estimate to deduce that

$$0 \le \int_{\{|x|\ge r\}} W(t,x)dx \le C(p,r,\rho)t.$$

Therefore,

$$\lim_{t\to 0} \int_{\{|x|\ge r\}} W(t,x)dx = 0. \tag{4.22}$$

It then follows from (4.14), (4.15), (4.17) and (4.22) that W is a very singular solution to (1.1) in the sense of Definition 1.1. Consequently, $W \in \mathcal{S}$, whence $W \le V$ by (4.1). Recalling (4.16), we realize that $W = V$ and the proof of Lemma 4.2 is complete. □

Lemma 4.3 *There is a nonnegative and non-increasing function*

$$g \in L^1((0,+\infty), r^{N-1}dr) \cap \mathcal{C}^\infty((0,+\infty)),$$

satisfying (1.12), (1.13) and such that

$$V(t,x) = t^{-a/2} g\left(|x|t^{-1/2}\right), \quad (t,x) \in (0,+\infty)\times\mathbb{R}^N.$$

Proof. Consider $u \in \mathcal{S}$ and $\lambda \in (0,+\infty)$. It is straightforward to check that the function $\overline{u}$ defined by

$$\overline{u}(t,x) = \lambda^a u(\lambda^2 t, \lambda x), \quad (t,x) \in (0,+\infty)\times\mathbb{R}^N,$$

also belongs to $\mathcal{S}$. The set $\mathcal{S}$ being invariant with respect to the above scaling transformation, we easily deduce that

$$V(t,x) = \lambda^{-a} V\left(\frac{t}{\lambda^2}, \frac{x}{\lambda}\right), \quad (t,x) \in (0,+\infty)\times\mathbb{R}^N, \quad \lambda > 0. \tag{4.23}$$

On the other hand, the equation (1.1) being rotationally invariant, it is clear that $\mathcal{S}$ is rotationally invariant and $V(t,.)$ is therefore radially symmetric with respect to the space variable for every $t \in (0,+\infty)$. Putting

$$g(r) = V(1,r,0,\ldots,0), \quad r \in (0,+\infty),$$

and using (4.23) yield

$$V(t,x) = t^{-a/2} g\left(|x| t^{-1/2}\right), \quad (t,x) \in (0,+\infty) \times \mathbb{R}^N.$$

We then proceed as in [2, Section 3] to check that g has the required properties. □

Proof of Theorem 1.2. Since f (cf. Theorem 3.8) and g (cf. Lemma 4.3) are solutions to (1.12), (1.13), it follows from [18, Theorem 2.1] that $f = g$, whence $U = V$. Now, if $\vartheta \in \mathcal{S}$, it follows from (3.22) and (4.1) that $U \le \vartheta \le V = U$, whence $\vartheta = U$. Therefore, $\mathcal{S} = \{U\}$ and Theorem 1.2 is proved. □

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