Asymptotic profiles of solutions to viscous Hamilton-Jacobi equations

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Abstract

The large time behavior of solutions to the Cauchy problem for the viscous Hamilton-Jacobi equation $u_t - \Delta u + |\nabla u|^q = 0$ is classified. If $q > q_c := (N+2)/(N+1)$, it is shown that non-negative solutions corresponding to integrable initial data converge in $W^{1,p}(\mathbb{R}^N)$ as $t \to \infty$ toward a multiple of the fundamental solution for the heat equation for every $p \in [1, \infty]$ (diffusion-dominated case). On the other hand, if $1 < q < q_c$, the large time asymptotics is given by the very singular self-similar solutions of the viscous Hamilton-Jacobi equation.

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For non-positive and integrable solutions, the large time behavior of solutions is more complex. The case $q \geq 2$ corresponds to the diffusion-dominated case. The diffusion profiles in the large time asymptotics appear also for $q_c < q < 2$ provided suitable smallness assumptions are imposed on the initial data. Here, however, the most important result asserts that under some conditions on initial conditions and for $1 < q < 2$, the large time behavior of solutions is given by the self-similar viscosity solutions to the non-viscous Hamilton-Jacobi equation $z_t + |\nabla z|^q = 0$ supplemented with the initial datum $z(x,0) = 0$ if $x \neq 0$ and $z(0,0) < 0$.

Résumé

Nous classifions le comportement asymptotique des solutions du problème de Cauchy pour l'équation de Hamilton-Jacobi avec diffusion $u_t - \Delta u + |\nabla u|^q = 0$. Si $q > q_c := (N+2)/(N+1)$, nous montrons que, lorsque $t \to \infty$, les solutions intégrables et positives convergent dans $W^{1,p}(\mathbb{R}^N)$ vers un multiple de la solution fondamentale de l'équation de la chaleur pour tout $p \in [1,\infty]$ (diffusion dominante). Ensuite, si $1 < q < q_c$, le comportement asymptotique est décrit par la solution très singulière auto-similaire de l'équation de Hamilton-Jacobi avec diffusion.

En ce qui concerne les solutions intégrables et négatives, la situation est plus complexe. Le terme de diffusion est de nouveau dominant si $q \geq 2$, ainsi que lorsque $q_c < q < 2$ pourvu que la donnée initiale soit suffisamment petite. Ensuite, pour $1 < q < 2$, nous identifions une classe de données initiales pour laquelle le comportement asymptotique des solutions est donné par une solution de viscosité auto-similaire de l'équation de Hamilton-Jacobi $z_t + |\nabla z|^q = 0$ avec la condition initiale (non continue) $z(x,0) = 0$ si $x \neq 0$ et $z(0,0) < 0$.

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Mots-clés : Equation de Hamilton-Jacobi diffusive, comportement asymptotique auto-similaire, estimations unilatérales du Laplacien.

1 Introduction

We investigate the large time behavior of integrable solutions to the Cauchy problem for the viscous Hamilton-Jacobi equation

$$u_t - \Delta u + |\nabla u|^q = 0\,, \quad x \in \mathbb{R}^N\,, \quad t > 0\,, \tag{1.1}$$

$$u(x,0) = u_0(x)\,, \quad x \in \mathbb{R}^N\,, \tag{1.2}$$

where $q > 1$. The dynamics of the solutions to (1.1)-(1.2) is governed by two competing effects, namely those resulting from the diffusive term $-\Delta u$ and those corresponding to the "hyperbolic" nonlinearity $|\nabla u|^q$. Our aim here is to figure out whether one of these two effects rules the large time behavior, according to the values of q and the initial data u_0. Since the nonlinear term $|\nabla u|^q$ is non-negative, it acts as an absorption term for non-negative solutions and as a source term for non-positive solutions. We thus consider separately non-negative and non-positive solutions. Let us outline our main results now.

For non-negative initial data, it is already known that diffusion dominates the large time behavior for $q > q_c := (N+2)/(N+1)$ and that the nonlinear term only becomes effective for $q < q_c$ [1, 4, 6, 8]. We obtain more precise information in Theorems 2.1 and 2.2 below. In particular, if $q \in (1, q_c)$ and the initial datum decays sufficiently rapidly at infinity, there is a balance between the diffusive and hyperbolic effects: the solution $u(t)$ behaves for large t like the very singular solution to (1.1), the existence and uniqueness of which have been established in [5, 3, 23].

For non-positive initial data, there are two critical exponents $q = q_c$ and $q = 2$, as already noticed in [21], and the picture is more complicated. More precisely, the diffusion governs the large time dynamics for any initial data if $q \geq 2$ and for sufficiently small initial data if $q \in (q_c, 2)$, and we extend the result from [21, Proposition 2.2] in that case (cf. Theorem 2.3, below). On the other hand, when $q \in (1, 2)$, we prove that, for sufficiently large initial data, the large time behavior is governed by the nonlinear reaction term. This fact is also true for any initial datum $u_0 \not\equiv 0$ if $N \leq 3$ and q is sufficiently close to 1. We actually conjecture that the nonlinear reaction term always dominates in the large time for any non-zero initial datum as soon as $q \in (1, q_c)$.

Let us finally mention that, when $q \in (q_c, 2)$, there is at least one (self-similar) solution for which there is a balance between the diffusive and hyperbolic effects for large times [7].

Before stating more precisely our results, let us recall that for every initial datum $u_0 \in W^{1,\infty}(\mathbb{R}^N)$ the Cauchy problem (1.1)-(1.2) has a unique global-in-time solution which is classical for positive times, that is

$$u \in \mathcal{C}(\mathbb{R}^N \times [0, \infty)) \cap \mathcal{C}^{2,1}(\mathbb{R}^N \times (0, \infty))\,.$$

In addition, this solution satisfies the estimates

$$\|u(t)\|_\infty \leq \|u_0\|_\infty \quad \text{and} \quad \|\nabla u(t)\|_\infty \leq \|\nabla u_0\|_\infty \quad \text{for all } t > 0. \tag{1.3}$$

Moreover, by the maximum principle, $u_0 \geq 0$ implies that $u \geq 0$ and $u_0 \leq 0$ ensures that $u \leq 0$. We refer the reader to [1, 4, 17] for the proofs of all these preliminary results. In addition, a detailed analysis of the well-posedness of (1.1)-(1.2) in the Lebesgue spaces $L^p(\mathbb{R}^N)$ may be found in the recent paper [7].

Notations. The notation to be used is mostly standard. For $1 \leq p \leq \infty$, the L^p-norm of a Lebesgue measurable real-valued function v defined on $\mathbb{R}^N$ is denoted by $\|v\|_p$. We will always denote by $\|\cdot\|_{\mathcal{X}}$ the norm of any other Banach space $\mathcal{X}$ used in this paper. Also, $W^{1,\infty}(\mathbb{R}^N)$ denotes the Sobolev space consisting of functions in $L^\infty(\mathbb{R}^N)$

whose first order generalized derivatives belong to $L^\infty(\mathbb{R}^N)$. The space of compactly supported and $\mathcal{C}^\infty$-smooth functions in $\mathbb{R}^N$ is denoted by $\mathcal{C}_c^\infty(\mathbb{R}^N)$, and $\mathcal{C}_0(\mathbb{R}^N)$ is the set of continuous functions u such that

$$\lim_{R\to\infty} \sup_{|x|\geq R} \{|u(x)|\} = 0\,.$$

For a real number r, we denote by $r^+ := \max\{r,0\}$ its positive part and by $r^- := \max\{-r,0\}$ its negative part. The letter C will denote generic positive constants, which do not depend on t and may vary from line to line during computations. Throughout the paper, we use the critical exponent

$$q_c := \frac{N+2}{N+1}\,.$$

2 Results and comments

As already outlined, the large time behavior of solutions to (1.1)-(1.2) is determined not only by the exponent q of the nonlinear term $|\nabla u|^q$ but also by the sign, size, and shape of the initial conditions. In the present paper, we attempt to describe this variety of different asymptotics of solutions, imposing particular assumptions on initial data. In order to present our results in the most transparent form, we divide this section into subsections.

2.1 Non-negative initial conditions

In Theorems 2.1 and 2.2 below, we always assume that

$$u_0 \text{ is a non-negative function in } L^1(\mathbb{R}^N)\cap W^{1,\infty}(\mathbb{R}^N)\,, \quad u_0 \not\equiv 0\,, \tag{2.1}$$

and we denote by $u = u(x,t)$ the corresponding non-negative solution of the Cauchy problem (1.1)-(1.2). In that case, we recall that $t \longmapsto \|u(t)\|_1$ is a non-increasing function and that $|\nabla u|$ belongs to $L^q(\mathbb{R}^N\times(0,\infty))$. In addition,

$$I_\infty := \lim_{t\to\infty}\int_{\mathbb{R}^N} u(x,t)\,dx = \int_{\mathbb{R}^N} u_0(x)\,dx - \int_0^\infty\int_{\mathbb{R}^N} |\nabla u(x,s)|^q\,dx\,ds \tag{2.2}$$

satisfies $I_\infty > 0$ if $q > q_c$ and $I_\infty = 0$ if $q \leq q_c$ (cf. [1, 4, 6], for details). Since we would have $I_\infty = \|u_0\|_1 > 0$ for the linear heat equation, we thus say that diffusion dominates the large time behavior when $I_\infty > 0$, that is, when $q > q_c$.

We first consider the diffusion-dominated case.

Theorem 2.1 *Suppose (2.1) and that $q > q_c$. For every $p\in[1,\infty]$,*

$$\lim_{t\to\infty} t^{(N/2)(1-1/p)}\|u(t) - I_\infty G(t)\|_p = 0 \tag{2.3}$$

and

$$\lim_{t\to\infty} t^{(N/2)(1-1/p)+1/2}\|\nabla u(t) - I_\infty \nabla G(t)\|_p = 0. \tag{2.4}$$

Here, $G(x,t) = (4\pi t)^{-N/2}\exp(-|x|^2/(4t))$ *is the fundamental solution of the heat equation.*

When $p = 1$, the relation (2.3) is proved in [8] and Theorem 2.1 extends the convergence of $u(t)$ towards a multiple of $G(t)$ to $W^{1,p}(\mathbb{R}^N)$, $p \in [1,\infty]$.

Remark 2.1 Theorem 2.1 holds true when $I_\infty = 0$ (i.e. for $q \le q_c$) as well, but in that case, the relation (2.3) says only that $\|u(t)\|_p$ tends to 0 as $t \to \infty$ faster than $t^{-(N/2)(1-1/p)}$.

Our next theorem is devoted to the balance case $1 < q < q_c$ when a particular self-similar solution of (1.1) appears in the large time asymptotics.

Theorem 2.2 *Suppose (2.1). Assume that* $q \in (1, q_c)$ *and, moreover, that*

$$\operatorname*{ess\,lim}_{|x|\to\infty} |x|^a u_0(x) = 0 \quad \textit{with} \quad a = \frac{2-q}{q-1}. \tag{2.5}$$

For every $p \in [1,\infty]$,

$$\lim_{t\to\infty} t^{(N/2)(1-1/p)+(a-N)/2}\|u(t) - W(t)\|_p = 0 \tag{2.6}$$

and

$$\lim_{t\to\infty} t^{(N/2)(1-1/p)+(a-N)/2+1/2}\|\nabla u(t) - \nabla W(t)\|_p = 0, \tag{2.7}$$

where $W(x,t) = t^{-a/2}W(xt^{-1/2},1)$ *is the very singular self-similar solution to (1.1).*

For the existence and uniqueness of the very singular solution to (1.1), we refer the reader to [5, 3, 23]. Notice also that the initial datum u_0 is integrable by assumption (2.5) since $a > N$ for $1 < q < q_c$.

Remark 2.2 In the critical case $q = q_c$, it is also expected that $u(t)$ converges towards a multiple of $G(t)$ with a correction in the form of an extra logarithmic factor resulting from the absorption term. This conjecture is supported by what is already known for non-negative solutions to the Cauchy problem $w_t - \Delta w + w^{(N+2)/N} = 0$ (see, e.g., [25] and the references therein).

2.2 Non-positive initial conditions

We now turn to non-positive solutions and assume that

$$u_0 \ \text{ is a non-positive function in } \ L^1(\mathbb{R}^N)\cap W^{1,\infty}(\mathbb{R}^N)\,, \quad u_0 \not\equiv 0\,. \tag{2.8}$$

We denote by $u = u(x,t)$ the corresponding non-positive solution of the Cauchy problem (1.1)-(1.2). In that case, we recall that $t \longmapsto \|u(t)\|_1$ is a non-decreasing function and put

$$I_\infty := \inf_{t\geq 0} \int_{\mathbb{R}^N} u(x,t)\, dx = -\sup_{t\geq 0} \|u(t)\|_1 \in [-\infty, -\|u_0\|_1]\,. \tag{2.9}$$

Substituting $u = -v$ in (1.1)-(1.2) we obtain that $v = v(x,t)$ is a non-negative solution to

$$v_t - \Delta v - |\nabla v|^q = 0, \quad v(x,0) = -u_0(x), \tag{2.10}$$

which has been studied in [7, 16, 17, 21].

We start again with the diffusion-dominated case.

Theorem 2.3 *Suppose (2.8).*

a) Assume that $q \geq 2$. Then $I_\infty > -\infty$ and $|\nabla u|$ belongs to $L^q(\mathbb{R}^N \times (0,\infty))$. In addition, I_∞ is given by (2.2) and the relations (2.3) and (2.4) hold true for every $p \in [1,\infty]$.

b) Assume that $q \in (q_c, 2)$. There exists $\varepsilon = \varepsilon(N,q)$ such that, if

$$\|u_0\|_1 \|\nabla u_0\|_\infty^{(N+1)q-(N+2)} < \varepsilon\,, \tag{2.11}$$

then the conclusions of part a) are still valid.

The fact that $I_\infty > -\infty$ under the assumptions of Theorem 2.3 is established in [21], together with the relation (2.3) for $p = 1$. We extend here this convergence to $W^{1,p}(\mathbb{R}^N)$, $p \in [1,\infty]$.

The smallness assumption imposed in (2.11) is necessary to obtain the heat kernel as the first term of the asymptotic expansion of solutions. This is an immediate consequence of the following theorem and the subsequent discussion.

Theorem 2.4 *Suppose (2.8) and that $q \in (q_c, 2)$.*

a) There exists a non-positive self-similar solution

$$V = V(x,t) = t^{-(2-q)/(2(q-1))} V(xt^{-1/2}, 1)$$

to (1.1) such that

$$\lim_{t\to\infty} t^{(N/2)(1-1/p)} \|V(t)\|_p = \infty \quad \text{and} \quad \lim_{t\to\infty} t^{(N/2)(1-1/p)+1/2} \|\nabla V(t)\|_p = \infty$$

for all $p \in [1,\infty]$.

b) There is a constant $K = K(q) \geq 0$ such that, if $u_0 \in W^{2,\infty}(\mathbb{R}^N)$ satisfies

$$\|u_0\|_\infty \ \left\|(\Delta u_0)^+\right\|_\infty^{1-2/q} > K \tag{2.12}$$

then

$$\lim_{t\to\infty} \|u(t)\|_\infty > 0. \tag{2.13}$$

The first part of Theorem 2.4 is proved in [7] while the second assertion is new. Let us point out here that, for the Hamilton-Jacobi equation $w_t + |\nabla w|^q = 0$, the L^∞-norm of solutions remains constant throughout time evolution, while it decays to zero for the linear heat equation. We thus realize that, under the assumptions of Theorem 2.4 b), the diffusive term is not strong enough to drive the solution to zero in L^∞ as $t \to \infty$ and the large time dynamics is therefore ruled by the Hamilton-Jacobi term $|\nabla u|^q$.

Unfortunately, the conditions (2.11) and (2.12) do not involve the same quantities. Still, we can prove that if u_0 fulfils

$$\|u_0\|_\infty \ \|D^2 u_0\|_\infty^{1-2/q} > K$$

(which clearly implies (2.12) since $q < 2$), the quantity $\|u_0\|_1 \|\nabla u_0\|_\infty^{(N+1)q-(N+2)}$ cannot be small. Indeed, there is a constant C depending only on q and N such that

$$\left(\|u_0\|_\infty \ \|D^2 u_0\|_\infty^{1-2/q} \right)^{q(N+1)/2} \le C \|u_0\|_1 \|\nabla u_0\|_\infty^{q(N+1)-(N+2)}. \tag{2.14}$$

For the proof of (2.14), put $B = \|u_0\|_\infty \|D^2 u_0\|_\infty^{1-2/q}$ and note that the Gagliardo-Nirenberg inequalities

$$\begin{aligned} \|u_0\|_\infty &\le C \ \|\nabla u_0\|_\infty^{N/(N+1)} \ \|u_0\|_1^{1/(N+1)}, \\ \|\nabla u_0\|_\infty &\le C \ \|D^2 u_0\|_\infty^{(N+1)/(N+2)} \ \|u_0\|_1^{1/(N+2)}, \end{aligned}$$

imply that

$$\begin{aligned} \|\nabla u_0\|_\infty^{(2-q)(N+2)} &\le C \ \|D^2 u_0\|_\infty^{(2-q)(N+1)} \ \|u_0\|_1^{2-q} \\ &= C \ B^{-q(N+1)} \ \|u_0\|_\infty^{q(N+1)} \ \|u_0\|_1^{2-q} \\ &\le C \ B^{-q(N+1)} \ \|\nabla u_0\|_\infty^{qN} \ \ \|u_0\|_1^2, \end{aligned}$$

whence the above claim.

We next show that the second assertion of Theorem 2.4 is also true when $q \in (1, q_c)$.

Theorem 2.5 *Suppose (2.8) and that $q \in (1, q_c]$. There is a constant $K = K(q) \ge 0$ such that, if $u_0 \in W^{2,\infty}(\mathbb{R}^N)$ fulfils (2.12), then (2.13) holds true.*

Furthermore, if $N \le 3$ and $1 < q < 4/\left(1 + \sqrt{1+2N}\right)$, then $K(q) = 0$.

We actually conjecture that $K(q) = 0$ for any $q \in (1, q_c)$, but we have yet been unable to prove it.

The last result confirms the domination of the Hamilton-Jacobi term for large times when (2.13) holds true and provides precise information on the large time behavior.

Theorem 2.6 *Let $q \in (1, 2)$. Assume that $u_0 \in \mathcal{C}_0(\mathbb{R}^N)$ fulfils (2.8) and is such that*

$$M_\infty := \lim_{t\to\infty} \|u(t)\|_\infty > 0\,. \tag{2.15}$$

Then

$$\lim_{t\to\infty} \|u(t) - Z_{M_\infty}(t)\|_\infty = 0\,, \tag{2.16}$$

where Z_{M_∞} *is given by*

$$Z_{M_\infty}(x,t) := -\left(M_\infty - (q-1)\ q^{-q/(q-1)}\ \left(\frac{|x|}{t^{1/q}}\right)^{q/(q-1)} \right)^+ \tag{2.17}$$

for $(x,t) \in \mathbb{R}^N \times (0,\infty)$*. In fact,* Z_{M_∞} *is the unique viscosity solution in* $\mathcal{BUC}(\mathbb{R}^N \times (0,\infty))$ *to*

$$z_t + |\nabla z|^q = 0 \quad \text{in} \quad \mathbb{R}^N \times (0,\infty) \tag{2.18}$$

with the bounded and lower semicontinuous initial datum $z(x,0) = 0$ *if* $x \neq 0$ *and* $z(0,0) = -M_\infty$*.*

The last assertion of Theorem 2.6 follows from [24]. Moreover, Z_{M_∞} is actually given by the Hopf-Lax formula

$$Z_{M_\infty}(x,t) = \inf_{y\in\mathbb{R}^N} \left\{ -M_\infty\ \mathbf{1}_{\{0\}}(y) + (q-1)\ q^{-q/(q-1)}|x-y|^{q/(q-1)}\ t^{-1/(q-1)} \right\}$$

for $(x,t) \in \mathbb{R}^N \times (0,\infty)$, where $\mathbf{1}_{\{0\}}$ denotes the characteristic function of the set $\{0\}$. Observe that Z_{M_∞} is a self-similar solution to (2.18) since $Z_{M_\infty}(x,t) = Z_{M_\infty}(xt^{-1/q},1)$.

If $N=1$, the convergence stated in Theorem 2.6 extends to the gradient of u.

Proposition 2.1 *Assume that* $N=1$ *and consider a non-positive function* u_0 *in* $W^{1,1}(\mathbb{R}) \cap W^{1,\infty}(\mathbb{R})$*. Under the assumptions and notations of Theorem 2.6, we have also*

$$\lim_{t\to\infty} t^{(1-1/p)/q}\ \|u_x(t) - Z_{M_\infty,x}(t)\|_p = 0$$

for $p \in [1,\infty)$*.*

In fact, if $N=1$ and $u_0 \in W^{1,1}(\mathbb{R}) \cap W^{1,\infty}(\mathbb{R})$, the function $U := u_x$ is a solution to the convection-diffusion equation

$$U_t - U_{xx} + (|U|^q)_x = 0\,, \quad x \in \mathbb{R}\,, \quad t > 0\,, \tag{2.19}$$

with initial datum $U(0) = u_{0x}$ and satisfies

$$\int_{\mathbb{R}} U(x,t)\ dx = \int_{\mathbb{R}} u_{0x}(x)\ dx = 0\,, \quad t \geq 0\,. \tag{2.20}$$

The large time behavior of non-negative or non-positive integrable solutions to (2.19) is now well-identified [12, 13] but this is far from being the case for solutions satisfying (2.20). In this situation, some sufficient conditions on $U(0)$ are given in [19] for the solution to (2.19) to exhibit a diffusion-dominated large time behavior. Also, convergence to N-waves is studied in [20] but, for solutions satisfying (2.20), no condition is given in that paper which guarantees that $U(t)$ really behaves as an N-wave for large times. As a consequence of our analysis, we specify such a condition and also provide several new information on the large time behavior of solutions to (2.19) satisfying (2.20). Results on the large time behavior of solutions to equation (2.19) satisfying the condition (2.20) are reviewed in the companion paper [2].

We finally outline the contents of the paper: the next section is devoted to some preliminary estimates. Theorems 2.1 and 2.3 (diffusion-dominated case) are proved in Section 4 and Theorem 2.2 in Section 5. The remaining sections are devoted to the "hyperbolic"-dominated case: Theorems 2.4 and 2.5 are proved in Section 5 and Theorem 2.6 and Proposition 2.1 in Section 6.

3 Preliminary estimates

Let us first state a gradient estimate for solutions to (1.1) which is a consequence of [4, Theorem 1] (see also [17, Theorem 2]). Note that, in this section, we do not impose a sign condition on the solution u to (1.1).

Proposition 3.1 *Assume that $u = u(x,t)$ is the solution to (1.1)-(1.2) corresponding to the initial datum $u_0 \in W^{1,\infty}(\mathbb{R}^N)$. For every $q > 1$, there is a constant $C_1 > 0$ depending only on q such that*

$$\|\nabla u(t)\|_\infty \le C_1 \; \|u_0\|_\infty^{1/q} \; t^{-1/q}\,, \quad \textit{for all} \quad t > 0\,. \tag{3.1}$$

Proof. Setting $v = u + \|u_0\|_\infty$, it readily follows from (1.1) and the maximum principle that v is a non-negative solution to (1.1). By [4, Theorem 1], there is a constant C depending only on q such that

$$\left\|\nabla v^{(q-1)/q}(t)\right\|_\infty \le C\; t^{-1/q}\,, \qquad t > 0\,.$$

Since $\nabla v = (q/(q-1))\; v^{1/q}\; \nabla v^{(q-1)/q}$ and $|u(x,t)| \le \|u_0\|_\infty$, we further deduce that

$$\|\nabla u(t)\|_\infty = \|\nabla v(t)\|_\infty \le C\; \|v(t)\|_\infty^{1/q}\; \left\|\nabla v^{(q-1)/q}(t)\right\|_\infty \le C\; \|u_0\|_\infty^{1/q}\; t^{-1/q}\,,$$

whence (3.1). □

Next, we derive estimates for the second derivatives of solutions to (1.1)-(1.2) when $q \in (1,2]$.

Proposition 3.2 *Under the assumptions of Proposition 3.1, if $q \in (1,2]$, the Hessian matrix $D^2 u = \left(u_{x_i x_j}\right)_{1 \le i,j \le N}$ of u satisfies*

$$D^2 u(x,t) \;\le\; \frac{\|\nabla u_0\|_\infty^{2-q}}{q\,(q-1)\,t}\; Id\,, \tag{3.2}$$

$$D^2 u(x,t) \;\le\; \frac{C_2\,\|u_0\|_\infty^{(2-q)/q}}{t^{2/q}}\; Id\,, \tag{3.3}$$

for $(x,t) \in \mathbb{R}^N \times (0,\infty)$, where C_2 is a positive constant depending only on q.

Furthermore, if $u_0 \in W^{2,\infty}(\mathbb{R}^N)$,

$$D^2 u(x,t) \le \|D^2 u_0\|_\infty\; Id\,. \tag{3.4}$$

In Proposition 3.2, Id denotes the identity matrix of $\mathcal{M}_N(\mathbb{R})$. Given two matrices A and B in $\mathcal{M}_N(\mathbb{R})$, we write $A \le B$ if $A\xi \cdot \xi \le B\xi \cdot \xi$ for every vector $\xi \in \mathbb{R}^N$.

For $q = 2$, the estimates (3.2) and (3.3) follow from the analysis of Hamilton [18] (since, if f is a non-negative solution to the linear heat equation $f_t = \Delta f$, the function $-\ln f$ solves (1.1) with $q = 2$). In Proposition 3.2 above, we extend that result to any $q \in (1,2]$.

Remark 3.1 The estimates (3.2) and (3.3) may also be seen as an extension to a multidimensional setting of a weak form of the Oleinik type gradient estimate for scalar conservation laws. Indeed, if $N = 1$ and $U = u_x$, then U is a solution to $U_t - U_{xx} + \left(|U|^q\right)_x = 0$ in $\mathbb{R} \times (0,\infty)$. The estimates (3.2) and (3.3) then read

$$U_x \le C\, \|U(0)\|_\infty^{2-q}\, t^{-1} \quad \text{and} \quad U_x \le C\, \|u_0\|_\infty^{(2-q)/q}\, t^{-2/q}$$

for $t > 0$, respectively, and we thus recover the results of [15, 20] in that case.

Proof of Proposition 3.2. For $1 \le i,j \le N$, we put $w_{ij} = u_{x_i x_j}$. It follows from equation (1.1) that

$$\begin{aligned} w_{ij,t} - \Delta w_{ij} &= -q \left(|\nabla u|^{q-2} \left(\sum_{k=1}^N u_{x_k}\, w_{jk} \right)\right)_{x_i} \\ &= -q\, |\nabla u|^{q-2} \sum_{k=1}^N w_{ik}\, w_{jk} - q\, |\nabla u|^{q-2} \sum_{k=1}^N u_{x_k}\, w_{jk,x_i} \\ &\quad - q\,(q-2)\, |\nabla u|^{q-4} \left(\sum_{k=1}^N u_{x_k}\, w_{ik}\right) \left(\sum_{k=1}^N u_{x_k}\, w_{jk}\right). \end{aligned} \tag{3.5}$$

Consider now $\xi \in \mathbb{R}^N \setminus \{0\}$ and set

$$h = \sum_{i=1}^N \sum_{j=1}^N w_{ij}\ \xi_i\ \xi_j\,.$$

Multiplying (3.5) by $\xi_i\ \xi_j$ and summing up the resulting identities yield

$$\begin{aligned} h_t - \Delta h &= -q\ |\nabla u|^{q-2}\ \sum_{k=1}^N \left(\sum_{i=1}^N w_{ik}\ \xi_i \right)^2 - q\ |\nabla u|^{q-2}\ \nabla u \cdot \nabla h \\ &\quad - q\ (q-2)\ |\nabla u|^{q-4}\ \left(\sum_{i=1}^N \sum_{j=1}^N u_{x_j}\ w_{ij}\ \xi_i \right)^2 . \end{aligned} \tag{3.6}$$

Thanks to the following inequalities

$$\begin{aligned} |\nabla u|^{q-4}\ \left(\sum_{i=1}^N \sum_{j=1}^N u_{x_j}\ w_{ij}\ \xi_i \right)^2 &\le |\nabla u|^{q-4}\ \sum_{j=1}^N |u_{x_j}|^2\ \sum_{j=1}^N \left(\sum_{i=1}^N w_{ij}\ \xi_i \right)^2 \\ &\le |\nabla u|^{q-2}\ \sum_{k=1}^N \left(\sum_{i=1}^N w_{ik}\ \xi_i \right)^2 , \end{aligned}$$

and

$$h^2 \le |\xi|^2\ \sum_{k=1}^N \left(\sum_{i=1}^N w_{ik}\ \xi_i \right)^2 ,$$

and since $q \le 2$, the right-hand side of identity (3.6) can be bounded from above. We thus obtain

$$\begin{aligned} h_t - \Delta h &\le -q\ (q-1)\ |\nabla u|^{q-2}\ \sum_{k=1}^N \left(\sum_{i=1}^N w_{ik}\ \xi_i \right)^2 - q\ |\nabla u|^{q-2}\ \nabla u \cdot \nabla h \\ &\le -q\ |\nabla u|^{q-2}\ \nabla u \cdot \nabla h - \frac{q\ (q-1)\ |\nabla u|^{q-2}}{|\xi|^2}\ h^2 . \end{aligned}$$

Consequently,

$$\mathcal{L} h \le 0 \quad \text{in} \quad \mathbb{R}^N \times (0,\infty) , \tag{3.7}$$

where the parabolic differential operator $\mathcal{L}$ is given by

$$\mathcal{L} z := z_t - \Delta z + q\ |\nabla u|^{q-2}\ \nabla u \cdot \nabla z + \frac{q\ (q-1)\ |\nabla u|^{q-2}}{|\xi|^2}\ z^2 .$$

On the one hand, since $q \in (1,2]$ and $|\nabla u(x,t)| \le \|\nabla u_0\|_\infty$, it is straightforward to check that

$$H_1(t) := \left(\frac{1}{\|h(0)\|_\infty} + \frac{q\ (q-1)\ t}{|\xi|^2\ \|\nabla u_0\|_\infty^{2-q}} \right)^{-1} , \qquad t > 0 ,$$

satisfies $\mathcal{L} H_1 \ge 0$ with $H_1(0) \ge h(x,0)$ for all $x \in \mathbb{R}^N$. The comparison principle then entails that $h(x,t) \le H_1(t)$ for $(x,t) \in \mathbb{R}^N \times (0,\infty)$, from which we conclude that

$$h(x,t) \le \|h(0)\|_\infty \le \|D^2 u_0\|_\infty\ |\xi|^2 ,$$

and
$$h(x,t) \leq \frac{|\xi|^2 \ \|\nabla u_0\|_\infty^{2-q}}{q\ (q-1)\ t}\,.$$
In other words, (3.2) and (3.4) hold true.

On the other hand, we infer from (3.1) that
$$H_2(t) := \frac{2\ C_1^{2-q}\ |\xi|^2}{q^2\ (q-1)}\ \frac{\|u_0\|_\infty^{(2-q)/q}}{t^{2/q}}\,, \qquad t>0\,,$$
satisfies $\mathcal{L}H_2 \geq 0$ with $H_2(0) = +\infty \geq h(x,0)$ for all $x \in \mathbb{R}^N$. We then use again the comparison principle as above and obtain (3.3). □

Remark 3.2 Since $q \in (1,2]$ and ∇u may vanish, the proof of Proposition 3.2 is somehow formal because of the negative powers of $|\nabla u|$ in (3.6). It can be made rigorous by first considering the regularised equation
$$u_t^\varepsilon - \Delta u^\varepsilon + \left(|\nabla u^\varepsilon|^2 + \varepsilon^2\right)^{p/2} = 0$$
for $\varepsilon \in (0,1)$, and then letting $\varepsilon \to 0$ as in [4].

In fact, we need a particular case of Proposition 3.2.

Corollary 3.1 *Under the assumptions of Proposition 3.2*

$$\Delta u(x,t) \ \leq\ \frac{C_3\ \|\nabla u_0\|_\infty^{2-q}}{t}\,, \tag{3.8}$$
$$\Delta u(x,t) \ \leq\ \frac{C_4\ \|u_0\|_\infty^{(2-q)/q}}{t^{2/q}}\,, \tag{3.9}$$

for $(x,t) \in \mathbb{R}^N \times (0,+\infty)$, where C_3 and C_4 are positive constants depending only on q and N.

Furthermore, if $u_0 \in W^{2,\infty}(\mathbb{R}^N)$,
$$\sup_{x\in\mathbb{R}^N} \Delta u(x,t) \leq \sup_{x\in\mathbb{R}^N} \Delta u_0(x)\,, \quad t \geq 0\,. \tag{3.10}$$

Proof. Consider $i \in \{1,\ldots,N\}$ and define $\xi^i = (\xi^i_j) \in \mathbb{R}^N$ by $\xi^i_i = 1$ and $\xi^i_j = 0$ if $j \neq i$. We take $\xi = \xi^i$ in (3.7) and obtain that $\mathcal{L}u_{x_ix_i} \leq 0$, that is,
$$(u_{x_ix_i})_t - \Delta u_{x_ix_i} + q\ |\nabla u|^{q-2}\ \nabla u.\nabla u_{x_ix_i} + q\ (q-1)\ |\nabla u|^{q-2}\ u^2_{x_ix_i} \leq 0$$
in $\mathbb{R}^N \times (0,\infty)$. Summing the above inequality over $i \in \{1,\ldots,N\}$ and recalling that
$$|\Delta u|^2 \leq N\ \sum_{i=1}^N u^2_{x_ix_i}\,,$$
we end up with
$$(\Delta u)_t - \Delta\left(\Delta u\right) + q\ |\nabla u|^{q-2}\ \nabla u.\nabla\left(\Delta u\right) + \frac{q\ (q-1)\ |\nabla u|^{q-2}}{N}\ |\Delta u|^2 \leq 0$$
in $\mathbb{R}^N \times (0,\infty)$. We next proceed as in the proof of Proposition 3.2 to complete the proof of Corollary 3.1. □

4 Diffusion-dominated case

The proofs of Theorems 2.1 and 2.3 rely on some properties of the non-homogeneous heat equation which we state now. Similar results have already been used in [8, 21].

Theorem 4.1 *Assume that $u = u(x,t)$ is the solution of the Cauchy problem to the linear non-homogeneous heat equation*

$$u_t = \Delta u + f(x,t), \quad x \in \mathbb{R}^N,\ t > 0, \tag{4.1}$$

$$u(x,0) = u_0(x), \quad x \in \mathbb{R}^N, \tag{4.2}$$

with $u_0 \in L^1(\mathbb{R}^N)$ and $f \in L^1(\mathbb{R}^N \times (0,\infty))$. Then

$$\lim_{t\to\infty} \|u(t) - I_\infty G(t)\|_1 = 0\,, \tag{4.3}$$

where

$$I_\infty := \lim_{t\to\infty} \int_{\mathbb{R}^N} u(x,t)\ dx = \int_{\mathbb{R}^N} u_0(x)\ dx + \int_0^\infty \int_{\mathbb{R}^N} f(x,t)\ dx\, dt\,.$$

Assume further that there is $p \in [1,\infty]$ such that $f(t) \in L^p(\mathbb{R}^N)$ for every $t > 0$ and

$$\lim_{t\to\infty} t^{1+(N/2)(1-1/p)}\ \|f(t)\|_p = 0\,. \tag{4.4}$$

Then

$$\lim_{t\to\infty} t^{(N/2)(1-1/p)} \|u(t) - I_\infty G(t)\|_p = 0\,, \tag{4.5}$$

and

$$\lim_{t\to\infty} t^{(N/2)(1-1/p)+1/2} \|\nabla u(t) - I_\infty \nabla G(t)\|_p = 0\,. \tag{4.6}$$

Proof. We first observe that the assumptions on u_0 and f warrant that I_∞ is finite, and we refer to [8] for the proof of (4.3). We next assume (4.4) and prove (4.6). Let $T > 0$ and $t \in (T,\infty)$. By the Duhamel formula,

$$\nabla u(t) = \nabla G(t-T) * u(T) + \int_T^t \nabla G(t-\tau) * f(\tau)\ d\tau.$$

It follows from the Young inequality that

$$\begin{aligned}
& t^{(N/2)(1-1/p)+1/2}\ \|\nabla u(t) - \nabla G(t-T) * u(T)\|_p \\
\leq\ & C\ t^{(N/2)(1-1/p)+1/2} \int_T^{(T+t)/2} (t-\tau)^{-(N/2)(1-1/p)-1/2}\ \|f(\tau)\|_1\ d\tau \\
+\ & C\ t^{(N/2)(1-1/p)+1/2} \int_{(T+t)/2}^t (t-\tau)^{-1/2}\ \|f(\tau)\|_p\ d\tau \\
\leq\ & C\ \left(\frac{t}{t-T}\right)^{(N/2)(1-1/p)+1/2} \int_T^\infty \|f(\tau)\|_1\ d\tau
\end{aligned}$$

$$
\begin{aligned}
&+ \; C \sup_{\tau\geq T} \left\{\tau^{(N/2)(1-1/p)+1}\, \|f(\tau)\|_p\right\} \int_{(T+t)/2}^{t} (t-\tau)^{-1/2}\, \tau^{-1/2}\, d\tau \\
&\leq \; C \left(\frac{t}{t-T}\right)^{(N/2)(1-1/p)+1/2} \int_T^\infty \|f(\tau)\|_1\, d\tau \\
&+ \; C \sup_{\tau\geq T} \left\{\tau^{(N/2)(1-1/p)+1}\, \|f(\tau)\|_p\right\}\, .
\end{aligned}
$$

Also, classical properties of the heat semigroup (see, e.g., [11]) ensure that

$$
\lim_{t\to\infty} t^{(N/2)(1-1/p)+1/2} \left\| \nabla G(t-T)*u(T) - \left(\int_{\mathbb{R}^N} u(x,T)\, dx\right) \nabla G(t-T) \right\|_p = 0\,,
$$

and

$$
\lim_{t\to\infty} t^{(N/2)(1-1/p)+1/2} \left\|\nabla G(t-T) - \nabla G(t)\right\|_p = 0
$$

for every $p\in[1,\infty]$. Since, by elementary calculations, we have

$$
\begin{aligned}
& \|\nabla u(t) - I_\infty\, \nabla G(t)\|_p \\
\leq \; & \|\nabla u(t) - \nabla G(t-T)*u(T)\|_p \\
+ \; & \left\| \nabla G(t-T)*u(T) - \left(\int_{\mathbb{R}^N} u(x,T)\, dx\right)\, \nabla G(t-T) \right\|_p \\
+ \; & \left| \int_{\mathbb{R}^N} u(x,T)\, dx - I_\infty \right|\, \|\nabla G(t-T)\|_p + |I_\infty|\, \|\nabla G(t-T) - \nabla G(t)\|_p\,,
\end{aligned}
$$

the previous relations imply that

$$
\begin{aligned}
& \limsup_{t\to\infty} t^{(N/2)(1-1/p)+1/2}\, \|\nabla u(t) - I_\infty\, \nabla G(t)\|_p \\
\leq \; & C \left(\int_T^\infty \|f(\tau)\|_1\, d\tau + \sup_{\tau\geq T}\left\{\tau^{(N/2)(1-1/p)+1}\, \|f(\tau)\|_p\right\} + \left| \int_{\mathbb{R}^N} u(x,T)\, dx - I_\infty \right| \right)\, .
\end{aligned}
$$

The above inequality being valid for any $T>0$, we may let $T\to\infty$ and conclude that (4.6) holds true. The assertion (4.5) then follows from (4.3) and (4.6) by the Gagliardo-Nirenberg inequality. □

Proof of Theorem 2.1. Since u is non-negative, we infer from [4, Eq. (17)] that there is a constant $C=C(q)$ such that

$$
\|\nabla u^{(q-1)/q}(t)\|_\infty \leq C\, \|u(t/2)\|_\infty^{(q-1)/q}\, t^{-1/2}\,, \quad t>0\,.
$$

Also, u is a subsolution to the linear heat equation and therefore satisfies

$$
\|u(t)\|_p \leq \|G(t)*u_0\|_p \leq C\, t^{-(N/2)(1-1/p)}\, \|u_0\|_1\,, \quad t>0\,,
$$

for every $p \in [1,\infty]$ by the comparison principle. Since $\nabla u = (q/(q-1))\, u^{1/q}\, \nabla u^{(q-1)/q}$, we obtain that

$$t^{(N/2)(1-1/p)+1}\, \| |\nabla u(t)|^q \|_p \le C\; t^{(N+2-q(N+1))/2} \underset{t\to\infty}{\longrightarrow} 0$$

for $p \in [1,\infty]$, because $q > (N+2)/(N+1)$. Theorem 2.1 then readily follows by Theorem 4.1 with $f(x,t) = -|\nabla u(x,t)|^q$. □

Proof of Theorem 2.3, part a). Since $q \ge 2$, we infer from [21] that I_∞ is finite and negative and that

$$\nabla u \in L^q(\mathbb{R}^N \times (0,\infty))\,. \tag{4.7}$$

Setting $b := \|\nabla u_0\|_\infty^{q-2}$, it follows from (1.3) that $u_t - \Delta u \ge -b\, |\nabla u|^2$ in $\mathbb{R}^N \times (0,\infty)$. The comparison principle then entails that $u \ge w$, where w is the solution to

$$w_t - \Delta w = -b\, |\nabla w|^2\,, \quad w(0) = u_0\,.$$

The Hopf-Cole transformation $h := e^{-bw} - 1$ then implies that h solves

$$h_t - \Delta h = 0\,, \quad h(0) = e^{-bu_0} - 1\,.$$

Therefore, for $t > 0$,

$$0 \le -bw(x,t) \le h(x,t) \le \|h(t)\|_\infty \le C\; t^{-N/2}\, \|h(0)\|_1 \le C\; t^{-N/2}\,,$$

since $u_0 \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$. Recalling that $0 \ge u \ge w$, we end up with

$$\|u(t)\|_\infty \le C\; t^{-N/2}\,, \quad t > 0\,. \tag{4.8}$$

It next follows from [17, Theorem 2] that

$$\|\nabla u(t)\|_\infty \le C\; \|u(t/2)\|_\infty\, t^{-1/2}\,, \quad t > 0\,,$$

which, together with (4.8), yields

$$\|\nabla u(t)\|_\infty \le C\; t^{-(N+1)/2}\,, \quad t > 0\,. \tag{4.9}$$

Recalling (1.3), we also have

$$\|\nabla u(t)\|_\infty \le C\; (1+t)^{-(N+1)/2}\,, \quad t \ge 0\,. \tag{4.10}$$

We next put

$$\mathcal{A}_1(t) := \sup_{\tau\in(0,t)} \left\{ \tau^{1/2}\, \|\nabla u(\tau)\|_1 \right\}\,,$$

which is finite by [7]. Since $q \geq 2$ and $N \geq 1$, we infer from the Duhamel formula and (4.10) that, for $\alpha \in (0,1/2)$,

$$\begin{aligned}
t^{1/2}\ \|\nabla u(t)\|_1 &\leq C\ \|u_0\|_1 + C\ t^{1/2} \int_0^t (t-\tau)^{-1/2}\ \|\nabla u(\tau)\|_q^q\ d\tau \\
&\leq C + C\ t^{1/2} \int_0^t (t-\tau)^{-1/2}\ (1+\tau)^{-(q-1)(N+1)/2}\ \|\nabla u(\tau)\|_1\ d\tau \\
&\leq C + C\ t^{1/2} \int_0^t (t-\tau)^{-1/2}\ (1+\tau)^{-1}\ \tau^{-1/2}\ \mathcal{A}_1(\tau)\ d\tau \\
&\leq C + C\ \alpha^{-1/2} \int_0^{(1-\alpha)t} (1+\tau)^{-1}\ \tau^{-1/2}\mathcal{A}_1(\tau)\ d\tau \\
&+ C\ t^{1/2}\ \mathcal{A}_1(t) \int_{(1-\alpha)t}^t (t-\tau)^{-1/2}\ \frac{2}{2+t}\ \tau^{-1/2}\ d\tau \\
&\leq C + C\ \alpha^{-1/2} \int_0^t (1+\tau)^{-1}\ \tau^{-1/2}\mathcal{A}_1(\tau)\ d\tau \\
&+ C\ \mathcal{A}_1(t) \int_{1-\alpha}^1 (1-\tau)^{-1/2}\ \tau^{-1/2}\ d\tau\,,
\end{aligned}$$

whence

$$\left(1 - C\ \alpha^{1/2}\right)\ \mathcal{A}_1(t) \leq C(\alpha)\ \left(1 + \int_0^t (1+\tau)^{-1}\ \tau^{-1/2}\mathcal{A}_1(\tau)\ d\tau\right)\,.$$

Consequently, there is $\alpha_0 \in (0,1/2)$ sufficiently small such that

$$\mathcal{A}_1(t) \leq \mathcal{B}_1(t) := C(\alpha_0)\ \left(1 + \int_0^t (1+\tau)^{-1}\ \tau^{-1/2}\mathcal{A}_1(\tau)\ d\tau\right)$$

for $t \geq 0$. Now, for $t \geq 0$,

$$\frac{d\mathcal{B}_1}{dt}(t) = C(\alpha_0)\ (1+t)^{-1}\ t^{-1/2}\mathcal{A}_1(t) \leq C(\alpha_0)\ (1+t)^{-1}\ t^{-1/2}\mathcal{B}_1(t)\,,$$

from which we deduce that

$$\mathcal{A}_1(t) \leq \mathcal{B}_1(t) \leq \mathcal{B}_1(0)\ \exp\left\{C(\alpha_0)\ \int_0^t (1+\tau)^{-1}\ \tau^{-1/2}\ d\tau\right\} \leq C(\alpha_0)\,.$$

We have thus proved that

$$\|\nabla u(t)\|_1 \leq C\ t^{-1/2}\,, \quad t > 0\,. \tag{4.11}$$

We finally infer from (4.9), (4.11) and the Hölder inequality that

$$t^{(N/2)(1-1/p)+1}\ \||\nabla u(t)|^q\|_p \leq C\ t^{(N+2-q(N+1))/2} \underset{t\to\infty}{\longrightarrow} 0$$

for $p \in [1, \infty]$, and we conclude as in the proof of Theorem 2.1. $\square$

PROOF OF THEOREM 2.3, PART B). Since $q \in (q_c, 2)$, we obtain from [21] that there is $\varepsilon > 0$ such that, if u_0 fulfils (2.11), then I_∞ is finite and negative and there are $C > 0$ and $\delta > 0$ such that

$$\|\nabla u(t)\|_q^q \le C\ t^{-1}\ (1+t)^{-\delta}\,, \quad t > 0\,. \tag{4.12}$$

In particular,

$$|\nabla u|^q \in L^1(\mathbb{R}^N \times (0,\infty)) \ \text{ and } \ \lim_{t\to\infty} t\ \||\nabla u(t)|^q\|_1 = 0\,. \tag{4.13}$$

We next claim that

$$\|\nabla u(t)\|_\infty \le C\ t^{-(N+1)/2}\,, \quad t > 0\,. \tag{4.14}$$

Indeed, we fix $r \in (q_c, q)$ such that $r < N/(N-1)$ and define $s = r/(r-1)$ and a sequence $(r_i)_{i\ge 0}$ by

$$r_0 = \frac{1}{q} \ \text{ and } \ r_{i+1} = \frac{(N+1)\ r - (N+2)}{2r} + \frac{q}{r}\ r_i\,, \quad i \ge 0\,.$$

We now proceed by induction to show that, for each $i \ge 0$, there is $K_i \ge 0$ such that

$$\|\nabla u(t)\|_\infty \le K_i\ \left(t^{-(N+1)/2} + t^{-r_i}\right)\,, \quad t > 0\,. \tag{4.15}$$

Thanks to (3.1), the assertion (4.15) is true for $i = 0$. Assume next that (4.15) holds true for some $i \ge 0$. We infer from (4.12), (4.15) and the Duhamel formula that

$$\begin{aligned}
\|\nabla u(t)\|_\infty &\le C\ \|u_0\|_1\ t^{-(N+1)/2} + C\ \int_0^{t/2} (t-\tau)^{-(N+1)/2}\ \|\nabla u(\tau)\|_q^q\ d\tau \\
&+ C\ \int_{t/2}^t (t-\tau)^{-(N/2)(1-1/r)-1/2}\ \|\nabla u(\tau)\|_{sq}^q\ d\tau \\
&\le C\ t^{-(N+1)/2}\ \left(\|u_0\|_1 + \int_0^{t/2} \|\nabla u(\tau)\|_q^q\ d\tau\right) \\
&+ C\ \int_{t/2}^t (t-\tau)^{-(N/2)(1-1/r)-1/2}\ \|\nabla u(\tau)\|_\infty^{q/r}\ \|\nabla u(\tau)\|_q^{q/s}\ d\tau \\
&\le C\ t^{-(N+1)/2} + C\ \mathcal{I}(t)\,,
\end{aligned}$$

where

$$\mathcal{I}(t) := \int_{t/2}^t (t-\tau)^{-(N/2)(1-1/r)-1/2}\ \left(\tau^{-(N+1)/2} + \tau^{-r_i}\right)^{q/r}\ \tau^{-1/s}\ d\tau\,.$$

Since $r < N/(N-1)$ and $q > q_c$, we have

$$\begin{aligned}
\mathcal{I}(t) &\leq C \int_{t/2}^{t} (t-\tau)^{-(N/2)(1-1/r)-1/2} \left(\tau^{-(q(N+1))/2r} + \tau^{-(qr_i)/r}\right) \tau^{-1/s}\, d\tau \\
&\leq C\, t^{-((N+1)r-(N+2))/2r} \left(t^{-(q(N+1))/2r} + t^{-(qr_i)/r}\right) \\
&\leq C \left(t^{-(N+1)/2}\, t^{-((N+1)q-(N+2))/2r} + t^{-r_{i+1}}\right) \\
&\leq C \left(t^{-(N+1)/2} + t^{-r_{i+1}}\right)
\end{aligned}$$

for $t \geq 1$. Consequently, for $t \geq 1$,

$$\|\nabla u(t)\|_\infty \leq K_{i+1} \left(t^{-(N+1)/2} + t^{-r_{i+1}}\right)\,,$$

while (1.3) implies that the same inequality is valid for $t \in [0,1]$ for a possibly larger constant K_{i+1}. Thus (4.15) is true for $i+1$, which completes the proof of (4.15). To obtain (4.14), it suffices to note that $r_i \to \infty$ since $q > r$.

Now, owing to (4.13) and (4.14), we are in a position to apply Theorem 4.1 and conclude that (2.3) and (2.4) holds true for $p = 1$ and $p = \infty$. The general case $p \in (1,\infty)$ then follows by the Hölder inequality. $\square$

5 Convergence towards very singular solutions

The goal of this section is to prove Theorem 2.2. Recall that we assume that $1 < q < q_c$ and that u_0 is a non-negative and integrable function satisfying in addition

$$\operatorname*{ess\,lim}_{|x|\to\infty} |x|^a\, u_0(x) = 0\,, \tag{5.1}$$

with $a = (2-q)/(q-1) \in (N,\infty)$. We define

$$R(u_0) := \inf\left\{R > 0\,,\ \ |x|^a\, u_0(x) \leq \gamma_q \ \text{ a.e. in } \{|x| \geq R\}\right\},$$

where $\gamma_q := (q-1)^{(q-2)/(q-1)}\, (2-q)^{-1}$ and observe that $R(u_0)$ is finite by (5.1).

Denoting by u the corresponding solution to (1.1) and introducing

$$\tau(u_0) := \left(\frac{(N+2)-q(N+1)}{(N+1)q-N}\right)^{1-q} R(u_0)^2\,,$$

we infer from [5, Lemma 2.2 & Proposition 2.4] that there is a constant C_1 depending only on N and q such that

$$t^{(a-N)/2}\, \|u(t)\|_1 + t^{a/2}\, \|u(t)\|_\infty + t^{(a+1)/2}\, \|\nabla u(t)\|_\infty \leq C_1 \tag{5.2}$$

for each $t > \tau(u_0)$ and

$$u(x,t) \leq \Gamma_q(|x| - R(u_0))\,, \qquad t > 0\,, \quad |x| > R(u_0)\,. \tag{5.3}$$

Here, Γ_q is given by $\Gamma_q(r) = \gamma_q \; r^{-a}$, $r \in (0,\infty)$.

Let us observe at this point that decay estimates for $\nabla u(t)$ in L^p can be deduced from (5.2) and the Duhamel formula.

Lemma 5.1 *For $p \in [1,\infty]$, there is a constant $C(p)$ depending only on N, q and p such that*

$$t^{((a+1)p-N)/2p} \; \|\nabla u(t)\|_{L^p} \le C(p) \quad \text{for} \quad t > \tau(u_0)\,. \tag{5.4}$$

PROOF. Indeed, since u is non-negative, it follows from [4, Theorem 1] that

$$\|\nabla u^{(q-1)/q}(t)\|_\infty \le C(q)\; t^{-1/q}$$

for $t > 0$, which, together with (5.2) and the Duhamel formula entails that, for $t > \tau(u_0)$,

$$\begin{aligned} \|\nabla u(t)\|_1 &\le \|\nabla G(t/2) * u(t/2)\|_1 + \int_{t/2}^t \|\nabla G(t-s) * |\nabla u|^q\|_1 \; ds \\ &\le C\; t^{-1/2}\; \|u(t/2)\|_1 + C \int_{t/2}^t (t-s)^{-1/2} \; \left\|\nabla u^{(q-1)/q}(s)\right\|_\infty^q \; \|u(s)\|_1 \; ds \\ &\le C\; t^{-(a+1-N)/2} + C \int_{t/2}^t (t-s)^{-1/2}\; s^{-(a+2-N)/2}\; ds \\ &\le C\; t^{-(a+1-N)/2}\,. \end{aligned}$$

Interpolating between (5.2) and the above estimate yields (5.4). □

In order to investigate the large time behavior of u, we use a rescaling method and introduce the sequence of rescaled solutions $(u_k)_{k\ge1}$ defined by

$$u_k(x,t) = k^a \; u(kx, k^2 t)\,, \qquad (x,t) \in \mathbb{R}^N \times [0,\infty)\,, \quad k \ge 1\,.$$

Lemma 5.2 *For $k \ge 1$, we have*

$$t^{(a-N)/2} \; \|u_k(t)\|_1 + t^{a/2} \; \|u_k(t)\|_\infty + t^{(a+1)/2} \; \|\nabla u_k(t)\|_\infty \le C_1 \tag{5.5}$$

for $t > \tau_k := \tau(u_0)\; k^{-2}$ and

$$u_k(x,t) \le \Gamma_q \left(|x| - \frac{R(u_0)}{k} \right) \quad \text{for} \quad |x| > \frac{R(u_0)}{k} \quad \text{and} \quad t > 0\,. \tag{5.6}$$

PROOF. It is straightforward to check that, for each $k \ge 1$, u_k is the solution to (1.1) with initial datum $u_k(0)$ and satisfies estimates (5.5) and (5.6) as a consequence of (5.2) and (5.3). □

We next use (1.1) and the non-negativity of u_k to control the behavior of $u_k(x,t)$ for large x uniformly with respect to k. For $k \ge 1$, $t > 0$ and $R \ge 0$, we put

$$I_k(R,t) := \int_{\{|x|\ge R\}} u_k(x,t) \; dx + \int_0^t \int_{\{|x|\ge R\}} |\nabla u_k(x,t)|^q \; dxdt\,. \tag{5.7}$$

Lemma 5.3 *For every $T > 0$, we have*

$$\lim_{R\to\infty} \sup_{k\geq 1} \sup_{t\in[0,T]} I_k(R,t) = 0\,. \tag{5.8}$$

PROOF. Let ϱ be a non-negative function in $\mathcal{C}^\infty(\mathbb{R}^N)$ such that $0 \leq \varrho \leq 1$ and

$$\varrho(x) = 0 \quad \text{if} \quad |x| \leq \frac{1}{2} \quad \text{and} \quad \varrho(x) = 1 \quad \text{if} \quad |x| \geq 1\,.$$

For $R > 0$ and $x \in \mathbb{R}^N$, we set $\varrho_R(x) = \varrho(x/R)$. As u_k is a non-negative solution to (1.1), we have

$$\begin{aligned} I_k(R,t) &\leq \int u_k(x,t)\ \varrho_R(x)\ dx + \int_0^t \int |\nabla u_k(x,s)|^q\ \varrho_R(x)\ dxds \\ &\leq \int u_k(x,0)\ \varrho_R(x)\ dx + \int_0^t \int u_k(x,s)\ |\Delta \varrho_R(x)|\ dxds \\ &\leq k^{a-N} \int_{\{|x|\geq kR/2\}} u_0(x)\ dx + \frac{|\Delta\varrho|_\infty}{R^2} \int_0^t \int_{\{R/2\leq|x|\leq R\}} u_k(x,s)\ dxds\,. \qquad (5.9) \end{aligned}$$

Owing to (5.1) and (5.6), we further obtain that, for $R \geq 1 + 4\ R(u_0)$,

$$\begin{aligned} I_k(R,t) &\leq k^{a-N} \int_{\{|x|\geq kR/2\}} \Gamma_q\left(\frac{|x|}{2}\right)\ dx \\ &\quad + \frac{|\Delta\varrho|_\infty}{R^2} \int_0^t \int_{\{R/2\leq|x|\leq R\}} \Gamma_q\left(|x| - \frac{R(u_0)}{k}\right)\ dxds \\ &\leq C\ R^{-(a-N)} + \frac{T\ |\Delta\varrho|_\infty}{R^2} \int_{\{R/2\leq|x|\leq R\}} \Gamma_q\left(\frac{R}{4}\right)\ dx \\ &\leq C(T,\varrho)\ R^{-(a-N)}\,. \end{aligned}$$

Lemma 5.3 then readily follows since $a > N$. □

We finally study the behavior of u_k for small times.

Lemma 5.4 *Let $r > 0$. There is a positive constant $C(r)$ depending only on q, N and r such that*

$$\int_{\{|x|\geq r\}} u_k(x,t)\ dx \leq C(r) \left(\sup_{|x|\geq kr/2} \{|x|^a\ u_0(x)\} + t \right) \tag{5.10}$$

for $t > \tau_k$ and $k \geq 4\ R(u_0)/r$.

PROOF. We fix $r > 0$ and use the same notations as in the proof of Lemma 5.3. Thanks to the properties of ϱ, we infer from (5.9) with $R = r$ that, for $t > \tau_k$ and

$k \geq 4\ R(u_0)/r$,

$$\begin{aligned}
\int_{\{|x|\geq r\}} u_k(x,t)\ dx &\leq \int u_k(x,t)\ \varrho_r(x)\ dx \\
&\leq k^{a-N} \int_{\{|x|\geq kr/2\}} u_0(x)\ dx \\
&+ \frac{|\Delta\varrho|_\infty}{r^2} \int_0^t \int_{\{r/2\leq|x|\leq r\}} u_k(x,s)\ dxds \\
&\leq C(\varrho,r) \left(\sup_{|x|\geq kr/2} \{|x|^a\ u_0(x)\} + t \right),
\end{aligned}$$

where we have used (5.6) to obtain the last inequality. □

PROOF OF THEOREM 2.2. Owing to Lemma 5.2 and Lemma 5.3 we may proceed as in [4, Theorem 3] to prove that there are a subsequence of (u_k) (not relabeled) and a non-negative function

$$u_\infty \in \mathcal{C}((0,\infty); L^1(\mathbb{R}^N)) \cap L^q((s,\infty)\times\mathbb{R}^N)) \cap L^\infty(s,\infty; W^{1,\infty}(\mathbb{R}^N))$$

satisfying

$$u_\infty(t) = G(t-s) * u_\infty(s) - \int_s^t G(t-\tau) * |\nabla u_\infty(\tau)|^q\ d\tau$$

and

$$\lim_{k\to\infty} \sup_{\tau\in[s,t]} \|u_k(\tau) - u_\infty(\tau)\|_1 = 0 \tag{5.11}$$

for every $s > 0$ and $t > s$.

It remains to identify the behavior of u_∞ as $t \to 0$. On the one hand, consider $r > 0$ and $t > 0$. Since $\tau_k \to 0$ as $k \to \infty$, we have $t > \tau_k$ for k large enough and it follows from Lemma 5.4, (5.1) and (5.11) that

$$0 \leq \int_{\{|x|\geq r\}} u_\infty(x,t)\ dx \leq C(r)\ t\,.$$

Consequently,

$$\lim_{t\to 0} \int_{\{|x|\geq r\}} u_\infty(x,t)\ dx = 0\,. \tag{5.12}$$

On the other hand, consider $M > 0$ and set $k_M := M^{1/(a-N)}$. For $k \geq k_M$, we denote by v_k the solution to (1.1) with initial datum $v_k(0)$ given by $v_k(x,0) := M\ k^N\ u_0(kx)$, $x \in \mathbb{R}^N$. Since $a > N$, we have $v_k(0) \leq u_k(0)$ for $k \geq k_M$ and the comparison principle warrants that

$$v_k(x,t) \leq u_k(x,t)\,, \qquad (x,t) \in \mathbb{R}^N \times [0,\infty)\,, \qquad k \geq k_M\,. \tag{5.13}$$

We next observe that $(v_k(0))$ converges narrowly towards $(M\ \|u_0\|_1)\ \delta$ as $k \to \infty$ (δ denoting the Dirac mass at $x=0$). We then proceed as in [4] to conclude that

$$\lim_{k\to\infty} \sup_{\tau\in[s,t]} \|v_k(\tau) - S_M(\tau)\|_1 = 0$$

for every $s>0$ and $t>s$, where S_M denotes the unique non-negative solution to (1.1) with initial datum $(M\ \|u_0\|_1)\ \delta$ [4]. Recalling (5.11) and (5.13), we realize that

$$S_M(x,t) \le u_\infty(x,t)\,, \qquad (x,t)\in\mathbb{R}^N\times(0,\infty)\,.$$

The above inequality being valid for any $M>0$, it is then straightforward to deduce that

$$\lim_{t\to 0} \int_{\{|x|\le r\}} u_\infty(x,t)\ dx = \infty\,. \tag{5.14}$$

In other words, u_∞ is a very singular solution to (1.1) and the uniqueness of the very singular solution to (1.1) (cf. [3, 23]) implies that $u_\infty = W$, where W is the very singular solution to (1.1), see Theorem 2.2. The uniqueness of the limit actually entails that the whole sequence $(u_k)_{k\ge 1}$ converges towards W in $\mathcal{C}([s,t];L^1(\mathbb{R}^N))$ for $s>0$ and $t>s$. Expressed in terms of u, we have thus shown that

$$\lim_{t\to\infty} t^{(a-N)/2}\ \|u(t)-W(t)\|_1 = 0\,. \tag{5.15}$$

Finally, it follows from (5.2), (5.15) and the Gagliardo-Nirenberg inequality that (2.6) holds true.

The last step of the proof is to obtain the convergence (2.7) for the gradients. Consider $p\in[1,\infty]$, $t>0$ and $\alpha\in(0,1)$. By the Duhamel formula, we have

$$\begin{aligned}
A_p(t) &:= t^{((a+1)p-N)/2p}\ \|\nabla(u-W)(t)\|_{L^p}\\
&\le t^{((a+1)p-N)/2p}\ \|\nabla G((1-\alpha)t)*(u-W)(\alpha t)\|_{L^p}\\
&+ t^{((a+1)p-N)/2p} \int_{\alpha t}^t \|\nabla G(t-s)*(|\nabla u(s)|^q - |\nabla W(s)|^q)\|_{L^p}\ ds\\
&\le C(\alpha)\ t^{(a-N)/2}\ \|(u-W)(\alpha t)\|_1\\
&+ C\ t^{((a+1)p-N)/2p} \int_{\alpha t}^t (t-s)^{-1/2}\ s^{-1/2}\ \|\nabla(u-W)(s)\|_{L^p}\ ds\,,
\end{aligned}$$

where we have used the fact that

$$\max\{\|\nabla u(s)\|_\infty, \|\nabla W(s)\|_\infty\} \le C\ s^{-(a+1)/2}$$

by (5.2) and the properties of W in order to obtain the last inequality. Consequently, by the definition of $A_p(t)$ and the change of variables $s\mapsto ts$, we obtain

$$A_p(t) \le C(\alpha)\ t^{(a-N)/2}\ \|(u-W)(\alpha t)\|_1$$

$$
\begin{aligned}
&+ \quad C\, t^{((a+1)p-N)/2p} \int_{\alpha t}^{t} (t-s)^{-1/2}\, s^{-1/2}\, s^{-((a+1)p-N)/2p}\, A_p(s)\, ds \\
&\leq \quad C(\alpha)\, t^{(a-N)/2}\, \|(u-W)(\alpha t)\|_1 \\
&+ \quad C \int_{\alpha}^{1} (1-s)^{-1/2}\, s^{-1/2}\, s^{-((a+1)p-N)/2p}\, A_p(st)\, ds\,.
\end{aligned}
$$

Now, introducing

$$A_p(\infty) := \limsup_{t\to+\infty} A_p(t) \geq 0\,,$$

which is finite by (5.4), we may let $t \to +\infty$ in the above inequality and use (5.15) to conclude that

$$A_p(\infty) \leq C \int_{\alpha}^{1} (1-s)^{-1/2}\, s^{-1/2}\, s^{-((a+1)p-N)/2p}\, ds\; A_p(\infty)\,.$$

Finally, the choice of $\alpha < 1$ sufficiently close to 1 readily yields that $A_p(\infty) = 0$, from which (2.7) follows. □

6 Proofs of Theorems 2.4 and 2.5

PROOF OF THEOREM 2.4, PART A). The required non-positive self-similar solution

$$V = V(x,t) = t^{-(2-q)/(2(q-1))} V\left(x\, t^{-1/2}, 1\right)$$

is constructed and studied in [7, Theorem 3.5]. In particular, it is shown that the self-similar profile $\mathcal{V}(x) := V(x,1)$ is a radially symmetric bounded $\mathcal{C}^2$ function. Moreover, the profile $\mathcal{V}$ and its first derivative $\mathcal{V}'$ both decay exponentially as $|x| \to \infty$ (see [7, Proposition 3.14]) □

PROOF OF THEOREM 2.4, PART B). Recall that by assumption (2.8), $u = u(x,t)$ is a non-positive solution to (1.1). For $t \geq 0$, we put $m(t) = \inf\{u(x,t)\,,\ x \in \mathbb{R}^N\} \leq 0$. The comparison principle ensures that $t \mapsto m(t)$ is a non-decreasing function of time and

$$m_\infty := \sup_{t\geq 0}\ m(t) \in (-\infty, 0]\,.$$

Since u is a classical solution to (1.1), it follows from (1.1) that

$$u(x,t) \leq u_0(x) + \int_0^t \Delta u(x,\tau)\, d\tau \leq u_0(x) + \int_0^t \sup_{y\in\mathbb{R}^N} \Delta u(y,\tau)\, d\tau$$

for every $x \in \mathbb{R}^N$ and $t \geq 0$. Therefore,

$$m(t) \leq -\|u_0\|_\infty + \int_0^t \sup_{y\in\mathbb{R}^N} \Delta u(y,\tau)\, d\tau\,,$$

and we infer from (3.9) and (3.10) that

$$m(t) \le -\|u_0\|_\infty + T \ \left\|(\Delta u_0)^+\right\|_\infty + C \ \|u_0\|_\infty^{(2-q)/q} \int_T^t \tau^{-2/q} \ d\tau$$

for $T > 0$ and $t > T$. Since $q < 2$, we may let $t \to \infty$ in the above inequality and obtain with the choice $T = \|u_0\|_\infty^{(2-q)/2} \ \|(\Delta u_0)^+\|_\infty^{-q/2}$ that there is a constant K depending only on q such that

$$m_\infty \le -\|u_0\|_\infty + K^{q/2} \ \left\|(\Delta u_0)^+\right\|_\infty^{(2-q)/2} \ \|u_0\|_\infty^{(2-q)/2} . \tag{6.1}$$

Therefore, if $\|u_0\|_\infty > K \ \|(\Delta u_0)^+\|_\infty^{(2-q)/q}$, we readily conclude from (6.1) that $m_\infty < 0$, whence (2.13). □

Proof of Theorem 2.5. The proof of the first assertion of Theorem 2.5 is the same as that of Theorem 2.4, part b), hence we skip it. We next assume that $N \le 3$ and that $1 < q < 4/(1+\sqrt{1+2N})$. For $t > 0$, we put

$$\ell(t) := \|u(t)\|_\infty \ \left\|(\Delta u(t))^+\right\|_\infty^{1-2/q} .$$

Since u is a non-positive subsolution to the linear heat equation, we infer from classical properties of the heat semigroup that

$$\|u(t)\|_\infty \ge \|G(t) * u_0\|_\infty \ge C \ t^{-N/2}$$

for t large enough. As $q < 2$, this estimate and (3.9) entail that, for t large enough,

$$\ell(t) \ge C \ t^{(4(2-q)-Nq^2)/2q^2} \underset{t\to\infty}{\longrightarrow} \infty ,$$

since $q < 4/(1+\sqrt{1+2N})$. Consequently, there exists t_0 large enough such that $\ell(t_0) > K(q)$ and we may apply the first assertion of Theorem 2.5 to $t \longmapsto u(t_0+t)$ to complete the proof. □

Under the assumptions of Theorem 2.4, part b) or Theorem 2.5, we may actually bound the L^1-norm of $u(t)$ from below and improve significantly [21, Proposition 2.1].

Proposition 6.1 *Assume that u_0 satisfies (2.8) and that*

$$M_\infty := \lim_{t\to\infty} \|u(t)\|_\infty > 0 .$$

Then there is a constant $C = C(N,q,u_0)$ such that

$$\|u(t)\|_1 \ge C \ t^{N/q} , \quad t \ge 0 . \tag{6.2}$$

PROOF. We fix $t > 0$. For $k \geq 1$, let $x_k \in \mathbb{R}^N$ be such that $\|u(t)\|_\infty - 1/k \leq -u(x_k, t)$. For $R > 0$, it follows from (3.1) and the time monotonicity of $\|u(t)\|_\infty$ that

$$\begin{aligned}
\|u(t)\|_1 &\geq -\int_{\{|x-x_k|\leq R\}} u(x,t)\ dx \\
&\geq \int_{\{|x-x_k|\leq R\}} \left(-u(x_k,t) - |x-x_k|\ \|\nabla u(t)\|_\infty\right)\ dx \\
&\geq C\left(\frac{R^N}{N}\left(\|u(t)\|_\infty - \frac{1}{k}\right) - \frac{C_1\ R^{N+1}}{N+1}\ \|u_0\|_\infty^{1/q}\ t^{-1/q}\right) \\
&\geq C\ R^N\ \left(M_\infty - \frac{1}{k} - C'\ R\ t^{-1/q}\right).
\end{aligned}$$

Letting $k \to \infty$ and choosing $R = \left(M_\infty\ t^{1/q}\right)/(2\ C')$ yields the claim (6.2). $\square$

7 Proof of Theorem 2.6 and Proposition 2.1

PROOF OF THEOREM 2.6.
STEP 1. Recall that, by (2.8), u_0 is a non-positive function. We assume further that u_0 is compactly supported in a ball $B(0, R_0)$ of $\mathbb{R}^N$ for some $R_0 > 0$.

For $\lambda \geq 1$, we introduce

$$u_\lambda(x,t) := u(\lambda x, \lambda^q t)\,, \quad (x,t) \in \mathbb{R}^N \times (0,\infty)\,,$$

which solves

$$u_{\lambda,t} + |\nabla u_\lambda|^q = \lambda^{q-2}\ \Delta u_\lambda \quad \text{in} \quad \mathbb{R}^N \times (0,\infty) \tag{7.1}$$

with initial datum $u_\lambda(0)$.

Lemma 7.1 *There is a constant $C = C(N, q, \|u_0\|_\infty)$ such that, for $t \geq 0$ and $\lambda \geq 1$,*

$$\|u_\lambda(t)\|_\infty + t^{1/q}\ \|\nabla u_\lambda(t)\|_\infty + t\ \|u_{\lambda,t}(t)\|_\infty \leq C\,. \tag{7.2}$$

PROOF. It first follows from (1.3) that

$$\|u_\lambda(t)\|_\infty = \|u(\lambda^q t)\|_\infty \leq \|u_0\|_\infty\,,$$

while Proposition 3.1 yields

$$\|\nabla u_\lambda(t)\|_\infty = \lambda\ \|\nabla u(\lambda^q t)\|_\infty \leq C_1\ \|u_0\|_\infty^{1/q}\ t^{-1/q}\,.$$

We next infer from [16, Theorem 5] that

$$\|u_{\lambda,t}(t)\|_\infty = \lambda^q\ \|u_t(\lambda^q t)\|_\infty \leq \lambda^q\ C(N,q)\ \|u_0\|_\infty\ \left(\lambda^q t\right)^{-1} = C(N,q)\ \|u_0\|_\infty\ t^{-1}\,,$$

which completes the proof. □

Owing to Lemma 7.1, we may apply the Arzelà-Ascoli theorem and deduce that there are a subsequence of (u_λ) (not relabeled) and a non-positive function $u_\infty \in \mathcal{C}(\mathbb{R}^N \times (0,\infty))$ such that

$$u_\lambda \longrightarrow u_\infty \quad \text{in} \quad \mathcal{C}(B(0,R)\times(t_1,t_2)) \tag{7.3}$$

for any $R>0$ and $0<t_1<t_2$. It also follows from (7.3) and Lemma 7.1 that $u_\infty(t)\in\mathcal{BUC}(\mathbb{R}^N)$ and satisfies

$$\|u_\infty(t)\|_\infty + t^{1/q}\ \|\nabla u_\infty(t)\|_\infty + t\ \|u_{\infty,t}(t)\|_\infty \le C \tag{7.4}$$

for each $t>0$. We next introduce the function $H_\lambda : \mathbb{R}\times\mathbb{R}^N\times\mathcal{S}_N(\mathbb{R}) \to \mathbb{R}$ defined by

$$H_\lambda(\xi_0,\xi,S) := \xi_0 + |\xi|^q - \lambda^{q-2}\ \mathrm{tr}(S)\,,$$

where $\mathcal{S}_N(\mathbb{R})$ denotes the subset of symmetric matrices of $\mathcal{M}_N(\mathbb{R})$ and $\mathrm{tr}(S)$ denotes the trace of the matrix S. On the one hand, we notice that (7.1) reads

$$H_\lambda(u_{\lambda,t},\nabla u_\lambda, D^2u_\lambda) = 0 \quad \text{in} \quad \mathbb{R}^N\times(0,\infty)$$

and that H_λ is elliptic. On the other hand, H_λ converges uniformly on every compact subset of $\mathbb{R}\times\mathbb{R}^N\times\mathcal{S}_N(\mathbb{R})$ towards $H_\infty : \mathbb{R}\times\mathbb{R}^N \to \mathbb{R}$ given by $H_\infty(\xi_0,\xi) := \xi_0 + |\xi|^q$. Therefore, for every $\tau>0$, $u_\infty(.+\tau)$ is the unique viscosity solution to (2.18) with initial datum $u_\infty(\tau)$ (see, e.g., [10, Proposition IV.1] and [9, Theorem 4.1]). In addition, since $u_\infty(\tau)$ is bounded and Lipschitz continuous by (7.4), we infer from [14, Section 10.3, Theorem 3] that $u_\infty(.+\tau)$ is given by the Hopf-Lax formula

$$u_\infty(x,t+\tau) = \inf_{y\in\mathbb{R}^N}\left\{u_\infty(y,\tau) + (q-1)\ q^{-q/(q-1)}\ |x-y|^{q/(q-1)}\ t^{-1/(q-1)}\right\} \tag{7.5}$$

for $(x,t)\in\mathbb{R}^N\times[0,\infty)$.

It remains to identify the behavior of $u_\infty(t)$ as $t\to 0$. Consider first $x\in\mathbb{R}^N$, $t\in(0,\infty)$ and $s\in(0,t)$. We infer from (3.9) and (7.1) that

$$\begin{aligned} u_\lambda(x,t) &\le u_\lambda(x,s) + \lambda^{q-2}\int_s^t \Delta u_\lambda(x,\sigma)\ d\sigma \\ &\le u_\lambda(x,s) + \lambda^{q-2}\int_s^t \lambda^2\ C\ \left(\lambda^q\sigma\right)^{-2/q}\ d\sigma \\ &\le u_\lambda(x,s) - C\ \lambda^{q-2}\ \left(t^{(q-2)/q} - s^{(q-2)/q}\right)\,. \end{aligned}$$

Since $q\in(1,2)$, we may pass to the limit as $\lambda\to\infty$ in the previous inequality and use (7.3) to deduce that $t\mapsto u_\infty(x,t)$ is non-increasing for every $x\in\mathbb{R}^N$. Since u_∞ is bounded by (7.4), we may thus define $u_\infty(0)$ by

$$u_\infty(x,0) := \sup_{t>0}\{u_\infty(x,t)\} \in (-\infty,0] \quad \text{for} \quad x\in\mathbb{R}^N\,. \tag{7.6}$$

In particular, $u_\infty(0)$ is a lower semicontinuous function as the supremum of continuous functions.

More information on $u_\infty(0)$ are consequences of the next result.

Lemma 7.2 *For each $t>0$, there is $\varrho(t)>0$ such that $u_\infty(x,t)=0$ if $|x|>\varrho(t)$ and*

$$\lim_{\lambda\to\infty} \|u_\lambda(t)-u_\infty(t)\|_\infty = 0\,. \tag{7.7}$$

Moreover, $\varrho(t)\to 0$ as $t\to 0$.

Taking Lemma 7.2 for granted, we see that (7.6) and Lemma 7.2 imply that $u_\infty(x,0)=0$ for $x\neq 0$ since $\varrho(t)\to 0$ as $t\to 0$. We set $\ell := -u_\infty(0,0)$, so that

$$u_\infty(x,0) = -\ell\ \mathbf{1}_{\{0\}}(x)\,, \quad x\in\mathbb{R}^N\,,$$

and fix $(x,t)\in\mathbb{R}^N\times(0,\infty)$. We will now proceed along the lines of [24] to show that $u_\infty(x,t)=Z_\ell(x,t)$ (recall that Z_ℓ is defined in (2.17)). Introducing the notation $\mu := (q-1)\ q^{-q/(q-1)}\ t^{-1/(q-1)}$, it follows from (7.6) and Lemma 7.2 that, for $0<\sigma<\tau$ and $|y|\le\varrho(\sigma)$,

$$\begin{aligned} u_\infty(y,\sigma)+\mu\ |x-y|^{q/(q-1)} &\ge u_\infty(y,\tau)+\mu\ |x-y|^{q/(q-1)} \\ &\ge u_\infty(0,\tau)+\mu\ |x|^{q/(q-1)}-\omega(\sigma)\,, \end{aligned}$$

with

$$\omega(\sigma) := \sup_{|y|\le\varrho(\sigma)} |u_\infty(y,\tau)-u_\infty(0,\tau)| + \mu \sup_{|y|\le\varrho(\sigma)} \left| |x-y|^{q/(q-1)} - |x|^{q/(q-1)} \right|\,,$$

while, for $0<\sigma<\tau$ and $|y|\ge\varrho(\sigma)$,

$$u_\infty(y,\sigma)+\mu\ |x-y|^{q/(q-1)} \ge 0\,.$$

The previous bounds from below and (7.5) entail that

$$u_\infty(x,t+\sigma) \ge \min\left\{0, u_\infty(0,\tau)+\mu\ |x|^{q/(q-1)}-\omega(\sigma)\right\}$$

for $0<\sigma<\tau$. Since $\varrho(\sigma)\to 0$ as $\sigma\to 0$ and $u_\infty\in\mathcal{C}(\mathbb{R}^N\times(0,\infty))$, we may pass to the limit as $\sigma\to 0$ in the above inequality and obtain

$$u_\infty(x,t) \ge \min\left\{0, u_\infty(0,\tau)+\mu\ |x|^{q/(q-1)}\right\}$$

for $\tau>0$. Letting $\tau\to 0$ yields

$$u_\infty(x,t) \ge \min\left\{0, -\ell+\mu\ |x|^{q/(q-1)}\right\} = Z_\ell(x,t)\,.$$

On the other hand, (7.5) and (7.6) ensure that

$$u_\infty(x,t+\tau) \le \inf_{y\in\mathbb{R}^N} \left\{ u_\infty(y,0) + \mu\ |x-y|^{q/(q-1)} \right\} = Z_\ell(x,t)\,,$$

whence $u_\infty(x,t) \le Z_\ell(x,t)$ by the continuity of u_∞ in $\mathbb{R}^N \times (0,\infty)$. We have thus shown that $u_\infty = Z_\ell$. In particular, $\|u_\infty(t)\|_\infty = \ell$ for $t \ge 0$. But (2.15) and (7.7) imply

$$\|u_\infty(t)\|_\infty = \lim_{\lambda\to\infty} \|u_\lambda(t)\|_\infty = \lim_{\lambda\to\infty} \|u(\lambda^q t)\|_\infty = M_\infty\,,$$

whence $\ell = M_\infty$ and $u_\infty = Z_{M_\infty}$. For $t>0$, the sequence $(u_\lambda(t))$ has thus only one possible cluster point in $L^\infty(\mathbb{R}^N)$ as $\lambda\to\infty$, from which we conclude that the whole family $(u_\lambda(t))$ converges to $Z_{M_\infty}(t)$ in $L^\infty(\mathbb{R}^N)$ as $\lambda\to\infty$. In particular, for $t=1$,

$$\lim_{\lambda\to\infty} \|u_\lambda(1) - Z_{M_\infty}(1)\|_\infty = 0\,.$$

Setting $\lambda = t^{1/q}$ and using the self-similarity of Z_{M_∞}, we are finally led to (2.16).

STEP 2. We now consider an arbitrary function $u_0 \in \mathcal{C}_0(\mathbb{R}^N)$ fulfilling (2.8) and such that (2.15) holds true. There is a sequence (u_0^n) of non-positive functions in $\mathcal{C}_c^\infty(\mathbb{R}^N)$ such that

$$u_0^n \longrightarrow u_0 \quad \text{in} \quad L^\infty(\mathbb{R}^N)\,.$$

For $n\ge 1$, we denote by u^n the solution to (1.1) with initial datum u_0^n and put

$$M_\infty^n := \lim_{t\to\infty} \|u^n(t)\|_\infty\,.$$

By [17, Corollary 4.3], we have

$$\|u^n(t) - u(t)\|_\infty \le \|u_0^n - u_0\|_\infty \quad \text{for} \quad t\ge 0\,,$$

from which we readily deduce that

$$|M_\infty^n - M_\infty| \le \|u_0^n - u_0\|_\infty\,.$$

Consequently, $M_\infty^n \longrightarrow M_\infty$ as $n\to\infty$ and (2.15) guarantees that $M_\infty^n > 0$ for n large enough. The analysis performed in the previous step then implies that

$$\lim_{t\to\infty} \|u^n(t) - Z_{M_\infty^n}(t)\|_\infty = 0$$

for n large enough. Therefore,

$$\begin{aligned}
\|u(t) - Z_{M_\infty}(t)\|_\infty &\le \|u(t) - u^n(t)\|_\infty + \|u^n(t) - Z_{M_\infty^n}(t)\|_\infty \\
&+ \|Z_{M_\infty^n}(t) - Z_{M_\infty}(t)\|_\infty \\
&\le \|u_0^n - u_0\|_\infty + \|u^n(t) - Z_{M_\infty^n}(t)\|_\infty + |M_\infty^n - M_\infty| \\
&\le 2\ \|u_0^n - u_0\|_\infty + \|u^n(t) - Z_{M_\infty^n}(t)\|_\infty\,,
\end{aligned}$$

whence

$$\limsup_{t\to\infty} \|u(t) - Z_{M_\infty}(t)\|_\infty \le 2\ \|u_0^n - u_0\|_\infty$$

for n large enough. Letting $n \to \infty$ then completes the proof of Theorem 2.6. $\square$

Proof of Lemma 7.2. Let h_0 be a non-positive function in $\mathcal{C}_c^\infty(\mathbb{R})$ such that $h_0(y) = -\|u_0\|_\infty$ if $y \in (-R_0, R_0)$ (recall that u_0 is compactly supported in $B(0, R_0)$). We denote by h the solution to the one-dimensional viscous Hamilton-Jacobi equation

$$\begin{aligned} h_t - h_{yy} + |h_y|^q &= 0 \quad \text{in} \quad \mathbb{R} \times (0,\infty)\,, \\ h(0) &= h_0 \quad \text{in} \quad \mathbb{R}\,. \end{aligned}$$

For $i \in \{1, \ldots, N\}$ and $(x,t) \in \mathbb{R}^N \times (0,\infty)$, we put $h^i(x,t) := h(x_i, t)$ and notice that h^i is the solution to (1.1) with initial datum $h^i(0) \le u_0$. The comparison principle then entails that

$$h(x_i,t) = h^i(x,t) \le u(x,t) \le 0\,, \quad (x,t) \in \mathbb{R}^N \times (0,\infty)\,. \tag{7.8}$$

We next introduce $w := h_y$ and notice that w is the solution to the one-dimensional convection-diffusion equation

$$\begin{aligned} w_t - w_{yy} + \left(|w|^q\right)_y &= 0 \quad \text{in} \quad \mathbb{R} \times (0,\infty)\,, \\ w(0) &= w_0 := h_{0,y} \quad \text{in} \quad \mathbb{R}\,. \end{aligned} \tag{7.9}$$

The comparison principle then entails that

$$b(y,t) \le w(y,t) \le a(y,t)\,, \quad (y,t) \in \mathbb{R} \times (0,\infty)\,, \tag{7.10}$$

where $b \le 0$ and $a \ge 0$ denote the solutions to (7.9) with initial data $b(0) = -w_0^- \le 0$ and $a(0) = w_0^+ \ge 0$. Since $w_0 \in L^1(\mathbb{R})$, it follows from [13] that

$$\lim_{t\to\infty} \|b(t) - \Sigma_{-B}(t)\|_1 = \lim_{t\to\infty} \|a(t) - \Sigma_A(t)\|_1 = 0\,, \tag{7.11}$$

where $B := \|b(0)\|_1$, $A := \|a(0)\|_1$, and, for $M \in \mathbb{R}$, Σ_M is the source solution to the one-dimensional conservation law

$$\begin{aligned} \Sigma_{M,t} + \left(|\Sigma_M|^q\right)_y &= 0 \quad \text{in} \quad \mathbb{R} \times (0,\infty)\,, \\ \Sigma(0) &= M\ \delta_0 \quad \text{in} \quad \mathbb{R}\,. \end{aligned}$$

Here, δ_0 denotes the Dirac mass in $\mathbb{R}$ centered at $y = 0$. The source solution Σ_M is actually given by

$$\Sigma_M(y,t) := y^{1/(q-1)}\ (qt)^{-1/(q-1)}\ \mathbf{1}_{[0,\xi_M(t)]}(y)\,, \quad \xi_M(t) := q\ \left(\frac{M}{q-1}\right)^{(q-1)/q}\ t^{1/q}\,,$$

if $M \geq 0$, and

$$\Sigma_M(y,t) := -|y|^{1/(q-1)}\ (qt)^{-1/(q-1)}\ \mathbf{1}_{[-\eta_M(t),0]}(y)\,, \quad \eta_M(t) := q\ \left(\frac{-M}{q-1}\right)^{(q-1)/q}\ t^{1/q}\,,$$

if $M \leq 0$ (see, e.g., [22]). In particular, Σ_M satisfies

$$\lambda\ \Sigma_M(\lambda y, \lambda^q t) = \Sigma_M(y,t) \quad \text{for} \quad (\lambda, y, t) \in (0,\infty)\times\mathbb{R}\times(0,\infty)\,. \tag{7.12}$$

Now, let $t > 0$ and set

$$\varrho(t) := N^{1/2}\ \max\{\xi_A(t), \eta_{-B}(t)\} \leq C\ t^{1/q}\,.$$

If $x \in \mathbb{R}^N$ is such that $|x| > \varrho(t)$, there is $i \in \{1,\ldots,N\}$ such that $|x_i| > \max\{\xi_A(t), \eta_{-B}(t)\}$, whence either $x_i > \xi_A(t)$ or $x_i < -\eta_{-B}(t)$. In the latter case, we infer from (7.8), (7.10) and (7.12) that

$$\begin{aligned}
0 \geq u_\lambda(x,t) &\geq h(\lambda x_i, \lambda^q t) = \int_{-\infty}^{\lambda x_i} w(y', \lambda^q t)\ dy' \\
&\geq \lambda \int_{-\infty}^{x_i} b(\lambda y', \lambda^q t)\ dy' \\
&\geq \lambda \int_{-\infty}^{x_i} (b(\lambda y', \lambda^q t) - \Sigma_{-B}(\lambda y', \lambda^q t))\ dy' \\
&\geq -\|(b - \Sigma_{-B})(\lambda^q t)\|_1\,.
\end{aligned}$$

Similarly, if $x_i > \xi_A(t)$, (7.8), (7.10) and (7.12) yield

$$0 \geq u_\lambda(x,t) \geq -\|(a - \Sigma_A)(\lambda^q t)\|_1\,.$$

Therefore, if $x \in \mathbb{R}^N$ is such that $|x| > \varrho(t)$, then

$$|u_\lambda(x,t)| \leq \max\{\|(a-\Sigma_A)(\lambda^q t)\|_1, \|(b-\Sigma_{-B})(\lambda^q t)\|_1\}\,. \tag{7.13}$$

Passing to the limit as $\lambda \to \infty$ in (7.13) and using (7.3) and (7.11) provide the first assertion of Lemma 7.2. We next use once more (7.3) and (7.13) to conclude that (7.7) holds true. $\square$

Proof of Proposition 2.1. We keep the notations of the proof of Theorem 2.6 and introduce

$$U_\lambda(x,t) := u_{\lambda,x}(x,t) = \lambda\ u_x(\lambda x, \lambda^q t)\,, \quad (x,t) \in \mathbb{R}\times(0,\infty)\,.$$

It follows from (7.1) and Lemma 7.1 that

$$U_{\lambda,t} + \left(|U_\lambda|^q\right)_x = \lambda^{q-2}\ U_{\lambda,xx}\,, \quad (x,t) \in \mathbb{R}\times(0,\infty)\,,$$

and

$$\|U_\lambda(t)\|_1 \le \|u_{0,x}\|_1 \quad \text{and} \quad t^{1/q}\, \|U_\lambda(t)\|_\infty \le C \tag{7.14}$$

for $t > 0$. We recall that, by Theorem 2.6, the family (u_λ) converges towards Z_{M_∞} in $\mathcal{C}(\mathbb{R}^N \times [t_1, t_2])$ for any $t_2 > t_1 > 0$. Owing to (7.14), we readily conclude that (U_λ) converges weakly-$\star$ towards $Z_{M_\infty,x}$ in $L^\infty(\mathbb{R}^N \times (t_1, t_2))$ for any $t_2 > t_1 > 0$. We may then proceed along the lines of [13, Section 3] to show that (U_λ) converges towards $Z_{M_\infty,x}$ in $L^1(\mathbb{R})$ as $\lambda \to \infty$. Expressing this convergence result in terms of $U = u_x$ and using (3.1) yield Proposition 2.1 by interpolation. $\square$

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