Test vectors for trilinear forms, when two representations are unramified and one is special

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April 7, 2007

1 Introduction

Let F be a finite extension of $\mathbb{Q}_p$, with ring of integers $\mathcal{O}_F$, and uniformizing parameter π_F, whose residual field has q elements. For $G = \mathrm{GL}_2(F)$, let (π_1, V_1), (π_2, V_2) and (π_3, V_3) be three irreducible, admissible, infinite dimensional representations of G. Using the theory of Gelfand pairs, Diprenda Prasad proves in [P] that that the space of G-invariant linear forms on $V_1 \otimes V_2 \otimes V_3$ has dimension at most one. He gives a precise criterion for this dimension to be one, that we will explain now.

Let D_F^* be the group of invertible elements of the quaternion division algebra D_F over F. When (π_i, V_i) is a discrete serie representation of G, denote by (π_i', V_i') the irreducible representation of D_F^* associated to (π_i, V_i) by the Jacquet-Langlands correspondance. Again, by the theory of Gelfand pairs, the space of D_F^*-invariant linear forms on $V_1' \otimes V_2' \otimes V_3'$ has dimension at most one.

Let σ_i be the two dimensional representations of the Weil-Deligne group of F associated to the irreducible representations π_i. The triple tensor product $\sigma_1 \otimes \sigma_2 \otimes \sigma_3$ is an eight dimensional symplectic representation of the Weil-Deligne group, and has local root number $\varepsilon(\sigma_1 \otimes \sigma_2 \otimes \sigma_3) = \pm 1$. When $\varepsilon(\sigma_1 \otimes \sigma_2 \otimes \sigma_3) = -1$, one can prove that the representations π_i's are all discrete serie representations of G.

Theorem 1. *(Prasad, theorem 1.4 of [P]) Let (π_1, V_1), (π_2, V_2), (π_3, V_3) be three irreducible, admissible, infinite dimensional representations of G such that the product of their central characters is trivial. If all the representations V_i's are cuspidal, assume that the residue characteristic of F is not 2. Then*

- $\varepsilon(\sigma_1 \otimes \sigma_2 \otimes \sigma_3) = 1$ *if and only if there exist a non zero G-invariant linear form on* $V_1 \otimes V_2 \otimes V_3$
- $\varepsilon(\sigma_1 \otimes \sigma_2 \otimes \sigma_3) = -1$ *if and only if there exist a non zero D_k^* invariant linear form on* $V_1' \otimes V_2' \otimes V_3'$.

Once you got a non zero G-invariant linear form ℓ on $V_1 \otimes V_2 \otimes V_3$, or a non zero D_k^*-invariant linear form ℓ' on $V_1' \otimes V_2' \otimes V_3'$, you want to find a vector in $V_1 \otimes V_2 \otimes V_3$ which is not in the kernel of ℓ, or a vector in $V_1' \otimes V_2' \otimes V_3'$ which is not in the kernel of ℓ'. Such a vector is called a test vector. At first sight, it appears to have strong connections with the new vectors v_1, v_2 and v_3 of the representations π_1, π_2 et π_3.

Theorem 2. *(Prasad, theorem 1.3 of [P]) When all the π_i's are unramified principal series representations of G, $v_1 \otimes v_2 \otimes v_3$ is a test vector for ℓ.*

Theorem 3. *(Gross and Prasad, proposition 6.3 of [G-P]) When all the π_i's are unramified twists of the special representation of G :*

- *if $\varepsilon(\sigma_1 \otimes \sigma_2 \otimes \sigma_3) = 1$, then $v_1 \otimes v_2 \otimes v_3$ is a test vector for ℓ,*
- *if $\varepsilon(\sigma_1 \otimes \sigma_2 \otimes \sigma_3) = -1$, let R' be the unique maximal order in D_F. Then the open compact subgroup $R'^* \times R'^* \times R'^*$ fixes a unique line in $V_1' \otimes V_2' \otimes V_3'$. Any vector on this line is a test vector for ℓ'.*

The proof by Gross and Prasad of the first statement of this theorem, actually contains another result:

Theorem 4. *When two of the π_i's are unramified twists of the special representation of G and the third one belongs to the unramified principal serie of G, $v_1 \otimes v_2 \otimes v_3$ is a test vector for ℓ.*

But the paper [G-P] ends up with an evidence that $v_1 \otimes v_2 \otimes v_3$ is not always a test vector for ℓ. Let $K = \mathrm{GL}(\mathcal{O}_F)$ be the maximal compact subgroup of G. If π_1 and π_2 are unramified and if π_3 has conductor $n \geq 1$, ℓ being G-invariant, v_1 and v_2 being K-invariant, one gets a K-invariant linear form

$$\begin{cases} V_3 & \longrightarrow & \mathbb{C} \\ v & \longmapsto & \ell(v_1 \otimes v_2 \otimes v) \end{cases}$$

which must be 0 since π_3 is ramified. Then $\ell(v_1 \otimes v_2 \otimes v_3) = 0$.

Now Gross and Prasad make the following suggestion. Let $\Gamma_0(\pi_F^n)$ be the congruence subgroup

$$\Gamma_0(\pi_F^n) = \{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in K \qquad c \equiv 0 \quad mod \, \pi_F{}^n \quad \}$$

and R be a maximal order $\mathrm{M}_2(F)$ such that $R^* \cap K = \Gamma_0(\pi_F^n)$. If v_2^* is a R^*-invariant vector inV_2, the linear form

$$\begin{cases} V_3 & \longrightarrow & \mathbb{C} \\ v & \longmapsto & \ell(v_1 \otimes v_2^* \otimes v) \end{cases}$$

is invariant under the action of $R^* \cap K = \Gamma_0(\pi_F^n)$, and one can still hope that $v_1 \otimes v_2^* \otimes v_3$ is a test vector for ℓ. In theorem 5 we will focus on the case $n = 1$, and prove that $v_1 \otimes v_2^* \otimes v_3$ is a test vector for ℓ, up to a condition on π_1 and π_2. This will almost complete the study of test vectors when the $\pi_i' s$ have ramification 0 or 1.

In the long term, the search for test vectors is motivated by the subconvexity problem for L-functions. Roughly speaking, one wants to bound some L-functions along the critical line $\Re(z) = \frac{1}{2}$. A recent and successful idea in this direction has been to relate triple products of automorphic forms to special values of L-functions on the critical line. In [B-R 1] and [B-R 2] Joseph Bernstein and Andre Reznikov did this in the *eigenvalue* aspect , and in [V] Akshay Venkatesh did it in the level aspect. More details about subconvexity and those related techniques will be found in [M-V]. Test vectors are key ingredients. Bernstein and Reznikov use an explicit test vector. Venkatesh uses a theoretical one, but explains that the bounds would be better with an explicit one (see paragraph 5 of [V]). Unfortunately, the difficulty of finding them increases with the ramification of the representations involved.

There is an extension of Prasad's result in [H-S], where Harris and Scholl prove that the dimension of the space of G-invariant linear forms on $V_1 \otimes V_2 \otimes V_3$ is one when π_1, π_2 and π_3 are principal series representations, either irreducible or reducible with their unique irreducible subspace, infinite dimensional. They apply the global setting of this to the construction of elements in the motivic cohomology of the product of two modular curves constructed by Beilinson.

I would like to thank Philippe Michel for suggesting this problem, and Wen-Ching Winnie Li who invited me to spend one semester at PennState University where I could write the first draft of this paper.

2 Strategy

2.1 Notations

Let (ρ, W) be a smooth representation of a closed subgroup H of G. Let Δ_H be the modular function on H. The induction of ρ from H to G is a representation π whose space is the space $\mathrm{Ind}_H^G(\rho)$ of functions f from G to W satisfying the two following conditions :

(1) $\forall h \in H \quad \forall g \in G \quad f(hg) = {\Delta_H}^{-\frac{1}{2}}(h)\rho(h)f(g)$,

(2) there exist an open compact subgroup K_f of G such that

$$\forall k \in K_f, \quad \forall g \in G, \quad f(gk) = f(g)$$

where G acts by right translation. The resulting function will be denoted $\langle \pi(g), f \rangle$ that is

$$\forall g, g_0 \in G \quad \langle \pi(g), f \rangle (g_0) = f(g_0 g).$$

With the additional condition that f must be compactly supported modulo H, one gets the *compact* induction denoted by ind_H^G. When G/H is compact, there is no difference between Ind_H^G and ind_H^G.

Let B the Borel subgroup of upper triangular matrices in G and T be the diagonal torus. Then we will use $\delta = {\Delta_B}^{-1}$ with $\delta\Big(\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}\Big) = |\frac{a}{d}|$ and Δ_T is trivial. The quotient $B \backslash G$ is compact and can be identified with $\mathbb{P}^1(F)$.

For a smooth representation V of G, V^* is the space of linear forms on V. The contragredient representation $\widetilde{\pi}$ is given by the action of G on $\widetilde{V}$, the subspace of smooth vectors in V^*. If H is a subgroup of G, $\widetilde{V} \subset \widetilde{V_{|H}} \subset V^*$.

More information about induced and contragredient representations will be found in [B-Z].

Let (π_1, V_1), (π_2, V_2) and (π_3, V_3) be three irreducible, admissible, infinite dimensional representations of G such that the product of their central characters is trivial. Assume that π_1 and π_2 are unramified principal series, and that π_3 has conductor $n \geq 1$. Then, according to theorem 1, there exist a non-zero, G-invariant linear form ℓ on $V_1 \otimes V_2 \otimes V_3$, and we are looking for a vector v in $V_1 \otimes V_2 \otimes V_3$ which is not in the kernel of ℓ. In order to follow Gross and Prasad suggestion, we will consider

$$\gamma = \begin{pmatrix} \pi_F^n & 0 \\ 0 & 1 \end{pmatrix} \qquad \text{and} \qquad R = \gamma^{-1} \mathrm{M}_2(\mathcal{O}_F) \gamma.$$

One can easily check that

$$R^* = \gamma^{-1} K \gamma \qquad \text{and} \qquad R^* \cap K = \Gamma_0(\pi_F^n).$$

If v_1, v_2 and v_3 denote the new vectors of π_1, π_2 and π_3, the vector

$$v_2^* = \pi_2(\gamma^{-1}) \cdot v_2$$

is invariant under the action of R^*. Hence we can write

$$v_1 \in {V_1}^K \qquad v_2^* \in {V_1}^{R^*} \qquad v_3 \in {V_3}^{R^* \cap K}$$

According to Gross and Prasad $v_1 \otimes v_2^* \otimes v_3$ should be a test vector for ℓ , for any $n \geq 1$. In this paper, we will focus on the case where $n = 1$. We will need the following condition regarding π_1 and π_2: since they are unramified principal series, they are induced from characters χ_1 and χ_2 of B, that are required to satisfy

$$\chi_1\begin{pmatrix} \pi_F & 0 \\ 0 & {\pi_F}^{-1} \end{pmatrix} \neq -1 \quad \text{or} \quad \chi_2\begin{pmatrix} \pi_F & 0 \\ 0 & {\pi_F}^{-1} \end{pmatrix} \neq -1 \tag{1}$$

We will prove

Theorem 5. *If $n = 1$, and (1) is satisfied, $v_1 \otimes v_2^* \otimes v_3$ is a test vector for ℓ.*

The proof will follow the same pattern as Prasad's proof of theorem 2 in [P], with the necessary changes.

2.2 Central characters

Let ω_1, ω_2 and ω_3 be the central caracters of π_1, π_2 and π_3. Notice that the condition $\omega_1\omega_2\omega_3 = 1$ derives from the G-invariance of ℓ. Since π_1 and π_2 are unramified, ω_1 and ω_2 are unramified too, and so is ω_3 because $\omega_1\omega_2\omega_3 = 1$. Let η_i, for $i \in \{1, 2, 3\}$ be unramified quasi-characters of F^* with $\eta_i^2 = \omega_i$ and $\eta_1\eta_2\eta_3 = 1$. Then

$$V_1 \otimes V_2 \otimes V_3 \simeq \big(V_1 \otimes \eta_1^{-1}\big) \otimes \big(V_2 \otimes \eta_2^{-1}\big) \otimes \big(V_3 \otimes \eta_3^{-1}\big)$$

as a representation of G. Hence it is enough to prove theorem 4 when the central characters of the representations are trivial.

When $n = 1$, it is also enough to prove theorem 5 when V_3 is the special representation Sp of G : take η_3 to be the unramified character such that $V_3 = \eta_3 \otimes \mathrm{Sp}$.

2.3 Prasad's exact sequences

Let us now explain how Prasad finds ℓ. It is equivalent to search ℓ or to search a non zero element in $\mathrm{Hom}_G\Big(V_1 \otimes V_2, \widetilde{V_3}\Big)$. Since the central characters of π_1 and π_2 are trivial, there are unramified characters μ_1 and μ_2 such that for $i = 1$ and $i = 2$

$$\pi_i = \mathrm{Ind}_B^G \chi_i \qquad \text{with} \qquad \chi_i\Big(\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}\Big) = \mu_i\Big(\frac{a}{d}\Big)$$

Hence

$$V_1 \otimes V_2 = \mathrm{Res}_G\, \mathrm{Ind}_{B \times B}^{G \times G}\Big(\chi_1 \times \chi_2\Big)$$

where G is diagonally embedded in $G \times G$ for the restriction. The action of G on $B \times B \backslash G \times G = \mathbb{P}^1(F) \times \mathbb{P}^1(F)$ has precisely two orbits : the first one is $\{(u, v) \in \mathbb{P}^1(F) \times \mathbb{P}^1(F) \quad | \quad u \neq v\}$, it is open and can be identified with $T\backslash G$, the second one is the diagonal embedding of $\mathbb{P}^1(F)$ in $\mathbb{P}^1(F)\times\mathbb{P}^1(F)$, it is closed and it can be identified with $B\backslash G$. Then, we have a short exact sequence of G-modules

$$0\to \mathrm{ind}_T^G\Big(\frac{\chi_1}{\chi_2}\Big) \xrightarrow{\mathbf{ext}} V_1\otimes V_2 \xrightarrow{\mathbf{res}} \mathrm{Ind}_B^G\Big(\chi_1\chi_2\delta^{\frac12}\Big)\to 0 \tag{2}$$

The surjection **res** is the restriction of functions from $G\times G$ to the diagonal part of $B\backslash G\times B\backslash G$, that is

$$\Delta_{B\backslash G}=\Big\{(g,bg)\quad|\quad b\in B,\quad g\in G\Big\}.$$

The injection **ext** takes a function $f\in\mathrm{ind}_T^G\Big(\frac{\chi_1}{\chi_2}\Big)$ to a function $F\in\mathrm{Ind}_{B\times B}^{G\times G}\Big(\chi_1\times\chi_2\Big)$ given by the relation

$$F\Big(g,\begin{pmatrix}0&1\\1&0\end{pmatrix}g\Big)=f(g).$$

Applying the functor $\mathrm{Hom}_G\Big(\cdot,\widetilde{V_3}\Big)$, one gets a long exact sequence

$$0\to\mathrm{Hom}_G\Big(\mathrm{Ind}_B^G\Big(\chi_1\chi_2\delta^{\frac12}\Big),\widetilde{V_3}\Big)\to\mathrm{Hom}_G\Big(V_1\otimes V_2,\widetilde{V_3}\Big)\to\mathrm{Hom}_G\Big(\mathrm{ind}_T^G\Big(\frac{\chi_1}{\chi_2}\Big),\widetilde{V_3}\Big)$$

$$\downarrow$$

$$\cdots\leftarrow\mathrm{Ext}_G^1\Big(\mathrm{Ind}_B^G\Big(\chi_1\chi_2\delta^{\frac12}\Big),\widetilde{V_3}\Big) \tag{3}$$

2.4 The simple case

The situation is easier when $n=1$ and $\mu_1\mu_2|\cdot|^{\frac12}=|\cdot|^{-\frac12}$. Then π_3 is special and there is a natural surjection

$$\mathrm{Ind}_B^G\Big(\chi_1\chi_2\delta^{\frac12}\Big)\longrightarrow\widetilde{V_3}$$

whose kernel is the one dimensional subspace of constant functions. Thanks to the exact sequence (2) one gets a surjection

$$\Psi: V_1\otimes V_2\longrightarrow\widetilde{V_3}$$

which corresponds to

$$\ell\begin{cases}V_1\otimes V_2\otimes V_3 & \longrightarrow & \mathbb{C}\\ v\otimes v'\otimes v'' & \longmapsto & \Psi(v\otimes v').v''\end{cases}$$

The surjection Ψ vanishes on $v_1\otimes v_2^*$ if and only if $\mathbf{res}(v_1\otimes v_2^*)$ has constant value on $\mathbb{P}^1(F)\simeq B\backslash G$. Easy computation proves that it is not constant : the new vectors v_1 and v_2 are functions from G to $\mathbb{C}$ such that

$$\forall i\in\{1,2\},\quad\forall b\in B,\quad\forall k\in K,\qquad v_i(bk)=\chi_i(b)\cdot\delta(b)^{\frac12}$$

and

$$\forall g\in G,\qquad v_2^*(g)=v_2(g\gamma^{-1}).$$

Then

$$(v_1\otimes v_2^*)\Big(\begin{pmatrix}1&0\\0&1\end{pmatrix}\Big)=v_1\Big(\begin{pmatrix}1&0\\0&1\end{pmatrix}\Big)v_2\Big(\gamma^{-1}\Big)=v_2\Big(\begin{pmatrix}\pi_F^{-1}&0\\0&1\end{pmatrix}\Big)=\mu_2(\pi_F)^{-1}|\pi_F|^{-\frac12}=\frac{\sqrt{q}}{\mu_2(\pi_F)}$$

and

$$(v_1 \otimes v_2^*)\Big(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\Big) = v_2\Big(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \pi_F^{-1} & 0 \\ 0 & 1 \end{pmatrix}\Big) = v_2\Big(\begin{pmatrix} 1 & 0 \\ 0 & \pi_F^{-1} \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\Big) = \frac{\mu_2(\pi_F)}{\sqrt{q}}.$$

The representation π_2 is principal so $\frac{\sqrt{q}}{\mu_2(\pi_F)} \neq \frac{\mu_2(\pi_F)}{\sqrt{q}}$ and

$$(v_1 \otimes v_2^*)\Big(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\Big) \neq (v_1 \otimes v_2^*)\Big(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\Big).$$

Hence, Ψ does not vanish on $v_1 \otimes v_2^*$. Then, v_1 being K-invariant and v_2^* being R^*-invariant, $\Psi(v_1 \otimes v_2^*)$ is a non zero $\Gamma_0(\pi_F^n)$-invariant element of $\widetilde{V_3}$, that is, a new vector for $\widetilde{\pi_3}$, and it does not vanish on v_3 :

$$\ell(v_1 \otimes v_2^* \otimes v_3) = \Psi(v_1 \otimes v_2^*).v_3 \neq 0$$

Then $v_1 \otimes v_2^* \otimes v_3$ is a test vector for ℓ.

2.5 The other case

If $n \geq 2$ or $\mu_1\mu_2|\cdot|^{\frac{1}{2}} \neq |\cdot|^{-\frac{1}{2}}$ then $\mathrm{Hom}_G\Big(\mathrm{Ind}_B^G\big(\chi_1\chi_2\delta^{\frac{1}{2}}\big), \widetilde{V_3}\Big) = 0$ and by corollary 5.9 of [P]

$$\mathrm{Ext}^1_G\Big(\mathrm{Ind}_B^G\big(\chi_1\chi_2\delta^{\frac{1}{2}}\big), \widetilde{V_3}\Big) = 0$$

Through the long exact sequence (3) we get an isomorphism

$$\mathrm{Hom}_G\Big(V_1 \otimes V_2, \widetilde{V_3}\Big) \simeq \mathrm{Hom}_G\Big(\mathrm{ind}_T^G\Big(\frac{\chi_1}{\chi_2}\Big), \widetilde{V_3}\Big)$$

and by Frobenius reciprocity

$$\mathrm{Hom}_G\Big(\mathrm{ind}_T^G\Big(\frac{\chi_1}{\chi_2}\Big), \widetilde{V_3}\Big) \simeq \mathrm{Hom}_T\Big(\Big(\frac{\chi_1}{\chi_2}\Big), \widetilde{V_{3|T}}\Big)$$

By lemmas 8 and 9 of [W], this latter space is one dimensional. Thus, we have a chain of isomorphic one dimensional vector spaces

$$\begin{array}{ccc} \ell & \in & \mathrm{Hom}_G\Big(V_1 \otimes V_2 \otimes V_3, \mathbb{C}\Big) \\ & & \downarrow \wr \\ \Psi & \in & \mathrm{Hom}_G\Big(V_1 \otimes V_2, \widetilde{V_3}\Big) \\ & & \downarrow \wr \\ \Phi & \in & \mathrm{Hom}_G\Big(\mathrm{ind}_T^G\big(\frac{\chi_1}{\chi_2}\big), \widetilde{V_3}\Big) \\ & & \downarrow \wr \\ \varphi & \in & \mathrm{Hom}_T\Big(\big(\frac{\chi_1}{\chi_2}\big), \widetilde{V_{3|T}}\Big) \end{array}$$

with generators ℓ, Ψ, Φ and φ corresponding via the isomorphisms. Notice that φ is a linear form on V_3 such that

$$\forall t \in T \qquad \forall v \in V_3 \qquad \varphi\big(\pi_3(t)v\big) = \frac{\chi_2(t)}{\chi_1(t)}\varphi(v) \tag{4}$$

Lemma 1. $\varphi(v_3) \neq 0$.

Proof : this is proposition 2.6 of [G-P] with the following translation :
- the local field F is the same,
- the quadratic extension K/F of Gross and Prasad is $F \times F$ (this case in included in their proof) and their group K^* is our torus T,
- the infinite dimensional representation V_1 of Gross and Prasad is our π_3,
- the one dimensional, unramified representation V_2 of Gross and Prasad is $\frac{\chi_1}{\chi_2}$.

Then the representation that Gross and Prasad call V is $\frac{\chi_1}{\chi_2} \otimes \pi_3$ and their condition (1.3) is exactly our condition (4). The character ω of Gross and Prasad, which is the central character of their V_1, is trivial for us. Let $\alpha_{K/F}$ be the quadratic character of F^* associated to the extension K/F by local class-field theory, and let σ and σ_3 be the representations of the Weil-Deligne group of F associated to $\frac{\chi_1}{\chi_2}$ and π_3. Thanks to [T] we know that $\varepsilon(\sigma \otimes \sigma_3) = \alpha_{K/F}(-1)$ because K is not a field, and we are in the first case of proposition 2.6.
The restriction of $\frac{\chi_1}{\chi_2} \otimes \pi_3$ to the group

$$M = \left\{ \begin{pmatrix} x & 0 \\ 0 & z \end{pmatrix} \quad | \quad x, y \in \mathcal{O}_F^* \right\} \times \Gamma_0(\pi_F^n)$$

fixes a unique line in V_3 : it is the line generated by the new vector v_3. According to Gross and Prasad, a non-zero linear form on V_3 which satisfies (4) cannot vanish on v_3. □

We still need to prove that $\ell(v_1 \otimes v_2^* \otimes v_3) \neq 0$. For the reason described at the end of section 2.4, it is enough to prove that

$$\begin{cases} V_3 & \longrightarrow & \mathbb{C} \\ v & \longmapsto & \ell(v_1 \otimes v_2^* \otimes v) \end{cases}$$

is non zero in $\widetilde{V_3}$. In order to do that we want to build a function F in $V_1 \otimes V_2$, of the form

$$F = \sum_{i \in I} a_i \Big\langle (\pi_1 \otimes \pi_2)(g_i), v_1 \otimes v_2^* \Big\rangle \tag{5}$$

which vanishes on the closed orbit of G in $\mathbb{P}^1(F) \times \mathbb{P}^1(F)$. Then, F is in the kernel of **res** so it is the image by **ext** of a function $f \in \mathrm{ind}_T^G\Big(\frac{\chi_1}{\chi_2}\Big)$. The important point is that f must be the characteristic function of the orbit of the unit in the decomposition of $T\backslash G$ under the action of $\Gamma_0(\pi_F^n)$, which means :

$$f(g) = \begin{cases} \frac{\chi_1(t)}{\chi_2(t)} & \text{if} \quad g = tk \quad \text{with} \quad t \in T \quad \text{and} \quad k \in \Gamma_0(\pi_F^n) \\ 0 & \text{else} \end{cases} \tag{6}$$

Then, the function

$$\begin{cases} G & \longrightarrow & \mathbb{C} \\ g & \longmapsto & f(g)\, \varphi\Big(\pi_3(g) v_3\Big) \end{cases}$$

is invariant by the action of T by left translation and we can do the following computation: on the one hand

$$\begin{aligned}\Big(\Psi(F)\Big)(v_3) &= \Big(\Phi(f)\Big)(v_3)\\ &= \int_{T\backslash G} f(g)\,\varphi\Big(\pi_3(g)v_3\Big)dg\\ &= \int_{(T\cap K)\backslash\Gamma_0(\pi_F^n)} \varphi\Big(\pi_3(k)v_3\Big)dk\\ &= \lambda\cdot\varphi(v_3).\end{aligned}$$

where λ is a non zero constant. Thanks to lemma 1 we know that $\varphi(v_3)\neq 0$ then

$$\Big(\Psi(F)\Big)(v_3)\neq 0.$$

On the other hand, it comes from (5) that

$$\begin{aligned}\Big(\Psi(F)\Big)(v_3) &= \sum_{i\in I} a_i\,\ell\Big(\pi_1(g_i)v_1\otimes\pi_2(g_i)v_2^*\otimes v_3\Big)\\ &= \sum_{i\in I} a_i\,\ell\Big(v_1\otimes v_2^*\otimes\pi_3(g_i^{-1})v_3\Big)\\ &= \Psi(v_1\otimes v_2^*)\Big(\Big(\sum_{i\in I} a_i\,\pi_3(g_i^{-1})\Big)v_3\Big)\end{aligned}$$

then $\Psi(v_1\otimes v_2^*)\neq 0$ and $v_1\otimes v_2^*\otimes v_3$ is a test vector for ℓ.

3 Calculations

3.1 The big function F and the little function f

The function F has to be $\mathbf{ext}(f)$, where f is the function described by formula (6). Since F is in $V_1\otimes V_2=\mathrm{Res}_G\,\mathrm{Ind}_{B\times B}^{G\times G}\Big(\chi_1\times\chi_2\Big)$ and $G=BK$, it is enough to know the values of F on $K\times K$.

Lemma 2. *$\forall(k,k')\in K\times K$,*

$$F(k,k')=\begin{cases}1 & \text{if}\quad k\in\Gamma_0(\pi_F^n)\quad\text{and}\quad k'\notin\Gamma_0(\pi_F)\\ 0 & \text{else}\end{cases}\tag{7}$$

Proof : F must vanish on

$$\Delta_{B\backslash G}=\Big\{(g,bg)\quad|\quad b\in B,\quad g\in G\Big\}$$

The other part of $B\backslash G\times B\backslash G$ can be identified with $T\backslash G$ via the bijection

$$\begin{cases}\Big(B\backslash G\times B\backslash G\Big)\setminus\Delta_{B\backslash G} & \rightarrow\quad T\backslash G\\ \Big(Bg, B\begin{pmatrix}0&1\\1&0\end{pmatrix}g\Big) & \longmapsto\quad Tg\end{cases}$$

through which, the orbit of the unit in $T\backslash G$ under the action of $\Gamma_0(\pi_F^n)$ corresponds to

$$\left\{ \left(Bk, B\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} k \right) \quad | \quad k \in \Gamma_0(\pi_F^n) \right\}$$

Pick any $(k,k') \in K \times K$. If $k' \in Bk$, then $k' \in \Gamma_0(\pi_F^n)$ if and only if $k \in \Gamma_0(\pi_F^n)$, and $k' \in \Gamma_0(\pi_F)$ if and only if $k \in \Gamma_0(\pi_F)$. Else, put

$$k = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad \text{and} \qquad k' = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}.$$

There exist $(b_1, b_2) \in B \times B$ such that

$$\begin{cases} k = b_1 k_0 \\ k' = b_2 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} k_0 \end{cases} \qquad \text{with} \qquad k_0 = \begin{pmatrix} c' & d' \\ c & d \end{pmatrix}.$$

Then

$$\begin{aligned} k_0 \in \Gamma_0(\pi_F^n) &\iff c \equiv 0 \quad mod\, {\pi_F}^n \quad \text{and} \quad c'd \in \mathcal{O}_F^* \\ &\iff c \equiv 0 \quad mod\, {\pi_F}^n \quad \text{and} \quad c' \in \mathcal{O}_F^* \\ &\iff k \in \Gamma_0(\pi_F^n) \quad \text{and} \quad k' \notin \Gamma_0(\pi_F). \end{aligned}$$

It follows that (k,k') corresponds to an element of the orbit of the unit in the decomposition of $T\backslash G$ under the action of $\Gamma_0(\pi_F^n)$ if and only if $k \in \Gamma_0(\pi_F^n)$ and $k' \notin \Gamma_0(\pi_F)$. $\square$

3.2 The big function F when $n = 1$

Now we have to find the coefficients a_i and elements g_i of (5) to get the right F. This can be done for $n = 1$. For the sake of simplicity, for any family (g_i) of elements of G, and (a_i) some complex numbers, denote

$$\Big(\sum_i a_i \cdot g_i\Big)(v_1 \otimes v_2^*) = \sum_i a_i \cdot \Big\langle (\pi_1 \times \pi_2)(g_i), v_1 \otimes v_2^* \Big\rangle$$

Let $\{\,\tau_0, \dots \tau_{q-1}\,\}$ be a set of representatives of $\mathcal{O}_F/\pi_F\mathcal{O}_F$ in $\mathcal{O}_F$, and A be the number

$$A = \Big(\frac{\mu_1(\pi_F)}{\sqrt{q}} - \frac{\sqrt{q}}{\mu_1(\pi_F)}\Big)^{-1}\Big(\frac{\mu_2(\pi_F)}{\sqrt{q}} - \frac{\sqrt{q}}{\mu_2(\pi_F)}\Big)^{-1}$$

which can be defined because the representations π_1 and π_2 are principal so $\mu_1(\pi_F)^2 - q \neq 0$ and $\mu_2(\pi_F)^2 - q \neq 0$.

Lemma 3. *When $n = 1$ and $1 + \mu_1(\pi_F)^2 \neq 0$ the function F is given by*

$$F = A \cdot \Big\{ \quad \frac{\sqrt{q}}{\mu_2(\pi_F)} \cdot \begin{pmatrix} 0 & 1 \\ \pi_F & 0 \end{pmatrix} + \frac{\mu_1(\pi_F)}{\sqrt{q}} \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$- \frac{1}{(1+\mu_1(\pi_F)^2)} \cdot \frac{\mu_1(\pi_F)}{\sqrt{q}} \cdot \frac{\mu_1(\pi_F)}{\mu_2(\pi_F)} \cdot \Big(\sum_{i=0}^{q-1} \begin{pmatrix} \pi_F & \tau_i \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ \pi_F & 0 \end{pmatrix} \Big)$$

$$- \frac{1}{(1+\mu_1(\pi_F)^2)} \cdot \frac{\mu_1(\pi_F)}{\sqrt{q}} \cdot \Big(\sum_{i=0}^{q-1} \begin{pmatrix} 1 & \frac{\tau_i}{\pi_F} \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & \frac{1}{\pi_F} \\ \pi_F & 0 \end{pmatrix} \Big) \quad \Big\} (v_1 \otimes v_2^*)$$

When $n = 1$ *and* $1 + \mu_2(\pi_F)^2 \neq 0$ *the function* F *is given by*

$$F = A \cdot \Big\{ \quad \frac{\sqrt{q}}{\mu_2(\pi_F)} \begin{pmatrix} 0 & 1 \\ \pi_F & 0 \end{pmatrix} + \frac{\mu_1(\pi_F)}{\sqrt{q}} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$- \frac{1}{(1+\mu_2(\pi_F)^2)} \cdot \frac{\mu_1(\pi_F)}{\sqrt{q}} \cdot \Big(\sum_{i=0}^{q-1} \begin{pmatrix} 1 & 0 \\ \tau_i & 1 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \Big)$$

$$- \frac{1}{(1+\mu_2(\pi_F)^2)} \cdot \frac{\mu_2(\pi_F)}{\sqrt{q}} \cdot \Big(\sum_{i=0}^{q-1} \begin{pmatrix} \frac{1}{\pi_F} & 0 \\ \tau_i & 1 \end{pmatrix} + \begin{pmatrix} 0 & \frac{1}{\pi_F} \\ 1 & 0 \end{pmatrix} \Big) \quad \Big\} (v_1 \otimes v_2^*)$$

Proof : for $g \in G$ and $k = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in K$ in order to compute

$$\Big\langle \pi_1(g), v_1 \Big\rangle (k) = v_1(kg) \quad \text{and} \quad \Big\langle \pi_2(g), v_2^* \Big\rangle (k) = v_2(kg\gamma^{-1})$$

write

$$kg = \begin{pmatrix} x & y \\ 0 & z \end{pmatrix} k_1 \quad \text{and} \quad kg\gamma^{-1} = \begin{pmatrix} x' & y' \\ 0 & z' \end{pmatrix} k_2$$

with k_1 and k_2 in K. Then

$$v_1(kg) = \frac{\mu_1(x)}{\mu_1(z)} \cdot \Big| \frac{x}{z} \Big|^{\frac{1}{2}} = \Big(\frac{\mu_1(\pi_F)}{\sqrt{q}} \Big)^{(\mathrm{val}\, x - \mathrm{val}\, z)} \qquad v_2(kg) = \Big(\frac{\mu_2(\pi_F)}{\sqrt{q}} \Big)^{(\mathrm{val}\, x' - \mathrm{val}\, z')}$$

The following tables give the pairs $\Big(\Big\langle \pi_1(g), v_1 \Big\rangle (k), \Big\langle \pi_2(g), v_2^* \Big\rangle (k) \Big)$. The entries, are : an element g in G, $val(c)$ and $val(d)$ where (c, d) is the second line of k.

The first table is inspired by the formula

$$T_{\pi_F} = K \begin{pmatrix} \pi_F & 0 \\ 0 & 1 \end{pmatrix} K = \sqcup_{i=1}^{q-1} \begin{pmatrix} \pi_F & \tau_i \\ 0 & 1 \end{pmatrix} K + \begin{pmatrix} 0 & 1 \\ \pi_F & 0 \end{pmatrix} K$$

g	$\mathrm{val}(c)=0$	$\mathrm{val}(c)\geq 1$
$\begin{pmatrix}\pi_F & \tau_i \\ 0 & 1\end{pmatrix}$ such that $c\tau_i + d \in {\mathcal{O}_F}^*$	$\left(\frac{\mu_1(\pi_F)}{\sqrt{q}}\right), 1$	$\left(\frac{\mu_1(\pi_F)}{\sqrt{q}}\right), 1$
$\begin{pmatrix}\pi_F & \tau_{i_0} \\ 0 & 1\end{pmatrix}$ such that $c\tau_{i_0} + d \in \pi_F\mathcal{O}_F$	$\left(\frac{\mu_1(\pi_F)}{\sqrt{q}}\right)^{-1}, 1$	$\emptyset$
$\begin{pmatrix}0 & 1 \\ \pi_F & 0\end{pmatrix}$	$\left(\frac{\mu_1(\pi_F)}{\sqrt{q}}\right), 1$	$\left(\frac{\mu_1(\pi_F)}{\sqrt{q}}\right)^{-1}, 1$

Fix

$$F_1 = \Big(\sum_{i=0}^{q-1}\begin{pmatrix}\pi_F & \tau_i \\ 0 & 1\end{pmatrix} + \begin{pmatrix}0 & 1 \\ \pi_F & 0\end{pmatrix}\Big)(v_1 \otimes v_2^*)$$

It comes out that $\forall (k,k') \in K\times K$

$$F_1(k,k') = q.\frac{\mu_1(\pi_F)}{\sqrt{q}} + \frac{\sqrt{q}}{\mu_1(\pi_F)} = \frac{\sqrt{q}}{\mu_1(\pi_F)}\cdot(1+\mu_1(\pi_F)^2)$$

Now consider $\gamma^{-1}F_1(k,k')$. On the one hand

$$\begin{aligned}\gamma^{-1}F_1 &= \Big(\sum_{i=0}^{q-1}\gamma^{-1}\begin{pmatrix}\pi_F & \tau_i \\ 0 & 1\end{pmatrix} + \gamma^{-1}\begin{pmatrix}0 & 1 \\ \pi_F & 0\end{pmatrix}\Big)(v_1 \otimes v_2^*) \\ &= \Big(\sum_{i=0}^{q-1}\begin{pmatrix}1 & \frac{\tau_i}{\pi_F} \\ 0 & 1\end{pmatrix} + \begin{pmatrix}0 & \frac{1}{\pi_F} \\ \pi_F & 0\end{pmatrix}\Big)(v_1 \otimes v_2^*).\end{aligned}$$

On the other hand, for any (k,k') in $K\times K$,

$$\Big(\gamma^{-1}F_1\Big)(k,k') = F_1(k\gamma^{-1}, k'\gamma^{-1}).$$

Take $k=\begin{pmatrix}a & b \\ c & d\end{pmatrix}$, and $k\gamma^{-1} = \begin{pmatrix}\frac{a}{\pi_F} & b \\ \frac{c}{\pi_F} & d\end{pmatrix}$. If $\mathrm{val}\, c = 0$, then $\mathrm{val}\, d + 1 \geq \mathrm{val}\, c$ and

$$k\gamma^{-1} = \begin{pmatrix}\frac{ad-bc}{c} & \frac{a}{\pi_F} \\ 0 & \frac{c}{\pi_F}\end{pmatrix}\begin{pmatrix}0 & -1 \\ 1 & \frac{\pi_F d}{c}\end{pmatrix}$$

with

$$\begin{pmatrix}0 & -1 \\ 1 & \frac{\pi_F d}{c}\end{pmatrix} \in K \quad\text{and}\quad (\chi_1\cdot\delta^{\frac{1}{2}})\begin{pmatrix}\frac{ad-bc}{c} & \frac{a}{\pi_F} \\ 0 & \frac{c}{\pi_F}\end{pmatrix} = \frac{\mu_1(\pi_F)}{\sqrt{q}}$$

If $\operatorname{val} c \geq 1$, then $\operatorname{val} d = 0$ and

$$k\gamma^{-1} = \begin{pmatrix} \frac{ad-bc}{\pi_F d} & b \\ 0 & d \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \frac{c}{\pi_F d} & 1 \end{pmatrix}$$

with

$$\begin{pmatrix} 1 & 0 \\ \frac{c}{\pi_F d} & 1 \end{pmatrix} \in K \quad \text{and} \quad (\chi_1 \cdot \delta^{\frac{1}{2}})(\begin{pmatrix} \frac{ad-bc}{\pi_F d} & b \\ 0 & d \end{pmatrix}) = \frac{\sqrt{q}}{\mu_1(\pi_F)}$$

The same calculation with k' leads to the following:

$$F_1(k\gamma^{-1}, k'\gamma^{-1}) = \frac{\sqrt{q}}{\mu_1(\pi_F)} \cdot (1 + \mu_1(\pi_F)^2) \cdot \begin{cases} \frac{\mu_1(\pi_F)}{\sqrt{q}} \cdot \frac{\mu_2(\pi_F)}{\sqrt{q}} & \text{if} \quad \operatorname{val} c = \operatorname{val} c' = 0 \\ \frac{\mu_1(\pi_F)}{\sqrt{q}} \cdot \frac{\sqrt{q}}{\mu_2(\pi_F)} & \text{if} \quad \operatorname{val} c = 0 \quad \text{and} \quad \operatorname{val} c' \geq 1 \\ \frac{\sqrt{q}}{\mu_1(\pi_F)} \cdot \frac{\mu_2(\pi_F)}{\sqrt{q}} & \text{if} \quad \operatorname{val} c \geq 1 \quad \text{and} \quad \operatorname{val} c' = 0 \\ \frac{\sqrt{q}}{\mu_1(\pi_F)} \cdot \frac{\sqrt{q}}{\mu_2(\pi_F)} & \text{if} \quad \operatorname{val} c \geq 1 \quad \text{and} \quad \operatorname{val} c' \geq 1 \end{cases}$$

Now, with the simple table

g	$\operatorname{val}(c) = 0$	$\operatorname{val}(c) \geq 1$
$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$1, \left(\frac{\mu_2(\pi_F)}{\sqrt{q}}\right)$	$1, \left(\frac{\mu_2(\pi_F)}{\sqrt{q}}\right)^{-1}$
$\begin{pmatrix} 0 & 1 \\ \pi_F & 0 \end{pmatrix}$	$\left(\frac{\mu_1(\pi_F)}{\sqrt{q}}\right), 1$	$\left(\frac{\mu_1(\pi_F)}{\sqrt{q}}\right)^{-1}, 1$

and

$$\begin{aligned} F = & A \cdot \Big(\frac{\sqrt{q}}{\mu_2(\pi_F)} \cdot \begin{pmatrix} 0 & 1 \\ \pi_F & 0 \end{pmatrix} + \frac{\mu_1(\pi_F)}{\sqrt{q}} \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\Big)(v_1 \otimes v_2^*) \\ & - \frac{A}{(1 + \mu_1(\pi_F)^2)} \cdot \frac{\mu_1(\pi_F)}{\sqrt{q}} \cdot \Big(\frac{\mu_1(\pi_F)}{\mu_2(\pi_F)} \cdot F_1 + \gamma^{-1} F_1\Big) \end{aligned}$$

one gets the first formula of lemma 3.

The second formula of lemma 3 is obtained by considering the decomposition

$$K = \sqcup_{j=0}^{q-1} \begin{pmatrix} 1 & 0 \\ \tau_j & 1 \end{pmatrix} \Gamma_0(\pi_F) \sqcup \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \Gamma_0(\pi_F)$$

Then, using the table

g	$\mathrm{val}(c)=0$ $\mathrm{val}(d)\geq 1$	$\mathrm{val}(d)=\mathrm{val}(c)=0$	$\mathrm{val}(c)\geq 1$ $\mathrm{val}(d)=0$
$\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$	$1, \left(\frac{\mu_2(\pi_F)}{\sqrt{q}}\right)$	$1, \left(\frac{\mu_2(\pi_F)}{\sqrt{q}}\right)$	$1, \left(\frac{\mu_2(\pi_F)}{\sqrt{q}}\right)^{-1}$
$\begin{pmatrix}1 & 0\\ \tau_j & 1\end{pmatrix}$ such that $\tau_j \neq 0$ and $d\tau_j + c \in {\mathcal{O}_F}^*$	$1, \left(\frac{\mu_2(\pi_F)}{\sqrt{q}}\right)$	$1, \left(\frac{\mu_2(\pi_F)}{\sqrt{q}}\right)$	$1, \left(\frac{\mu_2(\pi_F)}{\sqrt{q}}\right)$
$\begin{pmatrix}1 & 0\\ \tau_{j_0} & 1\end{pmatrix}$ such that $\tau_j \neq 0$ and $d\tau_{j_0} + c \in \pi_F\mathcal{O}_F$	$\emptyset$	$1, \left(\frac{\mu_2(\pi_F)}{\sqrt{q}}\right)^{-1}$	$\emptyset$
$\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$	$1, \left(\frac{\mu_2(\pi_F)}{\sqrt{q}}\right)^{-1}$	$1, \left(\frac{\mu_2(\pi_F)}{\sqrt{q}}\right)$	$1, \left(\frac{\mu_2(\pi_F)}{\sqrt{q}}\right)$

one gets a function

$$F_2 = \Big(\sum_{i=0}^{q-1}\begin{pmatrix}1 & 0\\ \tau_i & 1\end{pmatrix} + \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\Big)(v_1 \otimes v_2^*)$$

which satisfies $\forall (k,k') \in K \times K$

$$F_2(k,k') = q.\frac{\mu_2(\pi_F)}{\sqrt{q}} + \frac{\sqrt{q}}{\mu_2(\pi_F)} = \frac{\sqrt{q}}{\mu_2(\pi_F)} \cdot (1 + \mu_2(\pi_F)^2).$$

This is the same situation as the previous one : by computing $\gamma^{-1}F_2$ and choosing the right coefficients, one gets the second formula of lemma 3. $\square$

Conclusion : Thus, we could write the function F for $n = 1$ and $1 + \mu_1(\pi_F)^2 \neq 0$ or $1 + \mu_2(\pi_F)^2 \neq 0$. The latter condition is precisely condition 1 of theorem 5, which is now proved. Of course, it would be interesting to remove this condition and then to find F for any n.

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