Author manuscript, published in "Applied Mathematics and Computation 190, 2 (2007) 912-936"
DOI : 10.1016/j.amc.2007.01.107

The Korteweg de Vries Kawahara equation in a boundary domain and numerical results

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Abstract

We are concerned with the initial-boundary problem associated to the Korteweg de Vries Kawahara perturbed by a dispersive term which appears in several fluids dynamics problems. We obtain local smoothing effects that are uniform with respect to the size of the interval. We also propose a simple finite different scheme for the problem and prove its unconditional stability. Finally we give some numerical examples.

Key words: Evolution equations, gain in regularity, Sobolev space, numerical methods.
1991 MSC: : Primary 35Q53; Secondary 47J353

1 Introduction

We consider the non linear problem for the Korteweg de Vries - Kawahara(KdV-K) equation

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$$u_t + \eta\, u_{xxxxx} + u_{xxx} + u\, u_x + u_x = 0, \quad x \in [0,\, L[, \quad t \in [0,\, T[, \tag{1.1}$$
$$u(0,\, t) = g_1(t), \quad u_x(0,\, t) = g_2(t), \quad t \in [0,\, T[, \tag{1.2}$$
$$u(L,\, t) = 0, \quad u_x(L,\, t) = 0, \quad u_{xx}(L,\, t) = 0, \quad t \in [0,\, T[, \tag{1.3}$$
$$u(x,\, 0) = u_0(x) \tag{1.4}$$

where $u = u(x,\, t)$ is a real-valued function, $\eta \in \mathbb{R}$. Equation (1.1)-(1.4) is a version of the Benney-Lin equation

$$u_t + \eta\, u_{xxxxx} + \beta\,(u_{xxxx} + u_{xx}) + u_{xxx} + u u_x = 0 \tag{1.5}$$

with $-\infty < x < +\infty$, $t \in [0,\, T]$, T is an arbitrary positive time, $\beta > 0$ and $\eta \in \mathbb{R}$.
The above equation is a particular case from a benney-Lin equation derived by Benney [1] and later by Lin [2] (see also [3–5] and references therein). It describes one dimensional evolutions of small but finite amplitude long waves in various problems in fluid dynamics. This also can be seen as an hybrid of the well known fifth order Korteweg-de Vries(KdV) equation or Kawahara equation. In 1997, H. Biagioni and F. Linares [6] motivated by the results obtained by J. L. Bona *et al.* [7] showed that the initial value problem (1.5) is globally well-posed in $H^s(\mathbb{R})$, $s \geqslant 0$. The initial value problem (1.5) has been studied in the last few years, see for instance for comprehensive descriptions of results pertaining to the KdVK equation [6,8] and references therein.
Guided by experimental studied on water waves in channels [9–11], J. L. Bona and R. Winter [12,13] considered the Korteweg de Vries equation

$$u_t + u_{xxx} + u\, u_x + u_x = 0 \quad \text{for} \quad x,\, t \geqslant 0 \tag{1.6}$$
$$u(x,\, 0) = f(x) \quad \text{for} \quad x \geqslant 0 \tag{1.7}$$
$$u(0,\, t) = g(t) \quad \text{for} \quad t \geqslant 0 \tag{1.8}$$

and proved that such that a quarter plane problem is well posed. See [14–16] for theory involving nonlinearities having more general form. J. L. Bona and P. L. Bryant [9] had studied the same quarter plane problem for the regularized long-wave equation

$$u_t + u_x + u\, u_x - u_{xxt} = 0 \tag{1.9}$$

and proved it to be well posed. In [17,18], J. L. Bona and L. Luo studied an initial and boundary-value problem fr the nonlinear wave equation

$$u_t + u_{xxx} + [\, P(u)\,]_x = 0 \tag{1.10}$$

in the quarter plane $\{(x,\, t) :\ x \geqslant 0 \text{ and } t \geqslant 0\}$ with the initial data and boundary data specified at $t = 0$ and on $x = 0$, respectively. With suitable

restrictions on P and with conditions imposed on the initial data and boundary data which are quite reasonable with regard to potential applications, the aforementioned initial-boundary-value problem for (1.10) is show to be well posed.
In this work, motivated by the results obtained by T. Colin and M. Gisclon [19] we show that the Korteweg de Vries Kawahara equation has smoothing effects that are uniform with respect to the size of the interval and we also to propose a simple finite different scheme for the problem and prove its stability This paper is organized as follows: In section 2 outlines briefly the notation and terminology to be used subsequently and presents a statement of the principal result. In section 3 we consider the linear problem and we will find *a priori* estimates. In section 4 we obtain estimates that are independent of L for the nonhomogeneous linear system. In section 5 we prove the existence of a time T_{min} depending only on $||g||_{H^1(0,T)}$ and $||u_0||_{L^2((1+x^2)dx)}$ but not on L such that u^L exists on $[0,\, T_{min}]$ thanks to uniform (with respect to L) smoothing effects. In section 6 we present a finite different scheme for the initial-boundary-value problem (KdVK); we prove its stability and present some numerical experiments. The paper concludes with some commentary concerning aspects not covered in the present study. Our main result reads as follows

Theorem(Existence and Uniqueness). *Let* $\eta \leqslant 0$, $w_0 \in L^2((1+x^2)dx)$, $g \in H^1_{loc}(\mathbb{R}^+)$ *and* $0 < L < +\infty$. *Then there exists a unique weak maximal solution defined over* $[0,\, T_L]$ *to* (R). *Moreover, there exists* $T_{min} > 0$ *independent of* L, *depending only on* $||w_0||_{L^2(0,\,L)}$ *and* $||g||_{H^1(0,T)}$ *such that* $T_L \geqslant T_{min}$. *The solution* w *depends continuously on* w_0 *and* g *in the following sense: Let a sequence* $w_0^n \to w_0$ *in* $L^2((1+x^2)dx)$, *let a sequence* $g^n \to g$ *in* $H^1_{loc}(\mathbb{R}^+)$ *and denote by* w^n *the solution with data* $(w_0^n,\, g^n)$ *and* T_L^n *its existence time. Then*

$$\lim_{n\to+\infty} \inf T_L^n \geqslant T_L$$

and for all $t < T_L$, w^n *exists on the interval* $[0,\, T]$ *if* n *is large enough and* $u^n \to u$ *in* H_T.

2 Preliminaries

Let Ω a bounded domain in $\mathbb{R}$. For any real p in the interval $[1,\, \infty]$, $L^p(\Omega)$ denotes the collection of real-valued Lebesgue measurable p$^{\text{th}}$-power absolutely integrable functions defined on Ω. As usual, $L^\infty(\Omega)$ denotes the essentially bounded real-valued functions defined on Ω. These spaces get their usual norms,

$$||u||_{L^p(\Omega)} = \left[\int_\Omega |u(x)|^p\, dx\right]^{1/p} \quad \text{for} \quad 1 \leqslant p < +\infty$$

and

$$||u||_{L^\infty(\Omega)} = \sup_{x\in\Omega} \text{ess}\, |u(x)|$$

For an arbitrary Banach space X, the associated norm will be denoted $||\cdot||_X$. The following spaces will intervene in the subsequent analysis. For any real p in the interval $[1,\, \infty]$ and $-\infty \leqslant a < b \leqslant +\infty$ the notation $L^p(a,\, b:\; X)$ denotes the Banach space of measurable functions $u : (a,\, b) \to X$ whose norms are p^{th}-power integrable(essentially bounded if $p = +\infty$). These spaces get their norms,

$$||u||^p_{L^p(a,\, b\colon X)} = \int_a^b ||u(\,\cdot\,,\, t)||^p_X\, dt \quad \text{for} \quad 1 \leqslant p < +\infty.$$

and

$$||u||_{L^\infty(a,\, b\colon X)} = \text{supp}_{\; t\in(a,\, b)}\; ||u(\,\cdot\,,\, t)||_X \quad \text{for} \quad p = +\infty.$$

In this paper, we assume that for $0 < L < +\infty$ the initial data $u_0 \in L^2(0,\, L)$ and that $x\, u_0 \in L^2(0,\, L)$ and we introduce

$$||u_0||_{L^2((1+x^2)\, dx)} = \left[\int_0^L u_0^2\, (1 + x^2)\, dx\right]^{1/2}.$$

We introduce the following spaces:
$\mathbb{E} := \{f \in L^1(0,\, T:\; L^2((1 + x^2)\, dx)), \quad \sqrt{t}\, f \in L^2(0,\, T:\; L^2((1 + x^2)\, dx))\}$.
This space is endowed with the norm

$$\begin{aligned} ||f||_{\mathbb{E}} &= \int_0^T \left[\int_0^L f^2(x,\, t)\, (1 + x^2)\, dx\right]^{1/2} dt + \left[\int_0^T \int_0^L t\, [f(x,\, t)]^2\, (1 + x^2)\, dx\, dt\right]^{1/2} \\ &= \int_0^T ||f||_{L^2((1+x^2)\, dx)}\, dt + ||\sqrt{t}\, f||_{L^2(0,\, T\colon L^2((1+x^2)\, dx))} \\ &= ||f||_{L^1(0,\, T\colon L^2((1+x^2)\, dx))} + ||\sqrt{t}\, f||_{L^2(0,\, T\colon L^2((1+x^2)\, dx))} \end{aligned}$$

Let $T > 0$.

$$\begin{aligned} \mathbb{H}_T = \{&u \in C([0,\, T]:\; L^2((1 + x^2)\, dx)),\; u_x \in L^2(0,\, T:\; L^2((1 + x)\, dx)) \\ &\sqrt{t}\, u_x \in L^2([0,\, T]:\; L^2((1 + x)\, dx)),\; \sqrt{t}\, u_{xx} \in L^2([0,\, T]:\; L^2(0,\, L))\}. \end{aligned}$$

This space is endowed with the norm

$$||u||_{\mathbb{H}} = ||u||_{L^\infty(0,T:\,L^2((1+x^2)\,dx))} + ||u_x||_{L^2(0,T:\,L^2((1+x)\,dx))} + ||\sqrt{t}\,u_x||_{L^2(0,T:\,L^2((1+x)\,dx))} + ||\sqrt{t}\,u_{xx}||_{L^2(0,T:\,L^2(0,L))}.$$

Let Ω a bounded domain in $\mathbb{R}$. If $1 \leqslant p \leqslant +\infty$, and $m \geqslant 0$ is an integer, let $W^{m,p}(\Omega)$ be the Sobolev space of $L^p(\Omega)$-functions whose distributional derivatives up to order m also lie in $L^p(\Omega)$. The norm on $W^{m,p}(\Omega)$ is

$$||u||^p_{W^{m,p}(\Omega)} = \sum_{\alpha \leqslant m} ||\partial_x^\alpha u||^p_{L^p(\Omega)}.$$

The space $C^\infty(\overline{\Omega}) = \bigcap_{j\geqslant 0} C^j(\overline{\Omega})$ will be used, but its usual Fréchet-space topology will not be needed. $\mathcal{D}(\Omega)$ is the subspace of $C^\infty(\overline{\Omega})$ of functions with compact support in Ω. Its dual space, $\mathcal{D}'(\Omega)$, is the space of Schwartz distributions on Ω. When $p = 2$, $W^{m,p}(\Omega)$ will be denoted by $H^m(\Omega)$. This is a Hilbert space, and $H^0(\Omega) = L^2(\Omega)$. The notation $H^\infty(\Omega) = \bigcap_{j\geqslant 0} H^j(\Omega)$ will be used for the C^∞-functions on Ω, all of whose derivatives lie in $L^2(\Omega)$. Finally, $H^m_{\text{loc}}(\Omega)$ is the set of real-valued functions u defined on Ω such that, for each $\varphi \in \mathcal{D}(\Omega)$, $\varphi u \in H^m(\Omega)$. This space is equipped with the weakest topology such that all of the mappings $u \to \varphi u$, for $\varphi \in \mathcal{D}(\Omega)$, are continuous from $H^m_{\text{loc}}(\Omega)$ into $H^m(\Omega)$. With this topology, $H^m_{\text{loc}}(\Omega)$ is a Fréchet space. Let $\mathbb{R}^+$ denote the positive real numbers $(0, \infty)$. A simple but pertinent example of the localized Sobolev space is $H^m_{\text{loc}}(\mathbb{R}^+)$. Interpreting the foregoing definitions in this special case, $u \in H^m_{\text{loc}}(\mathbb{R}^+)$ if and only if $u \in H^m(0, T)$, for all finite $T > 0$. Moreover, $u_n \to u$ in $H^m_{loc}(\mathbb{R}^+)$ if and only if $u_n \to u$ in $H^m(0, T)$, for each $T > 0$.
We consider the inhomogeneous initial value problem

$$\begin{cases} \dfrac{\partial u}{\partial t}(t) = Au(t) + f(t), \qquad t > 0, \\ u(0) = x, \end{cases}$$

where $f : [0, T] \to X$ and A is the infinitesimal generator of a C_0 semigroup $S(t)$ so that the corresponding homogeneous equation, i. e., the equation with $f \equiv 0$, has a unique solution for every initial value $x \in D(A)$.
By c, a generic constant, not necessarily the same at each occasion, which depend in an increasing way on the indicated quantities.
The following result is going to be used several times in the rest of this paper.

Lemma 2.1 *We have the following inequalities*

$$||u||^2_{L^2((1+x)\,dx)} \leqslant 3\; ||u||^2_{L^2((1+x^2)\,dx)} \tag{2.1}$$
$$||u||_{L^\infty(0,\,T)} \leqslant (c+\sqrt{T}\,)\; ||u||_{H^1(0,\,T)} \tag{2.2}$$
$$||u||_{L^1(0,\,T:\,L^2((1+x^2)\,dx))} \leqslant \sqrt{T}\; ||u||_{L^2(0,\,T:\,L^2((1+x^2)dx))} \tag{2.3}$$

and, if $u \in H^1(0,\,L)$, $u(0)=0$ *then*

$$||u||_{L^\infty(0,\,L)} \leqslant 2\; ||u||_{L^2(0,\,L)}\; ||u_x||_{L^2(0,\,L)} \tag{2.4}$$

3 Uniform estimates on the solutions to the linear homogeneous problem

We consider the linear problem for the Korteweg de Vries Kawahara equation

$$u_t + \eta\, u_{xxxxx} + u_{xxx} + u_x = 0, \quad x \in [0,\,L[, \quad t \in [0,\,T[, \tag{3.1}$$
$$u(0,\,t) = 0, \quad u_x(0,\,t) = 0, \quad t \in [0,\,T[, \tag{3.2}$$
$$u(L,\,t) = 0, \quad u_x(L,\,t) = 0, \quad u_{xx}(L,\,t) = 0, \quad t \in [0,\,T[, \tag{3.3}$$
$$u(x,\,0) = u_0(x) \tag{3.4}$$

where $u = u(x,\,t)$ is a real-valued function, $\eta \in \mathbb{R}$.

Lemma 3.1 *Let* $u_0 \in L^2(0,\,L)$ *and* $\eta < 0$. *Then there exists a continuous function* $t \to c(t)$ *such that*

$$||u||_{L^\infty(0,\,T:\,L^2(0,\,L))} \leqslant c(T)\; ||u_0||_{L^2(0,\,L)}. \tag{3.5}$$
$$||u_{xx}(0,\,\cdot\,)||_{L^2(0,\,T)} \leqslant c(T)\; ||u_0||_{L^2(0,\,L)}. \tag{3.6}$$

Proof. Multiplying (3.1) by u and integrating over $x \in (0,\,L)$ we have

$$\int_0^L u\,u_t\,dx + \eta \int_0^L u\,u_{xxxxx}\,dx + \int_0^L u\,u_{xxx}\,dx + \int_0^L u\,u_x\,dx = 0. \tag{3.7}$$

Each term is treated separately integrating by parts

$$\int_0^L u\,u_t\,dx = \frac{1}{2}\,\frac{d}{dt}\int_0^L u^2\,dx, \qquad \eta\int_0^L u\,u_{xxxxx}\,dx = -\,\frac{1}{2}\,\eta\; u_{xx}^2(0,\,t),$$
$$\int_0^L u\,u_{xxx}\,dx = 0, \qquad \int_0^L u\,u_x\,dx = 0.$$

Replacing in (3.7) we have

$$\frac{1}{2}\,\frac{d}{dt}\int_0^L u^2\,dx - \,\frac{1}{2}\,\eta\,u_{xx}^2(0,\,t) = 0$$

then

$$\frac{d}{dt}\,||u||^2_{L^2(0,\,L)} - \eta\,u_{xx}^2(0,\,t) = 0. \tag{3.8}$$

Integrating (3.8) in $t \in (0,\,T)$ we have

$$||u||^2_{L^2(0,\,L)} - \eta\int_0^t u_{xx}^2(0,\,s)\,ds = ||u_0||^2_{L^2(0,\,L)}. \tag{3.9}$$

Then, using $\eta < 0$, the Gronwall inequality and straightforward calculus we have

$$||u||^2_{L^2(0,\,L)} - \eta\,||u_{xx}(0,\,\cdot\,)||^2_{L^2(0,\,T)} = c(T)\,||u_0||^2_{L^2(0,\,L)}. \tag{3.10}$$

and (3.5)-(3.6) follows. ■

Lemma 3.2 *Let $u_0 \in L^2(0,\,L)$ and $\eta < 0$. Then there exists a continuous function $t \to c(t)$ such that*

$$||u_x||_{L^2(0,\,T:\,L^2(0,\,L))} \leqslant c(T)\,||u_0||_{L^2(0,\,L)} \tag{3.11}$$
$$||u_{xx}||_{L^2(0,\,T:\,L^2(0,\,L))} \leqslant c(T)\,||u_0||_{L^2(0,\,L)}. \tag{3.12}$$
$$||u||_{L^\infty(0,\,T:\,L^2((1+x)\,dx))} \leqslant c(T)\,||u_0||_{L^2(0,\,L)}. \tag{3.13}$$

Remark 3.1 *From* (3.12) *we obtain that if $u_0 \in L^2(0,\,L)$ then $u \in H^1(0,\,L)$. This means that we have a gain of two derivatives in regularity, while in the Korteweg-de Vries equation only one derivative is gained.*

Proof. Multiplying the equation (3.1) by $x\,u$ and integrating over $x \in (0,\,L)$ we have

$$\int_0^L x\,u\,u_t\,dx + \eta\int_0^L x\,u\,u_{xxxxx}\,dx + \int_0^L x\,u\,u_{xxx}\,dx + \int_0^L x\,u\,u_x\,dx = 0. \tag{3.14}$$

Each term in (3.14) is treated separately

$$\int_0^L x\,u\,u_t\,dx = \frac{1}{2}\,\frac{d}{dt}\int_0^L x\,u^2\,dx, \quad \eta\int_0^L x\,u\,u_{xxxxx}\,dx = -\,\frac{5}{2}\,\eta\int_0^L u_{xx}^2\,dx,$$
$$\int_0^L x\,u\,u_{xxx}\,dx = \frac{3}{2}\int_0^L u_x^2\,dx, \qquad \int_0^L x\,u\,u_x\,dx = -\,\frac{1}{2}\int_0^L u^2\,dx.$$

Hence, replacing in (3.14) we have

$$\frac{d}{dt}\int_0^L x\,u^2\,dx - 5\,\eta\int_0^L u_{xx}^2\,dx + 3\int_0^L u_x^2\,dx - \int_0^L u^2\,dx = 0$$

then

$$\frac{d}{dt}\int_0^L x\,u^2\,dx - 5\,\eta\int_0^L u_{xx}^2\,dx + 3\int_0^L u_x^2\,dx = \int_0^L u^2\,dx = ||u||^2_{L^2(0,\,L)}. \tag{3.15}$$

where using (3.5) we obtain

$$\frac{d}{dt}\int_0^L x\,u^2\,dx - \eta\int_0^L u_{xx}^2\,dx + 3\int_0^L u_x^2\,dx \leqslant c(T)\;||u_0||^2_{L^2(0,\,L)}. \tag{3.16}$$

Integrating (3.16) in $t \in (0,\,T)$ we have

$$\begin{aligned}\int_0^L x\,u^2\,dx - \eta\int_0^t\int_0^L u_{xx}^2\,dx\,ds + 3\int_0^t\int_0^L u_x^2\,dx\,ds \\ \leqslant c(T)\;||u_0||^2_{L^2(0,\,L)} + \int_0^L x\,u_0^2\,dx \\ \leqslant c(T)\;||u_0||^2_{L^2(0,\,L)} + L\int_0^L u_0^2\,dx \\ = c(T)\;||u_0||^2_{L^2(0,\,L)} + L\;||u_0||^2_{L^2(0,\,L)} \leqslant c(T)\;||u_0||^2_{L^2(0,\,L)}\end{aligned}$$

then using that $\eta < 0$ and

$$\begin{aligned}||u||^2_{L^\infty(0,\,T:\,L^2(x\,dx))} - \eta\;||u_{xx}||^2_{L^2(0,\,T:\,L^2(0,\,L))} \\ + 3\,||u_x||^2_{L^2(0,\,T:\,L^2(0,\,L))} \leqslant c(T)\;||u_0||^2_{L^2(0,\,L)}\end{aligned} \tag{3.17}$$

we have that (3.11) and (3.12) follows. Moreover, adding (3.5) with the first term in (3.22), we obtain (3.13). ■

Lemma 3.3 *Let $u_0 \in L^2(0,\,L)$ and $\eta < 0$. Then there exists a continuous function $t \to c(t)$ such that*

$$\int_0^L u^2\,x^2\,dx \leqslant c(T)\;||u_0||^2_{L^2(0,\,L)}. \tag{3.18}$$

$$\int_0^T\int_0^L u_x^2\,x\,dx\,dt \leqslant c(T)\;||u_0||^2_{L^2(0,\,L)}. \tag{3.19}$$

$$\int_0^T\int_0^L u_{xx}^2\,x\,dx\,dt \leqslant c(T)\;||u_0||^2_{L^2(0,\,L)}. \tag{3.20}$$

$$||u||_{L^\infty(0,\,T:\,L^2((1+x^2)dx))} \leqslant c(T)\;||u_0||_{L^2(0,\,L)}. \tag{3.21}$$

$$||u_x||_{L^2(0,\,T:\,L^2((1+x)\,dx))} \leqslant c(T)\;||u_0||_{L^2(0,\,L)}. \tag{3.22}$$

Proof. Multiplying the equation (3.1) by $x^2\,u$ and integrating over $x \in (0,\,L)$ we have

$$\int_0^L x^2\,u\,u_t\,dx + \eta\int_0^L x^2\,u\,u_{xxxxx}\,dx + \int_0^L x^2\,u\,u_{xxx}\,dx + \int_0^L x^2\,u\,u_x\,dx = 0. \tag{3.23}$$

Each term is treated separately, integrating by parts

$$\int_0^L x^2\,u\,u_t\,dx = \frac{1}{2}\,\frac{d}{dt}\int_0^L x^2\,u^2\,dx,\quad \eta\int_0^L x^2\,u\,u_{xxxxx}\,dx = -\,5\,\eta\int_0^L x\,u_{xx}^2\,dx,$$
$$\int_0^L x^2\,u\,u_{xxx}\,dx = 3\int_0^L x\,u_x^2\,dx,\qquad \int_0^L x^2\,u\,u_x\,dx = \int_0^L x\,u^2\,dx.$$

Hence, in (3.23) we have

$$\frac{1}{2}\,\frac{d}{dt}\int_0^L x^2\,u^2\,dx - 5\,\eta\int_0^L x\,u_{xx}^2\,dx + 3\int_0^L x\,u_x^2\,dx - \int_0^L x\,u^2\,dx = 0$$

then

$$\frac{d}{dt}\int_0^L x^2\,u^2\,dx - 10\,\eta\int_0^L x\,u_{xx}^2\,dx + 6\int_0^L x\,u_x^2\,dx = 2\int_0^L x\,u^2\,dx \leqslant 2\,L\int_0^L u^2\,dx$$

hence

$$\frac{d}{dt}\int_0^L x^2\,u^2\,dx - 10\,\eta\int_0^L x\,u_{xx}^2\,dx + 6\int_0^L x\,u_x^2\,dx \leqslant 2\,L\int_0^L u^2\,dx$$

then, using (3.5) we obtain

$$\frac{d}{dt}\int_0^L x^2\,u^2\,dx - 10\,\eta\int_0^L x\,u_{xx}^2\,dx + 6\int_0^L x\,u_x^2\,dx \leqslant 2\,L\,c(T)\,||u_0||^2_{L^2(0,\,L)}. \tag{3.24}$$

Integrating (3.24) over $t \in (0,\,T)$ we have

$$\begin{aligned}\int_0^L x^2\,u^2\,dx - 10\,\eta\int_0^t\int_0^L x\,u_{xx}^2\,dx\,ds + 6\int_0^t\int_0^L x\,u_x^2\,dx\,ds \\ \leqslant 2\,L\,c(T)\,||u_0||^2_{L^2(0,\,L)} + \int_0^L x^2\,u_0^2\,dx \\ \leqslant 2\,L\,c(T)\,||u_0||^2_{L^2(0,\,L)} + L^2\int_0^L u_0^2\,dx \leqslant c(T)\,||u_0||^2_{L^2(0,\,L)}.\end{aligned}$$

We obtain

$$\int_0^L x^2\,u^2\,dx - 10\,\eta \int_0^t \int_0^L x\,u_{xx}^2\,dx\,ds \\ + 6 \int_0^t \int_0^L x\,u_x^2\,dx\,ds \leqslant c(T)\;||u_0||^2_{L^2(0,\,L)} \tag{3.25}$$

and (3.18)-(3.20) follows. From (3.5) and (3.18) we obtain (3.21) and from (3.11) with (3.19) we have (3.22). The result follows. ■

Lemma 3.4 *Let* $u_0 \in L^2(0,\,L)$ *and* $\eta < 0$. *Then there exists a continuous function* $t \to c(t)$ *such that*

$$||\sqrt{t}\;u_{xx}(0,\,\cdot)||_{L^2(0,T)} \leqslant c(T)\;||u_0||_{L^2(0,\,L)}. \tag{3.26}$$

$$||\sqrt{t}\;u_x||_{L^2(0,\,T:\,L^2(0,T))} \leqslant c(T)\;||u_0||_{L^2(0,\,L)}. \tag{3.27}$$

$$\int_0^t \int_0^L s\;u_{xx}^2\;x\,dx\,ds \leqslant c(T)\;||u_0||_{L^2(0,\,L)}. \tag{3.28}$$

$$||\sqrt{t}\;u||_{L^\infty(0,\,T:\,L^2(1+x)\,dx)} \leqslant c(T)\;||u_0||_{L^2(0,\,L)}. \tag{3.29}$$

Proof. In (3.10) we have

$$\frac{d}{dt}\int_0^L u^2\,dx - \eta\;u_{xx}^2(0,\,t) = c(T)\;||u_0||_{L^2(0,\,L)} \tag{3.30}$$

Multiplying (3.30) by t and integrating the resulting expression over $t \in [0,\,T]$ we have

$$\int_0^t s\left[\frac{d}{ds}\int_0^L u^2\,dx\right]ds - \eta \int_0^t s\;u_{xx}^2(0,\,s)\,ds \leqslant c(T)\;||u_0||_{L^2(0,\,L)}$$

then

$$t\int_0^L u^2\,dx - \eta \int_0^t s\;u_{xx}^2(0,\,s)\,ds \leqslant c(T)\;||u_0||_{L^2(0,\,L)} \\ + ||u||^2_{L^2(0,\,T:\,L^2(0,\,L)))}. \tag{3.31}$$

Using (3.5) we obtain

$$\int_0^L t\;u^2\,dx - \eta\;||\sqrt{t}\;u_{xx}(0,\,\cdot)||^2_{L^2(0,T)} \leqslant c(T)\;||u_0||^2_{L^2(0,\,L)} \tag{3.32}$$

and (3.26) follows. Multiplying (3.16) by t and integrating the resulting expression over $t \in [0,\,T]$ we have

$$\int_0^t s\left[\frac{d}{ds}\int_0^L x\,u^2\,dx\right]ds-\eta\int_0^t\int_0^L s\;u_{xx}^2\,dx\,ds+3\int_0^t\int_0^L s\;u_x^2\,dx\,ds$$
$$\leqslant c(T)\;||u_0||^2_{L^2(0,\,L)} \tag{3.33}$$

hence

$$t\int_0^L x\,u^2\,dx-\eta\;||\sqrt{t}\;u_{xx}||^2_{L^2(0,\,T:\;L^2(0,\,T))}$$
$$+3\;||\sqrt{t}\;u_x||^2_{L^2(0,\,T:\;L^2(0,\,T))}\leqslant c(T)\;||u_0||^2_{L^2(0,\,L)} \tag{3.34}$$

and (3.27) follows. Moreover, from (3.32) and (3.34) we obtain (3.29). ■

Lemma 3.5 *Let $u_0 \in L^2(0,\,L)$ and $\eta < 0$. Then there exists a continuous function $t \to c(t)$ such that*

$$\int_0^T\int_0^L s\;u_x^2\;x\,dx\,dt\leqslant c(T)\;||u_0||_{L^2(0,\,L)}. \tag{3.35}$$
$$\int_0^T\int_0^L s\;u_{xx}^2\;x\,dx\,dt\leqslant c(T)\;||u_0||_{L^2(0,\,L)}. \tag{3.36}$$
$$||\sqrt{t}\;u_x||_{L^2(0,\,T:\;L^2(1+x)\,dx)}\leqslant c(T)\;||u_0||_{L^2(0,\,L)}. \tag{3.37}$$

Proof. Multiplying (3.24) by t, and integrating by parts the first term we obtain

$$t\int_0^L x^2\;u^2\,dx-10\,\eta\int_0^t\int_0^L s\;x\;u_{xx}^2\,dx\,ds$$
$$+6\int_0^t\int_0^L s\;x\;u_x^2\,dx\,ds\leqslant c(T)\;||u_0||^2_{L^2(0,\,L)} \tag{3.38}$$

and (3.35), (3.36) follows. From (3.27) and (3.35) we obtain (3.37). ■

4 Non-homogeneous linear estimates

We consider the non-homogeneous linear problem for the Korteweg de Vries Kawahara equation

$$v_t+\eta\,v_{xxxxx}+v_{xxx}+v_x=f(x,\,t),\quad x\in[0,\,L[,\quad t\in[0,\,T[, \tag{4.1}$$
$$v(0,\,t)=0,\quad v_x(0,\,t)=0,\quad t\in[0,\,T[, \tag{4.2}$$
$$v(L,\,t)=0,\quad v_x(L,\,t)=0,\quad v_{xx}(L,\,t)=0,\quad t\in[0,\,T[, \tag{4.3}$$
$$v(x,\,0)=0 \tag{4.4}$$

where $v = v(x,\, t)$, $\eta \in \mathbb{R}$.

Lemma 4.1 *Let $f \in \mathbb{E}$. Then there exists a continuous function $t \to c(t)$ such that*

$$||v||_{L^\infty(0,\,T:\,L^2((\,1+x^2\,)\,dx))} \leqslant c(T)\,||f||_{L^1(0,\,T:\,L^2((\,1+x^2\,)\,dx))}. \tag{4.5}$$

Proof. Multiplying (4.1) by v, integrating over $(0,\,L)$ and performing similar calculus to (3.10) we have

$$\frac{d}{dt}\int_0^L v^2\,dx - \eta\,v_{xx}(0,\,t) \leqslant 2\int_0^L v\,f\,dx. \tag{4.6}$$

Multiplying (4.1) by $x^2\,v$, integrating over $(0,\,L)$ and performing similar calculus to (3.30) we have

$$\begin{aligned}\frac{d}{dt}\int_0^L x^2\,v^2\,dx - 10\,\eta\int_0^L x\,v_{xx}^2\,dx + 6\int_0^L x\,v_x^2\,dx&\\ \leqslant 2\int_0^L x^2\,v\,f\,dx + 2\,L\int_0^L v^2\,dx.&\end{aligned} \tag{4.7}$$

Adding (4.6) with (4.7) we have

$$\begin{aligned}\frac{d}{dt}\int_0^L v^2(\,1+x^2\,)\,dx - 10\,\eta\int_0^L x\,v_{xx}^2\,dx + 6\int_0^L x\,v_x^2\,dx&\\ \leqslant 2\int_0^L v\,f\,(\,1+x^2\,)\,dx + 2\,L\int_0^L v^2\,dx&\\ \leqslant 2\,||v||_{L^2((\,1+x^2\,)\,dx)}\,||f||_{L^2((\,1+x^2\,)\,dx)} + 2\,L\int_0^L v^2\,dx.&\end{aligned} \tag{4.8}$$

Integrating over $t \in (0,\,T)$ we obtain

$$\begin{aligned}\int_0^L v^2(\,1+x^2\,)\,dx - 10\,\eta\int_0^t\int_0^L x\,v_{xx}^2\,dx\,ds + 6\int_0^t\int_0^L x\,v_x^2\,dx\,ds&\\ \leqslant 2\int_0^t ||v||_{L^2((\,1+x^2\,)\,dx)}\,||f||_{L^2((\,1+x^2\,)\,dx)}\,ds&\\ +2\,L\int_0^t\int_0^L v^2\,dx\,ds&\\ \leqslant 2\,||v||_{L^\infty(0,\,T:\,L^2((\,1+x^2\,)\,dx))}\,||f||_{L^1(0,\,T:\,L^2((\,1+x^2\,)\,dx))}&\\ +2\,L\int_0^t\int_0^L v^2\,dx\,ds.&\end{aligned}$$

Hence

$$||v||_{L^\infty(0,T:\,L^2((1+x^2)\,dx))} - 10\,\eta \int_0^t\int_0^L x\,v_{xx}^2\,dx\,ds + 6\int_0^t\int_0^L x\,v_x^2\,dx\,ds$$
$$\leqslant 2\,||v||_{L^\infty(0,T:\,L^2((1+x^2)\,dx))}\;||f||_{L^1(0,T:\,L^2((1+x^2)\,dx))} + 2\,L\,||v||^2_{L^2(0,T:\,L^2(0,L))}$$

thus using estimates as in section 3; (3.5), (3.6), (3.12), (3.21) and straightforward calculus we obtain

$$||v||_{L^\infty(0,T:\,L^2((1+x^2)\,dx))} - 10\,\eta \int_0^t\int_0^L x\,v_{xx}^2\,dx\,ds + 6\int_0^t\int_0^L x\,v_x^2\,dx\,ds \leqslant c(T)\,||f||^2_{L^1(0,T:\,L^2((1+x^2)\,dx))} \tag{4.9}$$

and (4.5) follows. ■

Lemma 4.2 *Let $f \in \mathbb{E}$. Then there exists a continuous function $t \to c(t)$ such that*

$$||v_x||_{L^2(0,T:\,L^2((1+x)dx))} \leqslant c\,||f||_{L^1(0,T:\,L^2((1+x^2)dx))}. \tag{4.10}$$
$$||\sqrt{t}\,v_x||_{L^2(0,T:\,L^2((1+x)dx))} \leqslant c\,||f||_{L^1(0,T:\,L^2((1+x^2)dx))}. \tag{4.11}$$

Proof. From (4.1)-(4.4) we have(see [20] for the construction of this semigroup).

$$v(x,\,t) = \int_0^t S(t-s)\,f(x,\,s)\,ds.$$

Differentiating in x-variable we have

$$v_x(x,\,t) = \int_0^t \partial_x S(t-s)\,f(x,\,s)\,ds, \tag{4.12}$$

applying $||\cdot||_{L^2((1+x)dx)}$

$$||v_x(x,\,t)||_{L^2((1+x)dx)} \leqslant \int_0^t ||\partial_x S(t-s)\,f(x,\,s)||_{L^2((1+x)dx)}\,ds$$

multiplying by $\psi(t) \in L^2(0,\,T)$

$$||v_x(x,\,t)||_{L^2((1+x)dx)}\;\psi(t) \leqslant \left[\int_0^t ||\partial_x S(t-s)\,f(x,\,s)||_{L^2((1+x)dx)}\,ds\right]$$

integrating over $t \in (0,\, T)$

$$\int_0^T ||v_x(x,\, t)||_{L^2((1+x)\,dx)}\; \psi(t)\, dt \leqslant \int_0^T \left[\int_0^t ||\partial_x S(t-s)\; f(x,\, s)||_{L^2((1+x)\,dx)}\, ds\right] \psi(t)\, dt$$

applying $|\cdot|$

$$\left|\int_0^T ||v_x(x,\, t)||_{L^2((1+x)dx)}\, \psi(t)\, dt\right| \leqslant \int_0^T \left[\int_0^t ||\partial_x S(t-s)\; f(x,\, s)||_{L^2((1+x)dx)}\, ds\right] |\, \psi(t)\,|\, dt.$$

Using Fubini's Theorem and the Cauchy-Schwartz inequality we have

$$\begin{aligned}
&\left|\int_0^T ||v_x(x,\, t)||_{L^2((1+x)dx)}\, \psi(t)\, dt\right| \leqslant \int_0^T \int_0^T |\, \psi(t)\,|\; ||\partial_x S(t-s)\; f(x,\, s)||_{L^2((1+x)dx)}\, dt\, ds \\
&\leqslant \int_0^T \left[\int_0^T |\psi(t)|^2\, dt\right]^{1/2} \left[\int_0^T ||\partial_x S(t-s)\; f(x,\, s)||^2_{L^2((1+x)dx)}\, dt\right]^{1/2} ds \\
&\leqslant \left[\int_0^T |\psi(t)|^2\, dt\right]^{1/2} \int_0^T \left[\int_0^T ||\partial_x S(t-s)\; f(x,\, s)||^2_{L^2((1+x)dx)}\, dt\right]^{1/2} ds \\
&\leqslant M \left[\int_0^T |\psi(t)|^2\, dt\right]^{1/2} \int_0^T \left[\int_0^T ||f(x,\, s)||^2_{L^2((1+x)dx)}\, dt\right]^{1/2} ds \\
&\leqslant M \left[\int_0^T |\psi(t)|^2\, dt\right]^{1/2} \int_0^T ||f(x,\, s)||_{L^2((1+x)dx)} \left[\int_0^T dt\right]^{1/2} ds \\
&\leqslant M\; T^{1/2} \left[\int_0^T |\psi(t)|^2\, dt\right]^{1/2} \int_0^T ||f(x,\, s)||_{L^2((1+x)dx)}\, ds \\
&\overset{(2.1)}{\leqslant} \sqrt{3}\, M\; T^{1/2} \left[\int_0^T |\psi(t)|^2 dt\right]^{1/2} \int_0^T ||f(x,\, s)||_{L^2((1+x^2)\,dx)}\, ds \\
&\leqslant c\; \left[\int_0^T |\psi(t)|^2\, dt\right]^{1/2} ||f||_{L^1(0,T\colon L^2((1+x^2)dx))}.
\end{aligned}$$

Therefore

$$\begin{aligned}
&\left|\int_0^T ||v_x(x,\, t)||_{L^2((1+x)dx)}\, \psi(t)\, dt\right| \\
&\leqslant c\; \left[\int_0^T |\psi(t)|^2\, dt\right]^{1/2} ||f||_{L^1(0,T\colon L^2((1+x^2)dx))}. \qquad (4.13)
\end{aligned}$$

If $\psi(t) = ||v_x(x,\, t)||_{L^2((1+x)dx)}$ then

$$\left|\int_0^T ||v_x(x,\,t)||^2_{L^2((1+x)dx)}\,dt\right|$$
$$\leqslant c\,\left[\int_0^T ||v_x(x,\,t)||^2_{L^2((1+x)dx)}\,dt\right]^{1/2}\;||f||_{L^1(0,\,T:\,L^2((1+x^2)dx))}$$

then

$$||v_x(x,\,t)||^2_{L^2(0,\,T:\,L^2((1+x)dx))}\leqslant c\,||v_x(x,\,t)||_{L^2(0,\,T:\,L^2((1+x)dx))}\,||f||_{L^1(0,\,T:\,L^2((1+x^2)dx))}.$$

Moreover, multiplying by $\sqrt{t}$ and using similar calculus as in (4.10) we obtain (4.11). ■

Lemma 4.3 *Let $f\in\mathbb{E}$. Then there exists a continuous function $t\to c(t)$ such that*

$$||\sqrt{t}\,v_{xx}||_{L^2(0,\,T:\,L^2(0,\,L))}\leqslant c(T)\;||f||_{L^1(0,\,T:\,L^2(1+x^2)\,dx))}. \tag{4.14}$$
$$\int_0^T\int_0^L t\,v_x^2\,dx\,dt\leqslant c(T)\;||f||_{L^1(0,\,T:\,L^2(1+x^2)\,dx))}. \tag{4.15}$$

Proof. Multiplying (4.1) by $x\,v$ and integrating over $x\in[0,\,L]$ we have

$$\int_0^L x\,v\,v_t\,dx+\eta\int_0^L x\,v\,v_{xxxxx}\,dx+\int_0^L v_{xxx}\,dx$$
$$+\int_0^L x\,v\,v_x\,dx=\int_0^L x\,v\,f(x,\,t)\,dx$$

performing similar calculus and straightforward estimates as in (3.16) and using Lemma 2.1(2.1), we have

$$\frac{d}{dt}\int_0^L x\,v^2\,dx-\eta\int_0^L v_{xx}^2\,dx+3\int_0^L v_x^2\,dx$$
$$\leqslant c(T)\;||f||^2_{L^1(0,\,T:\,L^2(1+x^2)\,dx))}. \tag{4.16}$$

Multiplying by t and using straightforward calculus we obtain

$$t\int_0^L x\,v^2\,dx-\eta\;||\sqrt{t}\,v_{xx}||^2_{L^2(0,\,T:\,L^2(0,\,L))}$$
$$+3\int_0^t\int_0^L s\,v_x^2\,dx\,ds\leqslant c(T)\;||f||^2_{L^1(0,\,T:\,L^2(1+x^2)\,dx))} \tag{4.17}$$

the result follows. ■

Lemma 4.4 *Let $f \in \mathbb{E}$. Then there exists a continuous function $t \to c(t)$ such that*

$$||\sqrt{t}\, v||_{L^\infty(0,T:\, L^2((1+x^2)\,dx))} \leqslant c\, ||f||_{L^1(0,T:\, L^2((1+x^2)dx))}. \tag{4.18}$$

$$\int_0^T \int_0^L t\, v_x^2\, x\, dx\, dt \leqslant c(T)\, ||f||_{L^1(0,T:\, L^2((1+x^2)dx))}. \tag{4.19}$$

$$\int_0^T \int_0^L t\, v_{xx}^2\, x\, dx\, dt \leqslant c(T)\, ||f||_{L^1(0,T:\, L^2((1+x^2)dx))}. \tag{4.20}$$

$$||\sqrt{t}\, v_x||_{L^2(0,T:\, L^2(1+x)\,dx)} \leqslant c(T)\, ||f||_{L^1(0,T:\, L^2((1+x^2)dx))}. \tag{4.21}$$

$$||\sqrt{t}\, v_{xx}||_{L^2(0,T:\, L^2(1+x)\,dx)} \leqslant c(T)\, ||f||_{L^1(0,T:\, L^2((1+x^2)dx))}. \tag{4.22}$$

Proof. From (4.8) we have

$$\begin{aligned}
&\frac{d}{dt}\int_0^L v^2(\,1+x^2\,)\,dx - 10\,\eta\int_0^L x\,v_{xx}^2\,dx + 6\int_0^L x\,v_x^2\,dx \\
&\leqslant 2\,||v||_{L^2((\,1+x^2\,)\,dx)}\,||f||_{L^2((\,1+x^2\,)\,dx)} + 2\,L\int_0^L v^2\,dx.
\end{aligned} \tag{4.23}$$

Multiplying by t, we perform similar calculus as in (4.9) and we integrate over $t \in [0,\, T]$

$$\begin{aligned}
&||\sqrt{t}\, v||^2_{L^\infty(0,T:\, L^2((1+x^2)\,dx))} - 10\,\eta\int_0^t\int_0^L s\,x\,v_{xx}^2\,dx\,ds \\
&6\int_0^t\int_0^L s\,x\,v_x^2\,dx\,ds \leqslant c(T)\,||f||^2_{L^1(0,T:\, L^2((1+x^2)dx))}
\end{aligned} \tag{4.24}$$

and we obtain (4.18), (4.19) and (4.20). From (4.15) and (4.19) we have (4.21) and from (4.14) with (4.20) we have (4.22). ■

5 Non linear case

We now prove in this section a local existence result for the nonlinear system

$$u_t + \eta\, u_{xxxxx} + u_{xxx} + u\, u_x + u_x = 0, \quad x \in [0,\, L[, \quad t \in [0,\, T[, \tag{5.1}$$
$$u(0,\, t) = g_0(t), \quad u_x(0,\, t) = g_1(t), \quad t \in [0,\, T[, \tag{5.2}$$
$$u(L,\, t) = 0, \quad u_x(L,\, t) = 0, \quad u_{xx}(L,\, t) = 0, \quad t \in [0,\, T[, \tag{5.3}$$
$$u(x,\, 0) = u_0(x), \quad x \in [0,\, L] \tag{5.4}$$

where $u = u(x,\, t)$, $\eta \in \mathbb{R}$.
Let ξ_i be a smooth function defined over $\mathbb{R}^+$ such that

$$\xi(L) = \xi^{(1)}(L) = \xi^{(2)}(L) = 0 \qquad \text{and} \qquad \xi_i^{(k)}(0) = \begin{cases} 1,\, i = k \\ 0,\, i \neq k. \end{cases}$$

Definition 5.1 *A weak solution of* (5.1)-(5.4) *on* $[0,\, T]$ *is a function* $u(x,\, t) \in \mathbb{H}_T$ *such that*

$$w(x,\, t) = u(x,\, t) - \sum_{i=0}^{1} \xi_i(x)\, g_i(t) \tag{5.5}$$

satisfy

$$\begin{aligned} w(x,\, t) &= S(t) w_0(x) \\ &- \sum_{i=0}^{1} \int_0^t \Big[\xi_i\, \partial_s g_i + \eta\, \xi_i^{(5)}\, g_i + \xi_i^{(3)}\, g_i + \xi_i^{(1)}\, g_i + (w + \xi_i\, g_i)\, (w_x + [\xi_i^{(1)}]\, g_i) \Big]\, ds \end{aligned}$$

where $w_0(x) = u_0(x) - \sum_{i=0}^{1} \xi_i(x)\, g_i(0)$.

We consider the change of function (5.5) in (5.1)-(5.4). Hence this change of the function yields an equivalent problem, i.e., we transform the original problem into a problem with a Dirichlet boundary condition $g_i = 0 (i = 0,\, 1)$,

$$w_t + \eta\, w_{xxxxx} + w_{xxx} + w_x = -\, F(w,\, w_x,\, g_i), \quad x \in [0,\, L], \quad t \in [0,\, T[, \tag{5.6}$$
$$w(0,\, t) = 0, \quad w_x(0,\, t) = 0, \quad t \in [0,\, T[, \tag{5.7}$$
$$w(L,\, t) = 0, \quad w_x(L,\, t) = 0, \quad w_{xx}(L,\, t) = 0, \quad t \in [0,\, T[, \tag{5.8}$$
$$w(x,\, 0) = w_0(x) \equiv u_0(x) - u_0(x) - \sum_{i=0}^{1} \xi_i(x)\, g_i(0), \quad x \in [0,\, L] \tag{5.9}$$

where

$$F(w,\, w_x,\, g) \equiv \sum_{i=0}^{1} \Big[\xi_i\, \partial_s g_i + \eta\, \xi_i^{(5)}\, g_i + \xi_i^{(3)}\, g_i + \xi_i^{(1)}\, g_i + (w + \xi_i\, g_i)\, \Big(w_x + \xi_i^{(1)}\, g_i \Big) \Big]$$

We write (5.6) as

$$w(x,\, t) = S(t) w_0(x) - \int_0^t S(t-s)\, F(w,\, w_x,\, g_i)\, ds \tag{5.10}$$

where $S(t)$ is the linear semi group. We introduce the following functional Γ defined by

$$\Gamma(w_0,\, g_i,\, w) = S(t)w_0(x) - \int_0^t S(t-s)\, F(w,\, w_x,\, g_i)\, ds$$

Remark. We would like to construct a mapping $\Gamma : \mathbb{H}_T \to \mathbb{H}_T$ with the following property: Given $\xi = \Gamma(\psi)$ with $||\psi||_{\mathbb{H}_T} \leqslant R$ we have $||\Gamma(\psi)||_{\mathbb{H}_T} = ||\xi||_{\mathbb{H}_T} \leqslant R$. In fact, this property tell us that $\Gamma : \mathbb{B}_R(0) \to \mathbb{B}_R(0)$ where $\mathbb{B}_R(0)$ is a ball in the space $\mathbb{H}_T$. Then if we want to prove that there exists a unique solution u defined on $\mathbb{H}_T$, weak solution of (5.1)-(5.4), it is enough to apply the Banach's fixed point Theorem for $u \to \Gamma(u_0,\, g_i,\, u)$ on $\mathbb{B}_R(0)$, (which is a complete metric space) which yields local existence and uniqueness.

Lemma 5.1 *There exist a constant $c(T)$ depending on T. independent of L such that for all $u_0 \in L^2(0,\, L)$ and $\eta < 0$*

$$||S(t)u_0||_{\mathbb{H}} \leqslant c(T)\; ||u_0||_{L^2(0,\, L)} \tag{5.11}$$

and the map $T \to c(T)$ is continuous.

Proof. Is a consequence of inequalities (3.21) and (3.22). ■

For the non-homogeneous problem and using that $v(x,\, t) = \int_0^t S(t-s)\, f(x,\, s)\, ds$, we have

Lemma 5.2 *There exist a constant $c(T)$ depending on T. independent of L such that for all $f \in \mathbb{E}$ and $\eta < 0$*

$$\left|\left|\int_0^t S(t-s) f(s)\, ds\right|\right|_{\mathbb{H}} \leqslant c(T)\; ||f||_{\mathbb{E}} \tag{5.12}$$

and the map $T \to c(T)$ is continuous.

Proof. Is a consequence of inequalities (4.5), (4.10), (4.11) and (4.14). ■

Lemma 5.3 *For all $u \in H^1(0,\, L)$ such that $u(0) = 0$, we have*

$$||\,\sqrt{x}\; u\,||_{L^\infty(0,\, L)} \leqslant 5\; \Big[\; ||\,u\,||^{1/2}_{L^2(1+x)}\; ||\,u_x\,||^{1/2}_{L^2(1+x)} + ||\,u_x\,||_{L^2(1+x)}\;\Big] \tag{5.13}$$

Theorem 5.1 *Let $\eta < 0$. Suppose that $w_0,\, z_0 \in L^2(0,\, T)$, and g_i and h_i are in $H^1_{loc}(\mathbb{R}^+)$. Then there exists a continuous function $t \to c(t)$ such that for all $T \in [0,\, T_0]$ we have*

$$\begin{aligned} &||\; \Gamma(w_0,\, g_i,\, w) - \Gamma(z_0,\, h_i,\, z)\; ||_{\mathbb{H}} \\ &\leqslant c(T)\; ||w_0 - z_0||_{L^2(0,\, L)} + c(T)\; \sqrt{T}\; \Big[\; ||\, g_i - h_i\, ||_{H^1(0,\, T)} + ||w - z||_{\mathbb{H}} + ||z||_{\mathbb{H}}\;\Big] \\ &\quad + c(T)\; \sqrt{T}\; \Big[\; \Big(1 + \sqrt{T}\Big)\; ||w-z||_{\mathbb{H}}\; ||w||_{\mathbb{H}} + ||w-z||_{\mathbb{H}}\; ||w||_{\mathbb{H}}\;\Big] \end{aligned} \tag{5.14}$$

Proof. Let $\Gamma(w_0,\, g_i,\, w) = S(t)w_0(x) - \int_0^t S(t-s)\, F(w,\, w_x,\, g_i)\, ds$ then

$$\begin{aligned}\Gamma(w_0,\, g_i,\, w) - \Gamma(z_0,\, h_i,\, z) &= S(t)(w_0 - z_0)(x)\\ &\quad - \int_0^t S(t-s)\, [F(w,\, w_x,\, g_i) - F(z,\, z_x,\, h_i)]\, ds. \qquad (5.15)\end{aligned}$$

Let

$$\begin{aligned}K(s) &= F(w,\, w_x,\, g_i) - F(z,\, z_x,\, h_i)\\ &= \xi_i\, \xi_i^{(1)}\, (g_i^2 - h_i^2) + [\eta\, \xi_i^{(5)} + \xi_i^{(3)} + \xi_i^{(1)}]\, (g_i - h_i)\\ &\quad + \xi_i\, \partial_s(g_i - h_i) + \xi_i^{(1)}\, (w\, g_i - z\, h_i) + \xi_i\, (w_x\, g_i - z_x\, h_i)\\ &\quad + w\, w_x - z\, z_x\\ &= K_1 + K_2 + K_3\end{aligned}$$

where

$$\begin{aligned}K_1(s) &= \xi_i\, \xi_i^{(1)}\, (g_i^2 - h_i^2) + [\eta\, \xi_i^{(5)} + \xi_i^{(3)} + \xi_i^{(1)}]\, (g_i - h_i) + \xi_i\, \partial_s(g_i - h_i)\\ K_2(s) &= \xi_i^{(1)}\, (w\, g_i - z\, h_i) + \xi_i\, (w_x\, g_i - z_x\, h_i)\\ K_3(s) &= w\, w_x - z\, z_x.\end{aligned}$$

Hence in (5.15), using Lemma 5.1 and Lemma 5.2 we have

$$\begin{aligned}&||\Gamma(w_0,\, g_i,\, w) - \Gamma(z_0,\, h_i,\, z)||_{\mathbb{H}} \leqslant c(T)\, ||w_0 - z_0||_{L^2(0,\, L)} + c(T)\, ||K||_{\mathbb{E}} \qquad (5.16)\\ &\leqslant c(T)\, ||w_0 - z_0||_{L^2(0,\, L)} + c(T) \left[||K||_{L^1(0,\, T:\, L^2((1+x^2)\, dx))} + ||\sqrt{t}\, K||_{L^2(0,\, T:\, L^2((1+x^2)\, dx))} \right]\end{aligned}$$

We estimate separately $K_1,\, K_2,\, K_3$.

$$\begin{aligned}&||K_1||_{L^2((1+x^2)dx)}\\ &\leqslant ||\, \xi_i\, \xi_i^{(1)}\, (g_i^2 - h_i^2) + [\eta\, \xi_i^{(5)} + \xi_i^{(3)} + \xi_i^{(1)}]\, (g_i - h_i)\, ||_{L^2((1+x^2)dx)}\\ &\quad + ||\, \xi_i\, \partial_s(g_i - h_i)\, ||_{L^2((1+x^2)dx)}\\ &\leqslant ||\, [\, \xi_i\, \xi_i^{(1)}\, (g_i + h_i) + \eta\, \xi_i^{(5)} + \xi_i^{(3)} + \xi_i^{(1)}\,]\, (g_i - h_i)||_{L^2((1+x^2)dx)}\\ &\quad + ||\xi_i\, (\, \partial_s g_i - \partial_s h_i\,)\, ||_{L^2((1+x^2)\, dx)}\\ &\leqslant c\, (\, 1 + |\, g_i(t)\, | + |\, h_i(t)\, |\,)\, |\, g_i(t) - h_i(t)\, | + c\, |\, \partial_t g_i(t) - \partial_t h_i(t)\, |\\ &= c\, (\, 1 + |\, g_i(t)\, | + |\, h_i(t)\, |\,)\, |\, g_i(t) - h_i(t)\, | + c\, |\, g_i'(t) - h_i'(t)\, |. \qquad (5.17)\end{aligned}$$

Integrating over $t \in [0,\, T]$ we have

$$
\begin{aligned}
&||K_1||_{L^1(0,T:\, L^2((1+x^2)dx))}\\
&\leqslant c\int_0^t (\,1+|\,g_i(s)\,|+|\,h_i(s)\,|\,)\,|\,g_i(s)-h_i(s)\,|\,ds + c\int_0^t |\,g_i'(s)-h_i'(s)\,|\,ds\\
&\leqslant c\,\sqrt{T}\,||\,g_i-h_i\,||_{L^2(0,T)} + ||\,g_i\,||_{L^2(0,T)}\,||\,g_i-h_i\,||_{L^2(0,T)}\\
&+||\,h_i\,||_{L^2(0,T)}\,||\,g_i-h_i\,||_{L^2(0,T)} + c\,\sqrt{T}\,||\,g_i'-h_i'\,||_{L^2(0,T)}\\
&\leqslant c\,\sqrt{T}\,||\,g_i-h_i\,||_{H^1(0,T)} + ||\,g_i\,||_{L^2(0,T)}\,||\,g_i-h_i\,||_{H^1(0,T)}\\
&+||\,h_i\,||_{L^2(0,T)}\,||\,g_i-h_i\,||_{H^1(0,T)} + c\,\sqrt{T}\,||\,g_i-h_i\,||_{H^1(0,T)}.
\end{aligned}
$$

Using that g_i and h_i are in $H^1_{loc}(\mathbb{R}^+)$ we obtain

$$
||K_1||_{L^1(0,T:\, L^2((1+x^2)dx))} \leqslant c\,\sqrt{T}\,||\,g_i-h_i\,||_{H^1(0,T)}. \tag{5.18}
$$

Similarly, elevating square (5.17) we have

$$
\begin{aligned}
&||K_1||^2_{L^2((1+x^2)dx)}\\
&\leqslant c^2\,[\,(\,1+|\,g_i(t)\,|+|\,h_i(t)\,|\,)\,|\,g_i(t)-h_i(t)\,|+|\,g_i'(t)-h_i'(t)\,|\,]^2\\
&= c^2\,(\,1+|\,g_i(t)\,|+|\,h_i(t)\,|\,)^2\,|\,g_i(t)-h_i(t)\,|^2 + c^2\,|\,g_i'(t)-h_i'(t)\,|^2\\
&\quad + 2\,c^2\,(\,1+|\,g_i(t)\,|+|\,h_i(t)\,|\,)\,|\,g_i(t)-h_i(t)\,|\,|\,g_i'(t)-h_i'(t)\,|\\
&= c^2\,(\,1+|\,g_i(t)\,|^2+|\,h_i(t)|^2\,+2\,|\,g_i(t)\,|\\
&\quad +2\,|\,h_i(t)\,|+2\,|\,g_i(t)\,|\,|\,h_i(t)\,|\,)\,|\,g_i(t)-h_i(t)\,|^2\\
&\quad + c^2\,|\,g_i'(t)-h_i'(t)\,|^2 + 2\,c^2\,|\,g_i(t)-h_i(t)\,|\,|\,g_i'(t)-h_i'(t)\,|\\
&\quad + 2\,c^2\,|\,g_i(t)\,|\,|\,g_i(t)-h_i(t)\,|\,|\,g_i'(t)-h_i'(t)\,|\\
&\quad +2\,c^2\,|\,h_i(t)\,|\,|\,g_i(t)-h_i(t)\,|\,|\,g_i'(t)-h_i'(t)\,|\\
&= c^2\,|\,g_i(t)-h_i(t)\,|^2 + |\,g_i(t)\,|^2\,|\,g_i(t)-h_i(t)\,|^2 + |\,h_i(t)|^2\,|\,g_i(t)-h_i(t)\,|^2\\
&\quad + 2\,c^2\,|\,g_i(t)\,|\,|\,g_i(t)-h_i(t)\,|^2 + 2\,|\,h_i(t)\,|\,|\,g_i(t)-h_i(t)\,|^2\\
&\quad + 2\,|\,g_i(t)\,|\,|\,h_i(t)\,|\,|\,g_i(t)-h_i(t)\,|^2\\
&\quad + c^2\,|\,g_i'(t)-h_i'(t)\,|^2 + 2\,c^2\,|\,g_i(t)-h_i(t)\,|\,|\,g_i'(t)-h_i'(t)\,|\\
&\quad + 2\,c^2\,|\,g_i(t)\,|\,|\,g_i(t)-h_i(t)\,|\,|\,g_i'(t)-h_i'(t)\,|\\
&\quad + 2\,c^2\,|\,h_i(t)\,|\,|\,g_i(t)-h_i(t)\,|\,|\,g_i'(t)-h_i'(t)\,|.
\end{aligned} \tag{5.19}
$$

Multiplying (5.19) by t and integrating over $t \in [0,\, T]$ we have

$$\begin{aligned}
&\int_0^T t\,||K_1||^2_{L^2((1+x^2)dx)}\,dt\\
&\leqslant c^2\int_0^T t\,|\,g_i(t)-h_i(t)\,|^2\,dt+\int_0^T t\,|\,g_i(t)\,|^2\,|\,g_i(t)-h_i(t)\,|^2\,dt\\
&\quad+\int_0^T t\,|\,h_i(t)|^2\,|\,g_i(t)-h_i(t)\,|^2\,dt+2\,c^2\int_0^T t\,|\,g_i(t)\,|\,|\,g_i(t)-h_i(t)\,|^2\,dt\\
&\quad+2\int_0^T t\,|\,h_i(t)\,|\,|\,g_i(t)-h_i(t)\,|^2\,dt+2\int_0^T t\,|\,g_i(t)\,|\,|\,h_i(t)\,|\,|\,g_i(t)-h_i(t)\,|^2\,dt\\
&\quad+c^2\int_0^T t\,|\,g_i'(t)-h_i'(t)\,|^2\,dt+2\,c^2\int_0^T t\,|\,g_i(t)-h_i(t)\,|\,|\,g_i'(t)-h_i'(t)\,|\,dt\\
&\quad+2\,c^2\int_0^T t\,|\,g_i(t)\,|\,|\,g_i(t)-h_i(t)\,|\,|\,g_i'(t)-h_i'(t)\,|\,dt\\
&\quad+2\,c^2\int_0^T t\,|\,h_i(t)\,|\,|\,g_i(t)-h_i(t)\,|\,|\,g_i'(t)-h_i'(t)\,|\,dt\\
&\leqslant c^2\,T\,||\,g_i-h_i\,||^2_{L^2(0,T)}+T\,||\,g_i||^2_{L^\infty(0,T)}\,||\,g_i-h_i\,||^2_{L^2(0,T)}\\
&\quad+T\,||\,h_i||^2_{L^\infty(0,T)}\,||\,g_i-h_i\,||^2_{L^2(0,T)}+2\,c^2\,T\,||\,g_i||_{L^\infty(0,T)}\,||\,g_i-h_i\,||^2_{L^2(0,T)}\\
&\quad+2\,T\,||\,h_i||_{L^\infty(0,T)}\,||\,g_i-h_i\,||^2_{L^2(0,T)}\\
&\quad+2\,T\,||\,g_i||_{L^\infty(0,T)}\,||\,h_i||_{L^\infty(0,T)}\,||\,g_i-h_i\,||^2_{L^2(0,T)}\\
&\quad+c^2\,T\,||\,g_i'-h_i'\,||^2_{L^2(0,T)}\,+2\,c^2\,T\,||\,g_i-h_i\,||_{L^2(0,T)}\,||\,g_i'-h_i'\,||_{L^2(0,T)}\\
&\quad+2\,c^2\,T\,||\,g_i\,||_{L^\infty(0,T)}\,||\,g_i-h_i\,||_{L^2(0,T)}\,||\,g_i'-h_i'\,||_{L^2(0,T)}\\
&\quad+2\,c^2\,T\,||\,h_i\,||_{L^\infty(0,T)}\,||\,g_i-h_i\,||_{L^2(0,T)}\,||\,g_i'-h_i'\,||_{L^2(0,T)}
\end{aligned}\tag{5.20}$$

hence

$$\begin{aligned}
&||\sqrt{t}\,K_1||^2_{L^2(0,T:\,L^2((1+x^2)dx))}\\
&\leqslant c^2\,T\,||\,g_i-h_i\,||^2_{L^2(0,T)}+T\,||\,g_i||^2_{L^\infty(0,T)}\,||\,g_i-h_i\,||^2_{L^2(0,T)}\\
&\quad+T\,||\,h_i||^2_{L^\infty(0,T)}\,||\,g_i-h_i\,||^2_{L^2(0,T)}+2\,c^2\,T\,||\,g_i||_{L^\infty(0,T)}\,||\,g_i-h_i\,||^2_{L^2(0,T)}\\
&\quad+2\,T\,||\,h_i||_{L^\infty(0,T)}\,||\,g_i-h_i\,||^2_{L^2(0,T)}+2\,T\,||\,g_i||_{L^\infty(0,T)}\,||\,h_i||_{L^\infty(0,T)}\,||\,g_i-h_i\,||^2_{L^2(0,T)}\\
&\quad+c^2\,T\,||\,g_i'-h_i'\,||^2_{L^2(0,T)}\,+2\,c^2\,T\,||\,g_i-h_i\,||_{L^2(0,T)}\,||\,g_i'-h_i'\,||_{L^2(0,T)}\\
&\quad+2\,c^2\,T\,||\,g_i\,||_{L^\infty(0,T)}\,||\,g_i-h_i\,||_{L^2(0,T)}\,||\,g_i'-h_i'\,||_{L^2(0,T)}\\
&\quad+2\,c^2\,T\,||\,h_i\,||_{L^\infty(0,T)}\,||\,g_i-h_i\,||_{L^2(0,T)}\,||\,g_i'-h_i'\,||_{L^2(0,T)}.
\end{aligned}$$

Using that $H^1(0,\,T)\hookrightarrow L^\infty(0,\,T)$ and straightforward calculus we have

$$||\sqrt{t}\,K_1||_{L^2(0,T:\,L^2((1+x^2)dx))}\leqslant c\,\sqrt{T}\,||\,g_i-h_i\,||_{H^1(0,T)}.\tag{5.21}$$

From (5.18) and (5.21) we obtain

$$||K_1||_{\mathbb{E}}\leqslant c\,\sqrt{T}\,||\,g_i-h_i\,||_{H^1(0,T)}.\tag{5.22}$$

By other hand

$$
\begin{aligned}
&||K_2||_{L^2((1+x^2)\,dx)}\\
&\leqslant ||\, \xi_i^{(1)}\,(w-z)\,g_i||_{L^2((1+x^2)\,dx)} + ||\, \xi_i^{(1)}\,z\,(g_i-h_i)||_{L^2((1+x^2)\,dx)}\\
&\quad + ||\, \xi_i^{(1)}\,(w_x-z_x)\,g_i||_{L^2((1+x^2)\,dx)} + ||\, \xi_i^{(1)}\,z_x\,(g_i-h_i)||_{L^2((1+x^2)\,dx)}\\
&\leqslant c\,|g_i(t)|\;||w-z||_{L^2((1+x^2)\,dx)} + c\,|g_i(t)-h_i(t)|\;||z||_{L^2((1+x^2)\,dx)}\\
&\quad + c\,|g_i(t)|\;||w_x-z_x||_{L^2((1+x^2)\,dx)} + c\,|g_i(t)-h_i(t)|\;||z_x||_{L^2((1+x^2)\,dx)}
\end{aligned}
$$

Integrating over $t \in [0,\, T]$ and using the Hölder inequality

$$
\begin{aligned}
&\int_0^T ||K_2||_{L^2((1+x^2)\,dx)}\,dt\\
&\leqslant c\int_0^T |g_i(t)|\;||w-z||_{L^2((1+x^2)\,dx)}\,dt + c\int_0^T |g_i(t)-h_i(t)|\;||z||_{L^2((1+x^2)\,dx)}\,dt\\
&\quad + c\int_0^T |g_i(t)|\;||w_x-z_x||_{L^2((1+x^2)\,dx)}\,dt + c\int_0^T |g_i(t)-h_i(t)|\;||z_x||_{L^2((1+x^2)\,dx)}\,dt\\
&\leqslant c\,||g_i||_{L^2(0,T)}\;||w-z||_{L^2(0,T:\,L^2((1+x^2)\,dx))}\\
&\quad + c\,||g_i-h_i||_{L^2(0,T)}\;||z||_{L^2(0,T:\,L^2((1+x^2)\,dx))}\\
&\quad + c\,||g_i||_{L^2(0,T)}\;||w_x-z_x||_{L^2(0,T:\,L^2((1+x^2)\,dx))}\\
&\quad + c\,||g_i-h_i||_{L^2(0,T)}\;||z_x||_{L^2(0,T:\,L^2((1+x^2)\,dx))}\\
&\leqslant c\,||g_i||_{H^1(0,T)}\;||w-z||_{L^2(0,T:\,L^2((1+x^2)\,dx))}\\
&\quad + c\,||g_i-h_i||_{H^1(0,T)}\;||z||_{L^2(0,T:\,L^2((1+x^2)\,dx))}\\
&\quad + c\,||g_i||_{H^1(0,T)}\;||w_x-z_x||_{L^2(0,T:\,L^2((1+x^2)\,dx))}\\
&\quad + c\,||g_i-h_i||_{H^1(0,T)}\;||z_x||_{L^2(0,T:\,L^2((1+x^2)\,dx))}
\end{aligned}
$$

Using that g_i and h_i are in $H^1_{loc}(\mathbb{R}^+)$ we obtain

$$
\begin{aligned}
||K_2||_{L^1(0,T:\,L^2((1+x^2))} &\leqslant c\,||w_x-z_x||_{L^2(0,T:\,L^2((1+x^2)\,dx))}\\
&\quad + c\,||z_x||_{L^2(0,T:\,L^2((1+x^2)\,dx))}\\
&\leqslant c\,||w-z||_{\mathbb{H}} + c\,||z_x||_{\mathbb{H}}.
\end{aligned} \tag{5.23}
$$

The similar form, we obtain

$$
||\sqrt{t}\;K_2||_{L^1(0,T:\,L^2((1+x^2))} \leqslant c\,\sqrt{T}\;\left(\,||w-z||_{\mathbb{H}} + c\,||z_x||_{\mathbb{H}}\,\right). \tag{5.24}
$$

From (5.22) and (5.24) we obtain

$$
||K_2||_{\mathbb{E}} \leqslant c\,\sqrt{T}\;\left(\,||w-z||_{\mathbb{H}} + c\,||z_x||_{\mathbb{H}}\,\right). \tag{5.25}
$$

We estimate the term $K_3 = w\,w_x - z\,z_x = (w-z)\,w_x + z\,(w_x-z_x)$, then

$$||w\,w_x - z\,z_x||_{\mathbb{E}} = ||(w-z)\,w_x + z\,(w_x - z_x)||_{\mathbb{E}} \leqslant ||(w-z)\,w_x||_{\mathbb{E}} + ||z\,(w_x - z_x)||_{\mathbb{E}}$$

hence

$$\begin{aligned}
||K_3||_{\mathbb{E}} &\leqslant ||(w-z)\,w_x||_{\mathbb{E}} + ||z\,(w_x - z_x)||_{\mathbb{E}} \\
&= ||(w-z)\,w_x||_{L^1(0,T:\,L^2((1+x^2)dx))} + ||\sqrt{t}\,(w-z)\,w_x||_{L^2(0,T:\,L^2((1+x^2)dx))} \\
&+ ||z\,(w_x - z_x)||_{L^1(0,T:\,L^2((1+x^2)dx))} + ||\sqrt{t}\,z\,(w_x - z_x)||_{L^2(0,T:\,L^2((1+x^2)dx))} \\
&= ||(w-z)\,w_x||_{L^1(0,T:\,L^2((1+x^2)dx))} + ||z\,(w_x - z_x)||_{L^1(0,T:\,L^2((1+x^2)dx))} \\
&+ ||\sqrt{t}\,(w-z)\,w_x||_{L^2(0,T:\,L^2((1+x^2)dx))} + ||\sqrt{t}\,z\,(w_x - z_x)||_{L^2(0,T:\,L^2((1+x^2)dx))} \\
&= K_{3,1} + K_{3,2}
\end{aligned}$$

where

$$K_{3,1} = ||(w-z)\,w_x||_{L^1(0,T:\,L^2((1+x^2)dx))} + ||z\,(w_x - z_x)||_{L^1(0,T:\,L^2((1+x^2)dx))}$$

and

$$K_{3,2} = ||\sqrt{t}\,(w-z)\,w_x||_{L^2(0,T:\,L^2((1+x^2)dx))} + ||\sqrt{t}\,z\,(w_x - z_x)||_{L^2(0,T:\,L^2((1+x^2)dx))}.$$

Using that: if $a \geqslant 0$ and $b \geqslant 0$ then $\sqrt{a+b} \leqslant \sqrt{a} + \sqrt{b}$ we have

$$\begin{aligned}
K_{3,1} &= ||(w-z)\,w_x||_{L^1(0,T:\,L^2((1+x^2)dx))} + ||z\,(w_x - z_x)||_{L^1(0,T:\,L^2((1+x^2)dx))} \\
&= \int_0^T ||(w-z)\,w_x||_{L^2((1+x^2)dx)} dt + \int_0^T ||(w_x - z_x)\,z||_{L^2((1+x^2)dx)} dt \\
&= \int_0^T \sqrt{\int_0^L (w-z)^2\,w_x^2 (1+x^2) dx}\; dt + \int_0^T \sqrt{\int_0^L (w_x - z_x)^2\,z^2 (1+x^2) dx}\; dt \\
&= \int_0^T \sqrt{\int_0^L (w-z)^2\,w_x^2\,dx + \int_0^L (w-z)^2\,w_x^2\,x^2\,dx}\; dt \\
&+ \int_0^T \sqrt{\int_0^L (w_x - z_x)^2\,z^2\,dx + \int_0^L (w_x - z_x)^2\,z^2\,x^2\,dx}\; dt \\
&\leqslant \int_0^T \left[\sqrt{\int_0^L (w-z)^2\,w_x^2\,dx} + \sqrt{\int_0^L (w-z)^2\,w_x^2\,x^2\,dx} \right] dt \\
&+ \int_0^T \left[\sqrt{\int_0^L (w_x - z_x)^2\,z^2\,dx} + \sqrt{\int_0^L (w_x - z_x)^2\,z^2\,x^2\,dx} \right] dt \qquad (5.26)
\end{aligned}$$

but, using Lemma 2.1(2.4) follows that

$$\int_0^L (w-z)^2\, w_x dx \leqslant ||w-z||^2_{L^\infty(0,\,L)} \int_0^L w_x^2 dx \tag{5.27}$$
$$\leqslant ||w-z||^2_{L^\infty(0,\,L)} \int_0^L w_x^2(1+x)\, dx$$
$$\leqslant 2\, ||w-z||_{L^2((1+x)dx)}\, ||w_x-z_x||_{L^2((1+x)dx)}\, ||w_x||^2_{L^2((1+x)dx)}$$

and

$$\int_0^L (w-z)^2\, w_x^2\, x^2\, dx \leqslant ||x\,(w-z)^2||_{L^\infty(0,\,L)} \int_0^L x\, w_x^2\, dx$$
$$\leqslant ||\sqrt{x}\,(w-z)||^2_{L^\infty(0,\,L)} \int_0^L w_x^2\,(1+x)\, dx$$

then $||x\,(w-z)\,w_x||^2_{L^2(0,\,L)} \leqslant ||\sqrt{x}\,(w-z)||^2_{L^\infty(0,\,L)} \int_0^L w_x^2\,(1+x)\,dx$. Hence using Lemma 5.3 we have

$$||x(w-z)w_x||_{L^2(0,\,L)}$$
$$\leqslant ||\sqrt{x}\,(w-z)||_{L^\infty(0,\,L)} \sqrt{\int_0^L w_x^2\,(1+x)\,dx}$$
$$\leqslant 5\Big[\sqrt{||w-z||_{L^2((1+x)dx)}}\,\sqrt{||w_x-z_x||_{L^2((1+x)dx)}}$$
$$+||w-z||_{L^2((1+x)dx)}\Big]\, ||w_x||_{L^2((1+x)dx)} \tag{5.28}$$

This way, using (5.27) and (5.28) for the first term in (5.26) we obtain

$$\int_0^T \left[\sqrt{\int_0^L (w-z)^2\, w_x^2\, dx} + \sqrt{\int_0^L (w-z)^2\, w_x^2\, x^2\, dx}\right] dt$$
$$\leqslant \int_0^T \sqrt{\int_0^L (w-z)^2\, w_x^2\, dx}\; dt + \int_0^T \sqrt{\int_0^L (w-z)^2\, w_x^2\, x^2\, dx}\; dt$$
$$\leqslant \sqrt{2} \int_0^T ||w-z||^{1/2}_{L^2((1+x)dx)}\, ||w_x-z_x||^{1/2}_{L^2((1+x)dx)}\, ||w_x||_{L^2((1+x)dx)}\, dt$$
$$+\, 5 \int_0^T \Big[\, ||w-z||^{1/2}_{L^2((1+x)dx)}\, ||w_x-z_x||^{1/2}_{L^2((1+x)dx)} + ||w-z||_{L^2((1+x)dx)} \Big]\, ||w_x||_{L^2((1+x)dx)}\, dt$$
$$\leqslant c \int_0^T ||w-z||^{1/2}_{L^2((1+x)dx)}\, ||w_x-z_x||^{1/2}_{L^2((1+x)dx)}\, ||w_x||_{L^2((1+x)dx)}\, dt$$
$$+\, 5 \int_0^T ||w-z||_{L^2((1+x)dx)}\, ||w_x||_{L^2((1+x)dx)}\, dt$$
$$\leqslant c\, ||w-z||_{\mathbb{H}} \int_0^T ||w_x-z_x||^{1/2}_{L^2((1+x)dx)}\, ||w_x||_{L^2((1+x)dx)}\, dt$$
$$+\, 5\, ||w-z||_{\mathbb{H}} \int_0^T ||w_x||_{L^2((1+x)dx)}\, dt \leqslant c(\,\sqrt{T} + T^{1/4}\,)\, ||w-z||_{\mathbb{H}}\, ||w||_{\mathbb{H}}.$$

Therefore

$$\int_0^T \left[\sqrt{\int_0^L (w-z)^2 w_x^2 dx} + \sqrt{\int_0^L (w-z)^2 w_x^2 x^2 dx} \right] dt$$
$$\leqslant c\,(\,\sqrt{T} + T^{1/4}\,)\,||w-z||_{\mathbb{H}}\,||w||_{\mathbb{H}}. \tag{5.29}$$

Using the similar technique for the second term we obtain

$$\int_0^T \left[\sqrt{\int_0^L (w_x - z_x)^2\, z^2\, dx} \; + \sqrt{\int_0^L (w_x - z_x)^2\, z^2\, x^2\, dx} \;\right] dt$$
$$\leqslant c\,(\,\sqrt{T} + T^{1/4}\,)\,||w-z||_{\mathbb{H}}\,||w||_{\mathbb{H}}. \tag{5.30}$$

This way from (5.29) and (5.30) we have

$$K_{3,\,1} \leqslant c\,(\,\sqrt{T} + T^{1/4}\,)\,||w-z||_{\mathbb{H}}\,||w||_{\mathbb{H}}. \tag{5.31}$$

Now we estimate the term $K_{3,\,2}$. Using the Lemma 5.4, we estimate the following term

$$\begin{aligned}
&||x(w-z)\,w_x||_{L^2(0,\,L)} \\
&= ||\sqrt{x}\,(w-z)\,\sqrt{x}\,w_x||_{L^2(0,\,L)} \\
&\leqslant ||\sqrt{x}\,(w-z)||_{L^\infty(0,\,L)}\;||\sqrt{x}\,w_x||_{L^2(0,\,L)} \\
&\leqslant 5\left(\sqrt{||w-z||_{L^2((1+x)dx)}}\;\sqrt{||w_x - z_x||_{L^2((1+x)dx)}}\right. \\
&\quad \left. + ||w-z||_{L^2((1+x)dx)}\right)\,||\sqrt{x}\,w_x||_{L^2(0,\,L)} \\
&\leqslant 5\,\sqrt{||w-z||_{L^2((1+x)dx)}}\;\sqrt{||w_x - z_x||_{L^2((1+x)dx)}}\;||\sqrt{x}\,w_x||_{L^2(0,\,L)} \\
&\quad + 5\,||w-z||_{L^2((1+x)dx)}\,||\sqrt{x}\,w_x||_{L^2(0,\,L)} \\
&\leqslant 5\,\sqrt{||w-z||_{L^2((1+x)dx)}}\;\sqrt{||w_x - z_x||_{L^2((1+x)dx)}}\;||w_x||_{L^2((1+x)dx)} \\
&\quad + 5\,||w-z||_{L^2((1+x)dx)}\,||w_x||_{L^2((1+x)dx)},
\end{aligned} \tag{5.32}$$

hence from (5.32)

$$
\begin{aligned}
& ||\sqrt{t}\, x\, (w-z)\, w_x||_{L^2(0,\,T:\,L^2(0,\,L))} \\
& = ||\sqrt{t}\,\sqrt{x}\,(w-z)\,\sqrt{x}\, w_x||_{L^2(0,\,T:\,L^2(0,\,L))} \\
& \leqslant 5\,\sqrt{||w-z||_{L^\infty(0,\,T:\,L^2((1+x)dx))}}\;||\sqrt{t}\, w_x||_{L^\infty(0,\,T:\,L^2((1+x)dx)} \\
& \quad \times \sqrt{||w_x - z_x||_{L^2(0,\,T:\,L^2((1+x)dx))}}\;\sqrt{T} \\
& \quad + 5\;||w-z||_{L^\infty(0,\,T:\,L^2((1+x)dx))}\;||\sqrt{t}\, w_x||_{L^2(0,\,T:\,L^2((1+x)dx)} \\
& \leqslant c\;||w-z||_{\mathbb{H}}^{1/2}\;||w||_{\mathbb{H}}\;\sqrt{||w_x - z_x||_{L^2(0,\,T:\,L^2((1+x)dx))}}\;\sqrt{T} \\
& \quad + c\;||w-z||_{\mathbb{H}}\;\sqrt{T}\;||w||_{\mathbb{H}} \\
& \leqslant c\,\sqrt{T}\;||w-z||_{\mathbb{H}}\;||w||_{\mathbb{H}}, \quad \forall\;\; 0 \leqslant t \leqslant T. \qquad (5.33)
\end{aligned}
$$

This way from (5.33), the first term in $K_{3,\,2}$ is estimated by

$$||\sqrt{t}\,(w-z)\,w_x||_{L^2(0,\,T:\,L^2((1+x^2)dx))} \leqslant c\,\sqrt{T}\;||w-z||_{\mathbb{H}}\;||w||_{\mathbb{H}} \qquad (5.34)$$

and the similar form we estimate the second term in $K_{3,\,2}$.

$$||\sqrt{t}\,(w_x - z_x)\,z||_{L^2(0,\,T:\,L^2((1+x^2)dx))} \leqslant c\,\sqrt{T}\;||w-z||_{\mathbb{H}}\;||w||_{\mathbb{H}}. \qquad (5.35)$$

then

$$||K_3||_{\mathbb{E}} \leqslant c\,\sqrt{T}\;||w-z||_{\mathbb{H}}\;||w||_{\mathbb{H}}. \qquad (5.36)$$

Therefore replacing in (5.17) the terms in (5.22), (5.25), (5.36) we have

$$
\begin{aligned}
& ||\;\Gamma(w_0,\,g_i,\,w) - \Gamma(z_0,\,h_i,\,z)\;||_{\mathbb{H}} \\
& \leqslant c(T)\;||w_0 - z_0||_{L^2(0,\,L)} + c(T)\,[\;||K_1||_{\mathbb{E}} + ||K_2||_{\mathbb{E}} + ||K_3||_{\mathbb{E}}\;] \\
& \leqslant c(T)\;||w_0 - z_0||_{L^2(0,\,L)} + c(T)\;\sqrt{T}\;\Big[\;||\,g_i - h_i\,||_{H^1(0,\,T)} + ||w-z||_{\mathbb{H}} + ||z||_{\mathbb{H}}\;\Big] \\
& \quad + c(T)\;\sqrt{T}\;\Big[\;\Big(1+\sqrt{T}\Big)\;||w-z||_{\mathbb{H}}\;||w||_{\mathbb{H}} + ||w-z||_{\mathbb{H}}\;||w||_{\mathbb{H}}\;\Big] \qquad (5.37)
\end{aligned}
$$

where we obtain

$$
\begin{aligned}
& ||\;\Gamma(w_0,\,g_i,\,w) - \Gamma(z_0,\,h_i,\,z)\;||_{\mathbb{H}} \\
& \leqslant c(T)\;||w_0 - z_0||_{L^2(0,\,L)} + c(T)\;\sqrt{T}\;\Big[\;||\,g_i - h_i\,||_{H^1(0,\,T)} + ||w-z||_{\mathbb{H}} + ||z||_{\mathbb{H}}\;\Big] \\
& \quad + c(T)\;\sqrt{T}\;\Big[\;\Big(1+\sqrt{T}\Big)\;||w-z||_{\mathbb{H}}\;||w||_{\mathbb{H}} + ||w-z||_{\mathbb{H}}\;||w||_{\mathbb{H}}\;\Big] \qquad (5.38)
\end{aligned}
$$

The result follows. ∎

Theorem 5.2 *Let $\eta < 0$. Let $g_i \in H^1_{loc}(\mathbb{R}^+)$. Then there exists a time $T_1 \in\,]0,\,T_0]$ such that the map $\Gamma : \mathbb{B}_R(0) \to \mathbb{B}_R(0)$ with $u \to \Gamma(w_0,\,g_i,\,w)$ maps the ball $\mathbb{B}_R(0)$ into itself.*

Proof. Using (5.14) with $z_0 = 0$, $h_i = 0$, and $z = 0$ we have

$$|| \,\Gamma(w_0,\, g_i,\, w)\, ||_{\mathbb{H}} \leqslant c(T)\, ||w_0||_{L^2(0,\,L)} + c(T)\,\sqrt{T}\, \Big[\, ||\, g_i\, ||_{H^1(0,\,T)} + ||w||_{\mathbb{H}}\, \Big] + c(T)\,\sqrt{T}\, \Big[\, \Big(1+\sqrt{T}\Big)\, ||w||^2_{\mathbb{H}} + ||w||^2_{\mathbb{H}}\, \Big]. \qquad (5.39)$$

Let

$$\frac{R}{2} = c(T_0)\, ||w_0||_{L^2(0,\,L)} + c(T_0)\,\sqrt{T_0}\, ||\, g_i\, ||_{H^1(0,\,T)} \qquad (5.40)$$

then, if $w \in \mathbb{B}_R(0)$ we have

$$|| \,\Gamma(w_0,\, g_i,\, w)\, ||_{\mathbb{H}} \leqslant \frac{R}{2} + c(T)\,\sqrt{T}\, \Big[\, \Big(1+\sqrt{T}\Big)\, R^2 + 2\, R\, \Big]. \qquad (5.41)$$

We choose T such that

$$c(T)\,\sqrt{T}\, \Big[\, \Big(1+\sqrt{T}\Big)\, R^2 + 2\, R\, \Big] \leqslant \frac{R}{2} \qquad (5.42)$$

hence $||\Gamma\, u||_{\mathbb{H}} \leqslant R$ and the Theorem follows. ∎

Theorem 5.3 *Let $\eta < 0$. Assume that $g_i \in H^1_{loc}(\mathbb{R}^+)$. Then, there exists a time $T_2 \in]0,\, T_1]$ such that the application $w \to \Gamma(w_0,\, g_i,\, w)$ is a contraction over $(\mathbb{B}_R(0),\, ||\cdot||_{\mathbb{H}})$.*

Proof. Using (5.14) with $z_0 = w_0$, $h_i = g_i$, we have

$$\begin{aligned} &|| \,\Gamma(w_0,\, g_i,\, w) - \Gamma(w_0,\, g_i,\, z)\, ||_{\mathbb{H}} \\ &\leqslant c(T)\,\sqrt{T}\, [\, ||w-z||_{\mathbb{H}} + ||w||_{\mathbb{H}}\,] \\ &\quad + c(T)\,\sqrt{T}\, \Big[\, \Big(1+\sqrt{T}\Big)\, ||w-z||_{\mathbb{H}}\, ||w||_{\mathbb{H}} + ||w-z||_{\mathbb{H}}\, ||w||_{\mathbb{H}}\, \Big]. \end{aligned} \qquad (5.43)$$

If $w,\, z \in \mathbb{B}_R(0)$ we obtain

$$\begin{aligned} &|| \,\Gamma(w_0,\, g_i,\, w) - \Gamma(w_0,\, g_i,\, z)\, ||_{\mathbb{H}} \\ &\leqslant c(T)\,\sqrt{T}\, [\, ||w-z||_{\mathbb{H}} + R\,] \\ &\quad + c(T)\,\sqrt{T}\, \Big[\, \Big(1+\sqrt{T}\Big)\, ||w-z||_{\mathbb{H}}\, R + ||w-z||_{\mathbb{H}}\, R\, \Big] \\ &\leqslant c(T)\,\sqrt{T}\, \Big[\, [\, 2 + 2\, R\,] + \sqrt{T}\, R\, \Big]\, ||w-z||_{\mathbb{H}} \end{aligned} \qquad (5.44)$$

such that, if T is small enough namely

$$c(T)\,\sqrt{T}\,\Big[\,[\,2+2\,R\,]+\sqrt{T}\,R\,\Big]\leqslant 1$$

then the application $w \to \Gamma(w_0,\, g_i,\, w)$ is a contraction over $(\mathbb{B}_R(0),\, ||\cdot||_{\mathbb{H}})$. ■

Theorem 5.4 *If $\eta < 0$, there there exists a unique w defined in $\mathbb{H}_T$, weak solution of* (1.1)-(1.4).

Proof. To apply the Banach's fixed point Theorem for $(w_0,\, g,\, w) \to \Gamma(w_0,\, g,\, w)$ on $\mathbb{B}_R(0)$, which is a complete metric space and yields local existence and uniqueness. ■

Theorem 5.5 *If $\eta < 0$. Then the solution w depends continuously on $w_0 \in L^2((1+x^2)dx)$ and $g_i \in H^1_{loc}(\mathbb{R}^+)$.*

Proof. From (5.14) for small times, one gets

$$\begin{aligned}&||\,w-z\,||_{\mathbb{H}}\\&\leqslant c(T)\,||w_0-z_0||_{L^2(0,\,L)}+c(T)\,\sqrt{T}\,\Big[\,||\,g_i-h_i\,||_{H^1(0,\,T)}+||w-z||_{\mathbb{H}}+||z||_{\mathbb{H}}\,\Big]\\&\quad+c(T)\,\sqrt{T}\,\Big[\,\Big(1+\sqrt{T}\Big)\,||w-z||_{\mathbb{H}}\,||w||_{\mathbb{H}}+||w-z||_{\mathbb{H}}\,||w||_{\mathbb{H}}\,\Big]\end{aligned}\qquad(5.45)$$

Then if $w_0 \to z_0$ in $L^2((1+x^2)dx)$ and if $g_i \to h_i$ in $H^1(0,\,T)$ one gets that $w \to z$ in $\mathbb{H}_T$. The proof follows. ■

These results were obtained locally in time. But, since the time interval where this result holds depends only on $||w_0||_{L^2(0,\,L)}$ and $||g_i||_{H^1(0,\,T)}$, it can be extended as long the solution exists. Indeed we obtain

Theorem 5.6 (Existence and Uniqueness) *Let $\eta < 0$, $u_0 \in L^2((1+x^2)dx)$, $g_i \in H^1_{loc}(\mathbb{R}^+)$ for $i = 0,\, 1$, and $0 < L < +\infty$. Then there exists a unique weak maximal solution defined over $[0,\, T_L]$ for $(1.1) - (1.4)$. Moreover, there exists $T_{min} > 0$ independent of L, depending only on $||u_0||_{L^2(0,\,L)}$ and $||g_i||_{H^1(0,\,T)}$ such that $T_L \geqslant T_{min}$. The solution u depends continuously on u_0 and g_i in the following sense: Let a sequence $u_0^n \to u_0$ in $L^2((1+x^2)dx)$, let a sequence $g_i^n \to g_i$ in $H^1_{loc}(\mathbb{R}^+)$ and denote by u^n the solution with data $(u_0^n,\, g_i^n)$ and T_L^n its existence time. Then*

$$\liminf_{n\to+\infty}\, T_L^n \geqslant T_L \qquad (5.46)$$

and for all $t < T_L$, u^n exists on the interval $[0,\, T]$ if n is large enough and $u^n \to u$ in $\mathbb{H}_T$.

6 Numerical Methods

We consider finite differences based on unconditionally stable schemes similar to the described in [19] for the KdV equations.

Description of the scheme. We note by v_i^n the approximate value of $u(i\Delta x, n\Delta t)$, solution of the nonlinear problem (1.1)-(1.4), where Δx is the space-step, and Δt is the time-step, for $i = 0, \ldots, N$, and $n = 0, \ldots, M$. Define the discrete space

$$X_N = \{u = (u_0, u_1, \ldots, u_N) \in \mathbb{R}^{N+1} \mid u_0 = u_1 = 0 \text{ and } u_N = u_{N-1} = u_{N-2} = 0\}$$

and $(D^+u)_i = \dfrac{u_{i+1} - u_i}{\Delta x}$ and $(D^-u)_i = \dfrac{u_i - u_{i-1}}{\Delta x}$ the classical difference operators. In order to obtain a positive matrix we have to chose a particular discretization. The numerical scheme for the nonlinear problem (1.1)-(1.4) reads as follows :

$$\frac{v^{n+1} - v^n}{\Delta t} + Av^{n+1} + \frac{\alpha}{2} D^-[v^n]^2 = 0, \tag{6.1}$$

where $A = \eta D^+D^+D^+D^-D^- + D^+D^+D^- + \frac{1}{2}(D^+ + D^-)$, with $\alpha = 1$ for the nonlinear case, and $\alpha = 0$ for the linear case.

Remark 6.1 *We remark that A is well defined as a linear application $X_N \to \mathbb{R}^{N+1}$, in the sense that we do not need additional point on the outside of $[0, L]$ to compute Au. Thus A is represented by a 6-diagonal possitive defined matrix of $(N+1) \times (N+1)$:*

$$Au = \begin{pmatrix} c_1 & d_1 & e_1 & f_1 & & & \\ b_2 & c_2 & d_2 & e_2 & \ddots & 0 & \\ a_3 & b_3 & c_3 & \ddots & \ddots & \ddots & \\ & \ddots & \ddots & \ddots & \ddots & \ddots & f_{n-3} \\ & & \ddots & \ddots & \ddots & \ddots & e_{n-2} \\ & 0 & & a_{n-1} & b_{n-1} & c_{n-1} & d_{n-1} \\ & & & & a_n & b_n & c_n \end{pmatrix} \tag{6.2}$$

where $a_i = -\dfrac{\eta}{\Delta x^5}$, for $i = 3, \ldots, n$, $f_i = \dfrac{\eta}{\Delta x^5}$, for $i = 1, \ldots, n-3$, and

$$b_i = \begin{cases} 5\frac{\eta}{\Delta x^5} - \frac{1}{\Delta x^3} - \frac{1}{2\Delta x}, \text{ for } i = 2, \dots, n-1, \\ 3\frac{\eta}{\Delta x^5} - \frac{1}{\Delta x^3} - \frac{1}{2\Delta x}, \text{ for } i = n, \end{cases}$$

$$c_i = \begin{cases} -9\frac{\eta}{\Delta x^5} + \frac{3}{\Delta x^3}, \quad \text{for } i = 1, n-1, \\ -10\frac{\eta}{\Delta x^5} + \frac{3}{\Delta x^3}, \text{ for } i = 2, \dots, n-2, \\ -3\frac{\eta}{\Delta x^5} + \frac{1}{\Delta x^3}, \quad \text{for } i = n, \end{cases}$$

$$d_i = \begin{cases} 10\frac{\eta}{\Delta x^5} - \frac{3}{\Delta x^3} + \frac{1}{2\Delta x}, \text{ for } i = 1, \dots, n-2, \\ 5\frac{\eta}{\Delta x^5} - \frac{2}{\Delta x^3} + \frac{1}{2\Delta x}, \quad \text{for } i = n-1, \end{cases}$$

$$e_i = \begin{cases} -5\frac{\eta}{\Delta x^5} + \frac{1}{\Delta x^3}, \text{ for } i = 1, \dots, n-3, \\ -4\frac{\eta}{\Delta x^5} + \frac{1}{\Delta x^3}, \text{ for } i = n-2, \end{cases}$$

We consider the linear operators D^+ and D^- as matrices of size $(N+1) \times (N+1)$ and we note the following internal product $(z, w) = \sum_{1=0}^{N} z_i w_i$ and $(z, w)_x = (z, xw) = \sum_{1=0}^{N} i\Delta x z_i w_i$, and the norms in $\mathbb{R}^{N+1}$: $|z| = \sqrt{(z, z)}$ and $|z|_x = \sqrt{(z, z)_x}$. Then, we have the following lemma :

Lemma 6.1 *For all $z, w \in \mathbb{R}^{N+1}$, we have*

$$(D^+z, w) = z_N w_N - z_0 w_0 - (z, D^- w), \tag{6.3}$$

$$(D^+z, z) = \frac{1}{2}\left(\frac{z_N^2}{\Delta x} - \frac{z_0^2}{\Delta x} - \Delta x |D^+ z|^2\right), \tag{6.4}$$

$$(D^+z, w)_x = N z_N w_N - (z, D^- w)_x + \Delta x (z, D^- w) - (z, w), \tag{6.5}$$

$$(D^+z, z)_x = \frac{1}{2}\left(N z_N^2 - \Delta x |D^+ z|_x^2 - |z|^2\right). \tag{6.6}$$

Proof. Equations (6.3) and (6.5) are result of summing by parts. Equation (6.4) is result of using $(a-b)a = \frac{1}{2}(a^2 - b^2) + \frac{1}{2}(a-b)^2$. with $z_i = a$ and $z_{i+1} = b$, and summing over $i = 0, \dots, N$. The last equality (6.6) is result of the same identity with $z_i = a$ and $z_{i+1} = b$, multiplying by $i\Delta x$ and summing over $i = 0, \dots, N$. ■

In order to obtain estimates for the solution of the numerical scheme for the linear case, we have the following lemmas describing the quadratic forms associated to the different matrices.

Lemma 6.2 *For all $u \in X_N$, we have*

$$\frac{1}{2}((D^+ + D^-)u, u) = 0, \tag{6.7}$$

$$(D^+D^+D^-u, u) = \frac{\Delta x}{2}|D^+D^-u|^2, \tag{6.8}$$

$$(D^+D^+D^+D^-D^-u, u) = -\frac{1}{2\Delta x}\left[D^-D^-u\right]_0^2 - \frac{\Delta x}{2}|D^+D^-D^-u|^2. \tag{6.9}$$

Remark 6.2 *Since $u \in X_N$, the first term in the right-hand side of* (6.9) *is given by $\frac{1}{\Delta x}\left[D^-D^-u\right]_0^2 = (\Delta x)u_2^2$.*

Proof. The matrix $\frac{1}{2}(D^+ + D^-)$ is clearly antisymmetric and we have (6.7). Using (6.4) with $z = D^-u$ we obtain (6.8), and using the same identity with $z = D^-D^-u$ we obtain (6.9). ■

Corollary 6.1 *If $\eta \leqslant 0$, then $I + \Delta t A$ is positive definite, and for any $u^n \in X_N$ there exists a unique solution u^{n+1} of* (6.1).

Proof. From Lemma 6.2 we have for all $u \in X_N$ with $u \neq 0$,

$$((I + \Delta t A)u, u) \geqslant |u|^2 + \frac{\Delta t \Delta x}{2}|D^+D^-u|^2 \\ -\frac{\eta \Delta x \Delta t}{2}|D^+D^-D^-u|^2 - \frac{\eta \Delta t}{2\Delta x}\left[D^-D^-u\right]_0^2 > 0, \tag{6.10}$$

when $\eta \leqslant 0$. ■

The following estimate shows that the numerical scheme (6.1) with $\alpha = 0$ is l^2-stable and unconditionally stable.

Proposition 6.1 *Let $\eta \leqslant 0$. For any $v^n \in X_N$ satisfying the linear scheme* (6.1) *with $\alpha = 0$, there exists $C(T) > 0$ such that $|v^n| \leqslant |v^0|$.*

Proof. Multiplying the numerical scheme (6.1) by v^{n+1} we obtain

$$|v^{n+1}|^2 + \Delta t(Av^{n+1}, v^{n+1}) = (v^{n+1}, v^n), \tag{6.11}$$

and then, using the same identity of the proof of the Lemma **6.1** with $a = v^{k+1}$ and $b = v^k$ and summing for $k = 0, \ldots, n-1$ we have

$$|v^n|^2 + \sum_{k=0}^{n-1}|v^{k+1} - v^k|^2 + 2\Delta t\sum_{k=1}^{n}(Av^k, v^k) = |v^0|^2.$$

From (6.10) this last equality becomes

$$
\begin{aligned}
|v^n|^2 &+ \Delta t \sum_{k=0}^{n-1} \Delta t \left| \frac{v^{k+1} - v^k}{\Delta t} \right|^2 + \Delta x \sum_{k=1}^{n} \Delta t |D^+ D^- v^k|^2 \\
&- \eta \Delta x \sum_{k=1}^{n} \Delta t |D^+ D^- D^- v^k|^2 - \frac{\eta}{\Delta x} \sum_{k=1}^{n} \Delta t \left[D^- D^- v^k \right]_0^2 \\
&\leqslant |v^0|^2. \qquad (6.12)
\end{aligned}
$$

■

In order to obtain the unconditional stability for the nonlinear version of the scheme, we will find a discrete estimate that is equivalent to that of (3.28) (see Proposition **3.1**). Let us denote by x the sequence $x_i = i\Delta x$. We have :

Lemma 6.3 *For all $u \in X_N$, we have*

$$
\begin{aligned}
\frac{1}{2}((D^+ + D^-)u, xu) &= \frac{1}{4}\Delta x^2 |D^+ u|^2 - \frac{1}{2}|u|^2, \\
(D^+ D^+ D^- u, xu) &= \frac{3}{2}|D^- u|^2 + \frac{\Delta x}{2}|D^+ D^- u|_x^2 - \frac{\Delta x^2}{2}|D^+ D^- u|^2, \\
(D^+ D^+ D^+ D^- D^- u, xu) &= -\frac{5}{2}|D^- D^- u|^2 - \frac{\Delta x}{2}|D^+ D^- D^- u|_x^2 \\
&\quad + \Delta x^2 |D^+ D^- D^- u|^2 - \frac{1}{2}\left[D^- D^- u \right]_0^2,
\end{aligned}
$$

where $(xu)_i = i\Delta x u_i$.

Proof. Using (6.3), (6.4) and (6.5) we have

$$
\begin{aligned}
((D^+ + D^-)u, xu) &= (D^- u, u)_x - (u, D^- u)_x + \Delta x (u, D^- u) - |u|^2 \\
&= -\Delta x (D^+ u, u) - |u|^2 = \frac{\Delta x^2}{2}|D^+ u|^2 - |u|^2,
\end{aligned}
$$

and then we have the first identity of the Lemma. Following the same idea and applying the identities of Lemma **6.1** is easy to prove the rest of the identities. ■

Proposition 6.2 *Let $\eta \leqslant 0$. For any $v^n \in X_N$ satisfying the linear scheme (6.1) with $\alpha = 0$, there exists $C(T) > 0$ such that $|v^n|_x \leqslant C(T)|v^0|_x$ and*

$$
\begin{aligned}
\left(\sum_{k=1}^{n} \Delta t |D^- v^k|^2 \right)^{\frac{1}{2}} &\leqslant C(T)|v^0|, \\
\left(\sum_{k=1}^{n} \Delta t |D^+ D^- v^k|^2 \right)^{\frac{1}{2}} &\leqslant C(T)|v^0|, \qquad \text{if } \eta < 0,
\end{aligned}
$$

Proof. We multiply the numerical scheme (6.1) with $\alpha = 0$ by xv^{n+1}. Then, applying Lemma **6.3**, and the same identity of the proof of Lemma **6.1** with $a = \sqrt{x}v^{k+1}$ and $b = \sqrt{x}v^k$, we deduce

$$\begin{aligned}
|v^n|_x^2 + \Delta t \sum_{k=0}^{n-1} \Delta t \left| \frac{v^{k+1} - v^k}{\Delta t} \right|_x^2 + \left(3 + \frac{\Delta x^2}{2}\right) \sum_{k=1}^{n} \Delta t |D^- v^k|^2 \\
+ \Delta x \sum_{k=1}^{n} \Delta t |D^+ D^- v^k|_x^2 - \left(5\eta + \Delta x^2\right) \sum_{k=1}^{n} \Delta t |D^+ D^- v^k|^2 \\
- \eta \Delta x \sum_{k=1}^{n} \Delta t |D^+ D^- D^- v^k|_x^2 + 2\eta \Delta x^2 \sum_{k=1}^{n} \Delta t |D^+ D^- D^- v^k|^2 \\
- \eta \left[D^- D^- v^k\right]_0^2 \quad = \quad |v^0|_x^2 + \sum_{k=1}^{n} \Delta t |v^k|^2. \qquad (6.13)
\end{aligned}$$

On the other hand, Noting that

$$\begin{aligned}
|D^+ D^- v^k|_x^2 - \Delta x |D^+ D^- v^k|^2 = \sum_{i=1}^{N-1} \frac{(i-1)}{\Delta x} (v_{i+1}^k - 2v_i^k + v_{i-1}^k)^2 \geqslant 0, \\
|D^+ D^- D^- v^k|_x^2 - 2\Delta x |D^+ D^- v^k|^2 \\
+ \frac{1}{\Delta x} \left[D^- D^- v^k\right]_0^2 = \sum_{i=2}^{N-1} \frac{(i-2)}{\Delta x} (v_{i+1}^k - 3v_i^k + 3v_{i-1}^k - v_{i-2}^k)^2 \geqslant 0,
\end{aligned}$$

and replacing these inequalities in (6.13), we deduce

$$\begin{aligned}
|v^n|_x^2 + \Delta t \sum_{k=0}^{n-1} \Delta t \left| \frac{v^{k+1} - v^k}{\Delta t} \right|_x^2 + 3 \sum_{k=1}^{n} \Delta t |D^- v^k|^2 \\
- 5\eta \sum_{k=1}^{n} \Delta t |D^+ D^- v^k|^2 \leqslant |v^0|_x^2 + \sum_{k=1}^{n} \Delta t |v^k|^2.
\end{aligned}$$

Finally, using the inequalities of Proposition **6.1** and the fact that we have in a boundary domain $(0, L)$, we may conclude the proof. ■

Now, let us introduce the non-homogeneous linear scheme approximating the solution of the $(\text{KdVK})_{NH}$ problem :

$$\frac{v^{n+1} - v^n}{\Delta t} + A v^{n+1} = f^n.$$

The existence proof of the continuous case studied in the previous sections applies in the discrete non-homogeneous linear case and the discrete non-linear case for any discretization of the non-linear part, in particular for $f_n = \frac{1}{2} D^- [u^n]^2$. Thus, we obtain the following result of convergence :

Theorem 6.1 *For any $u^n \in X_N$ satisfying the non-linear scheme* (6.1)*, with $\alpha = 1$, and $\eta \leq 0$, there exists $\varepsilon_0 > 0$ such that, if $\Delta t \leq \varepsilon_0$, then there exists $T > 0$ and a constant $C = C(T) > 0$ (independent of Δt and Δx) such that :*

$$\sup_{k=0,\dots,p} |v^k|^2 + \Delta t \sum_{k=0}^{p} |D^- v^k|^2 - \eta \Delta t \sum_{k=0}^{p} |D^- D^- v^k|^2 \leq C |v_0|^2.$$

This result means that the scheme is unconditionally stable. Let us observe that in agreement with the gain of regularity of the (KdVK) equation, we obtain an additional estimate respect to the analogous numerical scheme of the KdV equation, studied in detail in [19]. On the other hand, in [21] it is described a similar scheme with application to KdV and Benney-Lin equations.

7 Some numerical results

First we compare our numerical solution with an explicit solution obtained by the Adomian decomposition method for a KdVK equation with initial condition in the unbounded domain $x \in \mathbb{R}$ ([22,23]). The tanh method to obtain explicit solutions of the KdvK equation is proposed by Wazwaz [24] in a slightly different way. To compare our numerical solution in a bounded domain $(0, L)$ with an explicit solution in all $x \in \mathbb{R}$ we consider solitons moving between $x = 0$ and $x = L$ no touching the boundaries. Let the KdvK equation

$$u_t - u_{xxxxx} + u_{xxx} + uu_x + u_x = 0, \qquad x \in \mathbb{R}, \quad t > 0 \tag{7.1}$$

with the initial condition

$$u(x, 0) = \frac{105}{169} \mathrm{Sech}^4 \left(\frac{1}{2\sqrt{13}} (x - x_0) \right)$$

where it is known that the explicit solution is given by the following travelling wave (see [22,23]):

$$u(x, t) = \frac{105}{169} \mathrm{Sech}^4 \left(\frac{1}{2\sqrt{13}} (x - \frac{205t}{169} - x_0) \right).$$

This result can be verified through substitution.

We make the simulations in Fortran90, using a factorization $A = LU$ with a generalization of the Thomas algoritm for a 6-diagonal matrix like (6.2), and a posteriori error correction using the residual. We choose $x_0 = 20.0$, $L = 200.0$,

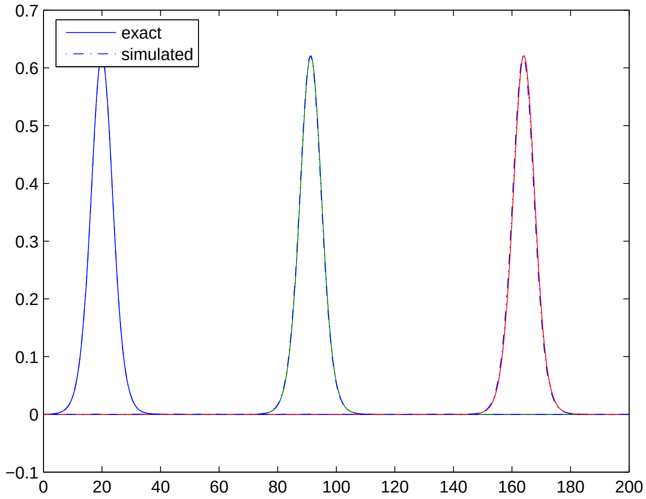

Fig. 1. Comparison between the exact solution and the simulated for $\Delta x = 10^{-3}$ $\Delta t = 1.2 \times 10^{-4}$ ($T = 0.0$ sec, $T = 60.0$ sec, and $T = 120.0$ sec).

$T = 120.0$ and we fix $\Delta t/\Delta x = 0.12$. We compute different simulations on the time interval $[0, T]$ for $n = 2\times10^1, 2\times10^2, \dots, 2\times10^5$. The comparison between exact solution and the best simulation (with $n = 200000$) is represented in Figure 1 for three different times. The error, that is the norm $L^\infty(0, T, L^2(0, L))$ of the difference between the exact solution and the simulation for different n is represented in Figure 2.

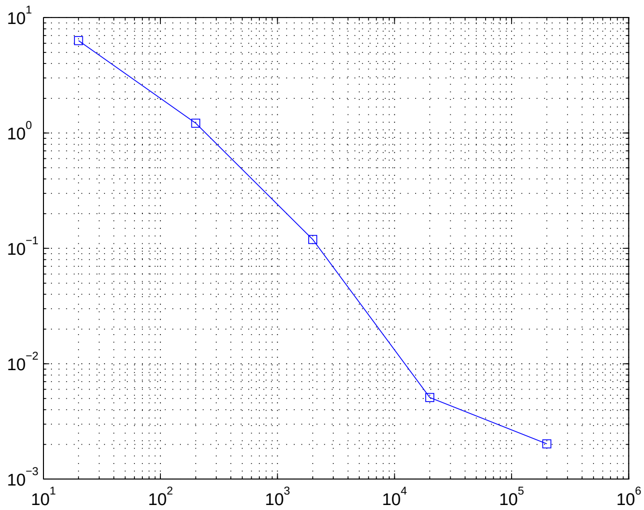

Fig. 2. Decreasing of the error between exact solution and simulated solution as function of $n = L/\Delta x$.

The second numerical test is an intersection of two solitons. We consider the equation (7.1) with the following initial condition:

$$u(x,0) = \frac{105}{169}\left\{\text{Sech}^4\left(\frac{1}{2\sqrt{13}}(x-20.0)\right) + \frac{1}{4}\text{Sech}^4\left(\frac{1}{\sqrt{13}}(x-60.0)\right)\right\}$$

This correspond to the superposition of two solitons with different speeds given by the nonlinear term $uu_x + u_x$ of the equation (7.1). For this example, we choose $L = 1000.0$, $T = 800.0$, $n = 100000$, $\Delta t = 8 \times 10^{-4}$, $\Delta x = 1 \times 10^{-2}$. Figure 3 shows three-dimensional plots of the solution.

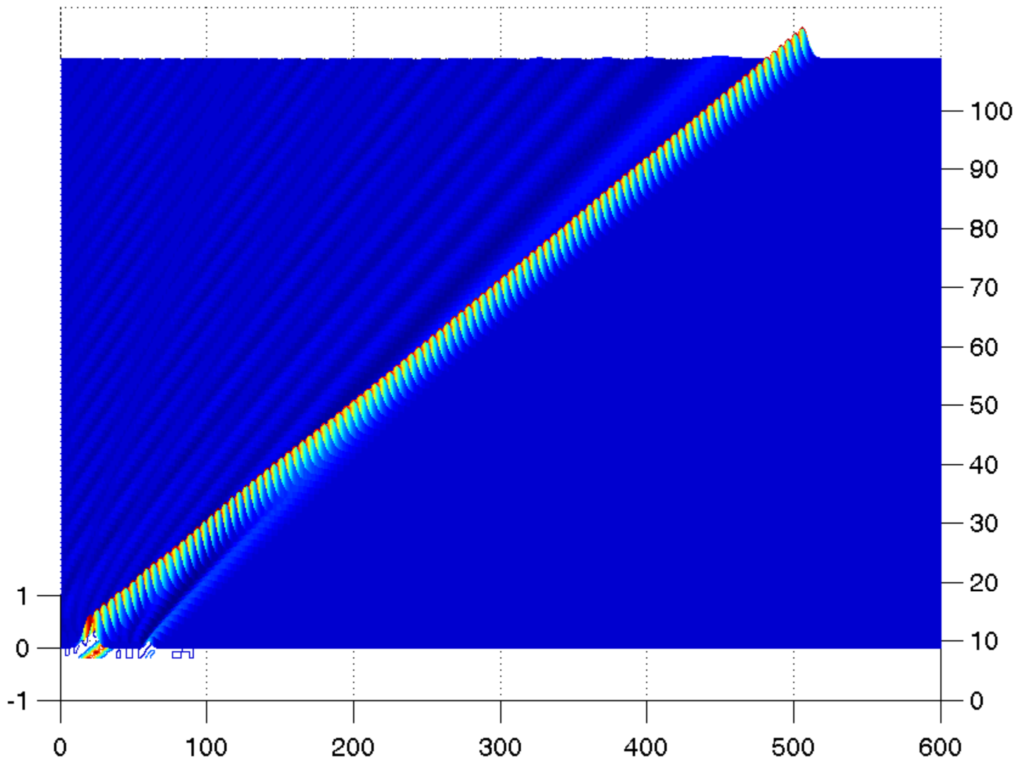

Fig. 3. Interaction of two solitons for the KdV-Kawahara equation.

Acknowledgments

JCC and OVV acknowledge support by Universidad del Bío-Bío. DIPRODE projects, # 052808 1/R and # 061008 1/R. MS has been supported by Fondecyt project # 1030718 and Fondap in Applied Mathematics (Project # 15000001). Finally, OVV and MS acknowledge support by CNPq/CONICYT Project, # 490987/2005-2 (Brazil) and # 2005-075 (Chile).

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