

Neumann Conditions on Fractal Boundaries

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Abstract

We consider some elliptic boundary value problems in a self-similar ramified domain of \mathbb{R}^2 with a fractal boundary with Laplace's equation and nonhomogeneous Neumann boundary conditions. The Hausdorff dimension of the fractal boundary is greater than one. The goal is twofold: first rigorously define the boundary value problems, second approximate the restriction of the solutions to subdomains obtained by stopping the geometric construction after a finite number of steps. For the first task, a key step is the definition of a trace operator. For the second task, a multiscale strategy based on transparent boundary conditions and on a wavelet expansion of the Neumann datum is proposed, following an idea contained in a previous work by the same authors. Error estimates are given and numerical results are presented.

self-similar domain, fractal boundary, partial differential equations
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1 Introduction

This work is concerned with some boundary value problems in some self-similar ramified domains of \mathbb{R}^2 with a fractal boundary. The domain called Ω is displayed in Figure 1. It is constructed in an infinite number of steps, starting from a simple hexagonal domain of \mathbb{R}^2 called Y^0 below; we call Y^n the domain obtained at step n : Y^{n+1} is obtained by gluing 2^{n+1} hexagonal sets to Y^n , each of them being a dilated copy of Y^0 with a dilation factor of a^{n+1} , for some fixed parameter a in $(0, 1)$. Finally, $\Omega = \cup_{n \in \mathbb{N}} Y^n$. There exists a critical parameter a^* , $\frac{1}{2} < a^* < 1$ such that, for all $a \leq a^*$, the previously mentioned subdomains of Ω do not overlap. We will always take a in the interval $[\frac{1}{2}, a^*]$. We say that Ω is self-similar, because $\Omega \setminus \overline{Y^n}$ is made out of 2^{n+1} dilated copies of Ω with the dilation factor of a^{n+1} . We will see that part of the boundary of Ω is a fractal set noted Γ^∞ below. Furthermore, if $a > \frac{1}{2}$ then the Hausdorff dimension of Γ^∞ is greater than one.

Such a geometry can be seen as a bidimensional idealization of the bronchial tree, for example. Indeed, this work is part of a wider project aimed at simulating the diffusion of medical sprays in lungs. Since the exchanges between the lungs and the circulatory system take place only in the last generations of the bronchial tree (the smallest structures), reasonable models for the diffusion of, e.g., oxygen may involve a nonhomogeneous Neumann or Robin condition on the top boundary Γ^∞ . Similarly, the lungs are mechanically coupled to the diaphragm, which also

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implies nonhomogeneous boundary conditions on Γ^∞ , if one is interested in a coupled fluid-structure model.

For partial differential equations in domain with fractal boundaries or fractal interfaces, variational techniques have been developed, involving new results on functional analysis; see [13, 9, 10, 3]. A nice theory on variational problems in fractal media is given in [14]. Some numerical simulations are described in [19, 18].

The present work is the continuation of two previous articles [1] and [2] where the domain Ω was already infinitely ramified and self-similar, but where Γ^∞ had a fractal dimension of one and was even contained in a straight line. In this case, it was possible to rather accurately study the traces properties of Sobolev spaces, although the domain Ω was not a $\epsilon - \delta$ domain as defined by Jones [6], see also [7] and [11]. These trace results allowed for the study of Poisson problems in Ω with some generalized nonhomogeneous Neumann conditions on Γ^∞ . In [2], we proposed a multilevel method in order to find the solution in Y^n by expanding the Neumann datum on the Haar wavelet basis and solving a sequence of boundary value problems in the elementary domain Y^0 with what we called *transparent boundary conditions* on the top part of the boundary of Y^0 . These conditions involve a nonlocal Dirichlet to Neumann operator, which can be obtained as the limit of a sequence of operators constructed by a simple induction relation, thanks to the self-similarity in the geometry and the scale invariance of the equation. It is important to emphasize that an arbitrary level of accuracy can be obtained. This construction is reminiscent of some of the techniques involved in the theoretical analysis of finitely ramified fractals (see [16, 17] and [15, 5] for numerical simulations).

The present paper deals with the more realistic case when the Hausdorff dimension is greater than one and has two goals:

1. Rigorously define a trace operator on Γ^∞ . The definition of the trace operator will be quite different from that in [1], because the Hausdorff dimension of Γ^∞ is greater than one. The trace operator will be a key ingredient for the study of Poisson problems in Ω with nonhomogeneous Neumann boundary conditions.
2. Show that the solution of the latter can be approximated in Y^n by essentially the same program as in [2]. For this reason, the description of this and the asymptotic analysis of the error will be short and we will refer to [2] for all the proofs.

The paper is organized as follows: the geometry is presented in section 2. In section 3, we give theoretical results on Sobolev spaces concerning in particular Poincaré's inequalities and trace results. The Poisson problems are studied in section 4. In section 5, we propose a strategy for approximating the restrictions of solutions to Y^n , $n \in \mathbb{N}$; since the method is close to the one studied in [2], we describe it, in particular the modifications with respect to [2], but we omit all the proofs. Numerical results are presented in section 6. For the reader's ease, some of the proofs are postponed to section 7.

2 The Geometry

Let a be a positive parameter. Consider the points of \mathbb{R}^2 : $P_1 = (-1, 0)$, $P_2 = (1, 0)$, $P_3 = (-1, 1)$, $P_4 = (1, 1)$, $P_5 = (-1 + a\sqrt{2}, 1 + a\sqrt{2})$ and $P_6 = (1 - a\sqrt{2}, 1 + a\sqrt{2})$. Let Y^0 be the open hexagonal subset of \mathbb{R}^2 defined as the convex hull of the last six points.

$$Y^0 = \text{Interior} \left(\text{Conv}(P_1, P_2, P_3, P_4, P_5, P_6) \right).$$

Let F_i , $i = 1, 2$ be respectively the similitudes defined by the following:

$$F_i(x) = \begin{pmatrix} (-1)^i \left(1 - \frac{a}{\sqrt{2}}\right) + \frac{a}{\sqrt{2}}(x_1 + (-1)^i x_2) \\ 1 + \frac{a}{\sqrt{2}} + \frac{a}{\sqrt{2}}(x_2 + (-1)^{i+1} x_1) \end{pmatrix}.$$

The similitude F_i has the dilation ratio a and the rotation angle $(-1)^{i+1}\pi/4$.

It is easily seen that to prevent $F_1(Y^0)$ and $F_2(Y^0)$ from overlapping, one must choose $a \leq \sqrt{2}/2$. For $n \geq 1$, we call \mathcal{A}_n the set containing all the 2^n mappings from $\{1, \dots, n\}$ to $\{1, 2\}$. We define

$$\mathcal{M}_\sigma = F_{\sigma(1)} \circ \dots \circ F_{\sigma(n)} \quad \text{for } \sigma \in \mathcal{A}_n, \quad (1)$$

and the ramified open domain, see Figure 1,

$$\Omega = \text{Interior} \left(\overline{Y^0} \cup \left(\bigcup_{n=1}^{\infty} \bigcup_{\sigma \in \mathcal{A}_n} \mathcal{M}_\sigma(\overline{Y^0}) \right) \right). \quad (2)$$

Stronger constraints must be imposed on a to prevent the sets $\mathcal{M}_\sigma(\overline{Y^0})$, $\sigma \in \mathcal{A}_n$, $n > 0$, from overlapping. It can be shown by elementary geometrical arguments that the condition is

$$2\sqrt{2}a^5 + 2a^4 + 2a^2 + \sqrt{2}a - 2 \leq 0, \quad (3)$$

i.e. ,

$$a \leq a^* \simeq 0.593465.$$

We call Γ^∞ the self similar set associated to the similitudes F_1 and F_2 , *i.e.* the unique compact subset of \mathbb{R}^2 such that

$$\Gamma^\infty = F_1(\Gamma^\infty) \cup F_2(\Gamma^\infty).$$

The Hausdorff dimension of Γ^∞ can be computed since Γ^∞ satisfies the Moran condition (open set condition), see [12, 8]:

$$\dim_H(\Gamma^\infty) = -\log 2 / \log a.$$

For instance, if a tends to a^* , then $\dim_H(\Gamma^\infty)$ tends to 1.3284371.

In all what follows, we will take $\frac{1}{2} \leq a \leq a^*$.

We split the boundary of Ω into Γ^∞ , $\Gamma^0 = [-1, 1] \times \{0\}$ and $\Sigma = \partial\Omega \setminus (\Gamma^0 \cup \Gamma^\infty)$. For what follows, it is important to define the polygonal open domain Y^N obtained by stopping the above construction at step $N + 1$,

$$Y^N = \text{Interior} \left(\overline{Y^0} \cup \left(\bigcup_{n=1}^N \bigcup_{\sigma \in \mathcal{A}_n} \mathcal{M}_\sigma(\overline{Y^0}) \right) \right). \quad (4)$$

We introduce the open domains $\Omega^\sigma = \mathcal{M}_\sigma(\Omega)$ and $\Omega^N = \bigcup_{\sigma \in \mathcal{A}_N} \Omega^\sigma = \Omega \setminus \overline{Y^{N-1}}$. We also define the sets $\Gamma^\sigma = \mathcal{M}_\sigma(\Gamma^0)$ and $\Gamma^N = \bigcup_{\sigma \in \mathcal{A}_N} \Gamma^\sigma$. The one-dimensional Lebesgue measure of Γ^σ for $\sigma \in \mathcal{A}_N$ and of Γ^N are given by

$$|\Gamma^\sigma| = a^N |\Gamma^0| \quad \text{and} \quad |\Gamma^N| = (2a)^N |\Gamma^0|.$$

It is clear that if $a > 1/2$ then $\lim_{N \rightarrow \infty} |\Gamma^N| = +\infty$.

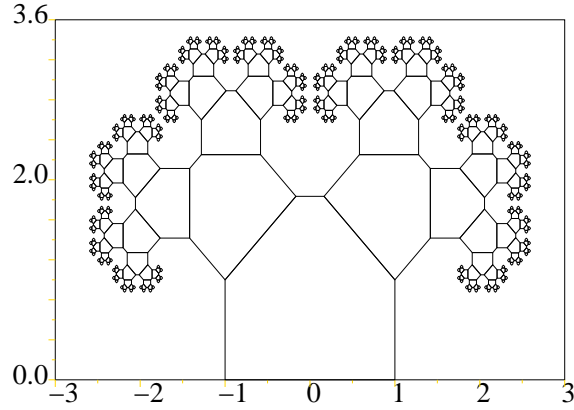
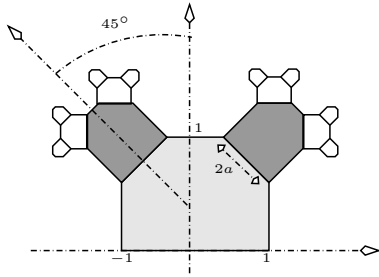


Figure 1: The ramified domain Ω (only a few generations are displayed).

3 Functional Setting

Let $H^1(\Omega)$ be the space of functions in $L^2(\Omega)$ with first order partial derivatives in $L^2(\Omega)$. We also define $\mathcal{V}(\Omega) = \{v \in H^1(\Omega); v|_{\Gamma^0} = 0\}$ and $\mathcal{V}(Y^n) = \{v \in H^1(Y^n); v|_{\Gamma^0} = 0\}$.

In this section, the proofs which can be obtained as easy modifications of those in [1] are omitted for brevity.

We will sometimes use the notation \lesssim to indicate that there may arise constants in the estimates, which are independent of the index n in Ω^n or Γ^n , Y^n , or the index σ in Ω^σ or Γ^σ .

3.1 Poincaré's inequality and consequences

Theorem 1 *There exists a constant $C > 0$, such that*

$$\forall v \in \mathcal{V}(\Omega), \quad \|v\|_{L^2(\Omega)}^2 \leq C \|\nabla v\|_{L^2(\Omega)}^2. \quad (5)$$

Proof. The proof is done by explicitly constructing a measure preserving and one to one mapping from Ω onto the fractured open set $\hat{\Omega}$ displayed in Figure 2. The main point is that the new domain $\hat{\Omega}$ has an infinity of vertical fractures and lies under the graph of a fractal function. This makes Poincaré's inequality easier to prove in $\hat{\Omega}$. For the ease of the reader, the details of the proof are postponed to §7. ■

Corollary 1 *There exists a positive constant C such that for all $v \in H^1(\Omega)$,*

$$\|v\|_{L^2(\Omega)}^2 \leq C \left(\|\nabla v\|_{L^2(\Omega)}^2 + \|v|_{\Gamma^0}\|_{L^2(\Gamma^0)}^2 \right). \quad (6)$$

Corollary 2 *There exists a positive constant C such that for all integer $n \geq 0$ and for all $\sigma \in \mathcal{A}_n$, for all $v \in H^1(\Omega^\sigma)$,*

$$\|v\|_{L^2(\Omega^\sigma)}^2 \leq C \left(a^{2n} \|\nabla v\|_{L^2(\Omega^\sigma)}^2 + a^n \|v|_{\Gamma^\sigma}\|_{L^2(\Gamma^\sigma)}^2 \right), \quad (7)$$

and for all $v \in H^1(\Omega^n)$

$$\|v\|_{L^2(\Omega^n)}^2 \leq C \left(a^{2n} \|\nabla v\|_{L^2(\Omega^n)}^2 + a^n \|v|_{\Gamma^n}\|_{L^2(\Gamma^n)}^2 \right). \quad (8)$$

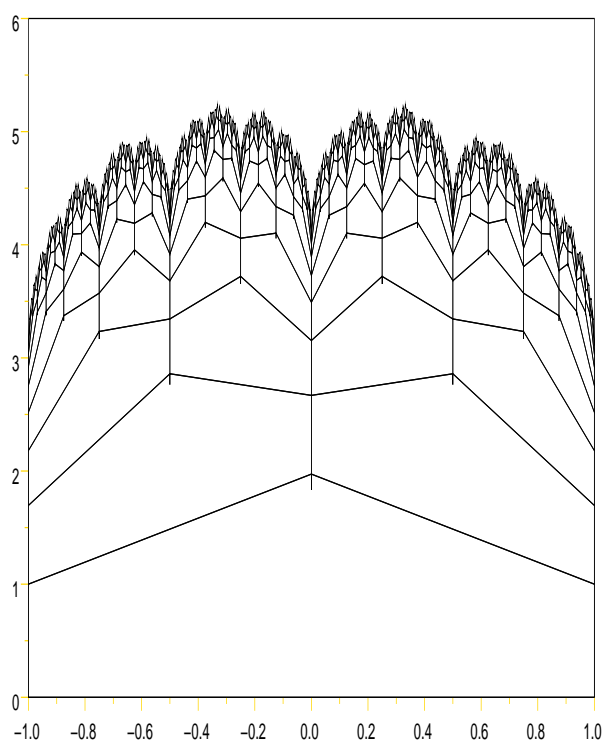


Figure 2: The open set $\widehat{\Omega}$

We need to estimate $\|v\|_{L^2(\Omega^n)}^2$ when $v \in H^1(\Omega)$:

Lemma 1 *There exists a positive constant C such that for all $v \in H^1(\Omega)$, for all $n \geq 0$,*

$$\|v\|_{L^2(\Omega^n)}^2 \leq C (2a^2)^n \left(\|\nabla v\|_{L^2(\Omega)}^2 + \|v|_{\Gamma^0}\|_{L^2(\Gamma^0)}^2 \right). \quad (9)$$

Proof. The proof uses the same arguments as that of Theorem 1. It is postponed to §7. ■
Condition (3) implies $2a^2 < 1$, because $2(a^*)^2 \sim 0.7044022575$. Thus, (9) implies the Rellich type theorem:

Theorem 2 (Compactness) *The imbedding of $H^1(\Omega)$ in $L^2(\Omega)$ is compact.*

The following lemma will be useful for defining a trace operator on Γ^∞ :

Lemma 2 *There exists a positive constant C such that for all $v \in H^1(\Omega)$, for all integers $p \geq 0$,*

$$\sum_{\sigma \in \mathcal{A}_p} \int_{\Gamma^\sigma} (v|_{\Gamma^\sigma})^2 \leq C(2a)^p \left(\|\nabla v\|_{L^2(\Omega)}^2 + \|v\|_{L^2(\Omega)}^2 \right). \quad (10)$$

Proof. We start from the trace inequality: for all $v \in H^1(\Omega)$,

$$\int_{\Gamma^0} (v|_{\Gamma^0})^2 \lesssim \|\nabla v\|_{L^2(\Omega)}^2 + \|v\|_{L^2(\Omega)}^2. \quad (11)$$

Rescaling yields

$$a^{-p} \int_{\Gamma^\sigma} (v|_{\Gamma^\sigma})^2 \lesssim \|\nabla v\|_{L^2(\Omega^\sigma)}^2 + a^{-2p} \|v\|_{L^2(\Omega^\sigma)}^2, \quad \forall \sigma \in \mathcal{A}_p. \quad (12)$$

Summing on all $\sigma \in \mathcal{A}_p$ and dividing by 2^p , we obtain

$$(2a)^{-p} \sum_{\sigma \in \mathcal{A}_p} \int_{\Gamma^\sigma} (v|_{\Gamma^\sigma})^2 \lesssim 2^{-p} \sum_{\sigma \in \mathcal{A}_p} \|\nabla v\|_{L^2(\Omega^\sigma)}^2 + (2a^2)^{-p} \sum_{\sigma \in \mathcal{A}_p} \|v\|_{L^2(\Omega^\sigma)}^2. \quad (13)$$

Then estimate (10) is obtained by combining (13) and (9). ■

Remark 1 *Note that $\frac{|\Gamma^p|}{|\Gamma^0|} = (2a)^p$, so (10) is equivalent to*

$$\frac{1}{|\Gamma^p|} \sum_{\sigma \in \mathcal{A}_p} \int_{\Gamma^\sigma} (v|_{\Gamma^\sigma})^2 \lesssim \|\nabla v\|_{L^2(\Omega)}^2 + \|v\|_{L^2(\Omega)}^2 \quad (14)$$

Corollary 3 *There exists a positive constant C such that for all $v \in H^1(\Omega)$, for all integers $p \geq 0$,*

$$\sum_{\sigma \in \mathcal{A}_p} \int_{\Gamma^\sigma} (v|_{\Gamma^\sigma} - \langle v|_{\Gamma^0} \rangle)^2 \leq C(2a)^p \|\nabla v\|_{L^2(\Omega)}^2, \quad (15)$$

where $\langle v|_{\Gamma^0} \rangle$ is the mean value of $v|_{\Gamma^0}$ on Γ^0 .

3.2 Traces

For defining traces on Γ^∞ , we need the classical result, see [4]:

Theorem 3 *There exists a unique Borel regular probability measure μ on Γ^∞ such that for any Borel set $A \subset \Gamma^\infty$,*

$$\mu(A) = \frac{1}{2}\mu(F_1^{-1}(A)) + \frac{1}{2}\mu(F_2^{-1}(A)). \quad (16)$$

The measure μ is called the *self-similar measure defined in the self similar triplet* $(\Gamma^\infty, F_1, F_2)$. Let L_μ^2 be the Hilbert space of the functions on Γ^∞ that are μ -measurable and square integrable with respect to μ , with the norm $\|v\|_{L_\mu^2} = \sqrt{\int_{\Gamma^\infty} v^2 d\mu}$.

A Hilbertian basis of L_μ^2 can be constructed with e.g. Haar wavelets.

Consider the sequence of linear operators $\ell^n : H^1(\Omega) \rightarrow L_\mu^2$,

$$\ell^n(v) = \sum_{\sigma \in \mathcal{A}_n} \left(\frac{1}{|\Gamma^\sigma|} \int_{\Gamma^\sigma} v \, dx \right) \mathbf{1}_{\mathcal{M}_\sigma(\Gamma^\infty)}, \quad (17)$$

where $|\Gamma^\sigma|$ is the Lebesgue measure of Γ^σ . Indeed, we have that

$$\begin{aligned} \|\ell^n(v)\|_{L_\mu^2}^2 &= \int_{\Gamma^\infty} \left(\sum_{\sigma \in \mathcal{A}_n} \left(\frac{1}{|\Gamma^\sigma|} \int_{\Gamma^\sigma} v \, dx \right) \mathbf{1}_{\mathcal{M}_\sigma(\Gamma^\infty)} \right)^2 d\mu \\ &= \sum_{\sigma \in \mathcal{A}_n} \left(\frac{1}{|\Gamma^\sigma|} \int_{\Gamma^\sigma} v \, dx \right)^2 \mu(\mathcal{M}_\sigma(\Gamma^\infty)) \\ &= \sum_{\sigma \in \mathcal{A}_n} 2^{-n} \left(\frac{1}{|\Gamma^\sigma|} \int_{\Gamma^\sigma} v \, dx \right)^2, \end{aligned} \quad (18)$$

where we have used $\mu(\mathcal{M}_\sigma(\Gamma^\infty)) = 2^{-n}$, which is a consequence of the definition of the invariant measure, see (16). From (18), using Cauchy-Schwarz inequality and the identity $|\Gamma^\sigma| = a^n |\Gamma^0|$, we obtain that

$$\|\ell^n(v)\|_{L_\mu^2}^2 \leq (2a)^{-n} \frac{1}{|\Gamma^0|} \sum_{\sigma \in \mathcal{A}_n} \int_{\Gamma^\sigma} v^2 \, dx = \frac{1}{|\Gamma^n|} \int_{\Gamma^n} v^2 \, dx. \quad (19)$$

From (19) and (10), we see that there exists a positive constant C such that for all $v \in H^1(\Omega)$, for all integer $n \geq 0$,

$$\|\ell^n(v)\|_{L_\mu^2}^2 \leq C \|v\|_{H^1(\Omega)}. \quad (20)$$

Proposition 1 *The sequence $(\ell^n)_n$ converges in $\mathcal{L}(H^1(\Omega), L_\mu^2)$ to an operator that we call ℓ^∞ .*

Proof. We aim at proving that there exists a positive constant C such that for all $v \in H^1(\Omega)$, for all integers n, m such that $0 \leq m \leq n$,

$$\|\ell^{n+1}(v) - \ell^m(v)\|_{L_\mu^2}^2 \leq C 2^{-m} \|\nabla v\|_{L^2(\Omega^m)}^2, \quad (21)$$

and the desired result will follow directly from (21).

An important observation for proving (21) is that, for all nonnegative integers p, q , $p < q$, for all $\sigma \in \mathcal{A}_{q-p}$,

$$\ell^q(v) \circ \mathcal{M}_\sigma = \ell^p(v \circ \mathcal{M}_\sigma), \quad \forall v \in H^1(\Omega). \quad (22)$$

Going back to $\|\ell^{n+1}(v) - \ell^m(v)\|_{L^2_\mu}^2$, we see that

$$\begin{aligned} \|\ell^{n+1}(v) - \ell^m(v)\|_{L^2_\mu}^2 &= \sum_{\eta \in \mathcal{A}_m} \sum_{\sigma \in \mathcal{A}_{n+1-m}} \|\mathbf{1}_{(\mathcal{M}_\eta \circ \mathcal{M}_\sigma)(\Gamma^\infty)} (\ell^{n+1}(v) - \ell^m(v))\|_{L^2_\mu}^2 \\ &= \sum_{\eta \in \mathcal{A}_m} \sum_{\sigma \in \mathcal{A}_{n+1-m}} 2^{-n-1} |(\ell^{n+1}(v)|_{(\mathcal{M}_\eta \circ \mathcal{M}_\sigma)(\Gamma^\infty)} - \ell^m(v)|_{(\mathcal{M}_\eta \circ \mathcal{M}_\sigma)(\Gamma^\infty)})|^2. \end{aligned}$$

Note that $\ell^{n+1}(v) - \ell^m(v)$ is constant on $(\mathcal{M}_\eta \circ \mathcal{M}_\sigma)(\Gamma^\infty)$, for $\eta \in \mathcal{A}_m$ and $\sigma \in \mathcal{A}_{n+1-m}$, and that, from (22),

$$\begin{aligned} \ell^{n+1}(v)((\mathcal{M}_\eta \circ \mathcal{M}_\sigma)(y)) &= (\ell^0(v \circ \mathcal{M}_\eta \circ \mathcal{M}_\sigma))(y), \\ \ell^m(v)((\mathcal{M}_\eta \circ \mathcal{M}_\sigma)(y)) &= (\ell^0(v \circ \mathcal{M}_\eta))(\mathcal{M}_\sigma(y)), \end{aligned}$$

for all $y \in \Gamma^\infty$. Therefore,

$$\|\ell^{n+1}(v) - \ell^m(v)\|_{L^2_\mu}^2 = \sum_{\eta \in \mathcal{A}_m} 2^{-m} \sum_{\sigma \in \mathcal{A}_{n+1-m}} 2^{m-n-1} |(\ell^0(v \circ \mathcal{M}_\eta))(\mathcal{M}_\sigma(y)) - (\ell^0(v \circ \mathcal{M}_\eta \circ \mathcal{M}_\sigma))(y)|^2, \quad (23)$$

where y is any point on Γ^∞ .

Calling $w = v \circ \mathcal{M}_\eta$ and $p = n + 1 - m$ for brevity, we have that

$$2^{-p} \sum_{\sigma \in \mathcal{A}_p} |(\ell^0(w))(\mathcal{M}_\sigma(y)) - (\ell^0(w \circ \mathcal{M}_\sigma))(y)|^2 = 2^{-p} \sum_{\sigma \in \mathcal{A}_p} |\ell^0((w - \langle w|_{\Gamma^0}) \circ \mathcal{M}_\sigma)(y)|^2, \quad (24)$$

where $\langle w|_{\Gamma^0}$ is the mean value of $w|_{\Gamma^0}$ on Γ^0 , because $\ell^0(w) = \ell^0(\langle w|_{\Gamma^0})$. From (24), we deduce that

$$\begin{aligned} 2^{-p} \sum_{\sigma \in \mathcal{A}_p} |(\ell^0(w))(\mathcal{M}_\sigma(y)) - (\ell^0(w \circ \mathcal{M}_\sigma))(y)|^2 &= 2^{-p} \sum_{\sigma \in \mathcal{A}_p} \left(\frac{1}{|\Gamma^\sigma|} \int_{\Gamma^\sigma} (w|_{\Gamma^\sigma} - \langle w|_{\Gamma^0}) \right)^2 \\ &\leq \frac{1}{|\Gamma^p|} \int_{\Gamma^p} (w|_{\Gamma^\sigma} - \langle w|_{\Gamma^0})^2 \end{aligned} \quad (25)$$

by Cauchy-Schwarz inequality. Therefore, from (15),

$$2^{-p} \sum_{\sigma \in \mathcal{A}_p} |(\ell^0(w))(\mathcal{M}_\sigma(y)) - (\ell^0(w \circ \mathcal{M}_\sigma))(y)|^2 \lesssim \|\nabla w\|_{L^2(\Omega)}^2. \quad (26)$$

Going back to (23) and using (26), we have that

$$\begin{aligned} \|\ell^{n+1}(v) - \ell^m(v)\|_{L^2_\mu}^2 &\lesssim 2^{-m} \sum_{\eta \in \mathcal{A}_m} \int_{\Omega} |\nabla(v \circ \mathcal{M}_\eta)|^2 \\ &= 2^{-m} \sum_{\eta \in \mathcal{A}_m} \int_{\Omega^\eta} |\nabla v|^2 = 2^{-m} \int_{\Omega^m} |\nabla v|^2, \end{aligned} \quad (27)$$

which is exactly (21). ■

Remark 2 The operator ℓ^∞ can be seen as a renormalized trace operator.

Remark 3 A different approach consists of passing to the limit in the sequence of quadratic forms $\mathcal{N}_n : H^1(\Omega) \rightarrow \mathbb{R}_+$, $v \mapsto \frac{1}{|\Gamma^n|} \int_{\Gamma^n} v^2$. The limit \mathcal{N}_∞ is a continuous quadratic form on $H^1(\Omega)$. In the spirit of [14], a continuous bilinear form a_∞ on $H^1(\Omega) \times H^1(\Omega)$ can then be defined by polarisation and the connection of a_∞ to a trace operator may be studied. This remark is not essential for what follows, since we will always work with ℓ^∞ .

4 A Class of Poisson Problems

Take $g \in L^2_\mu$ and $u \in H^{\frac{1}{2}}(\Gamma^0)$ and consider the variational problem: find $U(u, g) \in H^1(\Omega)$ such that

$$(U(u, g))|_{\Gamma^0} = u, \text{ and } \int_{\Omega} \nabla(U(u, g)) \cdot \nabla v = \int_{\Gamma^\infty} g \ell^\infty(v) d\mu, \forall v \in \mathcal{V}(\Omega). \quad (28)$$

If it exists, then $(U(u, g))$ satisfies $\Delta(U(u, g)) = 0$ in Ω , and $\partial_n(U(u, g)) = 0$ on Σ . We shall discuss the boundary condition on Γ^∞ after the following:

Theorem 4 For $g \in L^2_\mu$ and $u \in H^{\frac{1}{2}}(\Gamma^0)$, (28) has a unique solution.

Furthermore, if $g = \ell^\infty(\tilde{g})$, $\tilde{g} \in C^1(\overline{\Omega})$, if $w_q \in H^1(Y^q)$ is the solution of:

$$\begin{aligned} \Delta w_q &= 0 \text{ in } Y^q, & w_q|_{\Gamma^0} &= u, & \frac{\partial w_q}{\partial n} &= 0 \text{ on } \partial Y_q \setminus (\Gamma^0 \cup \Gamma^{q+1}), \\ \frac{\partial w_q}{\partial n} &= \frac{1}{|\Gamma^{q+1}|} \tilde{g}|_{\Gamma^{q+1}} \text{ on } \Gamma^{q+1}, \end{aligned}$$

then

$$\lim_{q \rightarrow \infty} \|(U(u, g))|_{Y^q} - w_q\|_{H^1(Y^q)} = 0. \quad (29)$$

Theorem 4 says in particular that (28) has an intrinsic meaning for a large class of data g . From the definition of w_q , we may say that $U(u, g)$ satisfies a renormalized/generalized Neumann condition on Γ^∞ with datum g .

Proof. We introduce $\hat{w}_q \in H^1(Y^q)$ as the solution of the variational problem

$$\begin{aligned} \hat{w}_q|_{\Gamma^0} &= u, \\ \int_{Y^q} \nabla \hat{w}_q \cdot \nabla v &= \int_{\Gamma^\infty} \ell^{q+1}(\tilde{g}) \ell^{q+1}(v) d\mu = \frac{1}{|\Gamma^{q+1}|} \sum_{\sigma \in \mathcal{A}_{q+1}} \frac{1}{|\Gamma^\sigma|} \int_{\Gamma^\sigma} \tilde{g} \int_{\Gamma^\sigma} v, \quad \forall v \in \mathcal{V}(\Omega). \end{aligned} \quad (30)$$

Then, $w_q - \hat{w}_q \in \mathcal{V}(Y^q)$ and

$$\int_{Y^q} \nabla(w_q - \hat{w}_q) \cdot \nabla v = \frac{1}{|\Gamma^{q+1}|} \sum_{\sigma \in \mathcal{A}_{q+1}} \int_{\Gamma^\sigma} \left(v(x) \left(\tilde{g}(x) - \frac{1}{|\Gamma^\sigma|} \int_{\Gamma^\sigma} \tilde{g} \right) \right) dx, \quad \forall v \in \mathcal{V}(\Omega). \quad (31)$$

But

$$\begin{aligned} & \frac{1}{|\Gamma^{q+1}|} \left| \sum_{\sigma \in \mathcal{A}_{q+1}} \int_{\Gamma^\sigma} \left(v(x) \left(\tilde{g}(x) - \frac{1}{|\Gamma^\sigma|} \int_{\Gamma^\sigma} \tilde{g} \right) \right) dx \right| \\ &= \frac{1}{|\Gamma^{q+1}|} \left| \int_{\Gamma^{q+1}} \left(v(x) \sum_{\sigma \in \mathcal{A}_{q+1}} \mathbf{1}_{\Gamma^\sigma}(x) \left(\tilde{g}(x) - \frac{1}{|\Gamma^\sigma|} \int_{\Gamma^\sigma} \tilde{g} \right) \right) dx \right| \\ &\leq \frac{1}{|\Gamma^{q+1}|} \left(\int_{\Gamma^{q+1}} |v(x)| dx \right) \sup_{y \in \Gamma^{q+1}} \left(\sum_{\sigma \in \mathcal{A}_{q+1}} \mathbf{1}_{\Gamma^\sigma}(y) \left| \tilde{g}(y) - \frac{1}{|\Gamma^\sigma|} \int_{\Gamma^\sigma} \tilde{g} \right| \right), \end{aligned} \quad (32)$$

and

$$\mathbf{1}_{\Gamma^\sigma}(y) \left| \tilde{g}(y) - \frac{1}{|\Gamma^\sigma|} \int_{\Gamma^\sigma} \tilde{g} \right| \leq |\Gamma^\sigma| \|\nabla \tilde{g}\|_\infty = a^{q+1} |\Gamma^0| \|\nabla \tilde{g}\|_\infty.$$

This yields that

$$\begin{aligned}
& \left| \frac{1}{|\Gamma^{q+1}|} \sum_{\sigma \in \mathcal{A}_{q+1}} \int_{\Gamma^\sigma} \left(v(x) \left(\tilde{g}(x) - \frac{1}{|\Gamma^\sigma|} \int_{\Gamma^\sigma} \tilde{g} \right) \right) dx \right| \\
& \leq \frac{a^{q+1} |\Gamma^0|}{|\Gamma^{q+1}|} \|\nabla \tilde{g}\|_\infty \int_{\Gamma^{q+1}} |v(x)| dx \\
& \leq a^{q+1} |\Gamma^0| \|\nabla \tilde{g}\|_\infty \left(\frac{1}{|\Gamma^{q+1}|} \int_{\Gamma^{q+1}} |v(x)|^2 dx \right)^{\frac{1}{2}},
\end{aligned} \tag{33}$$

where we have used Cauchy-Schwarz inequality in the last line. From (10), (5) and (33), we find that

$$\left| \frac{1}{|\Gamma^{q+1}|} \sum_{\sigma \in \mathcal{A}_{q+1}} \int_{\Gamma^\sigma} \left(v(x) \left(\tilde{g}(x) - \frac{1}{|\Gamma^\sigma|} \int_{\Gamma^\sigma} \tilde{g} \right) \right) dx \right| \lesssim a^{q+1} \|\nabla v\|_{L^2(Y^q)} \|\nabla \tilde{g}\|_\infty. \tag{34}$$

From (31) and (34), we obtain that

$$\|\nabla(w_q - \hat{w}_q)\|_{L^2(Y^q)} \lesssim a^{q+1} \|\nabla \tilde{g}\|_\infty. \tag{35}$$

The desired result will be proved if we show that $\|\nabla(U(u, g) - \hat{w}_q)\|_{L^2(Y^q)}$ tends to zero as q tends to infinity. Calling e_q the error $e_q = U(u, g)|_{Y^q} - \hat{w}_q \in \mathcal{V}(Y^q)$, we see that $\forall v \in \mathcal{V}(\Omega^0)$,

$$\begin{aligned}
& \int_{Y^q} \nabla e_q \cdot \nabla v \\
& = \int_{\Gamma^\infty} \ell^\infty(\tilde{g}) \ell^\infty(v) d\mu - \int_{\Gamma^\infty} \ell^{q+1}(\tilde{g}) \ell^{q+1}(v) d\mu - \int_{\Omega^{q+1}} \nabla U(u, g) \cdot \nabla v \\
& = \int_{\Gamma^\infty} (\ell^\infty(\tilde{g}) - \ell^{q+1}(\tilde{g})) \ell^\infty(v) d\mu - \int_{\Gamma^\infty} \ell^{q+1}(\tilde{g}) (\ell^{q+1}(v) - \ell^\infty(v)) d\mu - \int_{\Omega^{q+1}} \nabla U(u, g) \cdot \nabla v.
\end{aligned} \tag{36}$$

Passing to the limit in (21), we obtain that

$$\begin{aligned}
& \left| \int_{\Gamma^\infty} (\ell^\infty(\tilde{g}) - \ell^{q+1}(\tilde{g})) \ell^\infty(v) d\mu \right| \lesssim 2^{-\frac{q}{2}} \|\nabla \tilde{g}\|_{L^2(\Omega)} \|\nabla v\|_{L^2(\Omega)}, \\
& \left| \int_{\Gamma^\infty} \ell^{q+1}(\tilde{g}) (\ell^{q+1}(v) - \ell^\infty(v)) d\mu \right| \lesssim 2^{-\frac{q}{2}} \|\nabla \tilde{g}\|_{L^2(\Omega)} \|\nabla v\|_{L^2(\Omega)}.
\end{aligned} \tag{37}$$

Therefore

$$\lim_{q \rightarrow \infty} \sup_{v \in \mathcal{V}(\Omega), v \neq 0} \frac{\left| \int_{\Gamma^\infty} \ell^\infty(\tilde{g}) \ell^\infty(v) d\mu - \int_{\Gamma^\infty} \ell^{q+1}(\tilde{g}) \ell^{q+1}(v) d\mu \right|}{\|v\|_{H^1(\Omega)}} = 0. \tag{38}$$

Since $U(u, g) \in H^1(\Omega)$, we also have

$$\lim_{q \rightarrow \infty} \sup_{v \in \mathcal{V}(\Omega), v \neq 0} \frac{\int_{\Omega^{q+1}} \nabla U(u, g) \cdot \nabla v}{\|v\|_{H^1(\Omega)}} = 0. \tag{39}$$

From (36), (38) and (39) we deduce that $\lim_{q \rightarrow \infty} \|e_q\|_{H^1(Y^q)} = 0$, which concludes the proof. ■

5 Strategies for the Approximation of $U(u, g)|_{Y^{n-1}}$

Orientation In what follows, we propose strategies in order to approximate $U(u, g)|_{Y^{n-1}}$, where n is a fixed positive integer. We will mainly distinguish two cases:

- in the first case, g is either 0 or a wavelet in the Haar basis naturally associated with the dyadic construction of Γ^∞ . We show that $U(u, g)|_{Y^n}$ can be found by successively solving a finite number of boundary value problems in Y^0 with nonlocal boundary conditions on Γ^1 . These nonlocal boundary conditions involve a Dirichlet to Neumann operator which is not available but which can be approximated with an arbitrary accuracy.
- In the general case, we propose to expand g in the basis of the Haar wavelets. This yields an expansion for $U(u, g)|_{Y^{n-1}}$, for which error estimates are available.

This strategy has already been proposed and tested in [2] for a different choice of Y^0 , F_1 and F_2 , in the special case $a = \frac{1}{2}$. In this case, the Hausdorff dimension of Γ^∞ is one and no renormalization is needed.

For this reason, we will just describe the strategies and emphasize the dependency on the parameter a . We will omit all the proofs, which are easy modifications of those contained in [2].

5.1 The Case when $g = 0$.

We use the notation $\mathcal{H}(u) = U(u, 0)$. Call T the Dirichlet-Neumann operator from $H^{\frac{1}{2}}(\Gamma^0)$ to $(H^{\frac{1}{2}}(\Gamma^0))'$, $Tu = \partial_n \mathcal{H}(u)|_{\Gamma^0}$. We remark that $T \in \mathbb{O}$, where \mathbb{O} is the cone containing the self-adjoint, positive semi-definite, bounded linear operators from $H^{\frac{1}{2}}(\Gamma^0)$ to $(H^{\frac{1}{2}}(\Gamma^0))'$ which vanish on the constants.

If T is available, the self-similarity in the geometry and the scale-invariance of the equations imply that $\mathcal{H}(u)|_{Y^0} = w$ where w is such that

$$\Delta w = 0 \quad \text{in } Y^0, \quad \frac{\partial w}{\partial n} \Big|_{\partial Y^0 \setminus (\Gamma^0 \cup \Gamma^1)} = 0, \quad (40)$$

$$w|_{\Gamma^0} = u, \quad (41)$$

$$\frac{\partial w}{\partial n} + \frac{1}{a} (T(w|_{F_i(\Gamma^0)} \circ F_i)) \circ F_i^{-1} = 0 \quad \text{on } F_i(\Gamma^0), \quad i = 1, 2. \quad (42)$$

We call (42) a transparent boundary condition because it permits $\mathcal{H}(u)|_{Y^0}$ to be computed without error. We stress the fact that the problem (40,41,42) is well posed from the observation on T above. The construction may be generalized to $\mathcal{H}(u)|_{Y^{n-1}}$, $n \geq 1$:

Proposition 2 For $u \in H^{\frac{1}{2}}(\Gamma^0)$, $\mathcal{H}(u)|_{Y^{n-1}}$ can be found by successively solving $1+2+\dots+2^{n-1}$ boundary value problems in Y^0 :

- Loop: for $p = 0$ to $n - 1$,
 - Loop : for $\sigma \in \mathcal{A}_p$, (at this point, if $p \geq 1$, $(\mathcal{H}(u))|_{\Gamma^\sigma}$ is known)
 - Find $w \in H^1(Y^0)$ satisfying the boundary value problem (40), (42), and either (41) if $p = 0$, or $w|_{\Gamma^0} = \mathcal{H}(u)|_{\Gamma^\sigma} \circ \mathcal{M}_\sigma$ if $p > 0$.
 - Set $\mathcal{H}(u)|_{Y^0} = w$ if $p = 0$. If $p > 0$, set $\mathcal{H}(u)|_{\mathcal{M}_\sigma(Y^0)} = w \circ (\mathcal{M}_\sigma)^{-1}$.

We are left with approximating T : in Theorem 5 below, we show that T can be obtained as the limit of a sequence of operators constructed by a simple induction. The following result is the theoretical key to the method proposed below:

Proposition 3 *There exists a constant $\rho < 1$ such that for any $u \in H^{\frac{1}{2}}(\Gamma^0)$,*

$$\sum_{\sigma \in \mathcal{A}_p} \int_{\Omega^\sigma} |\nabla \mathcal{H}(u)|^2 \leq \rho^p \int_{\Omega} |\nabla \mathcal{H}(u)|^2, \quad \forall p > 0. \quad (43)$$

In order to approximate T , we introduce the mapping $\mathbb{M} : \mathbb{O} \mapsto \mathbb{O}$: for any $Z \in \mathbb{O}$,

$$\forall u \in H^{\frac{1}{2}}(\Gamma^0), \quad \mathbb{M}(Z)u = \frac{\partial w}{\partial n} \Big|_{\Gamma^0}, \quad (44)$$

where $w \in H^1(Y^0)$ satisfies (40), (41) and $\frac{\partial w}{\partial n} + \frac{1}{a} (Z(w|_{F_i(\Gamma^0)} \circ F_i)) \circ F_i^{-1} = 0$ on $F_i(\Gamma^0)$, $i = 1, 2$.

Theorem 5 *The operator T is the unique fixed point of \mathbb{M} . Moreover, if ρ , $0 < \rho < 1$, is the constant appearing in (43), then for all $Z \in \mathbb{O}$, there exists a positive constant C such that, for all $p \geq 0$,*

$$\|\mathbb{M}^p(Z) - T\| \leq C\rho^{\frac{p}{4}}, \quad (45)$$

Thanks to the linearity of (28), we are left with the approximation of $(U(0, g))|_{Y^{n-1}}$. We first distinguish the case when g belongs to the Haar basis associated to the dyadic decomposition of Γ^∞ .

5.2 The Case when g Belongs to the Haar Basis

The case when g is a Haar wavelet is particularly favorable because transparent boundary conditions may be used thanks to self-similarity. We assume that the operator T is available; this assumption is reasonable because thanks to Theorem 5, one can approximate T with an arbitrary accuracy.

Let us call $e_F = U(0, 1_{\Gamma^\infty})$.

We introduce the linear operator B , bounded from $(H^{\frac{1}{2}}(\Gamma^0))'$ to $L^2(\Gamma^0)$, by: $Bz = -\frac{\partial w}{\partial x_2} \Big|_{\Gamma^0}$, where $w \in \mathcal{V}(Y^0)$ is the unique weak solution to

$$\Delta w = 0 \quad \text{in } Y^0, \quad \frac{\partial w}{\partial n} = 0 \quad \text{on } \partial Y_0 \setminus (\Gamma^0 \cup \Gamma^1), \quad (46)$$

$$\frac{\partial w}{\partial x_2} \Big|_{F_i(\Gamma^0)} + \frac{1}{a} (T(w|_{F_i(\Gamma^0)} \circ F_i)) \circ F_i^{-1} = -z \circ F_i^{-1}, \quad i = 1, 2. \quad (47)$$

The self-similarity in the geometry and the scale-invariance of the equations are the fundamental ingredients for proving the following theorem:

Theorem 6 *The normal derivative y_F of e_F on Γ^0 belongs to $L^2(\Gamma^0)$ and is the unique solution to:*

$$\begin{aligned} y_F &= B y_F, \\ \int_{\Gamma^0} y_F &= -1. \end{aligned} \quad (48)$$

For all $n \geq 1$, the restriction of e_F to Y^{n-1} can be found by successively solving $1 + 2 + \dots + 2^{n-1}$ boundary value problems in Y^0 , as follows:

• Loop: for $p = 0$ to $n - 1$,

•• Loop : for $\sigma \in \mathcal{A}_p$, (at this point, if $p > 0$, $e_F|_{\Gamma^\sigma}$ is known)

••• Solve the boundary value problem in Y^0 : find $w \in H^1(\Omega)$ satisfying (46), with $w|_{\Gamma^0} = 0$ if $p = 0$, $w|_{\Gamma^0} = e_F|_{\Gamma^\sigma} \circ \mathcal{M}_\sigma$ if $p > 0$, and

$$\frac{\partial w}{\partial n} + \frac{1}{a} (T(w|_{F_i(\Gamma^0)} \circ F_i)) \circ F_i^{-1} = -\frac{1}{2^{p+1}a} y_F \circ F_i^{-1}, \quad \text{on } F_i(\Gamma^0), \quad i = 1, 2.$$

••• Set $e_F|_{Y^0} = w$ if $p = 0$, else set $e_F|_{\mathcal{M}_\sigma(Y^0)} = w \circ (\mathcal{M}_\sigma)^{-1}$.

When g is a Haar wavelet on Γ^∞ , the knowledge of T , e_F and y_F (with an arbitrary accuracy) permits $U(0, g)$ to be approximated with an arbitrary accuracy: call $g^0 = 1_{F_1(\Gamma^\infty)} - 1_{F_2(\Gamma^\infty)}$ the Haar mother wavelet, and define $e^0 = U(0, g^0)$. One may approximate $e^0|_{Y^{n-1}}$ by using the following:

Proposition 4 *We have $e^0|_{Y^0} = w$, where $w \in \mathcal{V}(Y^0)$ satisfies (46) and*

$$\frac{\partial w}{\partial n} + \frac{1}{a} (T(w|_{F_i(\Gamma^0)} \circ F_i)) \circ F_i^{-1} = \frac{(-1)^i}{2a} y_F \circ F_i^{-1} \text{ on } F_i(\Gamma^0), \quad i = 1, 2, \quad (49)$$

Furthermore, for $i = 1, 2$,

$$e^0|_{F_i(\Omega_0)} = \frac{(-1)^{i+1}}{2} e_F \circ F_i^{-1} + (\mathcal{H}(e^0|_{F_i(\Gamma_0)} \circ F_i)) \circ F_i^{-1}. \quad (50)$$

For a positive integer p , take $\sigma \in \mathcal{A}_p$. Call g^σ the Haar wavelet on Γ^∞ defined by $g^\sigma|_{\mathcal{M}_\sigma(\Gamma^\infty)} = g^0 \circ \mathcal{M}_\sigma^{-1}$, and $g^\sigma|_{\Gamma^\infty \setminus \mathcal{M}_\sigma(\Gamma^\infty)} = 0$; call $e^\sigma = U(0, g^\sigma)$, and y^σ (resp. y^0) the normal derivative of e^σ (resp. e^0) on Γ^0 . The following result shows that (e^σ, y^σ) can be found by induction:

Proposition 5 *The family (e^σ, y^σ) is defined by induction: assume that $\mathcal{M}_\sigma = F_i \circ \mathcal{M}_\eta$ for some $i \in \{1, 2\}$, $\eta \in \mathcal{A}_{p-1}$, $p > 1$. Then $e^\sigma|_{Y^0} = w$, where $w \in \mathcal{V}(Y^0)$ satisfies (46) and*

$$\frac{\partial w}{\partial n} + \frac{1}{a} (T(w|_{F_i(\Gamma^0)} \circ F_i)) \circ F_i^{-1} = -\frac{1}{2a} y^\eta \circ F_i^{-1}, \quad \text{on } F_i(\Gamma^0), \quad i = 1, 2. \quad (51)$$

Then, with $j = 1 - i$, $e^\sigma|_{\Omega \setminus Y^0}$ is given by

$$\begin{aligned} e^\sigma|_{F_i(\Omega)} &= \frac{1}{2} e^\eta \circ F_i^{-1} + (\mathcal{H}(e^\sigma|_{F_i(\Gamma_0)} \circ F_i)) \circ F_i^{-1}, \\ e^\sigma|_{F_j(\Omega)} &= (\mathcal{H}(e^\sigma|_{F_j(\Gamma_0)} \circ F_j)) \circ F_j^{-1}. \end{aligned} \quad (52)$$

If $\mathcal{M}_\sigma = F_i$, $i = 1, 2$, then y^η (resp. e^η) must be replaced by y^0 (resp. e^0) in (51), (resp. (52)).

What follows indicates that for $n \geq 1$ fixed, $\|\nabla e^\sigma\|_{L^2(Y^{n-1})}$, $\sigma \in \mathcal{A}_p$, decays exponentially as $p \rightarrow \infty$:

Theorem 7 *There exists a positive constant C such that for all integers n, p , $1 \leq n < p$, for all $\sigma \in \mathcal{A}_p$, the function e^σ satisfies*

$$\|\nabla e^\sigma\|_{L^2(Y^{n-1})} \leq C 2^{-n} \rho^{p-n}, \quad (53)$$

where ρ , $0 < \rho < 1$ is the constant appearing in Proposition 3.

5.3 The General Case

Consider now the case when g is a general function in L_μ^2 . It is no longer possible to use the self-similarity in the geometry for deriving transparent boundary conditions for $U(0, g)$. The idea is different: one expands g on the Haar basis, and use the linearity of (28) with respect to g for obtaining an expansion of $U(0, g)$ in terms of e_F , e^0 , and e^σ , $\sigma \in \mathcal{A}_p$, $p > 1$. Indeed, one can expand $g \in L_\mu^2$ as follows:

$$g = \alpha_F 1_{\Gamma^\infty} + \alpha_0 g^0 + \sum_{p=1}^{\infty} \sum_{\sigma \in \mathcal{A}_p} \alpha_\sigma g^\sigma. \quad (54)$$

The following result, which is a consequence of Theorem 7, says that $(U(0, g))|_{Y^{n-1}}$ can be expanded in terms of $e_F|_{Y^{n-1}}$, $e^0|_{Y^{n-1}}$, and $e^\sigma|_{Y^{n-1}}$, $\sigma \in \mathcal{A}_p$, $p \geq 1$:

$$U(0, g) = \alpha_F e_F + \alpha_0 e^0 + \sum_{p=1}^{\infty} \sum_{\sigma \in \mathcal{A}_p} \alpha_\sigma e^\sigma. \quad (55)$$

Moreover, a few terms in the expansion are enough to approximate $(U(0, g))|_{Y^{n-1}}$ with a good accuracy: we are going to use approximations of the form

$$U(0, g) \approx \alpha_F e_F + \alpha_0 e^0 + \sum_{p=1}^P \sum_{\sigma \in \mathcal{A}_p} \alpha_\sigma e^\sigma. \quad (56)$$

It is possible to prove error estimates:

Proposition 6 *Assume that $g \in L^2_\mu$ has the expansion (54). Consider the error*

$$r^P = U(0, g) - \alpha_F e_F - \alpha_0 e^0 - \sum_{p=1}^P \sum_{\sigma \in \mathcal{A}_p} \alpha_\sigma e^\sigma. \quad (57)$$

There exists a constant C (independent of g) such that for all integers n, P , with $1 \leq n < P$,

$$\|r^P\|_{H^1(Y^{n-1})} \leq C \sqrt{2^{-P}} \rho^{P-n} \|g\|_{L^2_\mu}, \quad (58)$$

where $\rho < 1$ is the constant in (43).

6 Numerical Results

To transpose the strategies described above to finite element methods, one needs to use self-similar triangulations of Ω : we first consider a regular family of triangulations \mathcal{T}_h^0 of Y^0 , with the special property that for $i = 1, 2$, the set of nodes of \mathcal{T}_h^0 lying on $F_i(\Gamma^0)$ is the image by F_i of the set of nodes of \mathcal{T}_h^0 lying on Γ^0 . Then one can construct self-similar triangulations of Ω by

$$\mathcal{T}_h = \cup_{p=0}^{\infty} \cup_{\sigma \in \mathcal{A}_p} \mathcal{M}_\sigma(\mathcal{T}_h^0),$$

with self-explanatory notations. With such triangulations and conforming finite elements, one can transpose all that has been done at the continuous level to the discrete level.

The case when g is a Haar wavelet In Figure 3, we display the contours of $e_F|_{Y^5}$ (top left), $e^0|_{Y^5}$ (top right), and $e^\nu|_{Y^5}$ for $\mathcal{M}_\nu = F_2$ (bottom left) and $\mathcal{M}_\nu = F_1 \circ F_1$ (bottom right). Again, we stress the fact that there is no error from the domain truncation, and that, for obtaining the result, we did not solve a boundary value problem in Y^5 , but a sequence of boundary value problems in Y^0 . Nevertheless, the function matches smoothly at the interfaces between the subdomains.

The general case In Figure 4, we plot three approximations of $U(0, g)|_{Y^5}$, where $g(s) = 1_{s < 0} \cos(3\pi s/2) - 1_{s > 0} \cos(3\pi s/2)$, where $s \in [-1, 1]$ is a parameterization of Γ^∞ : we have used the expansion in (56), with $P = 5$ in the top of Figure 4, $P = 3$ in the middle and $P = 2$ in the bottom. We see that taking $P = 2$ is enough for approximating $U(0, g)|_{Y^0}$, but not for $U(0, g)|_{Y^j}$, $j \geq 1$. Likewise $P = 3$ is enough for approximating $U(0, g)|_{Y^1}$, but not for $U(0, g)|_{Y^j}$, $j \geq 2$.

In Figure 5, we plot (in log scales) the errors $\|\sum_{p=i}^5 \sum_{\sigma \in \mathcal{A}_p} \alpha_\sigma e_h^\sigma\|_{L^2(Y^j)}$, for $i = 2, 3, 4$ and $j = 0, 1, 2, 3, 4$, where α^σ are the coefficients of the wavelet expansion of g . The behavior is the one predicted by Proposition 6.

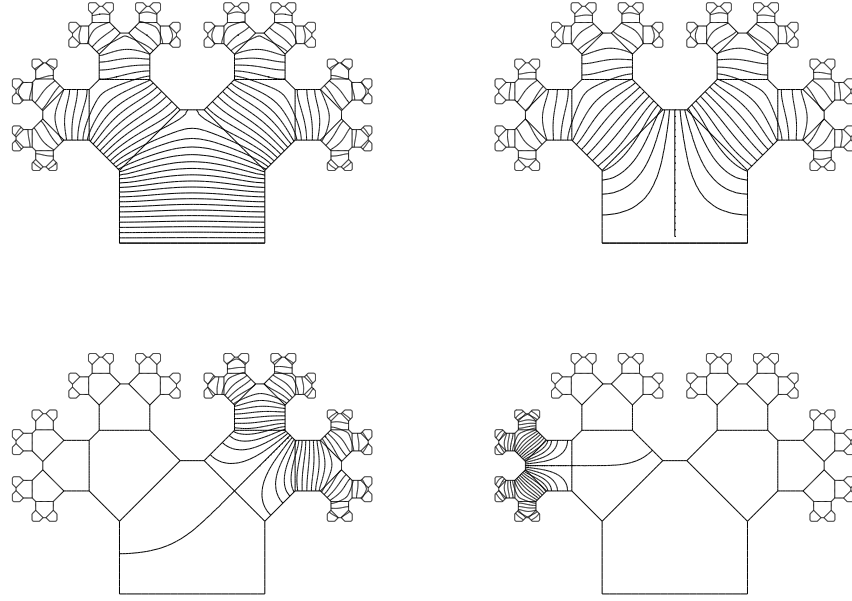


Figure 3: The contours of $e_F|_{Y^5}$ (top left), $e^0|_{Y^5}$ (top right), and $e^\nu|_{Y^5}$ for $\mathcal{M}_\nu = F_2$ (bottom left) and $\mathcal{M}_\nu = F_1 \circ F_1$ (bottom right).

7 Appendix

Proof of Theorem 1 We start by proving the desired inequality for functions in the space $\mathcal{V}(Y^N) = \{v \in W^1(Y^N) \text{ such that } v|_{\Gamma^0} = 0\}$ with a constant independent of N . It is enough to consider functions in $\{v \in C^\infty(\overline{Y^N}) \text{ such that } v|_{\Gamma^0} = 0\}$ since the last space is dense in $\mathcal{V}(Y^N)$. Consider the piecewise affine map H :

$$H(x_1, x_2) = \begin{cases} (x_1, x_2) & \text{if } x_2 \leq 1 + \sqrt{2}a(1 - |x_1|), \\ (1 + \frac{1}{\sqrt{2}a}(1 - x_2), 1 + \sqrt{2}a(x_1 - 1) + 2(x_2 - 1)) & \text{if } x_1 > 0, x_2 > 1 + \sqrt{2}a(1 - x_1), \\ (-1 - \frac{1}{\sqrt{2}a}(1 - x_2), 1 - \sqrt{2}a(x_1 + 1) + 2(x_2 - 1)) & \text{if } x_1 < 0, x_2 > 1 + \sqrt{2}a(1 + x_1), \end{cases}$$

It can be seen that H maps the set Y^0 to the fractured domain $\widehat{Y^0}$ displayed on figure 6. By computing the Jacobian of H (H is piecewise smooth), we see that that H is measure preserving. Let G_1 and G_2 be the affine maps in \mathbb{R}^2 defined by

$$\begin{aligned} G_1(x_1, x_2) &= \left(\frac{1}{2}(x_1 - 1), 1 - a(a - \sqrt{2})(x_1 + 1) + 2a^2x_2\right), \\ G_2(x_1, x_2) &= \left(\frac{1}{2}(x_1 + 1), 1 + a(a - \sqrt{2})(x_1 - 1) + 2a^2x_2\right). \end{aligned}$$

We define

$$\widehat{\mathcal{M}}_\sigma = G_{\sigma(1)} \circ \dots \circ G_{\sigma(n)} \quad \text{for } \sigma \in \mathcal{A}_n,$$

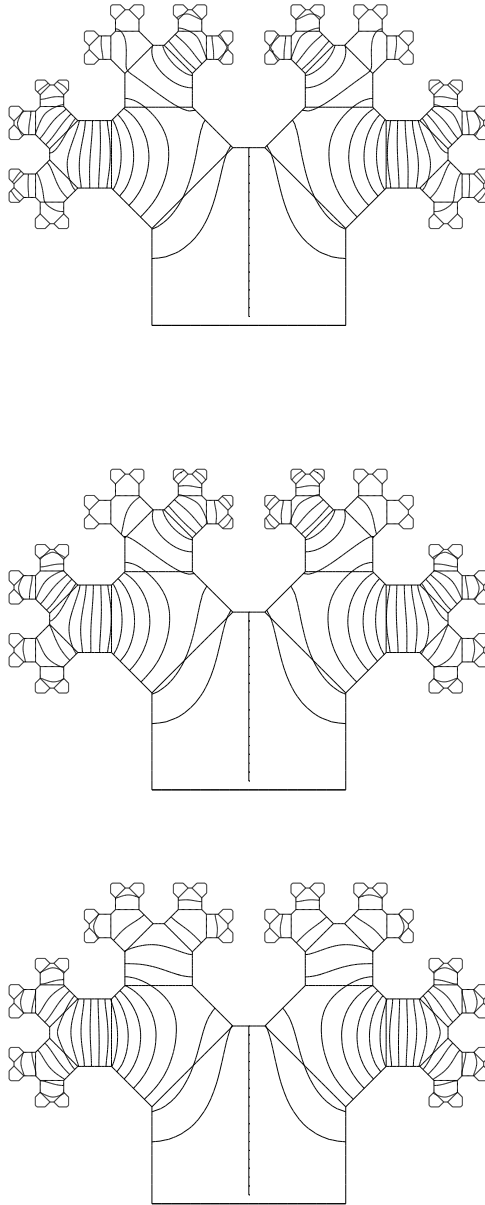


Figure 4: Contours of the approximations of $U(0, g)|_{Y^5}$ obtained by taking $P = 5$ (top), $P = 3$ (middle) and $P = 2$ (bottom) in (56).

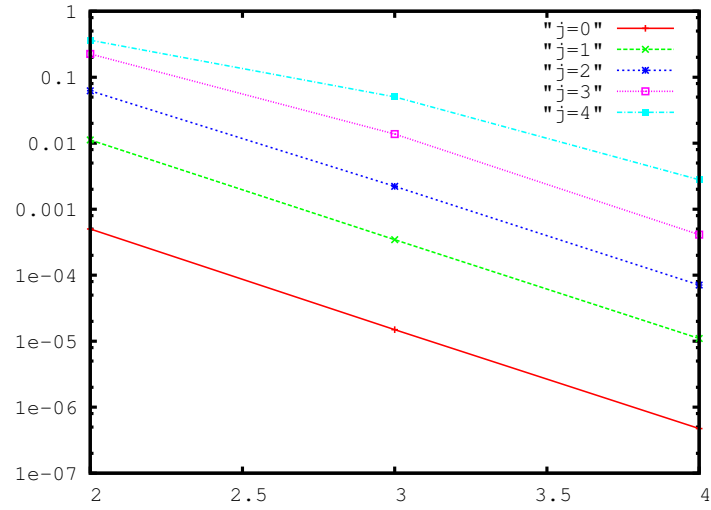


Figure 5: $\| \sum_{p=i}^5 \sum_{\sigma \in \mathcal{A}_p} \alpha^\sigma e_h^\sigma \|_{L^2(Y^j)}$ for $i = 2, 3, 4$ and $j = 0, 1, 2, 3, 4$.

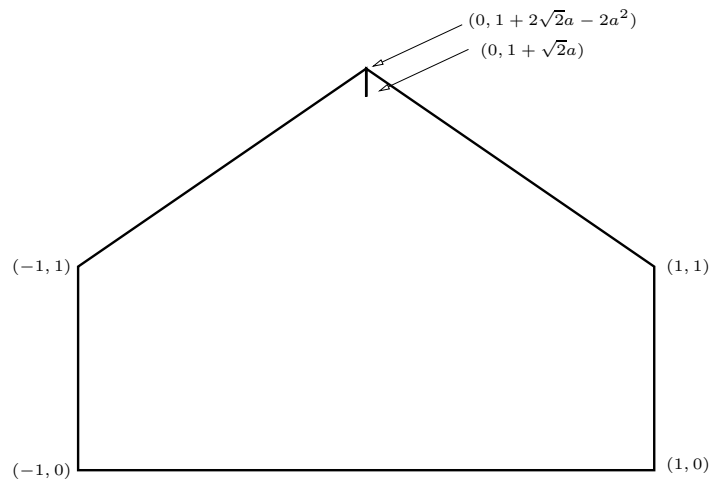


Figure 6: The fractured open set \widehat{Y}^0

the sets

$$\begin{aligned}\widehat{Y}^N &= \text{Interior} \left(\overline{\widehat{Y}^0} \cup \bigcup_{n=1}^N \bigcup_{\sigma \in \mathcal{A}_n} \widehat{\mathcal{M}}_\sigma(\overline{\widehat{Y}^0}) \right), \\ \widehat{\Omega} &= \text{Interior} \left(\overline{\widehat{Y}^0} \cup \bigcup_{n=1}^\infty \bigcup_{\sigma \in \mathcal{A}_n} \widehat{\mathcal{M}}_\sigma(\overline{\widehat{Y}^0}) \right),\end{aligned}$$

and the one to one mapping

$$\chi^N : \begin{cases} \widehat{Y}^N & \rightarrow Y^N, \\ x & \mapsto \mathcal{M}_\sigma \circ H^{-1} \circ \widehat{\mathcal{M}}_\sigma^{-1}(x) \quad \text{if } x \in \widehat{\mathcal{M}}_\sigma(\widehat{Y}^0). \end{cases}$$

Note that χ^N is a piecewise affine function and that the Jacobian of χ^N is almost everywhere 1. Moreover, take $\sigma \in \mathcal{A}_n$ with $n \leq N$, $(x_1, x_2) \in \widehat{\mathcal{M}}_\sigma(Y^0)$ and $h \in \mathbb{R}$ such that $(x_1, x_2 + h) \in \widehat{\mathcal{M}}_\sigma(Y^0)$. We aim at bounding $|\chi^N(x_1, x_2 + h) - \chi^N(x_1, x_2)|$: call $(z_1, z_2) = \widehat{\mathcal{M}}_\sigma^{-1}(x_1, x_2)$. It can be easily seen that $\widehat{\mathcal{M}}_\sigma^{-1}(x_1, x_2 + h) = (z_1, z_2 + (2a^2)^{-n}h)$. Therefore,

$$\begin{aligned}|\chi^N(x_1, x_2 + h) - \chi^N(x_1, x_2)| &= |\mathcal{M}_\sigma \circ H^{-1}(z_1, z_2 + (2a^2)^{-n}h) - \mathcal{M}_\sigma \circ H^{-1}(z_1, z_2)| \\ &\leq C_H a^n (2a^2)^{-n} |h| = C_H (2a)^{-n} |h|,\end{aligned}$$

where the constant C_H is the norm of H^{-1} , and where we have used the fact that \mathcal{M}_σ is a similitude with dilation ratio a^n . But $2a > 1$. Passing to the limit as h tends to zero, we see that

$$\left\| \frac{\partial \chi^N}{\partial x_2} \right\|_\infty \leq C_H. \quad (59)$$

Note that \widehat{Y}^N is contained in the rectangle $[-1, 1] \times [0, \zeta]$, where $\zeta = (1 + 2\sqrt{2}a - 2a^2) \sum_{n=0}^\infty (2a^2)^n = \frac{(1 + 2\sqrt{2}a - 2a^2)}{1 - 2a^2} \leq 8$ and it has $I^N = 2 + \sum_{n=0}^N 2^n$ vertical boundaries, (among which $\sum_{n=0}^N 2^n$ vertical fractures) see Figure 2. We order increasingly the abscissa $(\alpha_i)_{i=1, \dots, I^N}$ of these vertical segments. Notice also that \widehat{Y}^N can be seen as the epigraph of a function $\Phi^N : (-1, 1) \mapsto \mathbb{R}_+$, and that Φ^N is discontinuous at α_i , $i = 2, \dots, I^N - 1$, and linear in the intervals (α_i, α_{i+1}) , $i = 1, \dots, I^N - 1$. Another important and natural property is that the sequence (Φ^N) is nondecreasing with respect to N .

We call $\Phi^\infty = \lim_{N \rightarrow \infty} \Phi^N$. From the bound $|\frac{\partial \widehat{\mathcal{M}}_\sigma}{\partial x_2}| \leq (2a^2)^n$ if $\sigma \in \mathcal{A}_n$, we deduce that

$$\Phi^\infty - \Phi^N \leq (1 + 2\sqrt{2}a - 2a^2) \sum_{n=N}^\infty (2a^2)^n \lesssim (2a^2)^N. \quad (60)$$

Consider a function $v \in C_0^\infty(\overline{\widehat{Y}^N})$ such that $v|_{\Gamma^0} = 0$. Since the mapping χ^N is measure preserving,

$$\begin{aligned}\int_{Y^N} v^2 &= \int_{\widehat{Y}^N} v^2(\chi^N(x_1, x_2)) dx_2 dx_1 = \sum_{i=1}^{I^N-1} \int_{\alpha_i}^{\alpha_{i+1}} \int_0^{\Phi^N(x_1)} v^2(\chi^N(x_1, x_2)) dx_1 dx_2 \\ &= \sum_{i=1}^{I^N-1} \int_{\alpha_i}^{\alpha_{i+1}} \int_0^{\Phi^N(x_1)} \left(\int_0^{x_2} \frac{\partial(v \circ \chi^N)}{\partial x_2}(x_1, t) dt \right)^2 dx_2 dx_1 \\ &\leq \sum_{i=1}^{I^N-1} \int_{\alpha_i}^{\alpha_{i+1}} \int_0^{\Phi^N(x_1)} x_2 \int_0^{x_2} \left(\frac{\partial(v \circ \chi^N)}{\partial x_2}(x_1, t) \right)^2 dt dx_2 dx_1,\end{aligned}$$

by Cauchy-Schwarz inequality. Therefore, since Φ^N is bounded independently of N , there exists a constant C independent of N such that

$$\begin{aligned}
\int_{Y^N} v^2 &\leq \sum_{i=1}^{I_N-1} \int_{\alpha_i}^{\alpha_{i+1}} \frac{(\Phi^N(x_1))^2}{2} \int_0^{\Phi^N(x_1)} \left(\frac{\partial(v \circ \chi^N)}{\partial x_2}(x_1, t) \right)^2 dt dx_2 dx_1 \\
&\leq C \sum_{i=1}^{I_N-1} \int_{\alpha_i}^{\alpha_{i+1}} \int_0^{\Phi^N(x_1)} \left(\frac{\partial(v \circ \chi^N)}{\partial x_2}(x_1, t) \right)^2 dt dx_1 \\
&= C \int_{\widehat{Y^N}} \left(\frac{\partial(v \circ \chi^N)}{\partial x_2} \right)^2 \\
&\leq 2C \left(\int_{\widehat{Y^N}} \left(\frac{\partial v}{\partial x_1} \circ \chi^N \frac{\partial \chi_1^N}{\partial x_2} \right)^2 + \int_{\widehat{Y^N}} \left(\frac{\partial v}{\partial x_2} \circ \chi^N \frac{\partial \chi_2^N}{\partial x_2} \right)^2 \right).
\end{aligned}$$

Using (59), we obtain that

$$\begin{aligned}
\int_{Y^N} v^2 &\leq 2C \left(\int_{\widehat{Y^N}} \left(\frac{\partial v}{\partial x_1} \circ \chi^N \right)^2 + \int_{\widehat{Y^N}} \left(\frac{\partial v}{\partial x_2} \circ \chi^N \right)^2 \right) \\
&= 2C \left(\int_{Y^N} \left(\frac{\partial v}{\partial x_1} \right)^2 + \int_{Y^N} \left(\frac{\partial v}{\partial x_2} \right)^2 \right),
\end{aligned}$$

by the inverse change of variable, and the desired inequality is proved.

Proof of Lemma 1 We proceed exactly as in the proof of Theorem 1. We start by proving the desired inequality for functions in the space $\mathcal{V}(Y^N)$ for $N > n$, with a constant independent of N . Then, for functions of $\mathcal{V}(\Omega)$ the result will follow by letting N tend to ∞ .

$$\begin{aligned}
\int_{Y^N \cap \Omega^n} v^2 &= \int_{(\chi^N)^{-1}(Y^N \cap \Omega^n)} v^2(\chi^N(x_1, x_2)) dx_2 dx_1 = \sum_{i=1}^{I_N-1} \int_{\alpha_i}^{\alpha_{i+1}} \int_{\Phi^n(x_1)}^{\Phi^N(x_1)} v^2(\chi^N(x_1, x_2)) dx_1 dx_2 \\
&= \sum_{i=1}^{I_N-1} \int_{\alpha_i}^{\alpha_{i+1}} \int_{\Phi^n(x_1)}^{\Phi^N(x_1)} \left(\int_0^{x_2} \frac{\partial(v \circ \chi^N)}{\partial x_2}(x_1, t) dt \right)^2 dx_2 dx_1 \\
&\leq \sum_{i=1}^{I_N-1} \int_{\alpha_i}^{\alpha_{i+1}} \int_{\Phi^n(x_1)}^{\Phi^N(x_1)} x_2 \int_0^{x_2} \left(\frac{\partial(v \circ \chi^N)}{\partial x_2}(x_1, t) \right)^2 dt dx_2 dx_1 \\
&\lesssim \sum_{i=1}^{I_N-1} \int_{\alpha_i}^{\alpha_{i+1}} \int_{\Phi^n(x_1)}^{\Phi^N(x_1)} \int_0^{\Phi^N(x_1)} \left(\frac{\partial(v \circ \chi^N)}{\partial x_2}(x_1, t) \right)^2 dt dx_2 dx_1 \\
&\lesssim (2a^2)^n \sum_{i=1}^{I_N-1} \int_{\alpha_i}^{\alpha_{i+1}} \int_0^{\Phi^N(x_1)} \left(\frac{\partial(v \circ \chi^N)}{\partial x_2}(x_1, t) \right)^2 dt dx_2 dx_1,
\end{aligned}$$

because of (60), and we conclude exactly as in the proof of Theorem 1. Lemma 1 is proved for functions of $\mathcal{V}(\Omega)$. The proof of (9) for functions in $H^1(\Omega)$ is exactly of the same nature, except that we have to use the identity $v(\chi^N(x_1, x_2)) = v(x_1, 0) + \int_0^{x_2} \frac{\partial(v \circ \chi^N)}{\partial x_2}(x_1, t) dt$ instead of $v(\chi^N(x_1, x_2)) = \int_0^{x_2} \frac{\partial(v \circ \chi^N)}{\partial x_2}(x_1, t) dt$ for $v \in \mathcal{V}(\Omega)$.

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