Penalization approach for mixed hyperbolic systems with constant coefficients satisfying a Uniform Lopatinski Condition.

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Abstract

In this paper, we describe a new, systematic and explicit way of approximating solutions of mixed hyperbolic systems with constant coefficients satisfying a Uniform Lopatinski Condition via different Penalization approaches.

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1 Introduction.

In this paper, we describe a new, systematic and explicit way of approximating solutions of mixed hyperbolic systems with constant coefficients satisfying a Uniform Lopatinski Condition via different Penalization approaches. In applied Mathematics like, for instance, in the study of fluids dynamics, the method of penalization is used to treat boundary conditions in the case of complex geometries. By replacing the boundary condition by a singular perturbation of the PDE extended to a larger domain, this method allows the construction of an approximate, often more easily computable, solution. We consider mixed boundary value problems for hyperbolic systems:

$$\partial_t + \sum_{j=1}^{d} A_j \partial_j,$$

on $\{x_d \geq 0\}$, with boundary conditions on $\{x_d = 0\}$. The $n \times n$ real valued matrices A_j are assumed constant. Of course, we assume the coefficients to be constant as a first approach, aiming to generalize the results obtained here in future works. We assume that the boundary $\{x_d = 0\}$ is noncharacteristic, which means that $det A_d \neq 0$. We denote by
$y := (x_1, \dots, x_{d-1})$ and $x := x_d$. The problem writes:

$$\begin{cases} \mathcal{H}u = f, \quad \{x > 0\}, \\ \Gamma u|_{x=0} = \Gamma g, \\ u|_{t<0} = 0 \quad , \end{cases} \tag{1.1}$$

where the unknown $u(t,x) \in \mathbb{R}^n$, $\Gamma : \mathbb{R}^n \to \mathbb{R}^p$ is linear and such that $rg\ \Gamma = p$; which implies that Γ can be viewed as a $p \times n$ real valued constant matrix. Let us fix $T > 0$ once and for all for this paper. Let Ω_T^+ denotes the set $[0,T] \times \mathbb{R}^d_+$ and Υ_T denote the set $[0,T] \times \mathbb{R}^{d-1}$. f is a function in $H^k(\Omega_T^+)$, g is a function in $H^k(\Upsilon_T)$, where $k \geq 3$ or $k = \infty$, such that: $f|_{t<0} = 0$ and $g|_{t<0} = 0$. We make moreover the following Hyperbolicity assumption on $\mathcal{H}$:

Assumption 1.1. *For all* $(\eta, \xi) \in \mathbb{R}^{d-1} \times \mathbb{R} - \{0\}$, *the eigenvalues of*

$$\sum_{j=1}^{d-1} \eta_j A_j + \xi A_d$$

are real, semi-simple and of constant multiplicity.

Let us introduce now the frequency variable $\zeta := (\gamma, \tau, \eta)$, where $i\tau + \gamma$, with $\gamma \geq 0$, and $\tau \in \mathbb{R}$ stands for the frequency variable dual to t and $\eta = (\eta_1, \ldots, \eta_{d-1})$ where $\eta_j \in \mathbb{R}$ is the frequency variable dual to x_j. We note:

$$A(\zeta) := -(A_d)^{-1}\left((i\tau+\gamma)Id + \sum_{j=1}^{d-1} i\eta_j A_j\right).$$

Denote by M a $N \times N$, complex valued, matrix; $\mathbb{E}_-(M)$[resp $\mathbb{E}_+(M)$] is the linear subspace generated by the generalized eigenvectors associated to the eigenvalues of M with negative [resp positive] real part. If $\mathbb{F}$ and $\mathbb{G}$ denote two linear subspaces of $\mathbb{C}^N$ such that $\dim \mathbb{F} + \dim \mathbb{G} = N$, $det(\mathbb{F}, \mathbb{G})$ denotes the determinant obtained by taking orthonormal bases in each space. Up to the sign, the result is independent of the choice of the bases. We shall now explicit the Uniform Lopatinski Condition assumption:

Assumption 1.2. $(\mathcal{H}, \Gamma)$ *satisfies the Uniform Lopatinski Condition i.e for all* ζ *such that* $\gamma > 0$, *there holds:*

$$|det(\mathbb{E}_-(A), \ker \Gamma)| \geq C > 0. \tag{1.2}$$

The mixed hyperbolic system (1.1) has a unique solution in $H^k(\Omega_T^+)$, and, since $\mathcal{H}$ is hyperbolic with constant multiplicity, for all γ positive, the eigenvalues of A stay away from the imaginary axis. Moreover, as emphasized for instance by Chazarain and Piriou in [3] and Métivier in [8], there is a continuous extension of the linear subspace $\mathbb{E}_-(A)$ to $\{\gamma = 0, (\tau, \eta) \neq 0_{\mathbb{R}^d}\}$ that we will denote by $\tilde{\mathbb{E}}_-(A)$. $\tilde{\mathbb{E}}_+(A)$ extends as well continuously to $\{\gamma = 0, (\tau, \eta) \neq 0_{\mathbb{R}^d}\}$ and we will denote $\tilde{\mathbb{E}}_+(A)$ this extension. Moreover, there holds:

$$\tilde{\mathbb{E}}_-(A) \bigoplus \tilde{\mathbb{E}}_+(A) = \mathbb{C}^N.$$

We can refer the reader to [3], [6], [7], or [8] for detailed estimates concerning mixed hyperbolic problems satisfying a Uniform Lopatinski Condition. Moreover, we can refer to [10] for the proof of the continuous extension of the linear subspaces mentioned above in the hyperbolic-parabolic framework.

Remark 1.3. *As a consequence of the uniform Lopatinski condition, there holds, for all* $\zeta \neq 0$:

$$rg\ \Gamma = p = \dim \tilde{\mathbb{E}}_-(A(\zeta)).$$

1.1 A Kreiss Symmetrizer Approach.

We will now describe a penalization method involving a Kreiss Symmetrizer and a matrix constructed by Rauch in [12], in the construction of our singular perturbation. Note well that we have some freedom in both the choice of the Kreiss Symmetrizer and of Rauch's matrix. Let us denote respectively by $\hat{u}$, $\hat{f}$, and $\hat{g}$ the tangential Fourier-Laplace transform of u, f, and g. Since the Uniform Lopatinski Condition is holding for the mixed hyperbolic system (1.1), there is, see [9] a Kreiss symmetrizer S for the problem:

$$\begin{cases} \partial_x \hat{u} = A\hat{u} + \hat{f}, \quad \{x > 0\}, \\ \Gamma \hat{u}|_{x=0} = \Gamma \hat{g}, \end{cases} \tag{1.3}$$

That is to say there exists a matrix $S(\zeta)$, homogeneous of order zero in ζ, C^∞ in $\mathbb{R}^+ \times \mathbb{R}^d - \{0_{\mathbb{R}^{d+1}}\}$ and there are $\lambda > 0$, $\delta > 0$ and C_1 such that:

- S is hermitian symmetric.
- $\Re\,(SA) \geq \lambda Id$.
- $S \geq \delta Id - C_1 \Gamma^* \Gamma$.

An algebraic result proved by Rauch in [12] can be reformulated as follow, and a proof is recalled in section 2.2:

Lemma 1.4. *There is a hermitian symmetric, uniformly definite positive, $N \times N$ matrix B such that:*

$$\ker \Gamma = \mathbb{E}_+((S)^{-1}B).$$

Moreover B depends smoothly of ζ.

Remark 1.5. *This result is proved by constructing explicit matrices satisfying the desired properties. Thus, it is not merely an existence result and we can use the explicitly known matrix B in our construction of a penalization operator.*

Let us denote by $R := B^{\frac{1}{2}}$ and $S_R := R^{-1}SR^{-1}$. We will denote by $\mathbb{P}^-$ the projector on $\mathbb{E}_-(S_R)$ parallel to $\mathbb{E}_+(S_R)$ and by $\mathbb{P}^+$ the projector on $\mathbb{E}_+(S_R)$ parallel to $\mathbb{E}_-(S_R)$; $\underline{\mathbb{P}}^-$ and $\underline{\mathbb{P}}^+$ denoting the associated Fourier multiplier. We recall that, denoting by $\mathcal{F}$ the tangential Fourier transform, the Fourier multiplier $\underline{\mathbb{P}}^-(\partial_t, \partial_y, \gamma)$ [resp $\underline{\mathbb{P}}^+(\partial_t, \partial_y, \gamma)$] is then defined, for all $w \in H^k(\mathbb{R}^{d+1})$, and $\gamma > 0$, by:

$$\mathcal{F}\left(\underline{\mathbb{P}}^-(\partial_t, \partial_y, \gamma) w\right) = \mathbb{P}^-(\zeta)\mathcal{F}(w),$$

[resp

$$\mathcal{F}\left(\underline{\mathbb{P}}^{+}(\partial_t,\partial_y,\gamma)w\right)=\mathbb{P}^{+}(\zeta)\mathcal{F}(w)],$$

in the future we will rather write:

$$\mathcal{F}\left(\underline{\mathbb{P}}^{\pm}(\partial_t,\partial_y,\gamma)w\right)=\mathbb{P}^{\pm}(\zeta)\mathcal{F}(w).$$

We fix, once and for all, $\gamma>0$ big enough. Let us consider then the solution $\underline{u}^{\varepsilon}$ of the well-posed Cauchy problem on the whole space (1.4):

$$\begin{cases} \mathcal{H}\underline{u}^{\varepsilon}+\dfrac{1}{\varepsilon}\mathbb{M}\underline{u}^{\varepsilon}\mathbf{1}_{x<0}=f\mathbf{1}_{x>0}+\dfrac{1}{\varepsilon}\theta\mathbf{1}_{x<0}, \quad \{x\in\mathbb{R}\}, \\ \underline{u}^{\varepsilon}|_{t<0}=0, \end{cases} \tag{1.4}$$

where

$$\mathbb{M}:=-e^{\gamma t}A_d\underline{S}^{-1}\underline{R}\mathbb{P}^{-}\underline{R}e^{-\gamma t},$$
$$\theta:=-e^{\gamma t}A_d\underline{S}^{-1}\underline{R}\mathbb{P}^{-}\underline{\Gamma}\tilde{g},$$

and $\underline{S}(\partial_t,\partial_y)$ [resp $\underline{R}(\partial_t,\partial_y)$] denotes the Fourier multiplier associated to $S(\zeta)$ [resp $R(\zeta)$]. Let us define $\tilde{g}$ by:

$$\tilde{g}:=e^{-x^2}g.$$

In what follows, $\hat{g}$ will denote the Fourier-Laplace transform of $\tilde{g}$. Let us denote by

$$\tilde{\underline{u}}:=\underline{u}^{-}\mathbf{1}_{x<0}+u\mathbf{1}_{x\geq 0}=\underline{u}^{-}\mathbf{1}_{x\leq 0}+u\mathbf{1}_{x>0}.$$

u denotes the solution of (1.1), and thus belongs to $H^k(\Omega_T^+)$. $\underline{u}^{-}$ is a function belonging to $H^k(\Omega_T^-)$ and such that $\underline{u}^{-}|_{x=0}=u|_{x=0}$. More precisely, $\underline{u}^{-}$ can be computed by: $e^{\gamma t}\mathcal{F}^{-1}\left(R^{-1}(\underline{\hat{v}}^{-}+\mathbb{P}^{-}\Gamma\hat{g})\right)$, where $\underline{\hat{v}}^{-}$ is the solution of the problem:

$$\begin{cases} S_R\partial_x\underline{\hat{v}}^{-}-\mathbb{P}^{+}S_RA_R\underline{\hat{v}}^{-}=\mathbb{P}^{+}S_RA_R\mathbb{P}^{-}\Gamma\hat{g}, \quad \{x<0\}, \\ \underline{\hat{v}}^{-}|_{x=0}=\mathbb{P}^{+}R\hat{u}|_{x=0}, \end{cases}$$

and $\hat{u}$ denotes the Fourier-Laplace transform of the solution u of (1.1).

Theorem 1.6. *For all $k\geq 3$, if $f\in H^k(\Omega_T^+)$ and $g\in H^k(\Upsilon_T)$, then there holds:*

$$\|\underline{u}^{\varepsilon}-\underline{u}^{-}\|_{H^{k-3}(\Omega_T^-)}+\|\underline{u}^{\varepsilon}-u\|_{H^{k-3}(\Omega_T^+)}=\mathcal{O}(\varepsilon),$$

where $\underline{u}^{\varepsilon}$ denotes the solution of the Cauchy problem (1.4) and u denotes the solution of the mixed hyperbolic problem (1.1). If $g=0$ then:

$$\|\underline{u}^{\varepsilon}-\underline{u}^{-}\|_{H^{k-\frac{3}{2}}(\Omega_T^-)}+\|\underline{u}^{\varepsilon}-u\|_{H^{k-\frac{3}{2}}(\Omega_T^+)}=\mathcal{O}(\varepsilon).$$

Of course, since $\underline{u}^\varepsilon$ is defined for all $\{x \in \mathbb{R}\}$, its limit as $\varepsilon \to 0^+$, $\underline{\tilde{u}}$ is can be viewed as an "extension" of u on the fictive domain $\{x < 0\}$. The "extension" resulting from our method of penalization gives a continuous $\underline{\tilde{u}}$ across $\{x = 0\}$, while the method used in [2] gave simply: $\underline{\tilde{u}}|_{x<0} = 0$. We have the following Corollaries:

Corollary 1.7. *Assume for example that* $f \in H^\infty(\Omega_T^+)$ *and* $g \in H^\infty(\Upsilon_T)$ *then*

$$\|\underline{u}^\varepsilon - u\|_{H^s(\Omega_T^+)} = \mathcal{O}(\varepsilon); \quad \forall s > 0.$$

Corollary 1.8. *If* f *belongs to* $L^2(\Omega_T^+)$ *and* $g = 0$ *then:*

$$\lim_{\varepsilon \to 0^+} \|\underline{u}^\varepsilon - \underline{\tilde{u}}\|_{L^2(\Omega_T)} = 0.$$

One of the interest of this first approach lies in the rate of convergence of $\underline{u}^\varepsilon$ towards u. Indeed, in general, a boundary layer will form near the boundary in this kind of singular perturbation problem. For example in the paper by Bardos and Rauch [2], as confirmed by Droniou [4], a boundary layer forms. It is also the case in [11], as analyzed in our Appendix. There are also boundary layers phenomena in the parabolic context: see the approach proposed by Angot, Bruneau and Fabrie [1] for instance. However, surprisingly, and like in the penalization method proposed by Fornet and Guès in [5], our method allows the convergence to occur without formation of any boundary layer on the boundary. As a result, this leads to the kind of sharp stability estimate given in Theorem 1.6. These results concern the case where f and g are sufficiently regular. The reason is that we construct an approximate solution. In the case of g only in $L^2(\Upsilon_T)$, such a simple treatment does not work. However, let $\delta > 0$ be given. If we approximate f and g by smooth functions $f_\nu \in H^\infty(\Omega_T^+)$ and $g_\nu \in H^\infty(\Upsilon_T)$ such that $\|f - f_\nu\|_{L^2(\Omega_T^+)} < \delta$ and $\|g - g_\nu\|_{L^2(\Upsilon_T)} < \delta$, by the uniform Lopatinski condition, we get:

$$\|u_\nu - u\|_{L^2(\Omega_T^+)} < C\nu,$$

where u_ν is the solution of the mixed hyperbolic problem (1.1) with data f_ν and g_ν. We can now apply Corollary 1.8 to u_ν, and obtain by penalization a sequence u_ν^ε in $L^2(\Omega_T)$ such that: $\lim_{\varepsilon\to 0^+} u_\nu^\varepsilon = u_\nu$ in $L^2(\Omega_T^+)$. Finally, by choosing, ε sufficiently small, we get $\|u - u_\nu^\varepsilon\|_{L^2(\Omega_T^+)} < 2C\delta$. By choosing ε and ν as functions of δ, and noting $u_{(\delta)} = u_{\nu(\delta)}^{\varepsilon(\delta)}$, we have:

$$\lim_{\delta \to 0^+} \|u_{(\delta)} - u\|_{L^2(\Omega_T^+)} = 0. \tag{1.5}$$

1.2 A second Approach.

In the first approach we have just introduced, it is necessary to compute a Kreiss's Symmetrizer and a Rauch's matrix. In view of future numerical applications, we will now introduce another method preventing the computation of these matrices. The price to pay is that we need the preliminary computation of v, which is by definition the solution of the Cauchy problem on the free space:

$$\begin{cases} \mathcal{H}v = f, & (t,y,x) \in \Omega_T, \\ v|_{t<0} = 0 & \forall (y,x) \in \mathbb{R}^d. \end{cases} \tag{1.6}$$

Let us denote $\mathbf{P}^-(\zeta)$ the spectral projector on $\tilde{\mathbb{E}}_-(A(\zeta))$ parallel to $\tilde{\mathbb{E}}_+(A(\zeta))$, and $\mathbf{P}^+(\zeta)$ the spectral projector on $\tilde{\mathbb{E}}_+(A(\zeta))$ parallel to $\tilde{\mathbb{E}}_-(A(\zeta))$. Let us introduce $\underline{\mathbf{P}}^{\pm}(\partial_t, \partial_y, \gamma)$, the Fourier multiplier associated to $\mathbf{P}^{\pm}(\zeta)$. Let us denote by $\boldsymbol{\Pi}$ the projector on $\tilde{\mathbb{E}}_-(A(\zeta))$ parallel to $\mathrm{Ker}\Gamma$, which has a sense because of the Uniform Lopatinski Condition and denote $\underline{\boldsymbol{\Pi}}$ the associated Fourier multiplier. We define then $\tilde{h}$ by:

$$\tilde{h} := e^{-x^2}\left(\underline{\mathbf{P}}^-(e^{-\gamma t}v|_{x=0}) + \underline{\boldsymbol{\Pi}}e^{-\gamma t}(g - v|_{x=0})\right),$$

where g denotes the function involved in the boundary condition of the mixed hyperbolic problem (1.1). Now, let us consider the following singularly perturbed Cauchy problem on the whole space:

$$\begin{cases} \mathcal{H}u^\varepsilon + \dfrac{1}{\varepsilon}A_d e^{\gamma t}\underline{\mathbf{P}}^- e^{-\gamma t}u^\varepsilon \mathbf{1}_{x<0} = f\mathbf{1}_{x>0} + \dfrac{1}{\varepsilon}A_d e^{\gamma t}\tilde{h}\mathbf{1}_{x<0}, \\ u^\varepsilon|_{t<0} = 0 \quad . \end{cases} \tag{1.7}$$

Let us denote by

$$\tilde{u} := u^-\mathbf{1}_{x<0} + u\mathbf{1}_{x\geq 0} = u^-\mathbf{1}_{x\leq 0} + u\mathbf{1}_{x>0}.$$

u denotes the solution of (1.1) thus belonging to $H^k(\Omega_T^+)$ and u^- is a function belonging to $H^k(\Omega_T^-)$ and such that $u^-|_{x=0} = u|_{x=0}$. More precisely, u^- can be computed by: $e^{\gamma t}\mathcal{F}^{-1}(\mathcal{F}(\tilde{h}) + \hat{v}^-)$, where $\hat{v}^-$ is the solution of the problem:

$$\begin{cases} \partial_x(\mathbf{P}^+\hat{v}^-) - A(\mathbf{P}^+\hat{v}^-) = 0, & \{x<0\}, \\ \mathbf{P}^+\hat{v}^-|_{x=0} = \mathbf{P}^+\hat{u}|_{x=0}. \end{cases}$$

and $\hat{u}$ denotes the Fourier-Laplace transform of the solution u of (1.1). The problem (1.7) is well-posed and, for all $\varepsilon > 0$, there exists a unique $u^\varepsilon \in H^k(\Omega_T)$ solution. We will fix γ adequately big beforehand. We observe then the following result:

Theorem 1.9. *For all $k \geq 3$, if $f \in H^k(\Omega_T^+)$ and $g \in H^k(\Upsilon_T)$, then there holds:*

$$\|u^\varepsilon - u^-\|_{H^{k-3}(\Omega_T^-)} + \|u^\varepsilon - u\|_{H^{k-3}(\Omega_T^+)} = \mathcal{O}(\varepsilon),$$

where u^ε denotes the solution of the Cauchy problem (1.7) *and u denotes the solution of the mixed hyperbolic problem* (1.1)*.*

The singular perturbation involved in the definition of u^ε does not depend either of Kreiss's Symmetrizer or Rauch's matrix. As a result, for this method of penalization far less computations are necessary in order to obtain our singular perturbation. Note well that the proof of the energy estimates in Theorem 1.9 is completely different from the proof of the energy estimates in Theorem 1.6. Indeed, for our first approach our singularly perturbed problem was treated as a Cauchy problem, contrary to our second approach where it was interpreted as a transmission problem.

Corollary 1.10. *Assume for example that $f \in H^\infty(\Omega_T^+)$ and $g \in H^\infty(\Upsilon_T)$ then*

$$\|u^\varepsilon - u\|_{H^s(\Omega_T^+)} = \mathcal{O}(\varepsilon); \quad \forall s > 0.$$

Of course, we see that the same problem of regularity arises in Theorem 1.9 and Theorem 1.6. However, by a simple density argument, we can also prove here the exact analogous of (1.5).

Remark 1.11. *In the case where $f = 0$, then the solution v of* (1.6) *is $v = 0$ and thus, the perturbed cauchy problem* (1.7) *rewrites:*

$$\begin{cases} \mathcal{H}u^\varepsilon + \dfrac{1}{\varepsilon}A_d e^{\gamma t}\underline{\mathbf{P}}^- e^{-\gamma t}u^\varepsilon \boldsymbol{1}_{x<0} = \dfrac{1}{\varepsilon}A_d e^{\gamma t}e^{-x^2}\left(\underline{\mathbf{\Pi}}e^{-\gamma t}g\right)\boldsymbol{1}_{x<0}, & \{x \in \mathbb{R}\}, \\ u^\varepsilon|_{t<0} = 0 \quad . \end{cases}$$

2 Underlying approach leading to the proof of Theorem 1.6.

2.1 Some preliminaries.

Since the Uniform Lopatinski Condition holds, there is S, homogeneous of order zero in ζ, and such that there are $\lambda > 0$, $\delta > 0$ and C_1 and there holds:

- S is hermitian symmetric.
- $\Re(SA) \geq \lambda Id$.

- $S \geq \delta Id - C_1 \Gamma^* \Gamma$.

S is then called a Kreiss Symmetrizer for the problem:

$$\begin{cases} \partial_x \hat{u} = A\hat{u} + \hat{f}, \quad \{x > 0\}, \\ \Gamma \hat{u}|_{x=0} = \Gamma \hat{g}, \end{cases} \tag{2.1}$$

where $\hat{f}$ and $\hat{g}$ denotes respectively the Fourier-Laplace transforms of f and $\tilde{g}$; and $\hat{u}$ denotes the Fourier-Laplace transform of the solution u of the well-posed mixed hyperbolic problem (1.1). $\hat{u}$ is also solution, for all fixed $\zeta \neq 0$ of the following equation:

$$\begin{cases} S\partial_x \hat{u} = SA\hat{u} + S(A_d)^{-1}\hat{f}, \quad \{x > 0\}, \\ \Gamma \hat{u}|_{x=0} = \Gamma \hat{g}, \end{cases} \tag{2.2}$$

Remark 2.1. *Following our current assumptions, Γ is independent of $\zeta \neq 0$, however, more general boundary conditions, of the form:*

$$\Gamma(\zeta)\hat{u}|_{x=0} = \Gamma(\zeta)\hat{g},$$

can be treated. It would imply taking as boundary condition for (1.1):

$$\Gamma_\gamma u|_{x=0} = \Gamma_\gamma g,$$

with for γ big enough,

$$\Gamma_\gamma := \underline{\Gamma}(\partial_t, \partial_y)e^{-\gamma t},$$

where, $\underline{\Gamma}(\partial_t, \partial_y)$ denotes the Fourier multiplier associated to $\Gamma(\zeta)$, that is to say is defined by:

$$\mathcal{F}(\underline{\Gamma}(\partial_t, \partial_y)u) = \Gamma(\zeta)\mathcal{F}(u).$$

Referring for example to [3] and [7], Kreiss has proved that the existence of a Kreiss symmetrizer for the symbolic equation is sufficient to prove the well-posedness of the associated pseudodifferential equation (here (1.1)). Indeed, multiplying by $\hat{u}$ and integrating by parts the equation:

$$S\partial_x \hat{u} = SA\hat{u} + S(A_d)^{-1}\hat{f}$$

leads to the desired a priori estimates. For all $\zeta \neq 0$, $S(\zeta)$ is hermitian symmetric and definite positive on $\ker \Gamma$. Let us sum up the properties crucial in the proof of the well-posedness of our problem:

Proposition 2.2. *For all $\zeta = (\tau, \gamma, \eta)$ such that $\tau^2 + \gamma^2 + \sum_{j=1}^{d+1} \eta_j^2 = 1$, there holds:*

- *$S(\zeta)$ is hermitian symmetric.*
- $\Re\,(SA)\,(\zeta) := \frac{1}{2}(SA + (SA)^*)(\zeta)$ *is positive definite.*
- $-S(\zeta)$ *is definite negative on* $\ker\Gamma$ *and* $\ker\Gamma$ *is of same dimension as the number of negative eigenvalues in* $-S(\zeta)$.

Note that, by homogeneity of S, it is equivalent for the properties in Proposition 2.2 to hold for $|\zeta| = 1$ or for $|\zeta| > 0$. As a consequence of the first point and third point of Proposition 2.2, the Lemma 1.4 applies and gives a matrix B such that: $\ker\Gamma = \mathbb{E}_+(S^{-1}B)$. In the sequel, such a matrix B is fixed once for all.
The following chapter contains a proof of Lemma 1.4 assorted of a detailed construction of B.

2.2 Detailed proof of Lemma 1.4: Construction of the matrices B solving Lemma 1.4.

As we will emphasize in next chapter, Lemma 1.4 is a crucial feature in our first method of Penalization. The aim of this chapter is to give a more complete proof rather than simply recalling Rauch's result and, in the process, to precise how the matrices B solving Lemma 1.4 are constructed. For all $\zeta \neq 0$, $S(\zeta)$ is hermitian symmetric, uniformly definite positive on $\tilde{\mathbb{E}}_+(A(\zeta))$, and uniformly definite negative on $\tilde{\mathbb{E}}_-(A(\zeta))$; as a consequence, $S(\zeta)$ keeps exactly p positive eigenvalues and $N-p$ negative eigenvalues for all $\zeta \neq 0$. Basically, knowing that S is uniformly definite positive on $\ker\Gamma$; we search to express $\ker\Gamma$ in a way involving S. Consider $q \in \ker\Gamma$, since, for all $\zeta \neq 0$, $\mathbb{E}_-(S(\zeta)) \bigoplus \mathbb{E}_+(S(\zeta)) = \mathbb{C}^N$, we can split q in:

$$q := q^+ + q^-$$

with $q^+ \in \mathbb{E}_+(S(\zeta))$ and $q^- \in \mathbb{E}_-(S(\zeta))$.
Since $\dim ker\Gamma = \dim \mathbb{E}_+(S(\zeta)) = p$, these two linear subspaces are in bijection. Let us give the two main ideas behind this proof: one idea is to detail the bijection between $q \in ker\Gamma$ and $q^+ \in \mathbb{E}_+(S(\zeta))$ as it satisfies some constraints, the other is to come down to the model case where the eigenvalues of S are either 1 or -1. Let us denote:

$$\tilde{S}^{-1} = \begin{bmatrix} -Id_{N-p} & 0 \\ 0 & Id_p \end{bmatrix},$$

In a first step, we will prove the following result:

Proposition 2.3. *If we assume that $\mathbb{V}$ is a linear subspace of $\mathbb{C}^N$ of dimension p, and that there is $C > 0$ such that, for all $q \in \mathbb{V}$, there holds:*

$$\langle \tilde{S}^{-1} q, q\rangle \geq C\langle q, q\rangle,$$

then the two following equivalent properties hold:

- *There is a hermitian symmetric, positive definite matrix $\underline{\tilde{R}}$, such that:*

$$[q \in \mathbb{V}] \Leftrightarrow \left[\underline{\tilde{R}}^{-1} q \in \mathbb{E}_+(\underline{\tilde{R}}\tilde{S}\underline{\tilde{R}})\right],$$

  which is equivalent to:

$$\mathbb{V} = \mathbb{E}_+(\underline{\tilde{R}}^2 \tilde{S}).$$

- *There is a hermitian symmetric, positive matrix $\tilde{R}$, such that:*

$$[q \in \mathbb{V}] \Leftrightarrow \left[\tilde{R} q \in \mathbb{E}_+(\tilde{R}\tilde{S}^{-1}\tilde{R})\right],$$

  which is equivalent to:

$$\mathbb{V} = \mathbb{E}_+(\tilde{S}^{-1}\tilde{R}^2).$$

Moreover, we can link the two properties by taking:

$$\tilde{R}^2 = \tilde{S}\underline{\tilde{R}}^2\tilde{S}.$$

Proof. In this proof, we will show how to construct some matrices $\tilde{R}$ satisfying the required properties. There is a $(N-p) \times p$ matrix $\aleph$ of rank $N-p$ such that $\|\aleph\| \leq 1$ and:

$$\mathbb{V} = \{q \in \mathbb{C}^N, \quad q^- = \aleph q^+\},$$

where q^+[resp q^-] denotes the projector on $\mathbb{E}_+((\tilde{S})^{-1})$ [resp $\mathbb{E}_-((\tilde{S})^{-1})$]parallel to $\mathbb{E}_-((\tilde{S})^{-1})$ [resp $\mathbb{E}_+((\tilde{S})^{-1})$]. Indeed, $\dim \mathbb{V} = p = \dim \mathbb{E}_+((\tilde{S})^{-1})$, and $\mathbb{C}^N = \mathbb{E}_-((\tilde{S})^{-1}) \bigoplus \mathbb{E}_+((\tilde{S})^{-1})$. Moreover, there ic $C > 0$ such that, for all $q \in \mathbb{V}$, there holds:

$$\langle (\tilde{S})^{-1} q, q\rangle = -\langle q^-, q^-\rangle + \langle q^+, q^+\rangle \geq C\langle q, q\rangle.$$

and thus

$$|q^+|^2 - |\aleph q^+|^2 \geq C|q|^2,$$

which implies that $\|\aleph\| < 1$. We will show now that, for $\tilde{R}$ constructed as follow:

$$\tilde{R} = \begin{bmatrix} Id_{N-p} & -\aleph \\ -\aleph^* & Id_p \end{bmatrix},$$

there holds:

$$[q \in \mathbb{V}] \Leftrightarrow \left[\tilde{R}q \in \mathbb{E}_+(\tilde{R}\tilde{S}^{-1}\tilde{R})\right].$$

First,we see that the constructed $\tilde{R}$ is trivially hermitian symmetric and positive definite since $\|\aleph\| < 1$. First, we have:

$$\tilde{R}\tilde{S}^{-1}\tilde{R} = \begin{bmatrix} -Id_{N-p} + NN^* & 0 \\ 0 & Id_p - N^*N \end{bmatrix},$$

and

$$\tilde{R}q = \begin{pmatrix} q^- - \aleph q^+ \\ -\aleph^* q^- + q^+ \end{pmatrix}.$$

Thus, since $\|\aleph\| < 1$, there holds:

$$\left[\tilde{R}q \in \mathbb{E}_+(\tilde{R}\tilde{S}^{-1}\tilde{R})\right] \Leftrightarrow \left[q^- - \aleph q^+ = 0\right] \Leftrightarrow [q \in \mathbb{V}].$$

We will now prove that we have:

$$(\tilde{R})^{-1}\mathbb{E}_+(\tilde{R}\tilde{S}^{-1}\tilde{R}) = \mathbb{E}_+(\tilde{S}^{-1}\tilde{R}^2).$$

Since $\tilde{R}\tilde{S}^{-1}\tilde{R}$ is hermitian symmetric, the linear subspace $\mathbb{E}_+(\tilde{R}\tilde{S}^{-1}\tilde{R})$ is generated by the eigenvectors of $\tilde{R}\tilde{S}^{-1}\tilde{R}$ associated to positive eigenvalues. A basis of $(\tilde{R})^{-1}\mathbb{E}_+(\tilde{R}\tilde{S}^{-1}\tilde{R})$ is thus given by $((\tilde{R})^{-1}v_j)_j$ where v_j denotes an eigenvector of $\tilde{R}\tilde{S}^{-1}\tilde{R}$ associated to a positive eigenvalue λ_j. We have:

$$\tilde{R}\tilde{S}^{-1}\tilde{R}v_j = \lambda_j v_j.$$

Let us denote $w_j = (\tilde{R})^{-1}v_j$, we have then:

$$\tilde{R}\tilde{S}^{-1}\tilde{R}^2 w_j = \lambda_j \tilde{R} w_j \Leftrightarrow \tilde{S}^{-1}\tilde{R}^2 w_j = \lambda_j w_j.$$

As a result, w_j is an eigenvector of $S^{-1}\tilde{R}^2$ associated to the eigenvalue λ_j hence we obtain that:

$$(\tilde{R})^{-1}\mathbb{E}_+(\tilde{R}\tilde{S}^{-1}\tilde{R}) = \mathbb{E}_+(\tilde{S}^{-1}\tilde{R}^2).$$

We can also prove, the same way, that:

$$\underline{\tilde{R}}\mathbb{E}_+(\underline{\tilde{R}}\tilde{S}\underline{\tilde{R}}) = \mathbb{E}_+(\underline{\tilde{R}}^2\tilde{S}).$$

Now, taking

$$\tilde{R}^2 = \tilde{S}\underline{\tilde{R}}^2\tilde{S},$$

we can check that:

$$\mathbb{E}_+(\tilde{S}^{-1}\tilde{R}^2) = \mathbb{E}_+(\underline{\tilde{R}}^2\tilde{S}),$$

which concludes the proof. □

Lemma (**??**) is a Corollary of the following Proposition:

Proposition 2.4. *If S^{-1} denotes a smooth in $\zeta \neq 0$, matrix-valued function in the space of hermitian symmetric matrices with p positive eigenvalues and $N-p$ negative eigenvalues and $\ker\Gamma$ denotes a linear subspace of dimension p and there is $C > 0$ such that, for all $q \in \ker\Gamma$, there holds:*

$$\langle S^{-1}q, q\rangle \geq C\langle q, q\rangle,$$

then the two following equivalent properties hold:

- *There is a smooth in $\zeta \neq 0$, matrix-valued function $\underline{R}$, in the space of hermitian symmetric, positive matrices such that:*

  $$[q \in \mathrm{Ker}\Gamma] \Leftrightarrow \left[\forall\zeta \neq 0, \quad \underline{R}^{-1}(\zeta)q \in \mathbb{E}_+(\underline{R}(\zeta)S(\zeta)\underline{R}(\zeta))\right],$$

  which is equivalent to:

  $$\forall\zeta \neq 0, \quad \mathrm{Ker}\Gamma = \mathbb{E}_+(\underline{R}^2(\zeta)S(\zeta)).$$

- *There is a smooth in $\zeta \neq 0$, matrix-valued function R, in the space of hermitian symmetric, positive matrices such that:*

  $$[q \in \mathrm{Ker}\Gamma] \Leftrightarrow \left[\forall\zeta \neq 0, \quad R(\zeta)q \in \mathbb{E}_+(R(\zeta)S^{-1}(\zeta)R(\zeta))\right],$$

  which is equivalent to:

  $$\forall\zeta \neq 0, \quad \mathrm{Ker}\Gamma = \mathbb{E}_+(S^{-1}(\zeta)R^2(\zeta)).$$

Moreover, for all $\zeta \neq 0$, these two properties can be linked by taking:

$$(R(\zeta))^2 = S(\zeta)(\underline{R}(\zeta))^2S(\zeta).$$

Proof. We will show here that Proposition 2.4 can be deduced from Proposition 2.3. For all $\zeta \neq 0$, $S(\zeta)$ is a hermitian symmetric matrix, moreover S depends smoothly of ζ. As a consequence S^{-1} is also a hermitian symmetric matrix depending smoothly of ζ, and as such, there is a nonsingular matrix V such that:

$$\tilde{S}^{-1} = V^* \left(S^{-1}\right) V.$$

Let us denote Λ the diagonalized version of S^{-1} with eigenvalues sorted by increasing order, then there is Z depending smoothly of ζ such that, for all $\zeta \neq 0$, we have:

$$Z^*(\zeta) = Z^{-1}(\zeta),$$

and

$$\Lambda(\zeta) = Z^*(\zeta)\left(-S^{-1}\right)(\zeta)Z(\zeta).$$

As a consequence, V depends smoothly of ζ since, for all $\zeta \neq 0$:

$$V(\zeta) = (\underline{\Lambda}(\zeta))^{-\frac{1}{2}} Z(\zeta),$$

where $\underline{\Lambda}$ is the diagonal matrix obtained by taking the absolute value of each eigenvalue of Λ. For the sake of simplicity, let us omit the dependence in ζ. Now, for all $q \in V^{-1} \ker \Gamma$, there is $C > 0$, such that:

$$\langle \tilde{S}^{-1} q, q\rangle = \langle V^* S^{-1} V q, q\rangle = \langle S^{-1}(Vq), (Vq)\rangle \geq C\langle (Vq), (Vq)\rangle.$$

Moreover V is nonsingular, thus there is $C' > 0$, such that, for all $q \in V^{-1} \ker \Gamma$, there holds:

$$\langle \tilde{S}^{-1} q, q\rangle \geq C' \langle q, q\rangle.$$

Moreover $\dim V^{-1} \ker \Gamma = p$, using Proposition 2.3, for all fixed $\zeta \neq 0$, there is a hermitian symmetric, positive definite matrix $\underline{\tilde{R}}(\zeta)$, such that:

$$V^{-1}(\zeta) \ker \Gamma = \mathbb{E}_+((\underline{\tilde{R}}(\zeta))^2 \tilde{S}(\zeta)) = \underline{\tilde{R}}(\zeta)\mathbb{E}_+(\underline{\tilde{R}}(\zeta)\tilde{S}(\zeta)\underline{\tilde{R}})(\zeta).$$

We will now prove that we can construct $\underline{\tilde{R}}$ depending smoothly of ζ. First there is a $(N-p) \times p$ matrix $\aleph$ of rank $N-p$, depending smoothly of ζ, such that fore all $\zeta \neq 0$ $\|\aleph(\zeta)\| \leq 1$ and:

$$V^{-1}(\zeta) \ker \Gamma = \{q \in \mathbb{C}^N, \quad q^- = \aleph(\zeta) q^+\},$$

where q^+[resp q^-] denotes the projector on $\mathbb{E}_+((\tilde{S})^{-1})$ [resp $\mathbb{E}_-((\tilde{S})^{-1})$]parallel to $\mathbb{E}_-((\tilde{S})^{-1})$ [resp $\mathbb{E}_+((\tilde{S})^{-1})$]. $\underline{\tilde{R}}$ is given, for all $\zeta \neq 0$, by:

$$\underline{\tilde{R}}(\zeta) = \sqrt{\tilde{S}^{-1}(\zeta)\tilde{R}^2(\zeta)\tilde{S}^{-1}(\zeta)},$$

with $\tilde{R}$ given by:

$$\tilde{R}(\zeta) = \begin{bmatrix} Id_{N-p} & -\aleph(\zeta) \\ -\aleph^*(\zeta) & Id_p \end{bmatrix}.$$

Since $\tilde{S}^{-1} = V^* \left(S^{-1}\right) V$, there holds: $\tilde{S} = V^* S V$, and, as a consequence:

$$(V\underline{\tilde{R}})^{-1} \ker \Gamma = \mathbb{E}_+(\underline{\tilde{R}} V^* S V \underline{\tilde{R}}).$$

As $\underline{\tilde{R}} V^* S V \underline{\tilde{R}}$ is hermitian symmetric, a basis of the linear subspace $\mathbb{E}_+(\underline{\tilde{R}} V^* S V \underline{\tilde{R}})$ is given by the eigenvectors of $\underline{\tilde{R}} V^* S V \underline{\tilde{R}}$ associated to positive eigenvalues. This leads us to consider $v_j = (V\underline{\tilde{R}})^{-1} u_j$ satisfying:

$$\underline{\tilde{R}} V^* S V \underline{\tilde{R}} v_j = \lambda_j v_j.$$

We have:

$$\underline{\tilde{R}} V^* S V \underline{\tilde{R}} (V\underline{\tilde{R}})^{-1} u_j = \lambda_j (V\underline{\tilde{R}})^{-1} u_j.$$

hence:

$$(V\underline{\tilde{R}}) \underline{\tilde{R}} V^* S u_j = \lambda_j u_j.$$

Since $(V\underline{\tilde{R}})\underline{\tilde{R}} V^* = (\underline{\tilde{R}} V^*)^* (\underline{\tilde{R}} V^*)$ is hermitian symmetric and positive definite, we can then define its square root. We define $\underline{R}$ by:

$$\underline{R} = \sqrt{(\underline{\tilde{R}} V^*)^* (\underline{\tilde{R}} V^*)}.$$

Since both $\underline{\tilde{R}}$ and V depends smoothly of ζ, so does $\underline{R}$. Moreover, there holds:

$$\underline{R}^2 S u_j = \lambda_j u_j,$$

which gives:

$$\ker \Gamma = V \underline{\tilde{R}} \mathbb{E}_+(\underline{\tilde{R}} V^* S V \underline{\tilde{R}}) = \mathbb{E}_+(\underline{R}^2 S).$$

We have thus proved there is a smooth in $\zeta \neq 0$, matrix-valued function $\underline{R}$, in the space of hermitian symmetric, positive matrices such that:

$$[q \in \mathrm{Ker}\Gamma] \Leftrightarrow \left[\forall \zeta \neq 0, \quad \underline{R}^{-1}(\zeta) q \in \mathbb{E}_+(\underline{R}(\zeta) S(\zeta) \underline{R}(\zeta))\right],$$

which is equivalent to:

$$\forall \zeta \neq 0, \quad \mathrm{Ker}\Gamma = \mathbb{E}_+(\underline{R}^2(\zeta) S(\zeta)).$$

Now consider R defined, for all $\zeta \neq 0$, by:

$$R(\zeta) = \sqrt{S(\zeta)(\underline{R}(\zeta))^2 S(\zeta)},$$

$$R(\zeta) = \sqrt{(\underline{\tilde{R}}(\zeta)V^*(\zeta)S(\zeta))^*(\underline{\tilde{R}}(\zeta)V^*(\zeta)S(\zeta))}.$$

$\zeta \mapsto R(\zeta)$ is smooth and, for all ζ, $R(\zeta)$ is a hermitian symmetric, positive definite matrix. Moreover, there holds:

$$[q \in \mathrm{Ker}\Gamma] \Leftrightarrow \left[\forall \zeta \neq 0, \quad R(\zeta)q \in \mathbb{E}_+(R(\zeta)S^{-1}(\zeta)R(\zeta))\right],$$

which is equivalent to:

$$\forall \zeta \neq 0, \quad \mathrm{Ker}\Gamma = \mathbb{E}_+(S^{-1}(\zeta)R^2(\zeta)).$$

Let us detail the computation of $R(\zeta)$.

$$R(\zeta) = \sqrt{S(\zeta)V(\zeta)\underline{\tilde{R}}^2(\zeta)V^*(\zeta)S(\zeta)}.$$

Moreover

$$\underline{\tilde{R}}^2(\zeta) = \tilde{S}^{-1}(\zeta)\tilde{R}^2(\zeta)\tilde{S}^{-1}(\zeta),$$

we have thus:

$$R(\zeta) = \sqrt{\left(\tilde{R}(\zeta)\tilde{S}^{-1}(\zeta)V^*(\zeta)S(\zeta)\right)^*\left(\tilde{R}(\zeta)\tilde{S}^{-1}(\zeta)V^*(\zeta)S(\zeta)\right)},$$

which gives:

$$B(\zeta) = \left(\tilde{R}(\zeta)\tilde{S}^{-1}(\zeta)V^*(\zeta)S(\zeta)\right)^*\left(\tilde{R}(\zeta)\tilde{S}^{-1}(\zeta)V^*(\zeta)S(\zeta)\right).$$

We recall that $\tilde{R}$ is given, for all $\zeta \neq 0$, by:

$$\tilde{R}(\zeta) = \begin{bmatrix} Id_{N-p} & -\aleph(\zeta) \\ -\aleph^*(\zeta) & Id_p \end{bmatrix}.$$

and that for all $\zeta \neq 0$, $V(\zeta)$ is given by:

$$V(\zeta) = (\underline{\Lambda}(\zeta))^{-\frac{1}{2}}Z(\zeta),$$

where

$$\Lambda(\zeta) = Z^*(\zeta)\left(-S^{-1}\right)(\zeta)Z(\zeta)$$

with Λ is a diagonal matrix with real coefficients: $(\lambda_1, \dots, \lambda_N)$, and $\underline{\Lambda}$ denotes the diagonal matrix with diagonal coefficients $(|\lambda_1|, \dots, |\lambda_N|)$.

Remark 2.5. *In the construction of B the only freedom we have resides in the choice of $\aleph$.*

□

2.3 A change of dependent variables.

Let us denote by $R := B^{\frac{1}{2}}$ and $\hat{v} := R\hat{u}$. $\hat{v}$ is hence solution of (2.3):

$$\begin{cases} R^{-1}SR^{-1}\partial_x \hat{v} = R^{-1}SAR^{-1}\hat{v} + R^{-1}S(A_d)^{-1}\hat{f}, & \{x>0\}, \\ \Gamma R^{-1}\hat{v}|_{x=0} = \Gamma\hat{g}, \end{cases} \tag{2.3}$$

We will adopt the following notations: $S_R := R^{-1}SR^{-1}$, $A_R := RAR^{-1}$, and $\Gamma_R := \Gamma R^{-1}$. We first observe that:

$$\ker\Gamma_R = R\ker\Gamma = R\mathbb{E}_+((S)^{-1}R^2).$$

but $S_R^{-1} = RS^{-1}R$ thus

$$\ker\Gamma_R = R\mathbb{E}_+(R^{-1}S_R R) = \mathbb{E}_+(S_R).$$

This is where Lemma 1.4 is used in a crucial manner. Let us denote by $\mathbb{P}^-$ the projector on $\mathbb{E}_-(S_R)$ parallel to $\mathbb{E}_+(S_R)$ and by by $\mathbb{P}^+$ the projector on $\mathbb{E}_+(S_R)$ parallel to $\mathbb{E}_-(S_R)$; $\underline{\mathbb{P}}^-$ and $\underline{\mathbb{P}}^+$ denoting the associated Fourier multiplier. Since S_R is hermitian symmetric, $\mathbb{P}^-$ is in fact the orthogonal projector on $\mathbb{E}_-(S_R)$. The problem (2.3) can then be written:

$$\begin{cases} S_R\partial_x\hat{v} = S_R A_R\hat{v} + R^{-1}S(A_d)^{-1}\hat{f}, & \{x>0\}, \\ \mathbb{P}^-\hat{v}|_{x=0} = \mathbb{P}^-\Gamma\hat{g}, \end{cases}$$

This problem is well-posed because, as a direct Corollary of Proposition 2.2, we have:

Proposition 2.6. *For all ζ such that $\tau^2+\gamma^2+|\eta|^2 = 1$, there holds:*

- *$S_R(\zeta)$ is hermitian symmetric.*
- *$\Re(S_R A_R)(\zeta)$ is positive definite.*
- *$-S_R(\zeta)$ is definite negative on $\ker\Gamma_R$ and the dimension of $\ker\Gamma_R$ is the same as the number of negative eigenvalues of $-S_R(\zeta)$.*

Proof. For the sake of simplicity, let us omit the dependence in ζ in our notations.

- $S_R := R^{-1}SR^{-1}$, and both S and R are hermitian thus S_R is hermitian.

- $S_R A_R = R^{-1} S A R^{-1}$, thus for all $q \in \mathbb{C}^N$, there holds:

  $$2\langle \Re(S_R A_R) q, q\rangle = \langle S_R A_R q, q\rangle + \langle q, S_R A_R q\rangle = \langle R^{-1} S A R^{-1} q, q\rangle + \langle q, R^{-1} S A R^{-1} q\rangle,$$

  since R^{-1} is hermitian, we have then:

  $$= \langle S A R^{-1} q, R^{-1} q\rangle + \langle R^{-1} q, S A R^{-1} q\rangle = 2\langle \Re(SA) R^{-1} q, R^{-1} q\rangle.$$

  Since $\Re(SA)$ is positive definite and R is invertible, $\Re\,(S_R A_R)$ is thus positive definite.

- By construction of R, it satisfies $\ker \Gamma_R = \mathbb{E}_+(S_R)$, with S_R hermitian. As a consequence $-S_R$ is definite negative on $\ker \Gamma_R$ and the dimension of $\ker \Gamma_R$ is the same as the number of negative eigenvalues of $-S_R$.

□ Let us mention that, since R and S remains uniformly bounded in $\zeta \neq 0$, $\hat{f}$ and $R^{-1} S (A_d)^{-1} \hat{f}$ belongs to the same space. In a same spirit as [5], this suggests the following singular perturbation of (2.3):

$$S_R \partial_x \hat{v}^\varepsilon - \frac{1}{\varepsilon} \mathbb{P}^- \hat{v}^\varepsilon \mathbf{1}_{x<0} = S_R A_R \hat{v}^\varepsilon - \frac{1}{\varepsilon} \mathbb{P}^- \Gamma \hat{g} \mathbf{1}_{x<0} + R^{-1} S (A_d)^{-1} \hat{f}, \quad \{x \in \mathbb{R}\},$$

This is equivalent to perturb (2.2) as follow:

$$S \partial_x \hat{u}^\varepsilon - \frac{1}{\varepsilon} R \mathbb{P}^- R \hat{u}^\varepsilon \mathbf{1}_{x<0} = S A \hat{u}^\varepsilon - \frac{1}{\varepsilon} R \mathbb{P}^- \Gamma \hat{g} \mathbf{1}_{x<0} + S (A_d)^{-1} \hat{f}, \quad \{x \in \mathbb{R}\},$$

Finally, this induces the following perturbation for (1.1):

$$\begin{cases} \mathcal{H} \underline{u}^\varepsilon + \frac{1}{\varepsilon} \mathbb{M} \underline{u}^\varepsilon \mathbf{1}_{x<0} = f \mathbf{1}_{x>0} + \frac{1}{\varepsilon} \theta \mathbf{1}_{x<0}, \quad \{x \in \mathbb{R}\}, \\ \underline{u}^\varepsilon|_{t<0} = 0, \end{cases} \tag{2.4}$$

where

$$\mathbb{M} := -e^{\gamma t} A_d \underline{S}^{-1} \underline{R} \mathbb{P}^- \underline{R} e^{-\gamma t},$$
$$\theta = -e^{\gamma t} A_d \underline{S}^{-1} \underline{R} \mathbb{P}^- \underline{\Gamma} \tilde{g},$$

and $\underline{S}(\partial_t, \partial_y)$ [resp $\underline{R}(\partial_t, \partial_y)$] denotes the Fourier multiplier associated to $S(\zeta)$ [resp $R(\zeta)$].

3 Proof of Theorem 1.6.

First, we construct an approximate solution of equation (2.4) (which is also equation (1.4)), then prove suitable energy estimates that ensures $\underline{u}^\varepsilon$ and its approximate solution both converges towards the same limit as $\varepsilon \to 0^+$.

3.1 Construction of the approximate solution.

$\underline{u}^\varepsilon$ is the solution of the well-posed Cauchy problem:

$$\begin{cases} \mathcal{H}\underline{u}^\varepsilon + \dfrac{1}{\varepsilon}\mathbb{M}\underline{u}^\varepsilon \mathbf{1}_{x<0} = f\mathbf{1}_{x>0} + \dfrac{1}{\varepsilon}\theta\mathbf{1}_{x<0}, \quad \{x \in \mathbb{R}\}, \\ \underline{u}^\varepsilon|_{t<0} = 0. \end{cases}$$

$\underline{u}^\varepsilon$ is moreover the solution of the well-posed Cauchy problem:

$$\begin{cases} \underline{S}A_d^{-1}\mathcal{H}\underline{u}^\varepsilon + \dfrac{1}{\varepsilon}\underline{S}A_d^{-1}\mathbb{M}\underline{u}^\varepsilon \mathbf{1}_{x<0} = \underline{S}A_d^{-1}f\mathbf{1}_{x>0} + \dfrac{1}{\varepsilon}\underline{S}A_d^{-1}\theta\mathbf{1}_{x<0}, \quad \{x \in \mathbb{R}\}, \\ \underline{u}^\varepsilon|_{t<0} = 0. \end{cases}$$

The associated equation after tangential Fourier-Laplace transform writes :

$$S\partial_x \hat{\underline{u}}^\varepsilon - \frac{1}{\varepsilon}R\mathbb{P}^- R\hat{\underline{u}}^\varepsilon \mathbf{1}_{x<0} - SA\hat{\underline{u}}^\varepsilon = -\frac{1}{\varepsilon}R\mathbb{P}^-\Gamma\hat{g}\mathbf{1}_{x<0} + S(A_d)^{-1}\hat{f}\mathbf{1}_{x>0}, \quad \{x \in \mathbb{R}\}.$$

or alternatively:

$$\begin{cases} \hat{\underline{u}}^\varepsilon = R^{-1}\hat{v}^\varepsilon \\ S_R\partial_x\hat{v}^\varepsilon + \dfrac{1}{\varepsilon}\mathbb{P}^-\hat{v}^\varepsilon\mathbf{1}_{x<0} = S_RA_R\hat{v}^\varepsilon + \dfrac{1}{\varepsilon}\mathbb{P}^-\Gamma\hat{g}\mathbf{1}_{x<0} + R^{-1}S(A_d)^{-1}\hat{f}, \quad \{x \in \mathbb{R}\}, \end{cases}$$

We will use the following formulation as a transmission problem in our construction of an approximate solution:

$$\begin{cases} S_R\partial_x\hat{v}^{\varepsilon+} = S_RA_R\hat{v}^{\varepsilon+} + R^{-1}S(A_d)^{-1}\hat{f}, \quad \{x > 0\}, \\ S_R\partial_x\hat{v}^{\varepsilon-} + \dfrac{1}{\varepsilon}\mathbb{P}^-\hat{v}^{\varepsilon-} = S_RA_R\hat{v}^{\varepsilon-} + \dfrac{1}{\varepsilon}\mathbb{P}^-\Gamma\hat{g}, \quad \{x < 0\}, \\ \hat{v}^{\varepsilon+}|_{x=0^+} = \hat{v}^{\varepsilon-}|_{x=0^-}. \end{cases}$$

For Ω an open regular subset of $\mathbb{R}^{d+1}$, and $\rho \in \mathbb{N}$, let us introduce the weighted spaces $H^\varrho_\gamma(\Omega)$ defined by:

$$H^\varrho_\gamma(\Omega) = \{\varpi \in e^{\gamma t}L^2(\Omega), \|\varpi\|_{H^\varrho_\gamma(\Omega)} < \infty\};$$

where

$$\|\varpi\|^2_{H^\varrho_\gamma(\Omega)} = \sum_{\alpha, |\alpha| \leq \varrho} \gamma^{\rho - |\alpha|}\|e^{-\gamma t}\partial^\alpha\varpi\|^2_{L^2(\Omega)}.$$

We will construct an approximate solution $\underline{u}^\varepsilon_{app}$ of $\underline{u}^\varepsilon$. $\underline{u}^\varepsilon_{app}$ will be constructed as follow:

$$\underline{u}^\varepsilon_{app} = \underline{u}^{\varepsilon+}_{app}\mathbf{1}_{x>0} + \underline{u}^{\varepsilon-}_{app}\mathbf{1}_{x<0},$$

where $\underline{u}^{\varepsilon\pm}_{app}$ is an approximate solution of $\underline{u}^{\varepsilon\pm}$ satisfying the following ansatz:

$$\underline{u}^{\varepsilon\pm}_{app} = \sum_{j=0}^{M} \underline{U}^{\pm}_j(\zeta, x)\varepsilon^j,$$

where the profiles $\underline{U}^{\pm}_j$ belong to $H^{k-\frac{3}{2}j}_{\gamma}(\Omega^{\pm}_T)$, where $\Omega^{\pm}_T$ stands for $[0,T]\times\mathbb{R}^d_{\pm}$. Denote

$$\hat{v}^{\varepsilon}_{app} = R\mathcal{F}(e^{-\gamma t}\underline{u}^{\varepsilon}_{app}) := \hat{v}^{\varepsilon+}_{app}\mathbf{1}_{x>0} + \hat{v}^{\varepsilon-}_{app}\mathbf{1}_{x<0}.$$

$\hat{v}^{\varepsilon\pm}_{app}$ is then an approximate solution of $v^{\varepsilon\pm}$ and is of the form:

$$\hat{v}^{\varepsilon\pm}_{app} = \sum_{j=0}^{M} V^{\pm}_j(\zeta, x)\varepsilon^j;$$

where

$$V^{\pm}_j = R\mathcal{F}(e^{-\gamma t}\underline{U}^{\pm}_j),$$

and conversely

$$\underline{U}^{\pm}_j = e^{\gamma t}\mathcal{F}^{-1}\left(R^{-1}V^{\pm}_j\right).$$

The profiles $\underline{U}^{\pm}_j$ can be constructed inductively at any order. Let us show how the first profiles are constructed: Identifying the terms in ε^{-1} gives:

$$\mathbb{P}^- V^-_0 = \mathbb{P}^-\Gamma\hat{g}.$$

Hence, $\mathbb{P}^+ V^-_0$ remains to be computed in order to obtain the profile

$$\underline{U}^-_0 = e^{\gamma t}\mathcal{F}^{-1}\left(R^{-1}V^-_0\right).$$

Identifying the terms in ε^0 gives then that V^+_0 is solution of the well-posed problem:

$$\text{(3.1)} \qquad \begin{cases} S_R\partial_x V^+_0 = S_R A_R V^+_0 + R^{-1}S(A_d)^{-1}\hat{f}, & \{x>0\}, \\ \mathbb{P}^- V^+_0|_{x=0} = \mathbb{P}^-\Gamma\hat{g}. \end{cases}$$

The associated profile

$$\underline{U}^+_0 = e^{\gamma t}\mathcal{F}^{-1}\left(R^{-1}V^+_0\right)$$

belongs then to $H^k_{\gamma}(\Omega^+_T)$. Moreover, the problem (3.1) is Kreiss-Symmetrizable and thus the trace of the profile $\underline{U}^+_0$, see [3] for instance, satisfies:

$$\underline{U}^+_0|_{x=0} \in H^k_{\gamma}(\Upsilon_T).$$

Since V_0^+ has just be computed, $V_0^-|_{x=0}$ is given by: $V_0^+|_{x=0} - V_0^-|_{x=0} = 0$ and thus, there holds:

$$\mathbb{P}^- V_0^+|_{x=0} = \mathbb{P}^- V_0^-|_{x=0}.$$

Moreover

$$S_R \partial_x V_0^- - \mathbb{P}^- V_1^- = S_R A_R V_0^-, \quad \{x < 0\}.$$

Projecting this equation on $\mathbb{E}_+(S_R)$ collinearly to $\mathbb{E}_-(S_R)$ gives then:

$$S_R \partial_x \mathbb{P}^+ V_0^- - \mathbb{P}^+ S_R A_R V_0^- = 0, \quad \{x < 0\},$$

Since

$$\mathbb{P}^+ S_R A_R V_0^- = \mathbb{P}^+ S_R A_R \mathbb{P}^+ V_0^- + \mathbb{P}^+ S_R A_R \mathbb{P}^- \Gamma \hat{g},$$

we have then:

$$S_R \partial_x (\mathbb{P}^+ V_0^-) - \mathbb{P}^+ S_R A_R (\mathbb{P}^+ V_0^-) = \mathbb{P}^+ S_R A_R \mathbb{P}^- \Gamma \hat{g}, \quad \{x < 0\},$$

and as a consequence, $\mathbb{P}^+ V_0^-$ is solution of the following problem:

$$(3.2) \qquad \begin{cases} S_R \partial_x (\mathbb{P}^+ V_0^-) - \mathbb{P}^+ S_R A_R (\mathbb{P}^+ V_0^-) = \mathbb{P}^+ S_R A_R \mathbb{P}^- \Gamma \hat{g} \quad \{x < 0\}, \\ \mathbb{P}^+ V_0^-|_{x=0} = \mathbb{P}^+ V_0^+|_{x=0}. \end{cases}$$

Let us precise how (3.2) has to be interpreted: we denote $w = \mathbb{P}^+ V_0^-$. w is then totally polarized on $\mathbb{E}_+(S_R)$, and satisfies the problem:

$$(3.3) \qquad \begin{cases} \mathbb{P}^+ w = w \\ S_R \partial_x w - \mathbb{P}^+ S_R A_R w = \mathbb{P}^+ S_R A_R \mathbb{P}^- \Gamma \hat{g} \quad \{x < 0\}, \\ w|_{x=0} = \mathbb{P}^+ V_0^+|_{x=0}. \end{cases}$$

As we will see, the problem (3.3) is Kreiss-Symmetrizable and thus well-posed. Indeed, for all ζ such that $\tau^2 + \gamma^2 + |\eta|^2 = 1$, we have, omitting the dependencies in ζ in our notations:

- For all $q \in \mathbb{C}^N$, there holds:

$$\langle S_R q, q \rangle = \langle q, S_R q \rangle.$$

- Since $Re(S_R A_R)$ is positive definite and $\mathbb{P}^+$ is an orthogonal projector, there is $C > 0$ such that, for all $q \in \mathbb{E}_+(S_R)$, there holds:

$$\langle \mathbb{P}^+ S_R A_R \mathbb{P}^+ q, q \rangle + \langle q, \mathbb{P}^+ S_R A_R \mathbb{P}^+ q \rangle \geq C \langle q, q \rangle.$$

  Indeed, for all $q \in \mathbb{E}_+(S_R)$, there holds:

$$\langle \mathbb{P}^+ S_R A_R \mathbb{P}^+ q, q \rangle = \langle \mathbb{P}^+ S_R A_R \mathbb{P}^+ q, \mathbb{P}^+ q \rangle = \langle S_R A_R \mathbb{P}^+ q, \mathbb{P}^+ q \rangle.$$

- $-S_R$ is definite negative on $\ker \mathbb{P}^+$ that is to say, that there is $c > 0$ such that, for all $q \in \ker \mathbb{P}^+$, there holds:

$$\langle -S_R q, q\rangle \leq -c\langle q, q\rangle.$$

  Moreover $\ker \mathbb{P}^+$ has the same dimension as the number of negative eigenvalues in $-S_R$.

The profile $\underline{U_0^-}$ can then be computed by:

$$\underline{U_0^-} := e^{\gamma t}\mathcal{F}^{-1}\left(R^{-1}(w + \mathbb{P}^-\Gamma\hat{g})\right)$$

belongs to $H^k_\gamma(\Omega_T^-)$, moreover its trace $\underline{U_0^-}|_{x=0}$ belongs to $H^k_\gamma(\Upsilon_T)$. Consider now the equation:

$$\mathbb{P}^- V_1^- = S_R\partial_x V_0^- - S_R A_R V_0^-, \quad \{x < 0\}.$$

Since $\mathbb{P}^- V_1^-|_{x=0} = \mathbb{P}^- V_1^+|_{x=0}$, V_1^+ is solution of the well-posed problem:

$$\begin{cases} S_R\partial_x V_1^+ = S_R A_R V_1^+, \quad \{x > 0\}, \\ \mathbb{P}^- V_1^+|_{x=0} = S_R\partial_x V_0^-|_{x=0} - S_R A_R V_0^-|_{x=0}. \end{cases}$$

Due to the loss of regularity in the boundary condition, the associated profile

$$\underline{U_1^+} = e^{\gamma t}\mathcal{F}^{-1}\left(R^{-1}V_1^+\right)$$

belongs to $H^{k-\frac{3}{2}}_\gamma(\Omega_T^+)$, moreover its trace $\underline{U_1^+}|_{x=0}$ belongs to $H^{k-\frac{3}{2}}_\gamma(\Upsilon_T)$. Moreover, applying $\mathbb{P}^+$ to the equation:

$$\mathbb{P}^- V_2^- + S_R A_R V_1^- = S_R\partial_x V_1^-, \quad \{x < 0\},$$

we obtain:

$$\begin{cases} S_R\partial_x \mathbb{P}^+ V_1^- = \mathbb{P}^+ S_R A_R \mathbb{P}^+ V_1^- + \mathbb{P}^+ S_R A_R \mathbb{P}^- V_1^-, \quad \{x < 0\}, \\ \mathbb{P}^+ V_1^-|_{x=0} = \mathbb{P}^+ V_1^+|_{x=0}. \end{cases}$$

As before, let us take $\mathbb{P}^+ V_1^-$ as the unknown of the well-posed problem:

$$\begin{cases} S_R\partial_x(\mathbb{P}^+ V_1^-) - \mathbb{P}^+ S_R A_R(\mathbb{P}^+ V_1^-) = \mathbb{P}^+ S_R A_R\left(S_R\partial_x V_0^- - S_R A_R V_0^-\right), \ \{x < 0\}, \\ (\mathbb{P}^+ V_1^-)|_{x=0} = \mathbb{P}^+ V_1^+|_{x=0}. \end{cases}$$

This problem is Kreiss-Symmetrizable since, for all ζ such that $\tau^2 + \gamma^2 + |\eta|^2 = 1$, there holds:

- For all $q \in \mathbb{C}^N$, there holds:
$$\langle S_R q, q\rangle = \langle q, S_R q\rangle.$$
- There is $C > 0$ such that for all $q \in \mathbb{E}_+(S_R)$, there holds:
$$\langle \mathbb{P}^+ S_R A_R \mathbb{P}^+ q, q\rangle + \langle q, \mathbb{P}^+ S_R A_R \mathbb{P}^+ q\rangle \geq C\langle q, q\rangle.$$
- $-S_R$ is definite negative on $\ker \mathbb{P}^+$ that is to say, that there is $c > 0$ such that, for all $q \in \ker \mathbb{P}^+$, there holds:
$$\langle -S_R q, q\rangle \leq -c\langle q, q\rangle.$$
Moreover $\ker \mathbb{P}^+$ has the same dimension as the number of negative eigenvalues in $-S_R$.

However, due to a loss of regularity in both the source term and the boundary condition, the associated profile
$$\underline{U}_1^- = e^{\gamma t}\mathcal{F}^{-1}\left(R^{-1}\left(\mathbb{P}^+ V_1^- + S_R \partial_x V_0^- - S_R A_R V_0^-\right)\right)$$
belongs to $H_\gamma^{k-\frac{3}{2}}(\Omega_T^-)$. The construction of the following profiles can be pursued at any order the same way. In practice, we take:
$$u_{app}^{\varepsilon +} = \underline{U}_0^+,$$
$$u_{app}^{\varepsilon -} = \underline{U}_0^- + \varepsilon \underline{U}_1^-.$$
As a result, the approximate solution writes $\underline{u}_{app}^\varepsilon := \underline{u}_{app}^{\varepsilon +}\mathbf{1}_{x>0} + \underline{u}_{app}^{\varepsilon -}\mathbf{1}_{x<0}$; where $\underline{u}_{app}^{\varepsilon +}$ belongs to $H_\gamma^k(\Omega_T^+)$ and $\underline{u}_{app}^{\varepsilon -}$ belongs to $H_\gamma^{k-\frac{3}{2}}(\Omega_T^-)$. $\underline{u}_{app}^\varepsilon$ is then solution of a well-posed problem of the form:

$$\text{(3.4)} \qquad \begin{cases} \mathcal{H}\underline{u}_{app}^\varepsilon + \dfrac{1}{\varepsilon}\mathbb{M}\underline{u}_{app}^\varepsilon \mathbf{1}_{x<0} = f\mathbf{1}_{x>0} + \dfrac{1}{\varepsilon}\theta \mathbf{1}_{x<0} + \varepsilon \underline{r}^\varepsilon, & \{x \in \mathbb{R}\}, \\ \underline{u}_{app}^\varepsilon|_{t<0} = 0 & . \end{cases}$$

Where $\underline{r}^\varepsilon := \underline{r}^{\varepsilon +}\mathbf{1}_{x>0} + \underline{r}^{\varepsilon -}\mathbf{1}_{x<0}$, with $\underline{r}^{\varepsilon +} \in H_\gamma^{k-\frac{5}{2}}(\Omega_T^+)$ and $\underline{r}^{\varepsilon -} \in H_\gamma^{k-3}(\Omega_T^-)$.

Remark 3.1. *In the case where $g = 0$, the loss of regularity in the profiles is delayed by one step. As a result, in this case we obtain:*
$$\underline{u}_{app}^{\varepsilon +} \in H_\gamma^k(\Omega_T^+),$$
$$\underline{u}_{app}^{\varepsilon -} \in H_\gamma^k(\Omega_T^-),$$
$$\underline{r}^{\varepsilon +} \in H_\gamma^k(\Omega_T^+),$$
$$\underline{r}^{\varepsilon -} \in H_\gamma^{k-\frac{3}{2}}(\Omega_T^-).$$

3.2 Stability estimates

We will begin by proving energy estimates on the following equation:

$$S_R A_R \hat{e}^\varepsilon - S_R \partial_x \hat{e}^\varepsilon + \frac{1}{\varepsilon}\mathbb{P}^- \hat{e}^\varepsilon \mathbf{1}_{x<0} = \varepsilon \hat{r}^\varepsilon, \quad \{x \in \mathbb{R}\}, \tag{3.5}$$

where $\hat{e}^\varepsilon = R\left(\mathcal{F}(e^{-\gamma t}\underline{u}^\varepsilon) - \mathcal{F}(e^{-\gamma t}\underline{u}^\varepsilon_{app})\right) := \hat{w}^\varepsilon$; with $w^\varepsilon = \underline{u}^\varepsilon - \underline{u}^\varepsilon_{app}$. Refering to (3.4), w^ε is the solution of the Cauchy problem:

$$\begin{cases} \mathcal{H}w^\varepsilon + \dfrac{1}{\varepsilon}\mathbb{M}w^\varepsilon \mathbf{1}_{x<0} = \varepsilon \underline{r}^\varepsilon, \\ w^\varepsilon|_{t<0} = 0 \quad . \end{cases} \tag{3.6}$$

For a fixed positive ε, the perturbation is nonsingular and thus the principal part of the pseudodifferential operator $\mathcal{H} + \frac{1}{\varepsilon}\mathbb{M}$ is the same as the principal part of $\mathcal{H}$. Hence, there is a unique solution of the Cauchy problem (3.6): w^ε which belongs to $H^{k-3}_\gamma(\Omega_T)$. In order to simplify the notations, in this chapter we shall denote by L^2 and H^ϱ_γ the spaces: $L^2(\Omega_T)$ and $H^\varrho_\gamma(\Omega_T)$. We recall the definition of the weighted spaces: $H^\varrho_\gamma(\Omega_T)$ for $\rho \in \mathbb{N}$.

$$H^\varrho_\gamma(\Omega_T) = \{\varpi \in e^{\gamma t}L^2(\Omega_T), \|\varpi\|_{H^\varrho_\gamma(\Omega_T)} < \infty\};$$

where

$$\|\varpi\|^2_{H^\varrho_\gamma(\Omega_T)} = \sum_{\alpha, |\alpha| \leq \varrho} \gamma^{\rho - |\alpha|} \|e^{-\gamma t}\partial^\alpha \varpi\|^2_{L^2(\Omega_T)}.$$

For fixed positive ε, there holds:

$$\int_{-\infty}^{\infty} \partial_x \langle S_R \hat{e}^\varepsilon, \hat{e}^\varepsilon \rangle_{L^2} \, dx = 0.$$

$$\Leftrightarrow \int_{-\infty}^{\infty} 2Re\langle S_R \partial_x \hat{e}^\varepsilon, \hat{e}^\varepsilon \rangle_{L^2} \, dx = 0.$$

Using the equation, we have then:

$$\int_{-\infty}^{\infty} Re\langle S_R A_R \hat{e}^\varepsilon + \frac{1}{\varepsilon}\mathbb{P}^- \hat{e}^\varepsilon - \varepsilon \hat{r}^\varepsilon, \hat{e}^\varepsilon \rangle_{L^2} \, dx = 0.$$

which is equivalent to:

$$\int_{-\infty}^{\infty} Re\langle S_R A_R \hat{e}^\varepsilon, \hat{e}^\varepsilon \rangle_{L^2} \, dx = \frac{-1}{\varepsilon} \int_{-\infty}^{\infty} Re\langle \mathbb{P}^- \hat{e}^\varepsilon \varepsilon \hat{r}^\varepsilon, \hat{e}^\varepsilon \rangle_{L^2} \, dx$$

$$+\varepsilon \int_{-\infty}^{\infty} Re\langle \hat{r}^\varepsilon, \hat{e}^\varepsilon \rangle_{L^2)} \, dx.$$

But $Re\langle S_R A_R \hat{e}^\varepsilon, \hat{e}^\varepsilon\rangle = \langle Re\,(S_R A_R)\,\hat{e}^\varepsilon, \hat{e}^\varepsilon\rangle$ and $Re\,(S_R A_R)$ is positive definite, hence there is $C > 0$, independent of ε such that:

$$C\gamma \|\hat{e}^\varepsilon\|_{L^2}^2 + \frac{1}{\varepsilon}\int_{-\infty}^{\infty} Re\langle \mathbb{P}^- \hat{e}^\varepsilon, \hat{e}^\varepsilon\rangle \leq \int_{-\infty}^{\infty} Re\langle \varepsilon \hat{r}^\varepsilon, \hat{e}^\varepsilon\rangle_{L^2}\, dx.$$

Thus, because $\mathbb{P}^-$ is an orthogonal projector, for all positive λ, there holds:

$$C\gamma \|\hat{e}^\varepsilon\|_{L^2}^2 + \frac{1}{\varepsilon}\|\mathbb{P}^- \hat{e}^\varepsilon\|_{L^2}^2 \leq \frac{1}{2}\left(\frac{\gamma}{\lambda}\|\hat{e}^\varepsilon\|_{L^2}^2 + \frac{\lambda}{\gamma}\|\varepsilon \hat{r}^\varepsilon\|_{L^2}^2\right).$$

Choosing λ big enough we have $C - \frac{\varepsilon}{2\lambda} > 0$ and the following energy estimate:

$$\left(C - \frac{\varepsilon}{2\lambda}\right)\gamma \|\hat{e}^\varepsilon\|_{L^2}^2 + \frac{1}{\varepsilon}\|\mathbb{P}^- \hat{e}^\varepsilon\|_{L^2}^2 \leq \frac{\varepsilon^2 \lambda}{2\gamma}\|\hat{r}^\varepsilon\|_{L^2}^2.$$

This shows that $\hat{e}^\varepsilon$ converges towards zero in L^2 when ε tends to zero, with a rate in $\mathcal{O}(\varepsilon)$. We recall that $\hat{e}^\varepsilon$ is given by:

$$\hat{e}^\varepsilon := R\mathcal{F}\left(e^{-\gamma t}(\underline{u}_{app}^\varepsilon - \underline{u}^\varepsilon)\right),$$

and $\hat{r}^\varepsilon$ is given by:

$$\hat{r}^\varepsilon := R\mathcal{F}\left(e^{-\gamma t}\underline{r}^\varepsilon\right).$$

Moreover, since R and $\mathbb{P}^-$ are two uniformly bounded, uniformly definite positive matrices, there are two positive real numbers α and β such that, for all $\zeta \neq 0$ and $x \in \mathbb{R}$, there holds:

- $\alpha \|\mathcal{F}\left(e^{-\gamma t}(\underline{u}_{app}^\varepsilon - \underline{u}^\varepsilon)\right)\|_{L^2}^2 \leq \|\hat{e}^\varepsilon\|_{L^2}^2$.
- $\alpha \|\mathbb{P}^- \mathcal{F}\left(e^{-\gamma t}(\underline{u}_{app}^\varepsilon - \underline{u}^\varepsilon)\right)\|_{L^2}^2 \leq \|\mathbb{P}^- \hat{e}^\varepsilon\|_{L^2}^2$.
- $\|\hat{r}^\varepsilon\|_{L^2}^2 \leq \beta \|\mathcal{F}\left(e^{-\gamma t}\underline{r}^\varepsilon\right)\|_{L^2}^2$.

Applying then Plancherel's equality we obtain then:

$$\left(C - \frac{\varepsilon}{2\lambda}\right)\gamma \|\underline{u}_{app}^\varepsilon - \underline{u}^\varepsilon\|_{e^{\gamma t}L^2}^2 + \frac{1}{\varepsilon}\|\mathbb{M}\left(\underline{u}_{app}^\varepsilon - \underline{u}^\varepsilon\right)\|_{e^{\gamma t}L^2}^2 \leq \frac{\beta}{\alpha}\frac{\varepsilon^2\lambda}{2\gamma}\|\underline{r}^\varepsilon\|_{e^{\gamma t}L^2}^2.$$

We have thus proved there are two positive constants c and C such that:

$$c\gamma \|\underline{u}_{app}^\varepsilon - \underline{u}^\varepsilon\|_{e^{\gamma t}L^2}^2 + \frac{1}{\varepsilon}\|\mathbb{M}\left(\underline{u}_{app}^\varepsilon - \underline{u}^\varepsilon\right)\|_{e^{\gamma t}L^2}^2 \leq \frac{C\varepsilon^2}{\gamma}\|\underline{r}^\varepsilon\|_{e^{\gamma t}L^2}^2.$$

Let us denote by $\|.\|_{H_\gamma^\varrho}^* := \sqrt{\|.\|_{H_\gamma^\varrho(\Omega_T^+)}^2 + \|.\|_{H_\gamma^\varrho(\Omega_T^-)}^2}$. More generally, when $\underline{r}^\varepsilon \in H^\varrho$, there is two positive constants c_ρ and C_ρ such that:

$$c_\rho \gamma \|\underline{u}_{app}^\varepsilon - \underline{u}^\varepsilon\|_{H_\gamma^\varrho}^{*2} + \frac{1}{\varepsilon}\|\mathbb{M}\left(\underline{u}_{app}^\varepsilon - \underline{u}^\varepsilon\right)\|_{H_\gamma^\varrho}^{*2} \leq \varepsilon^2 \frac{C_\rho}{\gamma}\|\underline{r}^\varepsilon\|_{H_\gamma^\varrho}^{*2}.$$

As we have seen during the construction of the profiles, $\varrho = k - 3$ in general and $\rho = k - \frac{3}{2}$ in the case where $g = 0$.

3.3 End of the proof of Theorem 1.6.

As a consequence of our stability estimate, there holds:

$$\|\underline{u}^{\varepsilon}_{app} - \underline{u}^{\varepsilon}\|^2_{H^{k-3}(\Omega_T^-)} + \|\underline{u}^{\varepsilon}_{app} - \underline{u}^{\varepsilon}\|^2_{H^{k-3}(\Omega_T^+)} = \mathcal{O}(\varepsilon^2).$$

Moreover, by construction of $\underline{u}^{\varepsilon}_{app}$, there holds:

$$\|\underline{u}^{\varepsilon}_{app} - \underline{u}^{-}\|^2_{H^{k-3}(\Omega_T^-)} + \|\underline{u}^{\varepsilon}_{app} - u\|^2_{H^{k-3}(\Omega_T^+)} = \mathcal{O}(\varepsilon^2).$$

Hence, we have proved that:

$$\|\underline{u}^{\varepsilon} - \underline{u}^{-}\|^2_{H^{k-3}(\Omega_T^-)} + \|\underline{u}^{\varepsilon} - u\|^2_{H^{k-3}(\Omega_T^+)} = \mathcal{O}(\varepsilon^2).$$

By the same arguments, if $g = 0$, there holds:

$$\|\underline{u}^{\varepsilon} - \underline{u}^{-}\|^2_{H^{k-\frac{3}{2}}(\Omega_T^-)} + \|\underline{u}^{\varepsilon} - u\|^2_{H^{k-\frac{3}{2}}(\Omega_T^+)} = \mathcal{O}(\varepsilon^2).$$

This completes the proof of Theorem 1.6.

4 Proof of Theorem 1.9.

Like in the proof of Theorem 1.6, we begin by constructing formally an approximate solution of equation (1.7). We prove then suitable energy estimates that ensures both u^{ε} and its approximate solution converges towards $\tilde{u}$ as $\varepsilon \to 0^+$.

4.1 Construction of an approximate solution.

The goal of this Lemma is to replace the boundary condition $\Gamma u|_{x=0} = \Gamma g$ of problem (1.1) by a condition of the form $\underline{\mathbf{P}}^{-}(e^{-\gamma t}u)|_{x=0} = h$ with a suitable $h \in H^k(\Upsilon_T)$.

Lemma 4.1. *Let u denote the unique solution in $H^k(\Omega_T^+)$ of the mixed hyperbolic problem (1.1), $\underline{\mathbf{P}}^{+}(\partial_t, \partial_y, \gamma)\left(e^{-\gamma t}u\right)$ does not depend of the choice of the boundary operator Γ and of g. Let us introduce the function h of $H^k(\Upsilon_T)$ defined by:*

$$\underline{\mathbf{P}}^{-}\left(e^{-\gamma t}v|_{x=0}\right) + \underline{\mathbf{\Pi}}\left(e^{-\gamma t}(g - v|_{x=0})\right).$$

The solution u of the mixed hyperbolic problem (1.1) *is the unique solution of the following well-posed mixed hyperbolic problem* (4.1)*:*

$$\begin{cases} \mathcal{H}u = f, \quad \{x > 0\}, \\ \underline{\mathbf{P}}^{-}(\partial_t, \partial_y, \gamma)\left(e^{-\gamma t} u|_{x=0}\right) = h, \\ u|_{t<0} = 0 \quad . \end{cases} \tag{4.1}$$

In addition, the mapping $(f, g) \to h$ is linear continuous from $H^k(\Omega_T^+) \times H^k(\Upsilon_T)$ to $H^k(\Upsilon_T)$.

Proof. Let v denote a solution in $H^k(\Omega_T)$ of the equation:

$$\begin{cases} \mathcal{H}v = f, \quad (t, y, x) \in \Omega_T, \\ v|_{t<0} = 0 \quad . \end{cases}$$

We introduce then $\mathbb{U}$ which is, by definition, the solution of the following mixed hyperbolic problem:

$$\begin{cases} \mathcal{H}\mathbb{U} = 0, \quad \{x > 0\}, \\ \underline{\Gamma}(\partial_t, \partial_y, \gamma)\mathbb{U}|_{x=0} = \underline{\Gamma}(\partial_t, \partial_y, \gamma)g - \underline{\Gamma}(\partial_t, \partial_y, \gamma)v|_{x=0}, \\ \mathbb{U}|_{t<0} = 0 \quad . \end{cases}$$

The right hand side of the boundary condition is, a priori, in $H^{k-\frac{1}{2}}(\Upsilon_T)$. Hence the solution $\mathbb{U}$ belongs to $H^{k-\frac{1}{2}}(\Omega_T^+)$. By construction we have:

$$u = \mathbb{U} + v. \tag{4.2}$$

Hence, since $u \in H^k(\Omega_T^+)$ and $v \in H^k(\Omega_T^+)$, in fact we have:

$$\mathbb{U} \in H^k(\Omega_T^+).$$

Let $\hat{\mathbb{U}}$ denote the Fourier-Laplace transform in (t, y) of $\mathbb{U}$ (Fourier-Laplace transform tangential to the boundary) given by: $\mathcal{F}(e^{-\gamma t}\mathbb{U})$. It satisfies the following symbolic equation:

$$\begin{cases} \partial_x \hat{\mathbb{U}} = A(\zeta)\hat{\mathbb{U}}, \quad \{x > 0\}, \\ \Gamma(\zeta)\hat{\mathbb{U}}|_{x=0} = \Gamma(\zeta)\hat{g} - \Gamma(\zeta)\hat{v}|_{x=0}, \end{cases}$$

where $\hat{g}$ and $\hat{v}$ denotes respectively the tangential Fourier-Laplace transform of g and v. Since $A(\zeta)$ is independent of x, projecting the above equation on $\mathbb{E}_+(A(\zeta))$ gives then:

$$\partial_x \mathbf{P}^+ \hat{\mathbb{U}} = A(\zeta)\mathbf{P}^+ \hat{\mathbb{U}}.$$

Moreover $\mathbf{P}^+\hat{\mathbb{U}}|_{x=0} \in \mathbb{E}_-(A(\zeta)) \bigcap \mathbb{E}_+(A(\zeta))$ since $\lim_{x\to\infty} \mathbf{P}^+\hat{\mathbb{U}} = 0$. Hence, there holds:

$$\mathbf{P}^+\hat{\mathbb{U}} = 0,$$

and thus

$$\hat{\mathbb{U}} = \mathbf{P}^-\hat{\mathbb{U}}.$$

The boundary condition:

$$\Gamma(\zeta)\hat{\mathbb{U}}|_{x=0} = \Gamma(\zeta)\hat{g} - \Gamma(\zeta)\hat{v}|_{x=0}$$

is equivalent to:

$$\hat{\mathbb{U}}|_{x=0} \in \hat{g} - \hat{v}|_{x=0} + \mathrm{Ker}\Gamma.$$

We have thus:

$$\mathbf{P}^-\hat{\mathbb{U}}|_{x=0} \in \hat{g} - \hat{v}|_{x=0} + \ker\Gamma.$$

Let us denote by $\mathbf{\Pi}$ the projector on $\tilde{\mathbb{E}}_-(A)$ parallel to $\ker\Gamma$, which has a sense because the Uniform Lopatinski Condition holds.
Since $\hat{\mathbb{U}}|_{x=0} \in \tilde{\mathbb{E}}_-(A)$, and of the Uniform Lopatinski Condition, the above boundary condition is equivalent to:

$$\mathbf{\Pi}\mathbf{P}^-\hat{\mathbb{U}}|_{x=0} = \mathbf{\Pi}(\hat{g} - \hat{v}|_{x=0}),$$

and thus, because $\mathbf{P}^-\hat{\mathbb{U}}|_{x=0}$ belongs to $\mathbb{E}_-(A)$, we have:

$$\mathbf{P}^-\hat{\mathbb{U}}|_{x=0} = \mathbf{\Pi}(\hat{g} - \hat{v}|_{x=0}).$$

As a consequence, we obtain:

$$\mathbf{P}^-\hat{u}|_{x=0} = \mathbf{P}^-\hat{v}|_{x=0} + \mathbf{\Pi}(\hat{g} - \hat{v}|_{x=0}).$$

Hence, there holds:

$$\underline{\mathbf{P}}^-\left(e^{-\gamma t}u|_{x=0}\right) = \underline{\mathbf{P}}^-\left(e^{-\gamma t}v|_{x=0}\right) + \underline{\mathbf{\Pi}}\left(e^{-\gamma t}(g - v|_{x=0})\right) := h.$$

$\underline{\mathbf{P}}^+(\partial_t, \partial_y, \gamma)\left(e^{-\gamma t}u\right) = \underline{\mathbf{P}}^+(\partial_t, \partial_y, \gamma)\left(e^{-\gamma t}v\right)$, thus it does not depend of the choice of the boundary operator Γ and of g. Moreover, since $u|_{x=0} \in H^k(\Upsilon_T)$, it follows that
$g \in H^k(\Upsilon_T)$. Now, since the Uniform Lopatinski Condition holds, u satisfies the following energy estimate:

$$\frac{1}{\gamma}\|u\|^2_{e^{\gamma t}L^2(\Omega_T^+)} + \|u|_{x=0}\|^2_{e^{\gamma t}L^2(\Upsilon_T)} \leq \gamma\|f\|^2_{e^{\gamma t}L^2(\Omega_T^+)} + \|g\|_{e^{\gamma t}L^2(\Upsilon_T)},$$

More generally, we have:

$$\frac{1}{\gamma}\|u\|^2_{H^k_\gamma(\Omega^+_T)} + \|u|_{x=0}\|^2_{H^k_\gamma(\Upsilon_T)} \leq \gamma\|f\|^2_{H^k_\gamma(\Omega^+_T)} + \|g\|_{H^k_\gamma(\Upsilon_T)}.$$

where $\|\varpi\|^2_{H^k_\gamma} := \sum_{|\alpha|=0}^k \gamma^{k-|\alpha|}\|\partial^\alpha \varpi\|^2_{e^{\gamma t}L^2}$.
$h = \underline{\mathbf{P}}^-(e^{-\gamma t}u|_{x=0})$ hence

$$\|h\|^2_{L^2(\Upsilon_T)} \leq C\|e^{-\gamma t}u|_{x=0}\|^2_{L^2(\Upsilon_T)} = C\|u|_{x=0}\|^2_{e^{\gamma t}L^2(\Upsilon_T)};$$

and for $0 \leq j \leq d-1$, there holds:

$$\|\partial_j h\|^2_{L^2(\Upsilon_T)} \leq c_j\|\eta_j \mathbf{P}^-\mathcal{F}(e^{-\gamma t}u)|_{x=0}\| \leq c'_j\|u|_{x=0}\|_{H^1_\gamma(\Upsilon_T)}.$$

More generally, we have:

$$\|h\|^2_{H^k_\gamma(\Upsilon_T)} \leq C_k\gamma\|f\|^2_{H^k_\gamma(\Omega^+_T)} + C_k\|g\|^2_{H^k_\gamma(\Upsilon_T)}.$$

But γ is a positive real number fixed once and for all at the beginning of the paper, hence this proves that the mapping $(f,g) \to h$ is continuous from $H^k(\Omega^+_T) \times H^k(\Upsilon_T)$ to $H^k(\Upsilon_T)$. □

As we will see, Lemma 4.1 is central in our construction of an approximate solution. We will construct an approximate solution

$$u^\varepsilon_{app} := u^{\varepsilon+}_{app}\mathbf{1}_{x>0} + u^{\varepsilon-}_{app}\mathbf{1}_{x<0},$$

along the following ansatz:

$$u^{\varepsilon+}_{app} := \sum_{j=0}^M \varepsilon^j u^+_j(t,y,x),$$

with $u^+_j \in H^{k-\frac{3}{2}j}_\gamma(\Omega^+_T)$, $u^+_j|_{x=0} \in H^{k-\frac{3}{2}j}_\gamma(\Upsilon_T)$; and

$$u^{\varepsilon-}_{app} := \sum_{j=0}^M \varepsilon^j u^-_j(t,y,x),$$

with $u^-_j \in H^{k-\frac{3}{2}j}_\gamma(\Omega^-_T)$, $u^-_j|_{x=0} \in H^{k-\frac{3}{2}j}_\gamma(\Upsilon_T)$. As usual, we will refer to the terms $u^\pm_j$ as profiles. We will rather work on the reformulation of problem (1.7) as the transmission problem (4.3):

$$\begin{cases} \mathcal{H}u^{\varepsilon+} = f, \quad \{x>0\}, \\ \mathcal{H}u^{\varepsilon-} + \dfrac{1}{\varepsilon}A_d e^{\gamma t}\underline{\mathbf{P}}^- e^{-\gamma t}u^{\varepsilon-} = \dfrac{1}{\varepsilon}A_d e^{\gamma t}\tilde{h}, \quad \{x<0\}, \\ u^{\varepsilon+}|_{x=0} - u^{\varepsilon-}|_{x=0} = 0, \\ u^{\varepsilon\pm}|_{t<0} = 0 \quad . \end{cases} \tag{4.3}$$

Plugging $u_{app}^{\varepsilon+}$ and $u_{app}^{\varepsilon-}$ in (4.3) and identifying the terms with same power in ε, we obtain the following profiles equations:

- Identification of the terms of order ε^{-1} :

$$A_d e^{\gamma t}\underline{\mathbf{P}}^- e^{-\gamma t} u_0^- = A_d e^{\gamma t}\tilde{h}, \quad \{x<0\}. \tag{4.4}$$

- Identification of the terms of order ε^{0} :

$$\mathcal{H}u_0^- + A_d e^{\gamma t}\underline{\mathbf{P}}^- e^{-\gamma t} u_1^- = 0, \quad \{x<0\}. \tag{4.5}$$

$$\mathcal{H}u_0^+ = f, \quad \{x>0\}, \tag{4.6}$$

- Identification of the terms of order ε^{j} for $j \geq 1$:

$$\mathcal{H}u_j^- + A_d e^{\gamma t}\underline{\mathbf{P}}^- e^{-\gamma t} u_{j+1}^- = 0, \quad \{x<0\}. \tag{4.7}$$

$$\mathcal{H}u_j^+ = 0, \quad \{x>0\}, \tag{4.8}$$

- Translation of the continuity condition over the boundary on the profiles:

  For all $1 \leq j \leq M$, there holds:

$$u_j^+|_{x=0} - u_j^-|_{x=0} = 0. \tag{4.9}$$

Denote by $\hat{u}_j^{\pm} := \mathcal{F}(e^{-\gamma t}u_j^{\pm})$. We have then:

$$u_j^{\pm} := e^{\gamma t}\mathcal{F}^{-1}(\hat{u}_j^{\pm}).$$

We will now give the equations satisfied by the Fourier-Laplace transform of the profiles: $\hat{u}_j^{\pm}$. First, equation (4.4) is equivalent to:

$$\mathbf{P}^- \hat{u}_0^- = \mathcal{F}(\tilde{h}), \quad \{x<0\}.$$

We deduce from this equation that there holds:

$$\mathbf{P}^{-}\hat{u}_0^{-}|_{x=0} = \hat{h}.$$

Then, using (4.9) for $j = 0$, and (4.6) gives that, for γ big enough,

$$u_0^{+} = \mathcal{F}(e^{-\gamma t}\hat{u}_0^{+}),$$

where $\hat{u}_0^{+}$ is the solution of the well-posed first order ODE in x:

$$\begin{cases} \partial_x \hat{u}_0^{+} - A\hat{u}_0^{+} = \mathcal{F}(e^{-\gamma t}(A_d)^{-1} f), \quad \{x > 0\}, \\ \mathbf{P}^{-}\hat{u}_0^{+}|_{x=0} = h, \end{cases}$$

Thus u_0^{+} is solution of:

$$\begin{cases} \mathcal{H}u_0^{+} = f, \quad \{x > 0\}, \\ e^{\gamma t}\underline{\mathbf{P}}^{-} e^{-\gamma t} u_0^{+}|_{x=0} = h. \end{cases}$$

Thanks to Lemma 4.1, we recognize u_0^{+} as the solution of our starting mixed hyperbolic problem (1.1). Once u_0^{+} is known, so is $\hat{u}_0^{+}$ and thus $\hat{u}_0^{-}|_{x=0}$ is given by:

$$\hat{u}_0^{-}|_{x=0} = \hat{u}_0^{+}|_{x=0}.$$

Moreover,

$$u_0^{+}|_{x=0} = u_0^{-}|_{x=0} \in H^k_\gamma(\Upsilon_T).$$

By (4.5), there holds:

$$\partial_x \hat{u}_0^{-} - A\hat{u}_0^{-} + \mathbf{P}^{-}\hat{u}_1^{-} = 0, \quad \{x < 0\}.$$

As a consequence, $\mathbf{P}^{+}\hat{u}_0^{-}$ is given by the well-posed ODE:

$$\begin{cases} \partial_x(\mathbf{P}^{+}\hat{u}_0^{-}) - A(\mathbf{P}^{+}\hat{u}_0^{-}) = 0, \quad \{x < 0\}, \\ \mathbf{P}^{+}\hat{u}_0^{-}|_{x=0} = \mathbf{P}^{+}\hat{u}_0^{+}|_{x=0}. \end{cases}$$

Indeed, since $\ker \mathbf{P}^{+}(\zeta) = \mathbb{E}_{-}(A(\zeta))$, this problem satisfies the Uniform Lopatinski Condition: for all $\zeta \neq 0$, there holds:

$$\mathbb{E}_{-}(A(\zeta)) \bigoplus \mathbb{E}_{+}(A(\zeta)) = \mathbb{C}^N.$$

For γ big enough, by linearity of the inverse Fourier transform, u_0^{-} can then be computed by:

$$u_0^{-} := e^{\gamma t}\mathcal{F}^{-1}(\mathbf{P}^{-}\hat{u}_0^{-}) + e^{\gamma t}\mathcal{F}^{-1}(\mathbf{P}^{+}\hat{u}_0^{-}).$$

Following up with that process of construction, we can go on with the construction of the profiles at any order. Indeed, assume that all the profiles (u_j^+, u_j^-) up to order j have been computed. Then consider the equation obtained through identification:

$$\mathbf{P}^- \hat{u}_{j+1}^- = -\partial_x \hat{u}_j^- + A\hat{u}_j^-, \quad \{x < 0\}.$$

We see there is a loss of regularity between $\hat{u}_{j+1}^-$ and $\hat{u}_j^-$.
Let us say that $u_j^\pm \in H_\gamma^{m_j}(\Omega_T^\pm)$. Considering the traces, we have then: $u_j^\pm|_{x=0} \in H_\gamma^{m_j-\frac{1}{2}}(\Upsilon_T)$. We will show in this part how the Sobolev regularity of the profiles $u_{j+1}^\pm$, which is by definition m_{j+1}, can be computed knowing m_j. To begin with $\underline{\mathbf{P}}^- u_{j+1}^-$ belongs to $H_\gamma^{m_j-1}(\Omega_T^-)$. $\underline{\mathbf{P}}^- u_{j+1}^+|_{x=0}$, which belongs to $H_\gamma^{m_j-\frac{3}{2}}(\Upsilon_T)$, is known by $\underline{\mathbf{P}}^- u_{j+1}^+|_{x=0} = e^{\gamma t}\mathcal{F}^{-1}(\mathbf{P}^- \hat{u}_{j+1}^+|_{x=0})$, with:

$$\mathbf{P}^- \hat{u}_{j+1}^+|_{x=0} = \mathbf{P}^- \hat{u}_{j+1}^-|_{x=0}.$$

Hence, $\hat{u}_{j+1}^+ := \mathcal{F}(e^{-\gamma t} u_{j+1}^+)$ is the solution of the first order ODE in x :

$$\begin{cases} \partial_x \hat{u}_{j+1}^+ - A\hat{u}_{j+1}^+ = 0, \quad \{x > 0\}, \\ \mathbf{P}^- \hat{u}_{j+1}^+|_{x=0} = \mathbf{P}^- \hat{u}_{j+1}^-|_{x=0}. \end{cases}$$

Since $\ker \mathbf{P}^-(\zeta) = \mathbb{E}_+(A(\zeta))$, this problem satisfies the Uniform Lopatinski Condition: for all $\zeta \neq 0$, there holds:

$$\mathbb{E}_-(A(\zeta)) \bigoplus \mathbb{E}_+(A(\zeta)) = \mathbb{C}^N.$$

As a consequence, this problem is well-posed and, $u_{j+1}^+ \in H_\gamma^{m_j-\frac{3}{2}}(\Omega_T^+)$. Moreover, there holds:

$$u_{j+1}^+|_{x=0} = u_{j+1}^-|_{x=0} \in H_\gamma^{m_j-\frac{3}{2}}(\Upsilon_T).$$

Indeed, $\mathbf{P}^+ \hat{u}_{j+1}^+ \in H^\infty(\mathbb{R}_+^{d+1})$ hence $\underline{\mathbf{P}}^+ u_{j+1}^+|_{x=0} \in H_\gamma^{m_j-\frac{3}{2}}(\Upsilon_T)$ and thus $u_{j+1}^+|_{x=0} \in H_\gamma^{m_j-\frac{3}{2}}(\Upsilon_T)$. Furthermore, we have:

$$u_{j+1}^-|_{x=0} = u_{j+1}^+|_{x=0}.$$

Applying $\mathbf{P}^+$ on the following equation:

$$\mathbf{P}^- \hat{u}_{j+2}^- = -\partial_x \hat{u}_{j+1}^- + A\hat{u}_{j+1}^-, \quad \{x < 0\};$$

we obtain then the equation:

$$\partial_x(\mathbf{P}^+ \hat{u}_{j+1}^-) - A\mathbf{P}^+ \hat{u}_{j+1}^- = 0, \quad \{x < 0\}.$$

Remark 4.2.

$$\mathbf{P}^- \hat{u}^-_{j+2} = -\partial_x \hat{u}^-_{j+1} + A\hat{u}^-_{j+1}, \quad \{x<0\}.$$

shows that the "Fourier profile" $\hat{u}^-_{j+1}$ *must be so that* $-\partial_x \hat{u}^-_{j+1} + A\hat{u}^-_{j+1}$ *is polarized on* $\mathbb{E}_-(A)$*. It is indeed the case because we search for* $\hat{u}^-_{j+1}$ *satisfying:*

$$\partial_x(\mathbf{P}^+\hat{u}^-_{j+1}) - A\mathbf{P}^+\hat{u}^-_{j+1} = 0, \quad \{x<0\}.$$

u^-_{j+1} is given by:

$$u^-_{j+1} := e^{\gamma t}\mathcal{F}^{-1}(\mathbf{P}^-\hat{u}^-_{j+1}) + e^{\gamma t}\mathcal{F}^{-1}(\mathbf{P}^+\hat{u}^-_{j+1}).$$

with $\underline{\mathbf{P}}^+ u^-_{j+1} = e^{\gamma t}\mathcal{F}^{-1}(\mathbf{P}^+\hat{u}^-_{j+1})$ belongs to $H^{m_j-\frac{3}{2}}_\gamma(\Omega^+_T)$ and is the unique solution of the well-posed first order ODE:

$$\begin{cases} \partial_x(\mathbf{P}^+\hat{u}^-_{j+1}) - A(\mathbf{P}^+\hat{u}^-_{j+1}) = 0, \quad \{x<0\}, \\ \mathbf{P}^+\hat{u}^-_{j+1}|_{x=0} = \mathbf{P}^+\hat{u}^+_{j+1}|_{x=0}. \end{cases}$$

The profile u^-_{j+1} belongs to $H^{m_j-\frac{3}{2}}_\gamma(\Omega^-_T)$. This achieves to show that the knowledge of (u^+_j, u^-_j), allows us to compute (u^+_{j+1}, u^-_{j+1}).
Moreover $m_{j+1} = m_j - \frac{3}{2}$, that is to say that a construction of each supplementary profile consummate $\frac{3}{2}$ of Sobolev regularity. In practice, we take:

$$u^{\varepsilon+}_{app} = u^+_0,$$

$$u^{\varepsilon-}_{app} = u^-_0 + \varepsilon u^-_1.$$

As a result, the approximate solution writes $u^\varepsilon_{app} := u^{\varepsilon+}_{app}\mathbf{1}_{x>0} + u^{\varepsilon-}_{app}\mathbf{1}_{x<0}$; where $u^{\varepsilon+}_{app}$ belongs to $H^k_\gamma(\Omega^+_T)$ and $u^{\varepsilon-}_{app}$ belongs to $H^{k-\frac{3}{2}}_\gamma(\Omega^-_T)$. The so defined u^ε_{app} is solution of a well-posed problem of the form:

$$(4.10) \quad \begin{cases} \mathcal{H}u^\varepsilon_{app} + \frac{1}{\varepsilon}A_d e^{\gamma t}\underline{\mathbf{P}}^- e^{-\gamma t}u^\varepsilon_{app}\mathbf{1}_{x<0} = f\mathbf{1}_{x>0} + \frac{1}{\varepsilon}A_d e^{\gamma t}\tilde{h}\mathbf{1}_{x<0} + \varepsilon r^\varepsilon, \\ u^\varepsilon_{app}|_{t<0} = 0 \quad . \end{cases}$$

Where $r^\varepsilon := r^{\varepsilon+}\mathbf{1}_{x>0} + r^{\varepsilon-}\mathbf{1}_{x<0}$, with $r^{\varepsilon+} \in H^{k-\frac{5}{2}}_\gamma(\Omega^+_T)$ and $r^{\varepsilon-} \in H^{k-3}_\gamma(\Omega^-_T)$.

4.2 Asymptotic Stability of the problem as ε tends towards zero.

Denote by $v^\varepsilon = u^\varepsilon_{app} - u^\varepsilon$. By construction of u^ε_{app}, v^ε is solution of the following Cauchy problem:

$$\begin{cases} \mathcal{H}v^\varepsilon + \dfrac{1}{\varepsilon} A_d e^{\gamma t} \underline{\mathbf{P}}^- e^{-\gamma t} v^\varepsilon \mathbf{1}_{x<0} = \varepsilon r^\varepsilon, \\ v^\varepsilon|_{t<0} = 0 \quad . \end{cases} \tag{4.11}$$

For all positive ε, this problem is well-posed. In order to investigate the stability of this problem as ε goes to zero, we will reformulate it as a transmission problem. The restrictions of v^ε to $\{x > 0\}$ and $\{x < 0\}$, respectively denoted by $v^{\varepsilon+}$ and $v^{\varepsilon-}$ are solution the following transmission problem:

$$\begin{cases} \mathcal{H}v^{\varepsilon+} = \varepsilon r^{\varepsilon+}, \quad \{x > 0\}, \\ \mathcal{H}v^{\varepsilon-} + \dfrac{1}{\varepsilon} A_d e^{\gamma t} \underline{\mathbf{P}}^- e^{-\gamma t} v^{\varepsilon-} = \varepsilon r^{\varepsilon-}, \quad \{x < 0\}, \\ v^{\varepsilon+}|_{x=0} - v^{\varepsilon-}|_{x=0} = 0, \\ v^{\varepsilon\pm}|_{t<0} = 0 \quad . \end{cases} \tag{4.12}$$

Let us denote by V^ε the function, valued in $\mathbb{R}^{2N}$, defined for all $\{x > 0\}$ and $(t, y) \in [0, T] \times \mathbb{R}^{d-1}$ by:

$$V^\varepsilon(t, y, x) = \begin{pmatrix} V^{\varepsilon+}(t, y, x) \\ V^{\varepsilon-}(t, y, -x) \end{pmatrix}.$$

v^ε is solution of the Cauchy problem (4.11) iff V^ε is solution of the mixed hyperbolic problem on a half space (4.13) given below:

$$\begin{cases} \tilde{\mathcal{H}}V^\varepsilon + B^\varepsilon V^\varepsilon = \varepsilon R^\varepsilon, \quad \{x > 0\}, \\ \tilde{\Gamma} V^\varepsilon|_{x=0} = 0, \\ V^\varepsilon|_{t<0} = 0 \quad , \end{cases} \tag{4.13}$$

where

$$\tilde{\mathcal{H}} = \partial_t + \sum_{j=1}^{d-1} \begin{pmatrix} A_j & 0 \\ 0 & A_j \end{pmatrix} \partial_j + \begin{pmatrix} A_d & 0 \\ 0 & -A_d \end{pmatrix} \partial_x,$$

$$B^\varepsilon = \begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{\varepsilon} A_d e^{\gamma t} \underline{\mathbf{P}}^- e^{-\gamma t} \end{pmatrix},$$

$$R^\varepsilon(t, y, x) = \begin{pmatrix} r^{\varepsilon+}(t, y, x) \\ r^{\varepsilon-}(t, y, -x) \end{pmatrix},$$

and

$$\tilde{\Gamma} = \begin{pmatrix} Id & -Id \end{pmatrix}.$$

Returning to the construction of our approximate solution, we have $R^\varepsilon \in H_\gamma^{k-\frac{5}{2}}(\Omega_T^+) \times H_\gamma^{k-3}(\Omega_T^+)$ and is such that $R^\varepsilon|_{t<0} = 0$.
In fact $R^\varepsilon \in H_\gamma^{k'}(\Omega_T^+)$ with $k' = k-3$. For all positive ε, there exists a unique solution V^ε in $H_\gamma^k(\Omega_T^+)$ to the above problem. We will prove here that this solution converges, uniformly in ε, towards 0 in $H_\gamma^{k'}(\Omega_T^+)$, as ε vanishes. As in the proof of Kreiss Theorem, see [3] for instance, existence of solution for mixed hyperbolic systems like (1.7) or (4.13), are obtained through the proof of both direct and "dual" a priori estimates on an adjoint problem. This estimates results in the constant coefficient case of estimates on the Fourier-Laplace transform of the solution. Additionally, if this "Fourier" estimate can be proved, both direct and "dual" energy estimates are deduced from it. In a first step, let us recall formally how to conduct the Fourier-Laplace transform of a mixed hyperbolic problem:

$$\begin{cases} \mathcal{H}u = f, \quad \{x > 0\}, \\ \Gamma u|_{x=0} = g, \\ u|_{t<0} = 0 \quad , \end{cases}$$

Denote by $u_* := e^{-\gamma t}u$, u_* is in particular a solution of the following problem:

$$\begin{cases} \mathcal{H}u_* + \gamma u_* = e^{-\gamma t} f, \quad \{x > 0\}, \\ \Gamma u_*|_{x=0} = g \quad . \end{cases}$$

We take then the tangential (with respect to (t,y)) Fourier transform of this equation, which gives:

$$\begin{cases} A_d \partial_x \hat{u}_* + (\gamma + i\tau)\hat{u}_* + i\eta_j \displaystyle\sum_{j=1}^{d-1} A_j \hat{u}_* = \mathcal{F}\left(e^{-\gamma t} f\right), \quad \{x > 0\}, \\ \Gamma \hat{u}_*|_{x=0} = \hat{g} \quad . \end{cases}$$

Multiplying this equation by A_d^{-1}, we obtain that u^* is solution of the following ODE in x:

$$\begin{cases} \partial_x \hat{u}_* - A\hat{u}_* = (A_d)^{-1} \mathcal{F}\left(e^{-\gamma t} f\right), \quad \{x > 0\}, \\ \Gamma \hat{u}_*|_{x=0} = \hat{g} \quad . \end{cases}$$

Note that, $\hat{u}_*$ and u can be freely deduced from each other through the formulas:

$$\hat{u}_* = \mathcal{F}(e^{-\gamma t} u)$$

and

$$u = e^{\gamma t} \mathcal{F}^{-1}(\hat{u}_*).$$

We shall now introduce a rescaled solution $\underline{V}^\varepsilon$ of the solution V^ε of (4.13) defined as follows: $\underline{V}^\varepsilon(t,y,x) := V^\varepsilon(t,y,\varepsilon x)$, and the rescaled remainder: $\underline{R}^\varepsilon(t,y,x) := R^\varepsilon(t,y,\varepsilon x)$. Denoting by $\hat{\underline{V}}^\varepsilon = \mathcal{F}(e^{-\gamma t}\underline{V})$, the associated equation writes then:

$$\begin{cases} \partial_x \hat{\underline{V}}^\varepsilon - \varepsilon \tilde{A} \hat{\underline{V}}^\varepsilon + M \hat{\underline{V}}^\varepsilon = \varepsilon^2 \hat{R}^\varepsilon, \quad \{x > 0\}, \\ \tilde{\Gamma} \hat{\underline{V}}^\varepsilon|_{x=0} = 0 \quad . \end{cases}$$

where

$$M(\zeta) = \begin{pmatrix} 0 & 0 \\ 0 & \mathbf{P}^-(\zeta) \end{pmatrix}.$$

We remark that

$$\varepsilon \tilde{A}(\zeta) = \tilde{A}(\varepsilon \zeta) = \tilde{A}(\hat{\zeta}),$$

with $\hat{\zeta} = (\hat{\tau}, \hat{\gamma}, \hat{\eta}) := \varepsilon \zeta$. Moreover $\mathbf{P}^-$ is homogeneous of order zero in ζ. Let us denote $\tilde{R}^\varepsilon(\hat{\zeta}, x) := \hat{R}^\varepsilon(\zeta, x)$ and $\widetilde{\underline{V}}^\varepsilon(\hat{\zeta}, x) := \hat{\underline{V}}^\varepsilon(\zeta, x)$. Hence $\widetilde{\underline{V}}^\varepsilon$ is solution of the following problem:

$$\begin{cases} \partial_x \widetilde{\underline{V}}^\varepsilon + \left[-\tilde{A}(\hat{\zeta}) + M(\hat{\zeta}) \right] \widetilde{\underline{V}}^\varepsilon = \varepsilon^2 \tilde{R}^\varepsilon(\hat{\zeta}, x), \quad \{x > 0\}, \\ \tilde{\Gamma} \widetilde{\underline{V}}^\varepsilon|_{x=0} = 0 \quad . \end{cases}$$

As a consequence, the Uniform Lopatinski Condition for problem (4.13) writes: For all $\hat{\gamma} > 0$,

$$|det(\mathbb{E}_-(\tilde{A}(\hat{\zeta}) - M(\hat{\zeta}), \ker \Gamma)| \geq C > 0.$$

In view of the proof of the Proposition (4.3), we recall that the spaces $\mathbb{E}_\pm(A)$ have to be considered in the extended sense defined above.

Proposition 4.3. *Since $\mathcal{H}$ satisfies the hyperbolicity Assumption in Assumption 1.1, the Uniform Lopatinski Condition is satisfied for our present problem; that is to say that, for all $\hat{\zeta}$ such that $\hat{\gamma} > 0$ there holds:*

$$|det(\mathbb{E}_-(\tilde{A}(\hat{\zeta}) - M(\hat{\zeta}), \ker \Gamma)| \geq C > 0.$$

Proof. We will begin to show that the Uniform Lopatinski Condition writes as well that for all $\hat{\zeta} \neq 0$ there holds:

$$\mathbb{E}_+(A(\hat{\zeta}) - \mathbf{P}^-(\hat{\zeta})) \bigcap \mathbb{E}_-(A(\hat{\zeta})) = \{0\} \quad . \tag{4.14}$$

This notation keeps a sense for $\hat{\zeta}$ such that $\hat{\gamma} = 0$ because we will prove a posteriori that the involved linear subspaces continuously extends from $\{\hat{\zeta}, \hat{\gamma} > 0\}$ to $\{\hat{\zeta}, \hat{\gamma} = 0\}$. Then we will prove that, for all $\hat{\zeta}$, the property 4.14 holds true. The Uniform Lopatinski Condition writes actually, for all $\hat{\zeta} \neq 0$:

$$\mathbb{E}_-(\tilde{A}(\hat{\zeta}) - M(\hat{\zeta})) \bigcap \ker \tilde{\Gamma} = \{0\}.$$

and thus, since we have:

$$\mathbb{E}_-(\tilde{A}(\hat{\zeta}) - M(\hat{\zeta})) = \mathbb{E}_-(A(\hat{\zeta})) \times \mathbb{E}_+(A(\hat{\zeta}) - \mathbf{P}^-(\hat{\zeta})),$$

and by definition of $\tilde{\Gamma}$, the Uniform Lopatinski Condition writes then that, for all $\hat{\zeta} \neq 0$, there holds:

$$\mathbb{E}_+(A(\hat{\zeta}) - \mathbf{P}^-(\hat{\zeta})) \bigcap \mathbb{E}_-(A(\hat{\zeta})) = \{0\}.$$

Lemma 4.4.

$$\mathbb{E}_-\left(A(\hat{\zeta}) - \mathbf{P}^-(\hat{\zeta})\right) = \mathbb{E}_-\left(A(\hat{\zeta})\right),$$
$$\mathbb{E}_+\left(A(\hat{\zeta}) - \mathbf{P}^-(\hat{\zeta})\right) = \mathbb{E}_+\left(A(\hat{\zeta})\right).$$

Proof. For all $\hat{\zeta} \neq 0$, there is an invertible $N \times N$ matrix with complex coefficients $P(\hat{\zeta})$ such that: $P^{-1}AP$ is trigonal and the diagonal coefficients are sorted by increasing order of their real parts. Let us denote by ν the dimension of $\mathbb{E}_-(A)$. The above matrix P traduces the change of basis from the canonical basis of $\mathbb{C}^N$ into $(v_1, \dots, v_\nu, v_{\nu+1}, \dots, v_N)$, where

$$Span\left((v_k)_{1 \leq k \leq \nu}\right) = \mathbb{E}_-(A),$$

and

$$Span\left((v_k)_{\nu+1 \leq k \leq N}\right) = \mathbb{E}_+(A).$$

Moreover, there holds

$$P^{-1}\mathbf{P}^-P = D$$

where D is the diagonal matrix whose ν first diagonal terms are equal to 1 and the $N - \nu$ last diagonal terms are null.

$$P^{-1}(A - \mathbf{P}^-)P = P^{-1}AP - D.$$

$P^{-1}AP - D$ is also trigonal, with the same eigenvalues with positive real part as $P^{-1}AP$ and the same eigenvalues with negative real part as $P^{-1}AP - Id$. As a consequence, for all $\hat{\zeta} \neq 0$, there holds:

$$\mathbb{E}_- \left(A(\hat{\zeta}) - \mathbf{P}^-(\hat{\zeta}) \right) = \mathbb{E}_- \left(A(\hat{\zeta}) \right),$$

$$\mathbb{E}_+ \left(A(\hat{\zeta}) - \mathbf{P}^-(\hat{\zeta}) \right) = \mathbb{E}_+ \left(A(\hat{\zeta}) \right).$$

□

As a consequence of Lemma 4.4, the rescaled Uniform Lopatinski Condition for $\varepsilon > 0, \varepsilon \to 0$ happens to be exactly the same as the one written for bigger positive ε. Indeed, it writes, for all $\hat{\zeta} \neq 0$:

$$\mathbb{E}_+(A(\hat{\zeta})) \bigcap \mathbb{E}_-(A(\hat{\zeta})) = \{0\}.$$

□ The Lopatinski condition is satisfied, and, as a result, the following, uniform in ε, energy estimate holds for $\underline{V}^\varepsilon$, for all $\gamma \geq \gamma_{k'} > 0$:

$$\gamma \|\underline{V}^\varepsilon\|^2_{H^{k'}_\gamma(\Omega^+_T)} + \|\underline{V}^\varepsilon|^2_{x=0}\|_{H^{k'}_\gamma(\Upsilon_T)} \leq \frac{C}{\gamma} \|\varepsilon \underline{R}^\varepsilon\|^2_{H^{k'}_\gamma(\Omega^+_T)} \quad ;$$

which is equivalent to:

$$\gamma \|V^\varepsilon\|^2_{H^{k'}_\gamma(\Omega^+_T)} + \|V^\varepsilon|_{x=0}\|^2_{H^{k'}_\gamma(\Upsilon_T)} \leq \frac{C}{\gamma} \|\varepsilon R^\varepsilon\|^2_{H^{k'}_\gamma(\Omega^+_T)} \quad . \tag{4.15}$$

This proves the convergence of V^ε towards zero in $H^{k'}_\gamma(\Omega^+_T)$. The weight γ is fixed beforehand thus, in fact, the solution of (4.13) tends to zero in $H^{k'}(\Omega^+_T)$ at a rate at least in $\mathcal{O}(\varepsilon)$.

5 End of proof of Theorem 1.9.

Let us consider V^ε defined by:

$$V^\varepsilon(t, y, x) := \begin{pmatrix} u^{\varepsilon+}_{app}(t, y, x) - u^{\varepsilon+}(t, y, x) \\ u^{\varepsilon-}_{app}(t, y, -x) - u^{\varepsilon-}(t, y, -x) \end{pmatrix}.$$

This notation is perfectly fine because the so-defined function is solution of an equation of the form (4.13). Moreover, thanks to the stability estimate (4.15), there is γ_k positive such that, for all $\gamma > \gamma_k$, we have:

$$\gamma \|u^\varepsilon_{app} - u^\varepsilon\|^2_{H^{k-3}_\gamma(\Omega^+_T)} + \gamma \|u^\varepsilon_{app} - u^\varepsilon\|^2_{H^{k-3}_\gamma(\Omega^-_T)} + \|u^\varepsilon_{app} - u^\varepsilon\|^2_{H^{k-3}_\gamma(\Upsilon_T)} \leq \frac{C}{\gamma} \|\varepsilon R^{\varepsilon+}\|^2_{H^{k-3}_\gamma(\Omega^+_T)}.$$

Hence, it follows that:

$$\|u^\varepsilon_{app} - u^\varepsilon\|^2_{H^{k-3}(\Omega_T^+)} + \|u^\varepsilon_{app} - u^\varepsilon\|^2_{H^{k-3}(\Omega_T^-)} = \mathcal{O}(\varepsilon^2).$$

Moreover, by construction of u^ε_{app}, we have:

$$\|u^\varepsilon_{app} - u\|^2_{H^{k-3}(\Omega_T^+)} + \|u^\varepsilon_{app} - u^-\|^2_{H^{k-3}(\Omega_T^-)} = \mathcal{O}(\varepsilon^2).$$

As a result, we obtain that there holds:

$$\|u^\varepsilon - u\|^2_{H^{k-3}(\Omega_T^+)} + \|u^\varepsilon - u^-\|^2_{H^{k-3}(\Omega_T^-)} = \mathcal{O}(\varepsilon^2).$$

This concludes the proof of Theorem 1.9.

6 Appendix: answer to a question asked in [11].

In this chapter, we will show that the loss of convergence observed numerically in [11] in a neighborhood of the boundary is due to a boundary layer phenomenon. We consider the 1-D wave equation:

$$(6.1)\qquad \begin{cases} \partial_{tt}U - c^2\partial_{xx}U = 0, \quad (x,t) \in]0,\pi[\times\mathbb{R}^+, \\ U|_{x=0} = U|_{x=\pi} = 0, \\ U|_{t=0}(x) = sin(x), \\ \partial_t U|_{t=0} = 0. \end{cases}$$

As in [11], we define then $U^\varepsilon = U^{\varepsilon+}\mathbf{1}_{x>0} + U^{\varepsilon-}\mathbf{1}_{x<0}$ by:

$$(6.2)\qquad \begin{cases} \partial_{tt}U^{\varepsilon+} - c^2\partial_{xx}U^{\varepsilon+} = 0, \quad (x,t) \in]0,\pi[\times\mathbb{R}^+, \\ \partial_{tt}U^{\varepsilon-} - c^2\partial_{xx}U^{\varepsilon-} + \dfrac{1}{\varepsilon^2}U^{\varepsilon-} = 0, \quad (x,t) \in]-\infty,0[\times\mathbb{R}^+, \\ U^{\varepsilon+}|_{x=0} - U^{\varepsilon-}|_{x=0} = 0 \\ \partial_x U^{\varepsilon+}|_{x=0} - \partial_x U^{\varepsilon-}|_{x=0} = 0 \\ U^{\varepsilon+}|_{x=\pi} = 0. \\ U^{\varepsilon\pm}|_{t=0}(x) = sin(x), \quad \{\pm x > 0\}. \\ \partial_t U^{\varepsilon\pm}|_{t=0} = 0, \quad \{\pm x > 0\}. \end{cases}$$

We will now construct formally an approximate solution $U^{\varepsilon\pm}_{app}$ of $U^{\varepsilon\pm}$ satisfying the following ansatz:

$$U^{\varepsilon+}_{app} = \sum_{j=0}^{M} U_j^+(t,x)\varepsilon^j,$$

$$U_{app}^{\varepsilon -} = \sum_{j=0}^{M} U_j^- \left(t, x, \left(\frac{x}{\varepsilon}\right)\right) \varepsilon^j,$$

where the profiles $U_j^-(t,x,z) := \underline{U_j^-}(t,x) + U_j^{*-}(t,z)$, with

$$\lim_{z \to -\infty} e^{-\alpha z} U_j^{*-} = 0,$$

for some $\alpha > 0$. Since the stability estimates are trivial here, we will only focus on the construction of

$$U_{app}^{\varepsilon} := U_{app}^{\varepsilon +} \mathbf{1}_{x>0} + U_{app}^{\varepsilon -} \mathbf{1}_{x<0}.$$

Plugging $U_{app}^{\varepsilon \pm}$ into problem (6.2) and identifying the terms with same power of ε, we obtain the following equations:

$$\underline{U_0^-} = 0,$$

Moreover, $U_0^{*-} = 0$ as it is the only solution of the problem:

$$\begin{cases} U_0^{*-} - c^2 \partial_{zz} U_0^{*-} = 0, \quad \{z < 0\}, \\ \partial_z U_0^{*-}|_{z=0} = 0, \\ \lim_{z \to -\infty} U_0^{*-} = 0. \end{cases}$$

$U_{app}^{\varepsilon +}$ converges towards U_0^+ as $\varepsilon \to 0^+$. As awaited U_0^+ is the solution of the well-posed 1-D wave equation:

$$\begin{cases} \partial_{tt} U_0^+ - c^2 \partial_{xx} U_0^+ = 0, \quad (x,t) \in]0, \pi[\times \mathbb{R}^+, \\ U_0^+|_{x=0} = \underline{U_0^-}|_{x=0} + U_0^{*-}|_{z=0} = 0. \\ U_0^+|_{x=\pi} = 0. \\ U_0^+|_{t=0}(x) = sin(x), \quad \{x > 0\}. \\ \partial_t U_0^+|_{t=0} = 0, \quad \{x > 0\}. \end{cases}$$

Let us write the following profiles equations: First, we can see that, for all $j \geq 1$, there holds:

$$\underline{U_j^-} = 0.$$

where U_1^{*-} is the solution of the well-posed profile equation:

$$\begin{cases} U_1^{*-} - c^2 \partial_{zz} U_1^{*-} = -\partial_{tt} U_0^{*-} = 0, \quad \{z < 0\}, \\ \partial_z U_1^{*-}|_{z=0} = \partial_x U_0^+|_{x=0}, \\ \lim_{z \to -\infty} U_1^{*-} = 0. \end{cases}$$

Hence U_1^{*-} is given by:

$$U_1^{*-} = c\partial_x U_0^+|_{x=0} e^{\frac{z}{c}}.$$

We will show now that the profiles can be computed as any order. Assume that U_j^{*-} has been computed, U_j^+ is solution of the well-posed 1-D wave equation:

$$\begin{cases} \partial_{tt}U_j^+ - c^2\partial_{xx}U_j^+ = 0, \quad (x,t) \in]0,\pi[\times\mathbb{R}^+, \\ U_j^+|_{x=0} = U_j^{*-}|_{z=0}. \\ U_j^+|_{x=\pi} = 0. \\ U_j^+|_{t=0}(x) = 0, \quad \{x > 0\}. \\ \partial_t U_0^+|_{t=0} = 0, \quad \{x > 0\}. \end{cases}$$

U_{j+1}^{*-} is then solution of the well-posed profile equation:

$$\begin{cases} U_{j+1}^{*-} - c^2\partial_{zz}U_{j+1}^{*-} = -\partial_{tt}U_j^{*-}, \quad \{z < 0\}, \\ \partial_z U_{j+1}^{*-}|_{z=0} = \partial_x U_j^+|_{x=0}, \\ \lim\limits_{z\to-\infty} U_{j+1}^{*-} = 0. \end{cases}$$

Let us answer the question asked in [11]: $U^{\varepsilon-}$ is bound to present boundary layer behavior in $\{x = 0^-\}$, indeed its approximate solution is composed **exclusively** of boundary layer profiles, which describes quick transitions at the boundary using a fast scale in ε. As a result of the loss in convergence induced by the boundary layer, the following estimate holds:

$$\|U^\varepsilon - U\|_{L^2(]-\infty,\pi[\times\mathbb{R}^+)} = \mathcal{O}(\varepsilon^{\frac{1}{2}}).$$

In [11], their small parameter is $\mu = \varepsilon^2$, as a result, adopting the same notations as them, our estimate writes:

$$\|U^\mu - U\|_{L^2(]-\infty,\pi[\times\mathbb{R}^+)} = \mathcal{O}(\mu^{\frac{1}{4}}),$$

which is in agreement with the estimates given in [11]. Like in the penalization approach proposed by Bardos and Rauch [2] and underlined by Droniou in [4], the boundary layer only forms on one side of the boundary. The approximation $U^{\varepsilon+}$ of U, is computed by taking $U^{\varepsilon+}|_{x=0} = U^{\varepsilon-}|_{x=0}$, thus, in numerical applications, the boundary layer phenomenon also affects the rate of convergence of $U^{\varepsilon+}$ towards U,
as $\varepsilon \to 0^+$.

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