RESOLUTION OF SINGULARITIES OF THREEFOLDS IN POSITIVE CHARACTERISTIC I

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1. Introduction.

The purpose of this article and [13] is to prove theorem 2.1 below: resolution of singularities holds for algebraic varieties of dimension three over a field k of characteristic $p>0$ whenever k is differentially finite over some perfect subfield k_0. This condition is satisfied in particular when k is a function field over k_0. The resolution of singularities $\pi:\ \tilde{Z}\rightarrow Z$ which we obtain is projective, birational and an isomorphism away from the singular locus of any given variety Z. It should be emphasized however that our construction of π is purely existential: it neither respects embeddings of Z in a regular space, nor is given by any resolution algorithm.

For k an algebraically closed field of characteristic $p>5$, our result is a consequence of Abhyankar's theorem proved in 1966 [5] and of its refinement in [12]. Abhyankar's techniques have been greatly simplified in [17] under the same assumption on k. Although several written programs have appeared in the recent years to prove resolution of singularities [30] [28] or the weaker local uniformization theorem [43] over a perfect ground field, none has been completed to this date. On the other hand, dealing with nonperfect ground fields induces great technical difficulties which do not seem to have been systematically considered in any written program to this date. Some partial results in dimension three were already known [11] [38], but no such general statement as theorem 2.1

If the condition of π being "birational" is relaxed to "generically finite", the existence of $\pi:\ \tilde{Z}\rightarrow Z$ with $\tilde{Z}$ regular is known in all dimensions by de Jong's theorem [29]. It is worth noting that when k is not perfect, de Jong's theorem is somewhat weaker in the sense that the obtained π is not generically smooth in general. Our approach basically relies on two different kind of techniques:

On the one hand, we use Zariski's reduction to local uniformization of valuations as in [46]. This is still a subtle problem to handle in dimension three and we adapt Abhyankar's ramification techniques in [1] to dimension three in order to further reduce local uniformization to Artin-Schreier ($g\neq 0$ below) or purely inseparable ($g=0$ below) covers of degree p of a regular (three dimensional) germ of variety, i.e. for local rings

$$\mathcal{O}:=(S[X]/(h))_{(X,u_1,u_2,u_3)},\ h:=X^p-g^{p-1}X+f, \tag{1}$$

where S is a regular local ring of dimension three, essentially of finite type over k, with maximal ideal $m_S:=(u_1,u_2,u_3)$ and $f,g\in m_S$. Such reduction is the purpose of this first article and is the content of theorem 7.2. It is at the present state of our

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knowledge restricted to dimension at most three but does work for arbitrary ground fields of positive characteristic.

On the other, local uniformization is proved directly for singularities of the form (1) in [13] and this is the hard part. There, we rely on differential techniques introduced by the first author and J. Giraud which had already been successful in the purely inseparable case $g = 0$ in [11]. At this point, we emphasize that the Galois assumption in (1) is essential and that our proof does not extend in a trivial way when $h \in S[X]$ is an arbitrary monic polynomial of degree p. Dealing with Artin-Schreier coverings of degree p is what makes our result valid for small characteristics: S. Abhyankar cleverly avoids this difficulty in [5], one consequence being his restriction to $p \geq 7$. See also [35] for details.

We now conclude this introduction by sketching the logical organisation of the proof in this article and [13].

In the next four sections, we state or extend lower dimensional results that are necessary to reduce our resolution theorem to local uniformization of rank one, residually algebraic valuations. This is classical material. From section 6 on, we follow the lines of Abhyankar's proof in dimension two [1]. The basic ingredient used in here is Krull's structure theorem for the inertia group of extensions of valuation rings [32]. This leads to proving "pulling up" and "pushing down" theorems for local uniformization in elementary field extensions.

The only difficult case in pulling up is that of Artin-Schreier and purely inseparable extensions of degree p, dealt with in [13]. Our formulation of theorem 7.2 stresses the role played by *immediate* wildly ramified extensions of valued fields which are the main obstruction to proving local uniformization, see [31] and [33].

Pushing down theorems are less difficult because wildly ramified extensions can be avoided by this ramification theoretic construction. The main difficulty is concentrated in proposition 9.3 and deals with pushing down local uniformization in subextensions of the strict Henselisation of a given normal local ring of dimension three which are not necessarily Galois. We then have to refine substantially Abhyankar's argument in [1].

In [13], singularities of the form (1) are uniformized using Hironaka's methods when h has order smaller than p. What has to be achieved is to reduce to this situation.

In chapter 1 of [13], we introduce our main invariant $(\mathrm{ord}_x h, \Omega(x))$ at any point $x \in X := \mathrm{Spec}\mathcal{O}$. To begin with, it can be assumed that X is nonsingular away from $\eta^{-1}(E)$, E a divisor with normal crossings on $\mathrm{Spec}S$ and $\eta : X \to \mathrm{Spec}S$ (section **I**). Then $\Omega(x) = (\omega(x), \omega'(x)) \in \mathbf{N} \times \{1, 2, 3\}$ is built up in section **II.3** from certain Jacobian ideals $J(f, E)$ [11] [21] when the Hironaka characteristic polyhedron $\Delta(h; u_1, u_2, u_3; X) \subseteq \mathbf{R}^3_{\geq 0}$ is minimal [24].

The case $\omega(x) = 0$ is easily dealt with in **II.4.6** by a simple combinatorial algorithm (Hironaka's game). When $\omega(x) > 0$, the refinement $\omega'(x) \in \{1, 2, 3\}$ essentially stores in the information that $\omega(x)$ w.r.t. (1) is computed from g ($\omega'(x) = 1$), from f alone ($\omega'(x) = 2$) or from both f and g ($\omega'(x) = 3$). The main point is our definition of *permissible* blowing ups for $\Omega(x)$ and its non-increasing by such blowing ups (section **II.5**). Our notion of permissible blowing ups is more restrictive than that of Hironaka (i.e. regular and equimultiple centers for the hypersurface X). The main difficulty which is overcome here is to get some hold on the transformation laws for

Hironaka's characteristic polyhedra under permissible blowing ups in our sense. The cases $\omega'(x)=3$ (theorem **II.5.6**) and $\omega'(x)=1$ (theorems **I.1** and **I.2.7** of chapter 2) are easily dispatched once this behavior has been understood.

In chapter 2, we begin the definition of a secondary invariant $\kappa(x)\in\{0,1,2,3,4,5\}$ which is a *multivalued* function. The case $\kappa(x)\in\{0,1\}$ (**I.2.3** and **I.2.5**) corresponds to Abhyankar's good points: some reasonable algorithm makes $\Omega(x)$ drop. Section **II** recollects several cases where $\kappa(x)\in\{0,1\}$ in terms of the equation (1).

We then define $\kappa(x)\in\{2,3,4,5\}$ in the last two chapters in terms of equation (1); this is the reason why $\kappa(x)$ is multivalued. Roughly speaking, the cases $\kappa(x)\in\{2,3\}$ (resp. $\kappa(x)\in\{4,5\}$) mean that our Jacobian ideals $J(f,E)$ are transverse (resp. tangent) to E in a certain sense. Our strategy is then to make the maximal value of $\kappa(x)$ drop, and this is the technical heart of [13]. It is worth noting that in the case $\kappa(x)=3$, we are lead to use *nonpermissible* blowing ups by preparing the base $\mathrm{Spec}S$.

To conclude this overview of the proof, the authors want to point out that most of the constructions and computations in [13] are actually performed w.r.t. the graded ring $\mathrm{gr}_{m_S}S\simeq k(x)[U_1,U_2,U_3]$. This is sufficiently significant for the authors to make it hopeful that the unequal characteristic case could be dealt with by similar methods which are in progress.

2. Statement of main result and proof assuming the following sections.

The main result of this article is the following theorem. The proof given below recollects all results from the next sections and [13].

Theorem 2.1. *Let k be a field of positive characteristic which is differentially finite over a perfect field k_0, i.e. Ω^1_{k/k_0} has finite dimension. Let Z/k be a reduced quasiprojective scheme of dimension three with singular locus Σ. There exists a projective morphism $\pi:\ \tilde Z\to Z$, such that*

(i) *$\tilde Z$ is regular.*
(ii) *π induces an isomorphim $\tilde Z\backslash\pi^{-1}(\Sigma)\simeq Z\backslash\Sigma$.*
(iii) *$\pi^{-1}(\Sigma)\subset\tilde Z$ is a divisor with strict normal crossings.*

Proof. Let $Z_1,\dots,Z_s$ be the distinct irreducible components of Z. If $s\geq 2$, let $\eta: Z'\to Z$ be the blowing up along the scheme theoretic intersection $Z_1\cap(Z_2\cup\cdots\cup Z_s)$. Note that η induces an isomorphism away from the singular locus of Z and that the strict transform of Z_1 is a connected component of Z'. By induction on s, we reduce to the case when Z is irreducible. Since the theorem is true in dimension less than or equal to two [36], we assume that Z has dimension three.

Once (i) and (ii) have been proved, (iii) follows from proposition 4.1. By proposition 4.9, it is sufficient to prove local uniformization for any k-valuation ring V/k of $K(Z)/k$. By proposition 5.1, it can further be assumed that V has rank one and $\kappa(V)/k$ is algebraic.

By theorem 7.2, it can be assumed that V birationally dominates an Artin-Schreier or purely inseparable covering of degree p of an algebraic regular local ring of dimension three, essentially of finite type over k and that (2) of proposition 7.2 holds. Local uniformization in this last case is proved directly in [13], under the given assumption on k.

□

3. Notations and prerequisites.

From now on up to the end of this article, k denotes any field of characteristic $p > 0$.

Local rings and models.

Let R be a local ring. We denote the maximal ideal of R by m_R, by $\kappa(R) = R/m_R$ its residue field, and by $\hat{R}$ its formal completion at m_R. We denote by $QF(R)$ the quotient field of a domain R.

Suppose that $R \subseteq S$ is an inclusion of local rings. We say that R dominates S if $m_S \cap R = m_R$, which is denoted by $R < S$.

Suppose that K/k is a function field. We say that a subring R of K is an *affine* (resp. a *local*) model of K/k if $QF(R) = K$ and R is a k-algebra of finite type (resp. the localization at a prime ideal of a k-algebra of finite type). A model is called normal (resp. regular) if it is normal (resp. regular) as a ring.

Suppose that L/K is a finite field extension and R is a normal local ring such that $QF(R) = K$. We say that a subring S of L *lies over* R and R *lies below* S if S is the localization at a maximal ideal of the integral closure of R in L. In particular, S is then a normal local ring such that $QF(S) = L$.

Valuation rings.

A valuation ring V is called a k-valuation ring if $k \subseteq V$ and is denoted by V/k. The value group of a valuation ring V is denoted by VK, where $QF(V) = K$. The *rank* of V is the rank of VK as an ordered group, and the *rational rank* of V is the dimension of the $\mathbf{Q}$-vector space $VK \otimes_{\mathbf{Z}} \mathbf{Q}$.

Let K/k be a function field and V/k be a k-valuation ring with $QF(V) = K$. If X/k is an integral separated scheme with function field K, the unique point $p \in X$, if it exists, such that V dominates $\mathcal{O}_{X,p}$ is called the center of V on X.

An affine (resp. local) model of V/k is an affine (resp. local) model R of K/k such that $R \subseteq V$ (resp. $R < V$). The center of V on $X = \mathrm{Spec}R$ exists and is the prime ideal $m_V \cap R$. A regular local model of V/k is also called a local uniformization of V/k.

Monoidal transforms

Suppose that R is a Noetherian local domain. A *monoidal transform* $R < R_1$ is a birational extension of Noetherian local domains such that $R_1 = R[\frac{P}{x}]_m$ where P is a regular prime ideal of R (i.e. $\frac{R}{P}$ is a regular local ring), $0 \neq x \in P$ and m is a prime ideal of $R[\frac{P}{x}]$ such that $m \cap R = m_R$. Also $R < R_1$ is called a *quadratic transform* if $P = m_R$. If R is regular, and $R < R_1$ is a monoidal transform, then R_1 is regular and there exists a regular system of parameters (r.s.p. for short) $(x_1, \dots, x_d)$ of R and $r \leq d$ such that

$$R_1 = R\left[\frac{x_2}{x_1}, \dots, \frac{x_r}{x_1}\right]_m .$$

Suppose that K/k is a function field, V is a k-valuation ring with $QF(V) = K$ and R is a local model of V/k. Then $R < R_1$ is a monoidal transform *along* V if V dominates R_1.

Given a finite sequence

$$R_0 < R_1 < \cdots < R_n$$

such that R_i is a monoidal (resp. quadratic) transform of R_{i-1} for each i, $1 \leq i \leq n$, one says that R_n is an iterated monoidal (resp. quadratic) transform of R. Also, R_n is called an iterated monoidal (or quadratic) transform along V if V is a valuation ring birationally dominating $\cup_{i=1}^n R_i$.

Ramification theory of local rings.

Let L/K be a finite field extension. We will denote the group of K-automorphisms of L by $\mathrm{Gal}(L/K)$.

Suppose that L/K is a finite Galois (i.e. normal and separable) field extension with group $G := \mathrm{Gal}(L/K)$. Let R be a normal local ring such that $QF(R) = K$ and S be a normal local ring such that $QF(S) = L$ which lies above R. We can then define the *splitting group* and *inertia group* of S over R as follows:

$$G^s(S/R) := \{g \in G \mid g.S = S\}, \tag{2}$$

$$G^i(S/R) := \{g \in G^s(S/R) \mid \forall x \in S, g.x \equiv x \bmod m_S\}. \tag{3}$$

In other terms $G^i(S/R) = \mathrm{Ker}\,(G^s(S/R) \to \mathrm{Gal}(\kappa(S)/\kappa(R)))$ is a normal subgroup of $G^s(S/R)$. The *splitting field* K^s (resp. *inertia field* K^i) of S over R is the fixed field of $G^s(S/R)$ (resp. $G^i(S/R)$). We have a corresponding sequence of field inclusions

$$K \subseteq K^s \subseteq K^i \subseteq L,$$

K^s being the largest subfield of L such that S is the only local ring lying above $S \cap K^s$ (Proposition 1.46 [4]). We have a sequence of inclusion of local domains

$$R \subseteq R^s \subseteq R^i \subseteq S \tag{4}$$

where the *splitting ring* $R^s := S \cap K^s$ and the *inertia ring* $R^i := S \cap K^i$ of S over R lie above R. Then $R \subseteq R^s$ is unramified (i.e. local étale in the sense of [40] definition 2, p.80), with $R/m_R = R^s/m_{R^s}$, $R^s \subseteq R^i$ is unramified, R^i/m_{R^i} is Galois over $R^s/m_{R^s} = R/m_R$ with Galois group $G^s(S/R)/G^i(S/R)$, by Theorem 1.48 [4].

Finally, note that if K/k is a function field and $H \subseteq G$, the fixed ring $S^H = S \cap L^H$ is a normal local model of L^H/k if S is a normal local model of L/k (e.g. [37] lemma 1 on p. 262).

Ramification theory of valuation rings.

Let L/K be a finite extension of function fields over k, and let $W_1, \ldots, W_s$ be all extensions to L of a given k-valuation ring V of K such that $QF(V) = K$.

The *reduced ramification index* of W_j relative to V is defined to be ([47] p.53)

$$e_j := [W_j L : VK]. \tag{5}$$

The *relative degree* of W_j with respect to V is defined to be (*ibid.*)

$$f_j := [\kappa(W_j) : \kappa(V)]. \tag{6}$$

One says that V is *totally ramified* in L if V has a unique extension W to L (i.e. $s = 1$) and if the residue extension $\kappa(W)/\kappa(V)$ is purely inseparable.

In case L/K is Galois with group $G := \mathrm{Gal}(L/K)$ we define the splitting groups $G^s(W_j/V)$ and the inertia groups $G^i(W_j/V)$ as in the previous subsection. Note that $W_j(g.x) = W_j x$ for each $g \in G^s(W_j/V)$ and each $x \in L$, and that $W_j(g.x - x) > 0$ for each $g \in G^i(W_j/V)$ and each $x \in W_j$. The *ramification group* $G^r(W_j/V) \subseteq G^i(W_j/V)$ is defined by

$$G^r(W_j/V) := \{g \in G^i(W_j/V) \mid \forall x \in S, W_j(g.x - x) > W_j x\}.$$

The ramification group is thus the kernel of the well-defined map

$$G^i(W_j/V) \to \mathrm{Hom}(W_j L/VK, \kappa(W_j)^\times),\ g \mapsto \left(w \mapsto g x_w/x_w \bmod m_{W_j}\right), \tag{7}$$

where x_w is any element of L such that $W_j x_w = w$. There is an induced isomorphism

$$G^i(W_j/V)/G^r(W_j/V) \simeq \mathrm{Hom}(W_j L/VK, \kappa(W_j)^\times), \tag{8}$$

and $G^r(W_j/V)$ is a p-group ([47] theorems 24 and 25).

The groups $G^s(W_j/V)$, $G^i(W_j/V)$ and $G^r(W_j/V)$ are independent of j up to conjugation. The reduced ramification index e_j (resp. residue degree f_j) is then independent of j and denoted by e (resp. f). One deduces the equality

$$[L : K] = sefp^d, \tag{9}$$

where $d \geq 0$ and p^d divides the order of $G^r(W_j/V)$ ([47], corollary on p.78).

4. Reduction to local uniformization.

This section contains two known results: embedded resolution in a regular quasiprojective variety of dimension three and a refined version of Zariski's patching theorem, due to the first author.

We emphasize that the following proposition is considered as well known by experts in resolution of singularities. Unfortunately, the two authors do not know any published sufficiently general proof of it, except in the (essential) case when $\mathcal{I}$ is locally principal [10] (a self contained proof in the most general case where $Z := V(\mathcal{I})$ is a reduced closed subscheme of any excellent regular scheme of dimension three is in preparation, work of the first author in collaboration with U. Janssen and S. Saito). Reducing to this special case is called "Principalisation" and we give in proposition 4.2 below a sketch of proof of this result for self-completeness of the article. This is also intended to serve as an introduction to the more intricate techniques developed in [13]. We do not pretend any originality in this sketch of proof and apologize in advance if there existed a reference that we are not aware of. If the reader wants to attach names to this proposition, Abhyankar-Hironaka-Zariski would fit.

Proposition 4.1. *(Embedded resolution) Let K/k be a function field of transcendence degree three and let X/k be a regular quasiprojective model of K/k. Let $(0) \neq \mathcal{I} \subset \mathcal{O}_X$ be an ideal sheaf. There exists a finite composition of blowing ups*

$$X =: X(0) \leftarrow X(1) \leftarrow \cdots \leftarrow X(n)$$

with the following properties;

(i) *For each* j, $0 \leq j \leq n-1$, $X(j+1)$ *is obtained by blowing up a regular integral subscheme* $Y(j) \subset X(j)$ *with*

$$Y(j) \subseteq \{y_j \in X(j) \mid \mathcal{I}\mathcal{O}_{X(j),y_j} \text{ is not a normal crossings divisor}\}.$$

(ii) $\mathcal{I}\mathcal{O}_{X(n)}$ *is a normal crossings divisor.*

Proposition 4.2. *(Principalisation) Let* X/k *be a regular quasiprojective model of* K/k *and* $\mathcal{I} \subset \mathcal{O}_X$ *be an ideal sheaf. There exists a finite composition of blowing ups*

$$X =: X(0) \leftarrow X(1) \leftarrow \cdots \leftarrow X(n)$$

with the following properties;

(i) *For each* j, $0 \leq j \leq n-1$, $X(j+1)$ *is obtained by blowing up a regular integral subscheme* $Y(j) \subset X(j)$ *with*

$$Y(j) \subseteq \{y_j \in X(j) \mid \mathcal{I}\mathcal{O}_{X(j),y_j} \text{ is not locally principal}\}.$$

(ii) $\mathcal{I}\mathcal{O}_{X(n)}$ *is locally principal.*

Proof. Let $E_1, ..., E_m$ be the irreducible components of codimension one of $Z := V(\mathcal{I})_{\mathrm{red}}$, and $a(j) := \mathrm{ord}_{E_j}\mathcal{I}$, $1 \leq j \leq m$. Then $\mathcal{H} := \mathcal{O}_X(-\sum_{1 \leq i \leq m} a(i)E_i) \subseteq \mathcal{I}$, and

$$\mathcal{J} := \mathcal{H}^{-1}\mathcal{I} \subseteq \mathcal{O}_X$$

is such that $V(\mathcal{J})$ has codimension at least two in X. For $x \in X$, we denote $m(x) := \mathrm{ord}_x\mathcal{J}$ and let

$$\mu := \sup_{x \in X}\{m(x)\}, \ \Sigma := \{x \in X | m(x) = \mu\}.$$

Clearly

$$\mathcal{I} \text{ is locally principal} \Leftrightarrow \mu = 0 \Leftrightarrow \mathcal{J} = \mathcal{O}_X.$$

We thus assume from now on that $\mu \geq 1$. Then Σ is a closed subset of X of dimension zero or one. Let $\mathcal{E}$ be the idealistic exponent $(\mathcal{J}, \mu)$ ([26] or [27]). By definition, $\mathrm{Sing}(\mathcal{E}) := \Sigma$. A closed set Y is called *permissible* at $x \in \Sigma$ if Y is regular at x and $Y \subseteq \Sigma$ (in particular $\dim(Y) \leq 1$).

Basic invariants for embedded singularities.

Let $x \in \Sigma$ be a closed point. We denote $R := \mathcal{O}_{X,x}$ and $\mathfrak{m}$ its maximal ideal. Note that $\mathrm{gr}_{\mathfrak{m}}(R) = k(x)[Y_1, Y_2, Y_3]$, where (y_1, y_2, y_3) is a r.s.p. of R and Y_i denotes the image of y_i in $\mathfrak{m}/\mathfrak{m}^2$.

Let $J_x := \mathrm{cl}_\mu \mathcal{J}_x \subseteq k(x)[Y_1, Y_2, Y_3]_\mu$. We call *tangent cone* of $\mathcal{E}$ at x the affine subscheme $C_x(\mathcal{E}) \subset \mathbf{A}^3_{k(x)}$ with ideal $J_x\mathrm{gr}_{\mathfrak{m}}(R)$.

There exists a minimal $k(x)$-vector subspace $T_x \subseteq \mathfrak{m}/\mathfrak{m}^2 = k(x).Y_1 \oplus k(x).Y_2 \oplus k(x).Y_3$ such that

$$(J_x \cap k(x)[T_x])\mathrm{gr}_{\mathfrak{m}}(R) = J_x,$$

where $k(x)[T_x] = \mathrm{Sym}(T_x) \subseteq \mathrm{Sym}(\mathfrak{m}/\mathfrak{m}^2) = \mathrm{gr}_{\mathfrak{m}}(R)$, i.e. such that J_x is generated by elements in $k(x)[T_x]$. This minimal T_x is called the *directrix* of $\mathcal{E}$ at x. Let $\tau(x) := \dim_{k(x)}(T_x)$. The inclusion of vector spaces $J_x \subseteq \mathrm{Sym}^\mu(T_x)$ defines an embedding of cones

$$\mathrm{Dir}_x(\mathcal{E}) := \mathrm{Spec}\,(\mathrm{gr}_{\mathfrak{m}}(R)/T_x\mathrm{gr}_{\mathfrak{m}}(R)) \subseteq C_x(\mathcal{E}).$$

This subscheme also has an intrinsic definition [21], [25].

Near and very near points.

With notations as above, let $Y = V(y_1, \ldots, y_r)$ be permissible for $\mathcal{E}$ at x (so $r = 2$ or $r = 3$). Let $\mathcal{E}'$ be the *transform* of $\mathcal{E}$ in the blowing up X' of (X, x) along Y, i.e. $\mathcal{E}' = (\mathcal{J}', \mu)$ where $\mathcal{J}' = \mathcal{I}(Y)^{-\mu}\mathcal{J}$ is the weak transform of $\mathcal{J}$. We denote by Σ' the singular locus of $\mathcal{E}'$. Then

(a) For any point x' above x, $\mathrm{ord}_{x'}\mathcal{J}' \leq \mu$; if equality holds, we say that x' is near x.

(b) If x' is near x, then $\tau(x') \geq \tau(x)$; if equality holds, we say that x' is very near x.

To prove (a), let $f \in I$ with $\mathrm{ord}_x f = \mu$. By definition of permissibility and directrix, we have

$$F(Y_1, Y_2, Y_3) := \mathrm{in}_x f \in k(x)[Y_1, \ldots, Y_r] \cap k(x)[T_x]. \tag{10}$$

In the chart of X' where, say $y'_1 := y_1$ is an equation of the exceptional divisor,

$$y_1^{-\mu} f \equiv F(1, y'_2, y'_3) \mod (y'_1, y'_3) \text{ (resp.} \mod (y'_1)), \tag{11}$$

where $y'_2 = y_2/y_1$ and $y'_3 = y_3$ (resp. $y'_3 = y_3/y_1$) if $r = 2$ (resp. $r = 3$).

Now (11) proves (a), since $\mathrm{ord}_{x'}\mathcal{J}' \leq \deg F = \mu$. By Hironaka's theorem 2 [25],

$$x' \in \mathrm{Proj}(\mathrm{Dir}_x(\mathcal{E})) \subseteq \mathrm{Proj}(k(x)[Y_1, \ldots, Y_r])$$

if x' is very near to x. This holds if k is perfect or if $\mathrm{char}(k) \geq \dim(C_x(\mathcal{E}))$ which is the case here, since $\dim R = 3$. See also *ibid.* corollary 3.2. Hironaka's theorem leads to lemma 4.3 below which refines statement (b) above.

Using induction on μ, proposition 4.2 is now a consequence of its rephrased version in proposition 4.4 below.

□

Lemma 4.3. *Let $q: X' \to X$ be a blowing up along a permissible center Y at x as above. The following holds:*

1. *If $\tau(x) = 3$, then $Y = \{x\}$ and no $x' \in q^{-1}(x)$ is near x. Moreover x is an isolated point of Σ.*
2. *If $\tau(x) = 2$ and Y is a curve, then no $x' \in q^{-1}(x)$ is near x. Moreover Y and Σ coincide locally at x.*
3. *If $\tau(x) = 2$, $Y = \{x\}$ and $x' \in q^{-1}(x)$ is near x, then x' is uniquely determined, rational over x and has $\tau(x') \geq \tau(x) = 2$.*
4. *If $\tau(x) = 1$, Y is a curve and $x' \in q^{-1}(x)$ is near x, then x' is uniquely determined and rational over x. If Γ' is the one-dimensional component of $\Sigma' \cap q^{-1}(Y)$, then Γ' is either empty or a regular irreducible curve projecting isomorphically to Y by q.*
5. *If $\tau(x) = 1$ and $Y = \{x\}$, then those points $x' \in q^{-1}(x)$ near x all lie on the projective line*
$$L_x := \mathrm{Proj}(\mathrm{Dir}_x(\mathcal{E})) \subset q^{-1}(x) \simeq \mathbf{P}^2_{k(x)}.$$

Proof. Assertion (1) is clear from Hironaka's theorem 2 or corollary (3.2) [26]. Assertion (2) follows easily from (10) and (11).

To prove (3), we choose (y_1, y_2, y_3) in such a way that $T_x =< Y_2, Y_3 >$. By (11), the only point in $q^{-1}(x)$ which may be near x is the point $x' := (y'_1 = y_1, y'_2 = y_2/y_1, y'_3 =$

y_2/y_3). If x' is near x, for any $f \in \mathcal{J}_x$ with $\mathrm{ord}_x f = \mu$, with $F := \mathrm{in}_x f$, we have $F(Y'_2, Y'_3) \in \mathrm{cl}_\mu(\mathcal{J}') + (Y'_1)$. Therefore

$$T_{x'} + k(x')Y'_1 =< Y'_1, Y'_2, Y'_3 > . \qquad (12)$$

To prove (4), we may now choose (y_1, y_2, y_3) in such a way that $D_x = k(x).Y_2$, with $Y = (y_1, y_2)$. Similarly, the only point in $\in q^{-1}(x)$ which may be very near x is the point $x' := (y'_1 = y_1, y'_2 = y_2/y_1, y_3)$. For any $f \in \mathcal{J}_x$ with $\mathrm{ord}_x f = \mu$, we now have by (11)

$$y_1^{-\mu} f \equiv \lambda_f {y'_2}^\mu \mod (y'_1, y_3)$$

with $\lambda_f \neq 0$, hence $T_{x'} + < Y'_1, Y_3 >=< Y'_1, Y'_2, Y'_3 >$. Hence any irreducible component of Γ' maps surjectively to Y by q. By *ibid.* applied at the generic point of Y, Γ' is unique and $k(\Gamma') = k(Y)$, hence Γ' maps isomorphically to Y, since Y is regular.

To prove (5), we may now choose (y_1, y_2, y_3) in such a way that $T_x = k(x).Y_3$. For any $f \in \mathcal{J}_x$ with $\mathrm{ord}_x f = \mu$, we now have by (11)

$$y_1^{-\mu} f \equiv \lambda_f {y'_3}^\mu \mod (y'_1),$$

with $\lambda_f \neq 0$ and the conclusion follows.

□

Proposition 4.4. *Let X/k be a regular quasiprojective model of K/k and $\mathcal{E} = (\mathcal{I}, \mu)$ be an idealistic exponent on X, with $\dim V(\mathcal{I}) \leq 1$. There exists a finite composition of blowing ups*

$$X =: X(0) \leftarrow X(1) \leftarrow \cdots \leftarrow X(n)$$

with regular centers mapping to $V(\mathcal{I})$ such that the singular locus $\Sigma(n)$ of the transform $\mathcal{E}(n)$ of $\mathcal{E}$ in $X(n)$ is empty (in which case we say that $\mathcal{E}$ is desingularized in $X(n)$).

Proof. We apply the following algorithm where $\mathcal{E}(i) := (\mathcal{I}(i), \mu)$ is the transform of $\mathcal{E}$ in $X(i)$, $\Sigma(i)$ its singular locus and $X(0) := X$.

The algorithm.

1- If $\Sigma(i)$ has an irreducible component of dimension one which is singular, let $X(i+1)$ be the blowing up of $X(i)$ along any such singular point. Otherwise go to 2.

2- If $\Sigma(i)$ has regular components of dimension one which do not intersect transversally, let $X(i+1)$ be the blowing up of $X(i)$ along any such non-transverse intersection point. Otherwise go to 3.

3- If all components of dimension one of $\Sigma(i)$ are regular and intersect transversally, and some (distinct) two among them do intersect, let $X(i + 1)$ be the blowing up of $X(i)$ along any such intersecting curve. Otherwise go to 4.

4- If all connected components of $\Sigma(i)$ are regular and $\Sigma(i) \neq \emptyset$, let $X(i + 1)$ be the blowing up of $X(i)$ along any of these components. If $\Sigma(i) = \emptyset$, stop the algorithm.

We will prove that the above algorithm is finite, i.e. $\Sigma(i) = \emptyset$ for some $i \geq 0$ and the conclusion follows.

For each $i \geq 0$, let $s(i)$ be the defining step in the algorithm for $q(i) : X(i + 1) \to X(i)$, with $s(i) \in \{1, 2, 3, 4\}$. By lemma 4.3, we have $s(i + 1) \geq s(i)$ (note that under assumption (4) in lemma 4.3, the strict transform of Σ in X' is transverse to the exceptional divisor, hence to Γ' if all components of Γ meet transversally). Moreover, each exceptional curve in $\Sigma(i+1)$ created by the algorithm is regular. By embedded resolution of curves, we have $s(i) \geq 2$ for $i >> 0$.

If $s(i) = 2$ for some $i \geq 0$, any exceptional curve in $\Sigma(i+1)$ created by the algorithm is transverse to the strict transforms of each one dimensional irreducible component of $\Sigma(i)$. By embedded resolution of (reducible) curves, we have $s(i) \geq 3$ for $i >> 0$.

If $s(i) = 3$ for some $i \geq 0$, assumption (4) in lemma 4.3 holds. Let $C_1(i), \dots, C_{n(i)}(i)$ be all one dimensional irreducible components of $\Sigma(i)$. By (4) in lemma 4.3, we have

$$n(i+1) \leq n(i)$$

whenever $s(i) = 3$. Working above the generic point $\eta(i)$ of some one dimensional component of $\Sigma(i)$, we have $\mathcal{J}\mathcal{O}_{X(i),\eta(i)}$ principal for $i >> 0$ (this is a consequence of [47] appendix 5, theorem 3 and (E) on p. 391); hence $n(i)$ eventually drops for $i >> 0$. Therefore $s(i) = 4$ for $i \geq i_0$. The same argument together with lemma 4.3 gives the following structure for $\Sigma(i)$ for $i \geq i_0$:

(*) $\Sigma(i)$ is a disjoint union of closed points and projective lines, each of them defined over some finite extension of k and having normal crossings with the exceptional divisor $E(i)$ of $X(i) \to X(i_0)$.

Note that $E(i)$ has only normal crossings and at most two irreducible components at each $x(i) \in X(i)$. Finally, lemma 4.3 also implies that

(**) for each $x(i) \in \Sigma(i)$, we either have $\tau(x(i)) = 2$, or ($\tau(x(i)) = 1$ and $T_{x(i)} =: k(x(i)).Y_{x(i)}$ is transverse to each component of $E(i)$ at $x(i)$).

The above statement about transverseness of the directrix is an easy consequence of lemma 4.3 (4) and (5).

The invariant when $\tau(x(i)) = 1$.

For each point $x(i) \in \Sigma(i)$ with $\tau(x(i)) = 1$ and satisfying (**), we now define an invariant $(\beta(x(i)), \alpha(x(i))$. We take $i = 0$ in what follows to avoid extra indexing in i in what follows, a normal crossings divisor $E \subset X$ being specified. We let $J := \mathcal{J}_x$.

Let (y, u_1, u_2) be a r.s.p. of $R := \mathcal{O}_{X,x}$ be such that $T_x = k(x).Y$ and $\Sigma \subseteq V(y, u_1)$ locally at x. By (**), it can furthermore be assumed that E is such that $E \subseteq \mathrm{div}(u_1u_2)$ locally at x. The polyhedron $\Delta(\mathcal{E}; u_1, u_2; y) \subseteq \mathbf{R}^2_+$ is the convex hull of the set $E + \mathbf{R}^2_+$, where

$$E = \{(c/i, d/i) \in \mathbf{Q}^2_{\geq 0} | \exists f \in J\hat{R}, f = \gamma y^\mu + \sum_{i=1}^{\mu} \lambda_{i,a,b} y^{\mu-i} u_1^a u_2^b, \lambda_{i,c,d} \neq 0,\ \gamma \in \hat{R}\}.$$

The above expansion of f is made in $\hat{R} = k(x)[[y, u_1, u_2]]$ with $\lambda_{i,a,b} \in k(x)$. Note that γ is a unit whenever $\mathrm{ord}_x f = \mu$ by the Weierstrass preparation theorem. Moreover, we have $E \subseteq \{(x_1, x_2) \in \mathbf{R}^2_{\geq 0} | x_1 + x_2 > 1\}$ since $T_x = k(x).Y$.

Let $\mathbf{v} = (v_1, v_2)$ be a vertex of $\Delta(\mathcal{E}; u_1, u_2; y)$. For $f \in J$, we denote

$$\mathrm{in}_{\mathbf{v}}(f) := \overline{\gamma} Y^\mu + \sum_{c=iv_1, d=iv_2} \lambda_{i,c,d} Y^{\mu-i} U_1^c U_2^d \in k(x)[Y, U_1, U_2], \tag{13}$$

and

$$\mathrm{in}_{\mathbf{v}}(\mathcal{E}) := < \{\mathrm{in}_{\mathbf{v}}(f),\ f \in J\} > .$$

We say that a vertex $\mathbf{v}$ of $\Delta(\mathcal{E};u_1,u_2;y)$ is *solvable* if ($\mathbf{v}\in\mathbf{N}^2$ and there exists $\lambda_{\mathbf{v}}\in k(x)$ such that $\mathrm{in}_{\mathbf{v}}(\mathcal{E})=k(x).(Y+\lambda_{\mathbf{v}}U_1^{v_1}U_2^{v_2})^{\mu}$). We say that $\Delta(\mathcal{E};u_1,u_2;y)$ is *prepared* if no vertex of $\Delta(\mathcal{E};u_1,u_2;y)$ is *solvable*. Finally, we denote by

$$(\alpha(\mathcal{E};u_1,u_2;y),\beta(\mathcal{E};u_1,u_2;y)):=\min_{\mathbf{v}\in\Delta(\mathcal{E};u_1,u_2;y)}\{(v_1,v_2)_{\mathrm{lex}}\}. \tag{14}$$

We will use the notation (α,β) for short to denote the coordinates of this vertex of $\Delta(\mathcal{E};u_1,u_2;y)$ with smaller first coordinate when the context is clear.

It is easy to show [24] that there exists $z=y+\theta$, $\theta\in k(x)[[u_1,u_2]]$, such that $\Delta(\mathcal{E};u_1,u_2;z)$ is prepared: if a vertex $\mathbf{v}=(v_1,v_2)$ is solvable, change y to $y+\lambda_{\mathbf{v}}u_1^{v_1}u_2^{v_2}$ and note that $(\alpha+\beta,\alpha)_{\mathrm{lex}}$ increases. Note that the Newton polygon of θ is then contained in $\Delta(\mathcal{E};u_1,u_2;y)$. In particular, we have $T_x=k(x).Z$. We now define the main invariant:

$$i(x):=(\beta(x),\alpha(x)):=\min_{(z,u_1,u_2)}\{(\beta(\mathcal{E};u_1,u_2;z),\alpha(\mathcal{E};u_1,u_2;z))_{\mathrm{lex}}\}, \tag{15}$$

where the minimum is taken over all r.s.p.'s of R such that $\Delta(\mathcal{E};u_1,u_2;z)$ is prepared, $\Sigma\subseteq V(z,u_1)$ and $E\subseteq\mathrm{div}(u_1u_2)$ locally at x. The behaviour of our invariant under permissible blowing up is given in lemma 4.5 below.

We now conclude the proof. Assume that the above given algorithm does not stop. There exists an increasing function $\sigma:\ \mathbf{N}\to\mathbf{N}$ such that for each $i\geq 0$, $x_{\sigma(i)}\in\Sigma(i)$ and $x_{\sigma(i+1)}$ is a closed point on the exceptional divisor created by blowing up $(X_{\sigma(i)},x_{\sigma(i)})$ along the permissible center $Y_{\sigma(i)}$.

By lemma 4.3 (1), we have $\tau(x_{\sigma(i)})\leq 2$ for $i\geq 0$. Assume that $\tau(x_{\sigma(i_1)})=2$ for some $i_1\geq 0$. By lemma 4.3 (2) and (3), we have $\tau(x_{\sigma(i)})=2$ and $Y_{\sigma(i)}=\{x_{\sigma(i)}\}$ (coinciding locally at $x_{\sigma(i)}$) with $\Sigma(X_{\sigma(i)})$) for $i\geq i_1$. By (12), there exists a regular (possibly formal) curve Γ such that $x_{\sigma(i)}$ belongs to the strict transform of Γ for $i\geq i_1$. By standard arguments, we must have $\Gamma\subseteq\Sigma(i_1)$, so Γ is a formal branch of an irreducible component of $\Sigma(i_1)$ passing through $x(i_1)$: a contradiction, since $\{x_{\sigma(i_1)}\}$ is an isolated point of $\Sigma(X_{\sigma(i_1)})$. Therefore $\tau(x_{\sigma(i)})=1$ for all $i\geq 0$.

Using lemma 4.5 below, $\beta(x_{\sigma(i)})=\beta(x_{\sigma(i+1)})$ and $Y_{\sigma(i)}=\{x_{\sigma(i)}\}$ (coinciding locally at $x_{\sigma(i)}$) with $\Sigma(X_{\sigma(i)})$) for $i\geq i_1$. The last statement in (2) of lemma 4.5 also implies the existence of a regular (possibly formal) curve Γ such that $x_{\sigma(i)}$ belongs to the strict transform of Γ for $i\geq i_1$: a contradiction by the same argument as above (case $\tau(x_{\sigma(i_1)})=2$). This concludes the proof. □

Lemma 4.5. *Assume that $\tau(x)=1$ and x satisfies (**). Let $X'\to(X,x)$ be the blowing up along Σ and $x'\in q^{-1}(x)$ be very near x. The following holds.*

(1) *if Σ is a curve, then x' is uniquely determined and has*

$$i(x')\leq(\beta(x),\alpha(x)-1)_{\mathrm{lex}}.$$

(2) *if $\Sigma=\{x\}$, then $\beta(x')\leq\beta(x)$. If equality holds, then x' is rational over x and does not belong to the strict transform of* $\mathrm{div}(u_1)$.

Proof. To prove (1), note that x' has r.s.p. $(u_1,u_2,z'=z/u_1)$ if x' is near x by lemma 4.3 (4). Then $\Delta(\mathcal{E}';u_1,u_2;z')$ is the translated image of $\Delta(\mathcal{E};u_1,u_2;z)$ by one unit to the left, so $\Delta(\mathcal{E}';u_1,u_2;z')$ remains prepared. The conclusion follows.

We now prove (2). Since $\Sigma=\{x\}$, $I\not\subseteq(z,u_1)^{\mu}$, so

$$\alpha(\mathcal{E};u_1,u_2;z)<1. \tag{16}$$

In particular, we must have

$$\beta(\mathcal{E}; u_1, u_2; z) > 0, \tag{17}$$

since $\alpha(\mathcal{E}; u_1, u_2; z) + \beta(\mathcal{E}; u_1, u_2; z) > 1$.

By lemma 4.3 (5), x' is on the strict transform of $\mathrm{div}(z)$ and on the exceptional divisor of $X' \to X$. There are two charts to consider: the first one has origin the point $x'_0 := (u_1, u_2/u_1, z/u_1)$; the second contains a unique point $x'_1 := (u_1/u_2, u_2, z/u_2)$ on the strict transform of $\mathrm{div}(z)$.

If x'_0 is very near x, it is easy to see that $\Delta(\mathcal{E}'; u_1, u_2/u_1; z/u_1)$ is prepared, the correspondence between vertices of $\Delta(\mathcal{E}; u_1, u_2; z)$ and $\Delta(\mathcal{E}'; u_1, u_2/u_1; z/u_1)$ given by

$$(\alpha(x), \beta(x)) \mapsto (\alpha(x) + \beta(x) - 1, \beta(x)),$$

and the conclusion follows in this case.

If x'_1 is very near x, then similarly $\Delta(\mathcal{E}'; u_1/u_2, u_2; z/u_2)$ is prepared and we get

$$\beta(\mathcal{E}'; u_1/u_2, u_2; z/u_2) \leq \alpha(\mathcal{E}; u_1, u_2; z/u_1) + \beta(\mathcal{E}; u_1, u_2; z/u_1) - 1.$$

Then (16) implies $\beta(\mathcal{E}'; u_1/u_2, u_2; z/u_2) < \beta(\mathcal{E}; u_1, u_2; z/u_1)$.

The hard part is now to control $i(x')$ at points in the first chart distinct from x'_0. We denote $(u'_1 := u_1, u'_2 := u_2/u_1, z' := z/u_1)$. Let $P \in k(x)[u_1, u_2]$ be homogeneous, irreducible and unitary in u_2 such that (u'_1, v', z') is a r.s.p. at x', where $v' := P(1, u'_2)$. We have $R' := \mathcal{O}_{X',x'} = \mathcal{O}_{X,x}[u'_2, z']_{(u'_1, v', z')}$ and

$$\hat{R}' \simeq k(x')[[u'_1, v', z']], \ k(x') \simeq k(x)[u'_2]/(P(1, u'_2)).$$

When x' is rational over x, i.e. $P = u_2 + \lambda u_1$ for some $\lambda \in k(x)$, $\lambda \neq 0$, we replace (u_1, u_2, z) by $(u_1, v_2 := u_2 + \lambda u_1, z)$. By (14) and definition of vertex solvability, we have $\mathbf{v} := (\alpha(\mathcal{E}; u_1, v_2; z), \beta(\mathcal{E}; u_1, v_2; z)) = (\alpha(\mathcal{E}; u_1, u_2; z), \beta(\mathcal{E}; u_1, u_2; z))$ and this vertex is not solvable in $\Delta(\mathcal{E}; u_1, v_2; z)$. If $\Delta(\mathcal{E}; u_1, v_2; z)$ is not prepared, we will change z to $w := z + \theta$, $\theta \in (u_1)\hat{R}$ to get $\Delta(\mathcal{E}; u_1, v_2; w)$ prepared without changing $\mathrm{in}_{\mathbf{v}}(\mathcal{E})$. Then it can be assumed that $x' = x'_0$ and the conclusion follows.

We now assume that x' is not rational over x. The problem is that, in general, $\Delta(\mathcal{E}'; u'_1, v'; z')$ is not prepared. Let us denote

$$\delta := \inf\{c + d | (c, d) \in \Delta(\mathcal{E}; u_1, u_2; z)\} > 1. \tag{18}$$

Let μ_0 be the monomial valuation on $\hat{R} \simeq k(x)[[u_1, u_2, z]]$ given by

$$\mu_0(\sum_{abc} \lambda_{abc} u_1^a u_2^b z^c) = \inf\{c + \frac{a+b}{\delta} \mid \lambda_{abc} \neq 0\}.$$

We now compute the initial form of J w.r.t. μ_0 in $\mathrm{gr}_{\mu_0}(R) \simeq k(x)[U_1, U_2, Z]$, where Z (resp. U_1, U_2) is in degree one (resp. in degree $1/\delta$). To begin with, we have $\mu_0(J) = \mu$ by definition (18). Hence $\mathrm{in}_{\mu_0}(J)$ has weight μ and is generated as a vector space by forms

$$F := \mathrm{in}_{\mu_0} f = Z^\mu + \sum_{1 \leq i \leq \mu, \frac{\deg F_i}{i} = \delta} Z^{\mu - i} F_i(U_1, U_2). \tag{19}$$

The valuation μ_0 has a unique extension to $\hat{R}' \simeq k(x')[[u'_1, v', z']]$ which is denoted by μ_1 to avoid confusions, i.e.

$$\mu_1(\sum_{a'b'c'} \lambda_{a'b'c'} {u'_1}^{a'} {v'}^{b'} {z'}^{c'}) = \inf\{c' + \frac{a'}{\delta - 1} \mid \lambda_{a'b'c'} \neq 0\}.$$

Then $\mathrm{gr}_{\mu_1}(\hat{R}') \simeq k(x')[[v']][U'_1, Z']$, where Z' (resp. U'_1, v') is in degree one (resp. in degree $1/(\delta-1), 0$). Since J is generated by its elements of value μ w.r.t. μ_0, $J' := \mathcal{J}'_{x'}$ is generated by elements

$$F' := {U'_1}^{-\mu} F(U'_1, U'_1 u'_2, U'_1 Z') = Z'^{\mu} + \sum_{1 \leq i \leq \mu, \frac{\mathrm{deg} F_i}{i} = \delta} Z'^{\mu - i} {U'_1}^{i(\delta-1)} F_i(1, u'_2) \tag{20}$$

constructed from (19). Since there exists some F in (19) with $F_i \neq 0$ for some i, $1 \leq i \leq \mu$ by definition (18), we have

$$\alpha(\mathcal{E}'; u'_1, v'; z') = \delta - 1. \tag{21}$$

For any such pair (F, i) with $F_i \neq 0$, we denote $F_i =: U_1^{a(i)} G_i(U_1, U_2)$ with $G_i(0, U_2) \neq 0$, $a(i) \geq 0$. By construction, we have

$$\beta(\mathcal{E}'; u'_1, v'; z') \leq \frac{\mathrm{ord}_{v'}(G_i(1, u'_2))}{i},$$

and

$$\beta(\mathcal{E}'; u'_1, v'; z') \leq \frac{\mathrm{ord}_{v'}(G_i(1, u'_2))}{i} \leq \frac{\mathrm{deg} G_i}{i} \leq \beta(\mathcal{E}; u_1, v; z). \tag{22}$$

In particular, lemma 4.5 holds if $\Delta(\mathcal{E}'; u'_1, v'; z')$ is prepared: the middle equality in (22) is strict since $\mathrm{deg} G_i / i = \beta(x) > 0$ by (17) and $[k(x') : k(x)] > 1$. Otherwise, let $w' = z' + \theta'$, $\theta' \in k(x')[[u'_1, v']]$, be such that $\Delta(\mathcal{E}'; u'_1, v'; w')$ is prepared. By (21), we have $a := \mathrm{ord}_{u'_1} \theta' \geq \delta - 1$. If $a > \delta - 1$ (e.g. if $\delta \notin \mathbf{N}$), then $W' := \mathrm{in}_{\mu_1}(w') = Z'$, so

$$\beta(\mathcal{E}'; u'_1, v'; w') \leq \frac{\mathrm{ord}_{v'}(G_i(1, u'_2))}{i}, \tag{23}$$

and the conclusion follows. From now on, we assume that $a = \delta - 1$ and let

$$\Theta'(U'_1, v') := \mathrm{in}_{\mu_1}(\theta') = {U'_1}^{\delta - 1} \psi'(v'),$$

with $\psi'(v') \in k(x')[[v']]$. Changing z' to w' induces an automorphism of the $k(x')[[v']]$-module $\mathrm{in}_{\mu_1}(J'\hat{R}')$ given by

$$\Phi(U'_1, v', Z') \mapsto \Phi(U'_1, v', W' - \Theta'(U'_1, v')). \tag{24}$$

If $\dim_{k(x)}(\mathrm{in}_{\mu_0}(J)) \geq 2$, then $\mathrm{in}_{\mu_1}(J')$ has a minimal generator of the form

$$H' := \sum_{i_0 \leq i \leq \mu, \frac{\mathrm{deg} H_i}{i} = \delta} Z^{\mu - i} {U'_1}^{i(\delta-1)} H_i(1, u'_2)$$

with $i_0 \geq 1$ and $H_{i_0} \neq 0$, so (23) holds with G_i replaced with H_{i_0} and the conclusion follows. From now on, we assume in addition that $\dim_{k(x)} \mathrm{in}_{\mu_0}(\mathrm{in}_{\mu_0}(J)) = 1$. Then F in (19) is uniquely determined.

Let $\mu =: p^{\alpha} l$, with g.c.d.$(l, p) = 1$ and $I_0 := \{i, 1 \leq i \leq \mu \mid F_i \neq 0\}$. If $I_0 \not\subset p^{\alpha}\mathbf{N}$, then (23) holds for $i = i_0$, where $i_0 := \min\{i_0 \in I_0 \mid i_0 \notin p^{\alpha}\mathbf{N}\}$ and the conclusion follows. From now on, $I_0 \subset p^{\alpha}\mathbf{N}$ and let $i_0 := \min\{i_0 \in I_0\}$.

Suppose that $F_{p^\alpha} \in (k(x)[U_1, U_2])^{p^\alpha}$, say $F_{p^\alpha} = K(U_1, U_2)^{p^\alpha}$. After replacing z with $z + \frac{1}{l}K(u_1, u_2)$, it can be assumed that $i_0 > p^\alpha$. In particular we have $l \neq 1$, since $\Delta(\mathcal{E}; u_1, u_2; z)$ is prepared. By construction, we now have

$$(\delta - 1, \frac{\mathrm{ord}_{v'}(G_{i_0}(1, u'_2))}{i_0}) \in \Delta(\mathcal{E}'; u'_1, v'; w'),$$

from which one deduces $\beta(\mathcal{E}'; u'_1, v'; w') < \beta(\mathcal{E}; u_1, u_2; z))$ as in (22) and the conclusion follows.

There remains to deal with the case $i_0 = p^\alpha$ and $F_{p^\alpha} \notin (k(x)[U_1, U_2])^{p^\alpha}$, say

$$F_{p^\alpha} = K(U_1, U_2)^{p^{\alpha'}},$$

with $K(U_1, U_2) \notin (k(x)[U_1, U_2])^p$ and $\alpha' < \alpha$.

By (20) and (24), the coefficient c'_{p^α} of $W'^{\mu - p^\alpha}$ in $F'(U'_1, u'_2, W' - \Theta'(U'_1, v'))$ satisfies

$$\mathrm{in}_{\mu_1}(c'_{p^\alpha}) = U_1'^{p^\alpha(\delta-1)}(G_{p^\alpha}(1, u'_2) - l\psi'(v')^{p^\alpha}).$$

If k is differentially finite over some perfect subfield k_0, the estimates in chapter 1 **II.5.3.2**(i) of [13] applied to the polynomial $K(U_1, U_2)$ yield

$$\mathrm{ord}_{v'}(G_{p^\alpha}(1, u'_2) - l\psi'(v')^{p^\alpha}) \leq \frac{\mathrm{deg} G_{p^\alpha}}{[k(x') : k(x)]} + p^{\alpha'}$$

and we conclude that

$$\beta(\mathcal{E}'; u'_1, v'; w') \leq \frac{\beta(\mathcal{E}; u_1, u_2; z)}{[k(x') : k(x)]} + \frac{1}{p^{\alpha - \alpha'}}. \tag{25}$$

Since $\delta > 1$ and $\delta \in \mathbf{N}$, we have $\delta \geq 2$. On the other hand, $\delta \leq \alpha(\mathcal{E}; u_1, u_2; z) + \beta(\mathcal{E}; u_1, u_2; z)$, so by (16), we have $\beta(\mathcal{E}; u_1, u_2; z) > 1$. Since $[k(x') : k(x)] \geq 2$, (25) gives

$$\beta(\mathcal{E}'; u'_1, v'; w') \leq \frac{\beta(\mathcal{E}; u_1, u_2; z) + 1}{2} < \beta(\mathcal{E}; u_1, u_2; z).$$

For general k, let k_c be the field generated by all coefficients of K and P, and let ζ be a root of P over $k(x)$. There is an embedding $k_c(\zeta) \subset k(x')$ and we let

$$k'_1 := k_c(\zeta)^{\frac{1}{p}} \cap k(x') \subset k(x').$$

Then k'_1 is finitely generated, hence differentially finite over the prime subfield $k_0 := \mathrm{F}_p$, and there is an expansion

$$K(1, u'_2) \in k_c(\zeta)[[v']].$$

Hence it can be assumed that $\psi'(v')^{p^{\alpha-\alpha'-1}} \in k'_1[[v']]$ in the above argument. Since P is irreducible over $k(x)$, we have $[k'_1 : k_1] = [k(x') : k(x)]$. This proves that we may substitute for $k(x)$ its subfield k_1 in the above computations, so chapter 1 **II.5.3.2**(i) of [13], hence (25) holds in any case. □

Corollary 4.6. *Let K/k be a function field of transcendence degree three and V/k be a valuation ring with $QF(V) = K$ having a local uniformization R.*
Then for any local model R_0 of V/k, there exists a local uniformization R_1 of V/k such that $R_0 < R_1$.

Proof. Let $A_0 := k[x_1, \ldots, x_n]$ be an affine model of V/k such that R_0 is the localization of A_0 at some prime ideal. Write $x_i = \frac{f_i}{g_i}$, with $f_i, g_i \in R$, $g_i \neq 0$. By proposition 4.2, with $X := \mathrm{Spec} R$ and $\mathcal{I} := (f_i, g_i)$, there exists an iterated monoidal transform R' of R along V such that $x_i \in R'$. By induction on n, it can then be assumed that $A_0 \subseteq R'$. Then take $R_1 := R'$. □

Proposition 4.9 below is a refined version of Zariski's patching theorem ([46] Fundamental Theorem on p. 539) in characteristic zero. We first indicate why this result remains true over any base field k of positive characteristic. As written on p. 539 of [46], two preliminary results are required: theorem 7 (elimination of regular points in the fundamental locus of a birational map of threefolds) and the lemma in section 24 (local factorisation of birational morphisms of regular surfaces) of *loc.cit.*. Let us show that these two results can be extended to any base field.

To begin with, the lemma in section 24 has been proved in a very general setup by S.S. Abhyankar:

Proposition 4.7. *(*[2] *theorem 3) Let (R, M) and (R', M') be regular two dimensional local domains with a common quotient field K such that $R < R'$. Then R' is an iterated quadratic transform of R.*

In Zariski's proof, this lemma is used only on page 541 of [46] with $R := \mathcal{O}_{V,\eta}$ ($Q(\Gamma) := \mathcal{O}_{V,\eta}$ with Zariski's notations) and $R' = \mathcal{O}_{V',\eta'} =: Q(\Gamma')$, where V and V' are birationally equivalent projective varieties of dimension 3, Γ (resp. Γ') an irreducible curve on V (resp. on V') whose generic point η (resp. η') is regular on V (resp. on V'). Since $V' \cdots \to V$ is defined at η', proposition 4.7 applies.

Extending [46] theorem 7 to any base field is an easy consequence of proposition 4.1; Zariski's theorem 7 can be rephrased as follows:

Proposition 4.8. *Let $T : \; Z \cdots \to X$ be a birational map between three-dimensional integral projective varieties over k and $F \subset Z$ be its fundamental locus. There exists a composition of blowing ups with regular centers $q : \; Z' \to Z$ such that no point of $q^{-1}(F \cap Z_{\mathrm{reg}})$ is fundamental for $q \circ T$.*

Proof. Let $\tilde{Z}$ be the closure of the graph of the birational map T. Then $\tilde{Z}$ is projective, i.e. the blowing up of a certain ideal sheaf $\mathcal{I} \subset \mathcal{O}_Y$. We apply proposition 4.2 to the quasiprojective variety $U := Z_{\mathrm{reg}}$ and the ideal sheaf $\mathcal{I}_{|U}$. Each blowing up center $Y(j) \subset U(j)$ is either a closed point or a regular curve. For each j, $0 \leq j \leq n-1$, we define inductively a projective variety $Z(j)$ containing $U(j)$ as a dense open subset as follows, with $Z(0) := Z$ to begin with:

1- if $Y(j)$ is a closed point, let $Z(j+1)$ be the blowing up of $Z(j)$ along $Y(j)$;
2- if $Y(j)$ is a regular curve, let $\overline{Y(j)}$ be its Zariski closure in $Z(j)$. Let $Z(j+1) \to Z(j)$ be the minimal composition of point blowing ups making the strict transform $\overline{Y(j)}'$ of $\overline{Y(j)}$ regular followed by the blowing up along $\overline{Y(j)}'$.

Let $q : \; Z' := Z(n) \to Z$ be the obtained map. By the universal property of blowing up, the birational map $q \circ T$ is defined on $q^{-1}(F \cap Z_{\mathrm{reg}})$ and this concludes the proof. □

To get (ii) in theorem 2.1, we need the following refinement of Zariski's patching theorem which is due to the first author [12]. The proof in [12] is written for k algebraically closed, but only uses this assumption via propositions 4.7 and 4.8 above. Since these results have been seen to be valid for any ground field k, we thus have:

Proposition 4.9. *(Refined patching theorem* [12]*) Let K/k be a function field of transcendence degree three. Assume that any k-valuation ring V/k such that $QF(V) = K$ has a local uniformization.*

Then for any quasiprojective model Z/k of K/k, there exists a projective birational morphism $\pi: \tilde{Z} \to Z$, such that $\tilde{Z}$ is regular and π induces an isomorphim

$$\tilde{Z} \backslash \pi^{-1}(\Sigma) \simeq Z \backslash \Sigma,$$

where $\Sigma \subset Z$ is the singular locus of Z.

5. Reduction of local uniformization to the rank one, residually algebraic case.

In this section, we use known results in dimension less than or equal to two to reduce local uniformization in dimension three to the case of rank one, residually algebraic k-valuation rings.

Proposition 5.1. *Let K/k be a function field of dimension three and V/k be a k-valuation ring such that $QF(V) = K$ and either V has rank greater than one or $\kappa(V)/k$ is transcendental. Then V/k has a local uniformization.*

Proof. First assume that $\kappa(V)/k$ is transcendental. Let $t \in V$ be such that its image in $\kappa(V)$ is transcendental over k. Then $K/k(t)$ is an algebraic function field of transcendence degree two. By e.g. [36], there exists a local uniformization of $V/k(t)$, hence of V/k since $k(t) \subset V$.

Assume now that $\kappa(V)/k$ is algebraic and that V has rank greater than one. Let V be composed of the valuation ring V_0 and of the *rank one* k-valuation ring $\overline{V}$: V_0 is a residually transcendental k-valuation ring such that $QF(V) = K$, and $\overline{V}/k$ is a k-valuation ring of $\kappa(V_0)$ such that $QF(\overline{V}) = \kappa(V_0)$ and $\kappa(\overline{V}) = \kappa(V)$ (see [47], Chapter VI for background on composed valuations). We have

$$V := \{f \in V_0 \ / \ \overline{f} \in \overline{V}\},$$

where $\overline{f}$ denotes the image of f in $\kappa(V_0)$ by the canonical surjection. Let $\tau :=$ tr.deg$(\kappa(V_0)/k)$ (necessarily $\tau = 1$ or 2). Pick $t_i \in V_0$ such that $\overline{V}\overline{t_i} > 0, \ 1 \leq i \leq \tau$ and $\{\overline{t_1}, \ldots, \overline{t_\tau}\}$ is a transcendence basis of $\kappa(V_0)/k$. It has already been proved that there exists a local uniformization $R_0 := k(t_1, \ldots, t_\tau)[f_1, \ldots, f_n]_{\mathcal{P}_0}$ of V_0. Since $\overline{V}$ has rank one, there exists for each i, $1 \leq i \leq n$, a nonnegative integer a_i such that $\overline{t_1}^{a_i}\overline{f_i} \in \overline{V}$. Then

$$R_1 := k[t_1, \ldots, t_\tau, t_1^{a_1}f_1, \ldots, t_1^{a_n}f_n]_{m_V \cap k[t_1, \ldots, t_\tau, t_1^{a_1}f_1, \ldots, t_1^{a_n}f_n]}$$

is a local model of V/k satisfying $(R_1)_{m_{V_0} \cap R_1} = R_0$.

Let $\overline{R_1} := R_1/m_{V_0} \cap R_1$ and $\overline{K_1} := QF(\overline{R_1})$. Then $\overline{K_1}/k$ is a function field of transcendence degree $\tau \leq 2$ and $\overline{V} \cap \overline{K_1}$ is a k-valuation ring such that $QF(\overline{V} \cap \overline{K_1}) = \overline{K_1}$ dominating its local model $\overline{R_1}$. Again by [36], there exists a local uniformization $\overline{S}$ of $\overline{V} \cap \overline{K_1}$ dominating $\overline{R_1}$, say

$$\overline{S} := \overline{R_1}\,[\overline{g_1}, \ldots, \overline{g_{\overline{n}}}]_{m_{\overline{V}} \cap \overline{R_1}[\overline{g_1}, \ldots, \overline{g_{\overline{n}}}]}\,.$$

Write $\overline{g_i} = \overline{h_i}/\overline{h_0}$, $1 \leq i \leq \overline{n}$, where $\overline{h_0}, \overline{h_1}, \ldots, \overline{h_{\overline{n}}} \in \overline{R_1}$. For each i, $0 \leq i \leq \overline{n}$, let $h_i \in R_1$ be a lifting of $\overline{h_i}$, $g_i := h_i/h_0$ and

$$S := R_1[g_1, \ldots, g_{\overline{n}}]_{m_V \cap R_1[g_1, \ldots, g_{\overline{n}}]}.$$

We have $S_{m_{V_0} \cap S} = (R_1)_{m_{V_0} \cap R_1} = R_0$ and $S/m_{V_0} \cap S = \overline{S}$.

At this point, we have obtained a local model S of V/k such that the regular subspace $\mathrm{Spec}\overline{S} \subset \mathrm{Spec}S$ has maximal contact along V in the sense of Hironaka [21]; let

$$S =: S_0 < S_1 < \cdots < S_m < \cdots$$

be the sequence of monoidal transforms of S along V defined as follows. For $i \geq 0$, S_{i+1} is the monoidal transform at a smallest regular prime P_i satisfying $m_{V_0} \cap S_i \subset P_i$ which is *permissible*, i.e. such that $\mathrm{Spec}S_i$ is normally flat along P_i. Since principalisation of ideals holds in dimension τ ($\tau \leq 2$), $m_{V_0} \cap S_m$ itself becomes a permissible center for some $m \geq 0$. Therefore S_m is regular since $(S_m)_{m_{V_0} \cap S_m} = R_0$ is regular. Whence S_m is a local uniformization of V/k. □

6. Galois approximation of local rings

In this section, we state the connexion between the ramification theory of valuation rings and that of their normal local models.

Lemma 6.1. *Let L/K be a finite Galois extension of fields and let K', $K \subseteq K' \subseteq L$ be any intermediate field.*
Let S be a normal local ring such that $QF(S) = L$ and let $R' := S \cap K'$, $R := S \cap K$. We have

(1) $G^s(S/R') = G^s(S/R) \cap \mathrm{Gal}(L/K')$ *and* $G^i(S/R') = G^i(S/R) \cap \mathrm{Gal}(L/K')$.
(2) *If S is a valuation ring, then* $G^r(S/R') = G^r(S/R) \cap \mathrm{Gal}(L/K')$.

Proof. Part (1) of the lemma is a direct consequence of the definitions (2) and (3) ([4] proposition 1.49). For part (2), by (7), we have a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 1 & \to & G^r(S/R) & \to & G^i(S/R) & \to & \mathrm{Hom}(SL/RK, \kappa(S)^\times) & \to & 1 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 1 & \to & G^r(S/R') & \to & G^i(S/R') & \to & \mathrm{Hom}(SL/R'K', \kappa(S)^\times) & \to & 1. \end{array}$$

The inclusion $G^r(S/R') \subseteq G^r(S/R) \cap \mathrm{Gal}(L/K')$ is obvious. The reverse inclusion holds because the vertical arrow to the right is an inclusion. □

The following proposition is important since it gives the connexion between the ramification theory of valuation rings and the ramification theory of their normal local models. It is implicit in [1].

Proposition 6.2. *(Galois approximation) Let L/K be a Galois extension of function fields over k and let W/k be a k-valuation ring such that $QF(W) = L$. Let $V := W \cap K$.*
For any given normal local model R of V/k, let $\tilde{R}$ be the unique normal local model of W/k which lies above R, and let R^s (resp. R^i) be the splitting ring (resp. inertia ring) of $\tilde{R}$ over R.
There exists a normal local model R_0 of V/k such that for any normal local model R of V/k dominating R_0, the following holds.

(1) *We have*
$$G^s(W/V) = G^s(\tilde{R}/R) \text{ and } G^i(W/V) = G^i(\tilde{R}/R), \tag{26}$$
i.e. $R^s = \tilde{R}^{G^s(W/V)}$ *and* $R^i = \tilde{R}^{G^i(W/V)}$.
(2) *Let* $R^r := \tilde{R}^{G^r(W/V)}$. *Then*
$$\kappa(R^r) = \kappa(R^i), \tag{27}$$
and the action of $H := G^i(W/V)/G^r(W/V)$ on R^r is induced by a diagonal $\kappa(R^i)$-linear action on $\hat{R}^r \simeq \kappa(R^i)[[x_1, \ldots, x_n]]/I$.

Proof. Since V is the direct union of all its normal local models R, its integral closure $\overline{V}$ in L is the direct union of all corresponding integral closures $\overline{R}$ in L. Since the extensions of V to L are the localizations of $\overline{V}$ at its maximal ideals $m_1, \dots, m_s$, any R_0 such that for $1 \leq i \leq s$, the $m(W_i) \cap \overline{R_0}$'s are pairwise distinct satisfies the statement about splitting groups in (1) of the proposition.

Let now R be any normal local model of V/k such that $R_0 < R$. By definition of the inertia group, there is an inclusion $G^i(W/V) \subseteq G^i(\tilde{R}/R)$. Let $t_1, \dots, t_f$ be elements of W whose residues $\overline{t_1}, \dots, \overline{t_f}$ generate the $\kappa(V)$-vector space $\kappa(W)$. After possibly changing R_0, it can be assumed that $\kappa(V)/\kappa(R_0)$ is algebraic and that $t_1, \dots, t_f \in \tilde{R}_0$. Then any $g \in G^i(\tilde{R}/R) \subseteq G^s(W/V)$ maps to $\overline{g} \in \mathrm{Gal}(\kappa(W)/\kappa(V))$ such that $\overline{g}.\overline{t_i} = \overline{t_i}$, $1 \leq i \leq f$. Therefore $g \in G^i(W/V)$ and this concludes the proof of (1).

We now turn to the proof of (2). Let R_0 satisfy the conclusion of (1) in the proposition and let R be any normal local model of V/k such that $R_0 < R$. By proposition 1.46 [4], R^r is the only normal local subring of K^r, with quotient field K^r, which lies above R^i. The residue extension $\kappa(R^r)/\kappa(R^i)$ is purely inseparable since $\kappa(\tilde{R})/\kappa(R^i)$ is (Theorem 1.48 [4]). By (8), K^r/K^i is an Abelian extension of degree prime to p with group H.

Let $\eta \in R^r$, with minimal polynomial $P(X) \in K^i[X]$ over K^i. The field extension $K^i(\eta)/K^i$ is Abelian of order prime to p, since K^r/K^i is; whence $\deg P$ is prime to p and

$$P(X) = \prod_{g \in \mathrm{Gal}(K^i(\eta)/K^i)} (X - g.\eta).$$

By definition of $G^i(W/V)$, we have $g.\eta - \eta \in m_W \cap R^r = m_{R^r}$ for each $g \in G^i(W/V)$. Let $\overline{\eta}$ (resp. $\overline{P}$) be the image of η (resp. P) in $\kappa(R^r)$ (resp. $\kappa(R^r)[X]$). We get

$$\overline{P}(X) = (X - \overline{\eta})^{\deg P} \in \kappa(R^r)[X].$$

In particular, we have $\overline{\eta}^{\deg P} \in \kappa(R^i)$ since $P \in R^i[X]$. On the other hand, $\overline{\eta}^{p^\alpha} \in \kappa(R^i)$ for some $\alpha \geq 0$ since $\kappa(R^r)/\kappa(R^i)$ is purely inseparable. Therefore $\overline{\eta} \in \kappa(R^i)$, since $\deg P$ is prime to p, and this proves (27).

From now on, for $x \in m_{R^r}$, we write $\overline{x}$ for its initial form in $m_{R^r}/m_{R^r}^2$. Consider the following $\kappa(R^i)$-linear representation of H

$$\rho: \ H \to \mathrm{GL}\left(m_{R^r}/m_{R^r}^2\right), \ h \mapsto \left(\overline{x} \mapsto \overline{h.x}\right). \tag{28}$$

By (8), $\kappa(W)$ contains the group μ_ϵ of ϵ^{th}-roots of unity, where ϵ is the exponent of the Abelian group H, and ϵ is prime to p. Since $\kappa(W)/\kappa(V^i)$ is purely inseparable (Theorem 1.48 [4]), we also have $\mu_\epsilon \subseteq \kappa(V^i)$. Since $V^i = \cup_R R^i$, it can be assumed that $\mu_\epsilon \subseteq \kappa(R^i)$ after possibly changing R_0.

Now, any irreducible representation of H over $\kappa(R^i)$ has degree one, since H is Abelian and $\mu_\epsilon \subseteq \kappa(R^i)$. Therefore, ρ is diagonal up to choosing a basis $(\overline{x_1}, \dots, \overline{x_n})$ of $m_{R^r}/m_{R^r}^2$. We write $\rho(h).\overline{x_j} =: \chi_j(h)\overline{x_j}$, for $1 \leq j \leq n$ and $h \in H$, where $\chi_j \in \mathrm{Hom}(H, \kappa(R^i)^\times)$.

We now fix a field of representatives $\kappa(R^i) \subset \hat{R}^i \subseteq \hat{R}^r$. Let $\hat{K}^r := QF(\hat{R}^r)$ and $\hat{K}^i := QF(\hat{R}^i)$. Since W/V^i is totally ramified, we also have $\mathrm{Gal}(\hat{K}^r/\hat{K}^i) = H$ with the natural extension of the H-action to formal completions. Let

$$y_j := \frac{1}{\mid H \mid} \sum_{h \in H} \chi_j(h^{-1})(h.x_j) \in \hat{R}^r. \tag{29}$$

It is immediately checked that $\overline{y_j} = \overline{x_j}$ and that $h.y_j = \chi_j(h) y_j$ for each $h \in H$. After replacing x_j with y_j, it can therefore be assumed that

$$h.x_j = \chi_i(h) x_j \tag{30}$$

for each $h \in H$ and $1 \leq j \leq n$, i.e. the action is diagonal on $\hat{R}^r$.

□

Corollary 6.3. *Let L/K be a Galois extension of function fields of transcendence degree three over k and let W/k be a k-valuation ring such that $QF(W) = L$. Let $V := W \cap K$.*
Let K', $K \subseteq K' \subseteq L$ be an intermediate extension such that K' is contained in K^i, the inertia field of W over V. Let $V' := W \cap K'$.
If V/k has a local uniformization, then V'/k has a local uniformization.

Proof. By assumption, we have

$$G^i(W/V) \subseteq \mathrm{Gal}(L/K'). \tag{31}$$

Let $R_0 \subset K$ be a normal local model satisfying the conclusion of proposition 6.2 (1) w.r.t. the extension of valuation rings W/V. By corollary 4.6, there exists a local uniformization R of V/k such that $R_0 < R$. By proposition 6.2 (1) and (31), the corresponding $\tilde{R} \subset L$ satisfies the equality

$$G^i(W/V) = G^i(\tilde{R}/R) \subseteq \mathrm{Gal}(L/K').$$

Let $R' \subset K'$ be the unique local ring of K' lying above R which is dominated by V'. By construction, R' is contained in the inertia field $K^i(\tilde{R}/R)$, so that R' is local-étale over R by [40], theorem 2 on p.110. Then R' is regular, since R is. □

7. Abhyankar's strategy: reduction to Artin-Schreier and purely inseparable extensions.

In this section, we state the main reduction of local uniformization to the case of Artin-Schreier and purely inseparable extensions of degree p which is the purpose of [13]. The main ingredients in the proof of theorem 7.2 are postponed to the next two sections. Once more, the strategy is adapted from that of [1] in dimension two.

Definition 7.1. *An extension of valuation rings $V \subseteq W$, with $K := QF(V) \subseteq L := QF(W)$ is said to be immediate if the inclusions $VK \subseteq WL$ and $\kappa(V) \subseteq \kappa(W)$ are isomorphisms.*

When L/K is finite Galois, immediate *ramified* extensions of valuation rings are precisely those whose factor p^d in (9) is not trivial. See [33] for background and further scrutiny on these extensions. In order to emphasize the role played by such extensions in the local uniformization problem, we state the main reduction as follows:

Theorem 7.2. *Let k be a field of positive characteristic. Assume that for every function field K of transcendence degree three over k, for every k-valuation ring V/k of rank one such that $\kappa(V)/k$ is algebraic and $QF(V) = K$, and for every local uniformization R of V/k, the following holds:*

"For every pair $f, g \in m_R$ (with $g \neq 0$ if k is perfect) such that

(1) *the polynomial* $h := X^p - g^{p-1}X + f \in R[X]$ *is irreducible over* K, *and*

(2) *there exists a unique extension* W *of* V *to* $L := QF(S)$, *where*

$$S := (R[X]/(h))_{(m_R, X)},$$

and $V \subset W$ *is immediate,*

there exists a local uniformization of W/k*."*

Then for every function field K *of transcendence degree three over* k, *every* k*-valuation ring* V/k *of rank one such that* $\kappa(V)/k$ *is algebraic and* $QF(V) = K$ *has a local uniformization.*

Proof. Let K be a function field of transcendence degree three over k, and let V/k be a k-valuation ring of rank one such that $\kappa(V)/k$ is algebraic and $QF(V) = K$. Let $K_0 := k(x_1, x_2, x_3)$, where (x_1, x_2, x_3) is a transcendence basis of K/k, with $Vx_i \geq 0$ for $1 \leq i \leq 3$. Then $V_0 := V \cap K_0$ has a local uniformization R_0 on the regular projective model $\mathbf{P}^3_k$ of K_0.

Let K_1 be the separable algebraic closure of K_0 in K. Note that (x_1, x_2, x_3) can be chosen such that $K_1 = K_0$ if k is perfect by [37] theorem 26.3. The extension K/K_1 is a tower of purely inseparable extensions of degree p

$$K_1 := K_{1,0} \subset K_{1,1} \subset \cdots \subset K_{1,n} = K,$$

with $n \geq 0$. Assume that $V \cap K_1/k$ has a local uniformization R_1We claim that $V \cap K_{1,i}/k$ has a local uniformization $R_{1,i}$, which we prove by induction on i, $0 \leq i \leq n_1$. By proposition 8.3 and the induction step, $V \cap K_{1,i+1}/k$ has a local uniformization unless $V \cap K_{1,i} \subset V \cap K_{1,i+1}$ is immediate, which we assume now. Then there exists $\eta_{i+1} \in m_V \cap K_{1,i+1}$ such that $K_{1,i+1} = K_{1,i}(\eta_{i+1})$. By corollary 4.6, it can be assumed that $\eta^p_{i+1} \in R_{1,i}$. Then $V \cap K_{1,i+1}/k$ has a local uniformization *by assumption* in the statement of the theorem (with $R := R_{1,i}$ and $g := 0$, $f := -\eta_{i+1} \in m_R$), and this proves the claim. Hence it can be assumed that $K = K_1$ without loss of generality.

Let L/K_0 be a Galois closure of K/K_0 with Galois group G_0, and W be an extension of V to L. Let $G := \mathrm{Gal}(L/K) \subseteq G_0$. One denotes by G^i_0 (resp. G^i) and G^r_0 (resp. G^r) the corresponding inertia and ramification group of W/V_0 (resp. W/V). The corresponding fixed fields are denoted by K^i_0 (resp. K^i) and K^r_0 (resp. K^r). By lemma 6.1, we have $G^i = G^i_0 \cap G$ and $G^r = G^r_0 \cap G$. There is a diagram of field inclusions

$$\begin{array}{ccccccc} K & \to & K^i & \to & K^r & \to & L \\ \uparrow & & \uparrow & & \uparrow & & \\ K_0 & \to & K^i_0 & \to & K^r_0 & & . \end{array}$$

By corollary 6.3 applied to the field extension L/K_0, the valuation ring W and the intermediate field $K' := K^i_0$, $W \cap K^i_0/k$ has a local uniformization since V_0/k has.

By (8), K^r_0/K^i_0 is an Abelian extension of order prime to p, whence a tower of Abelian extensions of prime degrees $l_i \neq p$. Therefore, using induction on $[K^r_0 : K^i_0]$, $W \cap K^r_0/k$ has a local uniformization by proposition 8.3 (2) (whose assumption is satisfied by (9)).

Now K^r/K_0^r need not be Galois. However, since G_0^r is a p-group, it is nilpotent, that is, there is a composition series

$$(1) = H_0 \subset H_1 \subset \cdots \subset H_\alpha = G_0^r,$$

where for each j, $0 \leq j < \alpha$, H_j is invariant in G_0^r, $H_{j+1}/H_j \subseteq Z(G_0^r/H_j)$ and $H_{j+1}/H_j \simeq \mathbf{Z}/p$ (this is easily proved by induction on α, where $\mid G_0^r \mid =: p^\alpha$, given that the center of a p-group is nontrivial). In particular, we have

$$h_{j+1}^{-1} g h_{j+1} \subseteq gH_j \tag{32}$$

for each $g \in G_0^r$, $h_{j+1} \in H_{j+1}$. Let H'_j be the subgroup of G_0^r generated by G^r and H_j, $0 \leq j \leq \alpha$. Equation (32) implies that H'_j is invariant in H'_{j+1} and that H'_{j+1}/H'_j is a quotient of $H_{j+1}/H_j \simeq \mathbf{Z}/p$. By Galois correspondence, the extension K^r/K_0^r is a tower of Galois extensions of degree p. As in the purely inseparable case, the local uniformization in each intermediate extension of this tower is handled by proposition 8.3 (nonimmediate case) or by assumption in the statement of the theorem (immediate case). Therefore $W \cap K^r/k$ has a local uniformization.

By proposition 9.5 below applied to the field extension L/K, the valuation ring W and the intermediate field $K' := K^r$, V/k has a local uniformization since $W \cap K^r/k$ has. □

8. Refined monomialization and applications to local uniformization.

In this section, we prove the following refined version of proposition 4.1 along a valuation. A simple application yields proposition 8.3, which handles the local uniformization of extensions of prime degree which are not immediate (definition 7.1).

Proposition 8.1. *Let L/k be a function field of transcendence degree three and W/k be a k-valuation ring of L with $QF(W) = L$, of rank one, rational rank r $(1 \leq r \leq 3)$ and such that $\kappa(W)/k$ is algebraic. Let S_0 be a given normal local model of W/k and $f_0 \in S_0$, $f_0 \neq 0$.*
Assume that there exists a local uniformization of W/k.
There exists $f \in m_{S_0}$, $f \neq 0$, such that $f_0 \mid f$, and a local uniformization S of W/k with r.s.p. (x_1, x_2, x_3) having the following properties:

1. *$S_0 < S$ and $(S_0)_f = S_f$.*
2. *$\sqrt{fS} = \sqrt{m_{S_0}S} = (x_1 \cdots x_r)$ and $Wx_1, \ldots, Wx_r$ are linearly independent in $WL \otimes_{\mathbf{Z}} \mathbf{Q}$.*

Proof. There exists a local uniformization S_1 of W/k such that $S_0 < S_1$ by corollary 4.6. Since S_0 and S_1 are birational and of the same dimension (three), there exists $g \in m_{S_0}$, $g \neq 0$ such that $(S_0)_g = (S_1)_g$. We pick elements $f_1, \ldots, f_r \in m_{S_0}$ such that $Wf_1, \ldots, Wf_r$ are linearly independent in $WL \otimes_{\mathbf{Z}} \mathbf{Q}$. Then (1) holds with $f := g(f_0 \cdots f_r)$ for any local uniformization S of W/k such that $S_1 < S$ and $(S_1)_f = S_f$.

By proposition 4.1 applied to the ideal $fm_{S_0}S_1$, there exists a local uniformization S_2 of W/k with r.s.p. (y_1, y_2, y_3) such that $S_1 < S_2$, $(S_1)_f = (S_2)_f$ and each of the ideals fS_2 and $m_{S_0}S_2$ is monomial in y_1, y_2, y_3.
Let $E \subseteq \{1, 2, 3\}$ (resp. $F \subseteq \{1, 2, 3\}$) be the set of indices i such that (y_i) divides fS_2 (resp. $m_{S_0}S_2$) in S_2. Note that $F \subseteq E$, since $f \in m_{S_0}$, and that

$$\sharp(E) \geq r, \tag{33}$$

since the values $\{Wf_j\}_{1\leq j\leq r}$ are linearly independent (hence the values $\{Wy_i\}_{i\in E}$ generate $WL\otimes_{\mathbf{Z}}\mathbf{Q}$). There remains to achieve the extra property that $\sharp(E)=\sharp(F)=r$ which is the content of (2) in the statement of the proposition.

Suppose $F\subset E$. Take $i_1\in E\setminus F$, $i_2\in F$ and let $\{i_3\}:=\{1,2,3\}\setminus\{i_1,i_2\}$. Let $S_2^{(1)}$ be the monoidal transform at (y_{i_1},y_{i_2}) of S_2 along W.

If $Wy_{i_2}\geq Wy_{i_1}$, then

$$\left(y_{i_3}^{(1)}:=y_{i_3},y_{i_1}^{(1)}:=y_{i_1},y_{i_2}^{(1)}:=P\left(\frac{y_{i_2}}{y_{i_1}}\right)\right)$$

is a r.s.p. of $S_2^{(1)}$, where $P\in S_2[X]$ is monic and such that its image in $\kappa(S_2)[X]$ is irreducible. We denote by $E^{(1)},F^{(1)}$ the corresponding subsets of $\{1,2,3\}$. By construction,

$$E^{(1)}\setminus F^{(1)}=E\setminus(F\cup\{i_1\}),$$

so that we have achieved a reduction in $\sharp(E\setminus F)$.

If $Wy_{i_2}<Wy_{i_1}$,

$$\left(y_{i_3}^{(1)}:=y_{i_3},y_{i_1}^{(1)}:=\frac{y_{i_1}}{y_{i_2}},y_{i_2}^{(1)}:=y_{i_2}\right)$$

is a r.s.p. of $S_2^{(1)}$ and we have $E^{(1)}=E$, $F^{(1)}=F$. Since W has rank one, we have $nWy_{i_2}\geq Wy_{i_1}$ for some $n\geq 1$ so that we achieve a reduction in $\sharp(E\setminus F)$ after iterating n times. Therefore, after possibly replacing S_2 by an iterated monoidal transform along W, it can be assumed that $E=F$. Note that property (1) of the proposition is preserved by this construction.

By (33), we can now number the regular parameters of S_2 in such a way that $\{1,\dots,r\}\subseteq E=F$ and $Wy_1,\dots,Wy_r$ generate $WL\otimes_{\mathbf{Z}}\mathbf{Q}$. Suppose that $\sharp(E)>r$ (in particular $r\leq 2$) so that $r+1\in E$ after possibly renumbering $y_{r+1},\dots,y_3$.

By lemma 8.2 below, there exists an integer $n\geq 1$, a sequence of monoidal transforms along W

$$S_2=:S_2^{(0)}<S_2^{(1)}<\cdots<S_2^{(n)},$$

a matrix $A=(a_{ij})\in GL(r+1,\mathbf{Z})$ with nonnegative entries such that the following holds: the iterated monoidal transform $S_2^{(n-1)}$ along W has a r.s.p. $(y_1^{(n-1)},y_2^{(n-1)},y_3^{(n-1)})$; for each i, $1\leq i\leq r+1$, there is an expression

$$y_i=\prod_{j=1}^{r+1}\left(y_j^{(n-1)}\right)^{a_{ij}};$$

we have $y_3=y_3^{(n-1)}$ if $r=1$. Moreover, there exists $i_n,i'_n\in\{1,\dots,r+1\}$, $i_n\neq i'_n$ such that $Wy_{i_n}^{(n-1)}=Wy_{i'_n}^{(n-1)}$ and $S_2^{(n)}$ is the monoidal transform of $S_2^{(n-1)}$ at $(y_{i_n}^{(n-1)},y_{i'_n}^{(n-1)})$ along W.

Note that $S=S_2^{(n)}$ still has property (1) of the proposition. By construction, the corresponding subsets $E^{(n)},F^{(n)}$ of $\{1,2,3\}$ satisfy $E^{(n)}=F^{(n)}\subset E$. Therefore we can achieve a reduction of $\sharp(E)$ if $\sharp(E)>r$.

If $r=2$ or if ($r=1$ and $\sharp(E)=2$), we let $S:=S_2^{(n)}$. If $r=1$ and $\sharp(E)=3$, we apply twice the previous procedure obtained from lemma 8.2. In both cases, we achieve an iterated monoidal transform S of S_2 along W satisfying (1) and (2) in the statement of the proposition. □

The following lemma is an immediate application of the Perron algorithm ([45] theorem 1 on p. 862).

Lemma 8.2. *Let $R^{(0)}$ be a regular local ring of dimension $d \geq 2$ and r.s.p. $(x_1^{(0)}, \dots, x_d^{(0)})$. Let V be a valuation ring such that $R^{(0)} < V$ and $QF(V) = K := QF(R^{(0)})$. Assume that $Vx_1^{(0)}, \dots, Vx_r^{(0)}$ are linearly independent in $VK \otimes_{\mathbf{Z}} \mathbf{Q}$, generate an Archimedean subgroup of VK and that $Vx_{r+1}^{(0)} \in \oplus_{i=1}^r \mathbf{Q} Vx_i^{(0)}$.*
There exists an integer $n \geq 1$, a sequence of monoidal transforms along V

$$R^{(0)} < R^{(1)} < \cdots < R^{(n)},$$

where for each j, $1 \leq j \leq n$, $R^{(j)}$ has a r.s.p. $(x_1^{(j)}, x_2^{(j)}, \dots, x_d^{(j)})$ and there exists $i_j, i'_j \in \{1, \dots, r+1\}$, $i_j \neq i'_j$ such that $R^{(j)}$ is the monoidal transform of $R^{(j-1)}$ at $(x_{i_j}^{(j-1)}, x_{i'_j}^{(j-1)})$ along V and the following holds:

1. *for $1 \leq j \leq n$, we have $x_i^{(j)} = x_i^{(j-1)}$ for $i \in \{1, \dots, d\} \setminus \{i'_j\}$;*
2. *for $1 \leq j \leq n-1$, we have $x_{i'_j}^{(j)} = x_{i'_j}^{(j-1)} / x_{i_j}^{(j-1)}$;*
3. *we have $Vx_{i'_n}^{(n-1)} = Vx_{i_n}^{(n-1)}$.*

Proposition 8.1 together with the Perron algorithm imply the following result. It is worth observing that the proof of (2) of proposition 8.3 is not harder in the wildly ramified case $l = p$ than in the tamely ramified case $l \neq p$. See the statement of theorem 7.2, where emphasis is put on the extra difficulty caused by the wildly ramified *and* immediate case. See also the argument in [13], **II.4.6** of chapter 1 for another argument shortcutting the computations after (36) below.

Proposition 8.3. *Let L/K be an extension of function fields of transcendence degree three over k of prime degree l and let W/k be a k-valuation ring of rank one such that $QF(W) = L$. Let $V := W \cap K$.*
Assume that either

1. *$[L : K] = [\kappa(W) : \kappa(V)]$, or*
2. *$[L : K] = \mid WL/VK \mid$.*

If V/k has a local uniformization, then W/k has a local uniformization.

Proof. Under assumption (1) (resp. (2)), we pick $\eta \in W$ such that its image $\overline{\eta}$ in $\kappa(W)$ does not belong to $\kappa(V)$ (resp. $W\eta \notin VK$). Then $L = K(\eta)$, since $l = [L : K]$ is prime. Let

$$P(X) := X^l + f_1 X^{l-1} + \cdots + f_l \in K[X] \tag{34}$$

be the minimal polynomial of η over K, where $f_i \in V$ for $1 \leq i \leq l$. By proposition 4.6, there exists a local uniformization R of V/k such that $f_i \in R$ for $1 \leq i \leq l$. Then $S := R[\eta]_{m_W \cap R[\eta]}$ is a local model of W/k.

Under assumption (1), we have $m_W \cap R[\eta] = m_R R[\eta]$ since the morphism

$$\kappa(R)(\overline{\eta}) \simeq \frac{R[X]}{m_R R[X] + P(X)} \to \frac{R[\eta]}{m_R R[\eta]}, \ X \mapsto \eta.$$

is surjective. Therefore $m_S = m_R S$, so S is a regular local ring of dimension three, since R is.

Under assumption (2), we note that

$$W\eta^l = Wf_l < \min_{1\leq i\leq l-1} W(f_i\eta^{l-i}), \tag{35}$$

since $W\eta$ has order l in WL/VK. We deduce that

$$Wf_l^{\frac{l!}{l}} < \min_{1\leq i\leq l-1}\{Wf_i^{\frac{l!}{i}}\}.$$

Let

$$I := (\{f_i^{\frac{l!}{i}}\}_{1\leq i\leq l}) \subset R.$$

Let $E := \{j \ / \ f_j \neq 0\} \subseteq \{1,\dots,l\}$. Applying consecutively proposition 4.1 to the ideal I, then proposition 8.1 to $f_0 := \prod_{j\in E} f_j$, it can be assumed that R has a r.s.p. (x_1,x_2,x_3), such that the following holds: we have

$$IR = (f_l^{\frac{l!}{l}}) = \left(\prod_{1\leq j\leq r} x_j^{\alpha_j}\right)^{\frac{l!}{l}}, \tag{36}$$

where $Vx_1,\dots,Vx_r$ are linearly independent in $VK\otimes_{\mathbf{Z}}\mathbf{Q}$ and $\alpha_1,\dots,\alpha_r$ are positive integers. Moreover, f_j is a unit in R times a monomial in $x_1,\dots,x_r$ for each $j\in E$.

Let $\mathcal{S} := R[X]_{(m_R,X)}$, where X is an indeterminate. Let $\mathcal{W}_0$ be the discrete valuation ring $\mathcal{S}_{P(X)}$, and $\mathcal{W}$ be the rank two k-valuation ring which is composed of $\mathcal{W}_0$ and of the valuation ring W of $\kappa(\mathcal{W}_0)\simeq QF(S)=L$. Then $\mathcal{S}<\mathcal{W}$ and $\mathcal{S}$ has a r.s.p. $(x_0 := X, x_1,x_2,x_3)$. We have

$$\mathcal{W}x_0 = \frac{1}{l}\sum_{1\leq j\leq r}\alpha_j\mathcal{W}x_j$$

by (35) and (36).

By lemma 8.2, there exists an integer $n\geq 1$, a sequence of monoidal transforms along $\mathcal{W}$

$$\mathcal{S} =: \mathcal{S}^{(0)} < \mathcal{S}^{(1)} < \cdots < \mathcal{S}^{(n)}, \tag{37}$$

a matrix $A=(a_{ij})_{0\leq i,j\leq r}\in GL(r+1,\mathbf{Z})$ with nonnegative entries such that the following holds: the iterated monoidal transform $\mathcal{S}^{(n-1)}$ along $\mathcal{W}$ has a r.s.p. $(x_0^{(n-1)},\dots,x_3^{(n-1)})$; for each i, $0\leq i\leq r$, there is an expression

$$x_i = \prod_{j=0}^{r}\left(x_j^{(n-1)}\right)^{a_{ij}};$$

for each i, $r+1\leq i\leq 3$, we have $x_i = x_i^{(n-1)}$. Moreover, there exists $i_n,i'_n\in\{0,\dots,r\}$, $i_n\neq i'_n$ such that $\mathcal{W}x_{i_n}^{(n-1)} = \mathcal{W}x_{i'_n}^{(n-1)}$ and $\mathcal{S}^{(n)}$ is the monoidal transform of $\mathcal{S}^{(n-1)}$ at $(x_{i_n}^{(n-1)}, x_{i'_n}^{(n-1)})$ along $\mathcal{W}$.

We have expressions

$$x_0^l = \prod_{j=0}^{r}\left(x_j^{(n-1)}\right)^{la_{0j}}$$

and

$$f_l = \prod_{j=0}^{r}\left(x_j^{(n-1)}\right)^{\sum_{i=1}^{r}\alpha_i a_{ij}}.$$

Since the $x_j^{(n-1)}$'s have rationally independent values for $j \in \{0, \dots, r\} \setminus \{i'_n\}$ and $\mathcal{W}x_{i_n}^{(n-1)} = \mathcal{W}x_{i'_n}^{(n-1)}$, we recover from (35) that

$$la_{0j} = \sum_{i=1}^{r} \alpha_i a_{ij}, \text{ for } j \in \{0, \dots, r\} \setminus \{i_n, i'_n\}, \tag{38}$$

and

$$l(a_{0i_n} + a_{0i'_n}) = \sum_{i=1}^{r} \alpha_i (a_{ii_n} + a_{ii'_n}). \tag{39}$$

For $j \in \{0, \dots, r\} \setminus \{i_n, i'_n\}$, let $\beta_j := la_{0j}$. For $j \in \{i_n, i'_n\}$, let

$$\beta_j := \min\{\mathrm{la_{0j}}, \sum_{\mathrm{i=1}}^{\mathrm{r}} \alpha_\mathrm{i} \mathrm{a_{ij}}\}.$$

Since each f_i is either zero or a monomial in $x_1, \dots, x_r$ times a unit, we get from (36) and (38) that

$$g := \text{g.c.d.}_{\mathcal{S}^{(n-1)}} \left(x_0^l, \{f_i x_0^{l-i}\}_{1 \leq i \leq l} \right) = \prod_{j \in \{0, \dots, r\}} x_j^{\beta_j},$$

and from (36) and (39) that

$$\mathrm{ord}_{m_{\mathcal{S}^{(n-1)}}} x_0^l = \mathrm{ord}_{m_{\mathcal{S}^{(n-1)}}} f_l < \min_{1 \leq i \leq l-1} \{\mathrm{ord}_{m_{\mathcal{S}^{(n-1)}}} f_i x_0^{l-i}\}.$$

The previous considerations yield the following expression in $\mathcal{S}^{(n-1)}$:

$$P(x_0) = g \left((x_j^{(n-1)})^m - u(x_{j'}^{(n-1)})^m + \text{ terms of order at least } m+1 \right), \tag{40}$$

where $\{j, j'\} = \{i_n, i'_n\}$, $m := (la_{0i_n} - \beta_{i_n}) + (la_{0i'_n} - \beta_{i'_n}) \geq 0$ and $u \in S$ is a unit such that $u^{-1} f_l$ is a monomial in $x_1, \dots, x_r$. Note that $m > 0$ since A is a nonsingular matrix.

We now claim that $m = 1$. Namely, for $0 \leq j \leq n-1$, let

$$g^{(j)} := \text{g.c.d.}_{\mathcal{S}^{(j)}} (x_0^l, f_l).$$

Each of $x_0^l, u^{-1} f_l, g^{(j)}$ is a monomial in $x_0^{(j)}, \dots, x_r^{(j)}$ and we let

$$\frac{x_0^l}{g^{(j)}} =: \prod_{i \in E^{(j)}} \left(x_i^{(j)} \right)^{\epsilon_i^{(j)}}, \quad \frac{f_l}{g^{(j)}} =: \prod_{i \in F^{(j)}} \left(x_i^{(j)} \right)^{\epsilon_i^{(j)}},$$

where $E^{(j)}, F^{(j)} \subset \{0, \dots, r\}$ and $\epsilon_i^{(j)} > 0$. Note that $E^{(j)} \cap F^{(j)} = \emptyset$. We extend the definition by letting $\epsilon_i^{(j)} = 0$ whenever $i \notin E^{(j)} \cup F^{(j)}$.

Let

$$\delta^{(j)} := \text{g.c.d.}(\{\epsilon_i^{(j)}\}_{i \in E^{(j)} \cup F^{(j)}}).$$

By assumption (2) in the proposition, we have $\delta^{(0)} = 1$. For $1 \leq j \leq n-1$, the transformation law for the exponents $\epsilon_i^{(j)}$ are

$$\epsilon_i^{(j)} = \epsilon_i^{(j-1)} \text{ if } i \neq i_j, \tag{41}$$

and;

$$\epsilon_{i_j}^{(j)} = \epsilon_{i_j}^{(j-1)} + \epsilon_{i'_j}^{(j-1)} \text{ if } \{i_j, i'_j\} \subseteq E^{(j)} \text{ or } \{i_j, i'_j\} \subseteq F^{(j)}, \tag{42}$$

or

$$\epsilon_{i_j}^{(j)} = \mid \epsilon_{i_j}^{(j-1)} - \epsilon_{i_j{}'}^{(j-1)} \mid \text{ otherwise.} \tag{43}$$

The formula $d^{(j)} = d^{(j-1)}$ follows from (41) and from (42) or (43). Therefore $m = d^{(n-1)} = 1$ in (40) and the claim is proved.

Since $m = 1$ in (40), the iterated monoidal transform $S^{(n-1)}$ of $S \simeq \mathcal{S}/P(x_0)$ in $S^{(n-1)}$ which is induced by (37) is regular. Hence W/k has a local uniformization. □

9. Pushing down local uniformization in tamely ramified extensions.

In this section, we consider the following problem.

Problem 9.1. *(Pushing down local uniformization) Let L/K be an extension of function fields of transcendence degree three over k and let W/k be a k-valuation ring of rank one such that $QF(W) = L$ having a local uniformization.*

Has $W \cap K/k$ a local uniformization?

An affirmative answer to this problem would allow us to deduce local uniformization from de Jong's theorem [29] or from its weaker valuative version [34]. However, we do not know how to deal with the case when $W/W \cap K$ is wildly ramified. Since we are interested here in applications to the local uniformization problem, this wildly ramified case can be avoided via Abhyankar's construction performed in the proof of theorem 7.2. In proposition 9.5 below we answer in the affirmative the special case of problem 9.1 when $W/W \cap K$ is tamely ramified. See [6] for a survey and open problems related to simultaneous resolution along a valuation.

Problem 9.2. *(Lying below problem) Let L/K be an extension of function fields of transcendence degree three over k and let W/k be a k-valuation ring of rank one such that $QF(W) = L$ having a local uniformization.*

Does there exist a normal local model R of $W \cap K/k$ lying below some local uniformization S of W/k?

While problem 9.2 has trivially an affirmative answer when L/K is Galois with group G (in which case the invariant ring $R := S^G$ lies below S), it is by no ways obvious in the non Galois case. See [1] theorem 4.8 for an affirmative answer in dimension two and [19], [39] for refined statements. See also [15] (resp. [16]) in characteristic zero, all dimensions (resp. for a counterexample to the global version of problem 9.2 in dimension two and all characteristics). Finally, see [20] for the case $\dim_{\mathbf{Q}}(VK \otimes_{\mathbf{Z}} \mathbf{Q}) \geq \mathrm{tr.deg}(K/k) - 1$. On the other hand, we prove the following case, where V' is unramified over V but K'/K is not necessarily Galois, including the case $\dim_{\mathbf{Q}}(VK \otimes_{\mathbf{Z}} \mathbf{Q}) = 1$ which is not covered by Fu's theorem [20]:

Proposition 9.3. *Let L/K be a Galois extension of function fields of transcendence degree three over k and let W/k be a k-valuation ring of rank one such that $QF(W) = L$ and $\kappa(W)/k$ is algebraic. Let $V := W \cap K$.*
Let K', $K \subseteq K' \subseteq L$ be an intermediate extension such that K' is contained in K^i, the inertia field of W over V. Let $V' := W \cap K'$. Assume that V'/k has a local uniformization S'_0.

There exists a local uniformization S' of V'/k, with $S'_0 < S'$, and a local uniformization R of V/k lying below S'.

Proof. Let K^s be the splitting field of W over V. There is a Cartesian square of field extensions

$$\begin{array}{llll} K^s & \rightarrow & K^s.K' & \subseteq \quad K^i \\ \uparrow & & \uparrow & \\ K^s \cap K' & \rightarrow & K' & . \end{array}$$

By proposition 6.2, there exists a normal local model R_0 of V/k such that for any normal local model R of V/k dominating R_0, we have

$$G^s(W/V) = G^s(\tilde{R}/R) \text{ and } G^i(W/V) = G^i(\tilde{R}/R), \tag{44}$$

where $\tilde{R}$ is the unique normal local model of W/k lying above R. Let R' be the unique normal local model of V'/k lying above R. Since $K' \subseteq K^i$ and K^i is the inertia field of $\tilde{R}$ over R by (44), R' is local-étale over R by [40], theorem 2 on p.110, so that

$$R \text{ is regular } \Leftrightarrow R' \text{ is regular.} \tag{45}$$

By corollary 4.6, it can be assumed that $R'_0 < S'_0$. Then by lemma 6.1, we also have $G^i(\tilde{S'_0}/S'_0) = G^i(W/V')$, so that the unique normal local model S^i_0 of $W \cap K^i/k$ lying above S'_0 is regular. Now, the invariant ring $S^s_0 := (S^i_0)^{G^s(W/V)/G^i(W/V)} \subset K^s$ a normal local model of V^s/k, where $V^s := W \cap K^s$. By (44) and lemma 6.1, we have equalities $G^s(\tilde{R}_0/R^s_0) = G^s(\tilde{S'_0}/S^s_0) = G^s(W/V)$ and $G^i(\tilde{R}_0/R^s_0) = G^i(\tilde{S'_0}/S^s_0) = G^i(W/V)$. This shows that

$$G^i(S^i_0/S^s_0) = G^s(W/V)/G^i(W/V) = \mathrm{Gal}(K^i/K^s).$$

Therefore S^i_0 is local-étale over S^s_0 by [40], theorem 2 on p.110, so that S^s_0 is regular, since S^i_0 is. In particular, V^s/k has a local uniformization.

Assume that the statement of the proposition holds whenever $K' = K^s$. Then there exists a local uniformization S^s of V^s/k, with $S^s_0 < S^s$, and a local uniformization R of V/k lying below S^s. The unique normal local model S' of V'/k lying above R dominates S'_0 and is regular by (45). Therefore we have reduced the proposition to the case $K' = K^s$, which we assume from now on.

Let R_1 be a normal local model of V/k such that R'_1 dominates the given local uniformization S'_0 of V'/k. We now pick $f_1, \dots, f_r \in R_1$ such that $(Vf_1, \dots, Vf_r)$ form a basis of $VK \otimes_{\mathbf{Z}} \mathbf{Q}$, and let $f_0 := f_1 \cdots f_r$. By proposition 8.1, there exists $f \in m_{R'_1}$, $f \neq 0$ such that $f_0 \mid f$ and a local uniformization S' of V'/k with r.s.p. (x_1, x_2, x_3) having the following properties;

(1) $R'_1 < S'$ and $(R'_1)_f = S'_f$.
(2) $\sqrt{fS'} = \sqrt{m_{R'_1}S'} = (x_1 \cdots x_r)$, and $V'x_1, \dots, V'x_r$ are linearly independent in $V'K' \otimes_{\mathbf{Z}} \mathbf{Q}$.

For each i, $1 \leq i \leq r$, there is an expression

$$f_i = \gamma_i \prod_{j=1}^{r} x_j^{a_{ij}},$$

where γ_i is a unit in S'. Since $(V'f_i)_{1 \leq i \leq r}$ and $(V'x_j)_{1 \leq j \leq r}$ are bases of $V'K' \otimes_{\mathbf{Z}} \mathbf{Q}$, the matrix $A := (a_{ij})_{1 \leq i,j \leq r}$ is nonsingular. After possibly permuting two of the f_i's (if $r \geq 2$), it can be assumed that $\det A > 0$. Let $B =: (b_{ij})$ be the adjoint matrix of A, and let

$$F_i := \prod_{j=1}^{r} f_j^{b_{ji}} = \left(\prod_{j=1}^{r} \gamma_j^{b_{ji}} \right) x_i^{\det A} \in S' \cap K, \tag{46}$$

for $1 \leq i \leq r$.

If $r \leq 2$, let $\mathcal{P}' := (x_{r+1}, \dots, x_3) \in \mathrm{Spec}S'$, and $\mathcal{P}'_1 := \mathcal{P}' \cap R'_1$. Let $\pi : \mathrm{Spec}S' \to \mathrm{Spec}R'_1$ and $\mathcal{F} \subset \mathrm{Spec}R'_1$ be the fundamental locus of π. By properties (1) and (2) above, we have $\pi^{-1}(\mathcal{F}) = V(x_1 \cdots x_r)$. In particular, $V(\mathcal{P}'_1) \not\subseteq \mathcal{F}$, so that the induced map $\mathrm{Spec}(S'/\mathcal{P}') \to \mathrm{Spec}(R'_1/\mathcal{P}'_1)$ is birational. In algebraic terms, there exists $g_{r+1}, \dots, g_3 \in \mathcal{P}'_1$ such that

$$\mathcal{P}' = (g'_{r+1}, \dots, g'_3), \tag{47}$$

where

$$g_j =: g'_j \prod_{i=1}^{r} x_i^{c_{ij}}, \tag{48}$$

with $g'_j \in S'$ not divisible by x_i for $1 \leq i \leq r$, $r+1 \leq j \leq 3$. We let

$$G'_j := \frac{g_j^{\det A}}{\prod_{i=1}^{r} F_i^{c_{ij}}} = \gamma'_j (g'_j)^{\det A} \in S', \tag{49}$$

where $\gamma'_j \in S'$ is a unit, for $r+1 \leq j \leq 3$.

By (44), $G^s(\tilde{R_1}/R_1) = \mathrm{Gal}(L/K^s) = \mathrm{Gal}(L/QF(R'_1))$, so that $R'_1 \subset R_1^h$, where R_1^h is the Henselization of R_1 ([40], theorem 2 on p.110). In particular, R_1 lies dense in R'_1 for the $m_{R'_1}$-adic topology. Let

$$C := \max_{i,j}\{c_{ij}\} \tag{50}$$

and let $h_j \in R_1$ be such that $h_j \equiv g_j \ \mathrm{mod} m_{R'_1}^{C+1}$, for $r+1 \leq j \leq 3$. By (48), (50) and property (2) above, there is an expression

$$h_j =: h'_j \prod_{i=1}^{r} x_i^{c_{ij}},$$

with $h'_j \in S'$ not divisible by x_i for $1 \leq i \leq r$, $r+1 \leq j \leq 3$, and a congruence

$$h'_j \equiv g'_j \ \mathrm{mod}(x_1 \cdots x_r)S'.$$

Let

$$H_j := \frac{h_j^{\det A}}{\prod_{i=1}^{r} F_i^{c_{ij}}} \in S' \cap K.$$

Comparison with (49) produces the congruence

$$H_j \equiv \gamma'_j (g'_j)^{\det A} \ \mathrm{mod}(x_1 \cdots x_r)S', \tag{51}$$

for $r+1 \leq j \leq 3$. By (46), (47) and (51), we have

$$\sqrt{(\{F_i\}_{1 \leq i \leq r}, \{H_j\}_{r+1 \leq j \leq 3})S'} = m_{S'}. \tag{52}$$

Let $\overline{R}$ be the integral closure of $R_1[\{F_i\}_{1 \leq i \leq r}, \{H_j\}_{r+1 \leq j \leq 3}] \subset S' \cap K$. We have $R := \overline{R}_{m_{S'} \cap \overline{R}} < S'$ and $QF(R) = K$. By (52) and Zariski's Main Theorem ([40], theorem 1 on p.41), R lies below S'. Now, R is regular since S' is by (45) and the proposition follows. □

Lemma 9.4. *Let L/K be a Galois extension of function fields of transcendence degree three over k of prime degree $l \neq p$.*
Let W/k be a k-valuation ring of rank one such that $QF(W) = L$ and $\kappa(W)/k$ is algebraic. Let $V := W \cap K$.
If W/k has a local uniformization, then V/k has a local uniformization.

Proof. We have $L = K^i(W/V)$ -in which case the lemma follows from proposition 9.3 (Galois case)- except if $s = f = p^d = 1$, $e = l$ in (9), which we assume from now on. In particular $G := \mathrm{Gal}(L/K) = G^i(W/V) = \mathbf{Z}/l$ and $G^r(W/V) = (1)$.

Let μ_l be the group of l^{th}-roots of unity in $\overline{k}$ and ζ_l be a generator of μ_l. First assume that $\mu_l \not\subset K$. Let $K' := K(\zeta_l)$ and $L' := L(\zeta_l)$. Then L'/K' is still a Galois extension with group G, where G acts trivially on μ_l. Let W' be an extension of W to L' and $V' := W' \cap K'$. By (8), $\mu_l \subseteq \kappa(W) = \kappa(V)$, so that we have $G^i(W'/W) = G^i(V'/V) = (1)$, i.e. V (resp. W) totally splits in K' (resp. L').

By corollary 6.3, W'/k has a local uniformization since W has. If V'/k has local uniformization, then V has local uniformization too by proposition 9.3 (Galois case). In other terms, it can be assumed that $\mu_l \subset K$.

Let R_0 be a normal local model of V/k satisfying the conclusion of proposition 6.2, and let S be a local uniformization of W/k such that $\tilde{R_0} < S$. We have $k(\zeta_l) \subset S$. Also S is stable by G, since any conjugate of S is dominated by W, hence equal to S. Let $R := S^G$. Then R is a normal local model of V/k and S lies above R. By proposition 6.2, we have $G^i(S/R) = G$, $\kappa(R) = \kappa(S)$, and the action on $\hat{S} \simeq \kappa(S)[[x_1, x_2, x_3]]$ is given by

$$g.x_i = \zeta_l^{t_i} x_i,$$

where g is a generator of G and $t_i \not\equiv 0 \bmod l$ for some i. Note that it can be assumed that $x_1, x_2, x_3 \in S$ by (29), since $k(\zeta_l) \subset S$.

The lattice

$$N := \{\mathbf{v} := (v_1, v_2, v_3) \in \mathbf{Z}^3 \ \mid \ v_1 t_1 + v_2 t_2 + v_3 t_3 \equiv 0 \bmod l\}$$

has index l in $\mathbf{Z}^3$. By elementary linear algebra, there exists a basis $(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$ of $\mathbf{Z}^3$ having the following properties

(a) $(l\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$ is a basis of N,
(b) $\mathbf{N}^3 \subseteq \mathbf{N}\mathbf{v}_1 + \mathbf{N}\mathbf{v}_2 + \mathbf{N}\mathbf{v}_3$, and
(c) $\sum_{j=1}^{3} \mathbf{v}_{ij} W x_j \geq 0$ for $1 \leq i \leq 3$.

We define

$$y_i := \prod_{j=1}^{3} x_j^{\mathbf{v}_{ij}} \tag{53}$$

for $1 \leq i \leq 3$, and $\overline{S} := S[y_1, y_2, y_3]$. By (c), $S_1 := \overline{S}_{m_W \cap \overline{S}}$ is a local model of W and S_1 is regular by (b). By (a) and (53), we have

$$g.y_1 = \zeta_l^t y_1, \ g.y_2 = y_2, \ g.y_3 = y_3 \tag{54}$$

for some $t \not\equiv 0 \bmod l$. Let $R_1 := S_1^G$. Then R_1 is a normal local model of V/k and S_1 lies above R_1. By proposition 6.2, we have $\kappa(R_1) = \kappa(S_1)$, so that there exist $\gamma_2, \gamma_3 \in R_1$ such that S_1 has r.s.p. $(z_1 := y_1, z_2 := y_2 - \gamma_2, z_3 := y_3 - \gamma_3)$ satisfying

$$g.z_1 = \zeta_l^t z_1, \ g.z_2 = z_2, \ g.z_3 = z_3$$

by (54).

Therefore

$$\hat{S_1^G} = \hat{S_1}^G = \kappa(S_1)[[z_1^l, z_2, z_3]]$$

is a regular local ring. Thus S_1^G is a local uniformization of V/k.

□

Proposition 9.5. *Let L/K be a Galois extension of function fields of transcendence degree three over k and let W/k be a k-valuation ring of rank one such that $\kappa(W)/k$ is algebraic and $QF(W) = L$. Let $V := W \cap K$.*
Let K', $K \subseteq K' \subseteq L$ be an intermediate extension such that K' is contained in K^r, the ramification field of W over V. Let $V' := W \cap K'$.
If V'/k has a local uniformization, then V/k has a local uniformization.

Proof. Let K^i be the inertia field of W over V. There is a Cartesian square of field extensions

$$\begin{array}{ccccc} K^i & \rightarrow & K^i.K' & \subseteq & K^r \\ \uparrow & & \uparrow & & \\ K^i \cap K' & \rightarrow & K' & & . \end{array}$$

By lemma 6.1, $K'' := K^i.K'$ is the inertia field of W over V'. Let $V'' := W \cap K''$. Then V''/k has a local uniformization by corollary 6.3, since V'/k has. Since K^r/K^i is an Abelian extension of order prime to p, K''/K^i is a tower of Abelian extensions of prime degrees $l_i \neq p$. By successive applications of lemma 9.4, V^i/k has a local uniformization, where $V^i := W \cap K^i$. The conclusion then follows from proposition 9.3. □

References

[1] Abhyankar, S., Local uniformization on algebraic surfaces over ground fields of characteristic $p \neq 0$, *Ann. Math.* **63** (1956), 491-526.

[2] Abhyankar, S., On the valuations centered in a local domain, *Amer. J. Math.* **78** (1956), 321-348.

[3] Abhyankar, S., Simultaneous resolution for algebraic surfaces, *Amer. J. Math.* **78** (1956), 761-790.

[4] Abhyankar, S., Ramification theoretic methods in algebraic geometry, *Annals of Math. Studies* **43** (1959), Princeton University Press.

[5] Abhyankar, S., Resolution of singularities of embedded algebraic surfaces, second edition, *Springer Monographs in Math.* (1998), Springer Verlag.

[6] Abhyankar, S., Resolution of singularities and modular Galois theory, *Bull. Amer. Math. Soc.* **38**(2) (2001), 131-169.

[7] Abhyankar, S., Tame Coverings and fundamental groups of algebraic varieties, *Amer. J. Math.* **81** (1959), 46-94.

[8] Abramovich, D. and de Jong, A.J. Smoothness, semistablility, and toroidal geometry, *J. Alg. Geom.* **6** (1997), 789-801.

[9] Bogomolov, F., Pantev, T., Weak Hironaka Theorem, *Math. Res. Lett.* **3** (1996), 299-307.

[10] Cossart, V., Desingularization of embedded excellent surfaces, *Tohoku Math. J., II. Ser.* **33** (1981), 25-33.

[11] Cossart, V., Polyèdre caractéristique d'une singularité, *Thesis* (1987), Univ. Paris-Sud, Centre d'Orsay.

[12] Cossart, V., Modèle projectif régulier et désingularisation, *Math. Ann.* **293**(1) (1992), 115-122.

[13] Cossart, V. and Piltant, O., Resolution of singularities of threefolds in positive characteristics II, *preprint.*

[14] Cutkosky, S.D., Local monomialization and factorization of morphisms, *Astérisque* **260** (1999).

[15] Cutkosky, S.D., Simultaneous resolution of singularities, *Proc. Amer. Math. Soc.* **128**(7) (2000), 1905-1910.

[16] Cutkosky, S.D., Generically finite morphisms and simultaneous resolution of singularities, in *Comm. Algebra (Grenoble/Lyon 2001), Contemp. Math.* **331** (2003), 75-99.

[17] Cutkosky, S.D., Resolution of singularities for 3-folds in positive charcteristics, *preprint.*

[18] Cutkosky, S.D. and Piltant, O., Monomial resolutions of morphisms of algebraic surfaces, *Comm. Algebra* **28**(12) (2000), 5935-5959.

[19] Cutkosky, S.D. and Piltant, O., Ramification of valuations, *Adv. Math.* **183**(1) (2004), 1-79.

[20] Fu, D., Local weak simultaneous resolution for high rational ranks, *J. Algebra* **194**(2) (1997), 614-630.

[21] Giraud, J., Contact maximal en caractéristique positive, *Ann. Sc. ENS* 4ème *série* **8** (1975), 201-234.

[22] Hironaka, H., Resolution of singularities of an algebraic variety over a field of characteristic zero, *Ann. Math.* **79** (1964), 109-326.

[23] Hironaka, H., Desingularization of excellent surfaces, *Advanced Science Seminar in Algebraic Geometry* (1967), Bowdoin College, Brunswick, Maine, 1967.

[24] Hironaka, H., Characteristic polyhedra of singularities, *J. Math. Kyoto U.* **7**(3) (1967), 251-293.

[25] Hironaka, H., Additive groups associated with points of a projective space, *Ann. Math.* **92** (1970), 327-334.

[26] Hironaka, H., Idealistic exponents of singularity, in *J.J. Sylvester symposium, John Hopkins Univ. (Baltimore 1976)*, John Hopkins Univ. Press (1977), 52-125.

[27] Hironaka, H., Theory of infinitely near singular points, *J. Korean Math. Soc.* **40**(5) (2003), 901-920.

[28] Hironaka, H., Three key theorems on infinitely near singularities, in *Singularités Franco-Japonaises*, *Sémin. Congr.* **10** (20035), 87-126.

[29] de Jong, A.J., Smoothness, semistability and Alterations, *Publ. Math. I.H.E.S.* **83** (1996), 51-93.

[30] Kawanoue, H., Toward resolution of singularities over a field of positive characteristic Part I, *RIMS prerint* **1568** (2006).

[31] Knaf, H., Kuhlmann, F.V., Abhyankar places admit local uniformization in any characteristic, *Ann. Sc. ENS Sér. 4* **38**(6) (2005), 833-846.

[32] Krull, W., Galoissiche Theorie bewerteter körper, *Sitzungsberichte der Bayerischen Akaemie der Wissenschaften, München* (1930), 225-238.

[33] Kuhlmann, F.V., Valuation theoretic and model theoretic aspects of local uniformization, in *Resolution of Singularities - A Research Textbook in tribute to O. Zariski, Progr. Math* **181** (2000), Birkhauser, 381-456.

[34] Kuhlmann, F.V., Every place admits local uniformization in a purely wild extension of the function field, *preprint.*

[35] Lipman, J., Introduction to resolution of singularities, in *Algebraic Geometry, Arcata 1974, Amer. Math. Soc. Proc. Symp. Pure Math.* **29** (1975), 187-230.

[36] Lipman, J., Desingularization of two-dimensional schemes, *Ann. Math.* **107** (1978), 151-207.

[37] Matsumura, H., Commutative ring theory, *Cambridge studies in advanced mathematics* **8** (1986), Cambridge Univ. Press.

[38] Moh, T.T., On a Newton polygon approach to the uniformization of singularities of characteristic p, in *Algebraic geometry and singularities (La Ràbida 1991), Progr. Math* **134** (1996), Birkhauser, 49-93.

[39] Piltant, O., On the Jung method in positive characteristic, *Proc. of the Int. Conf. in Honor of F. Pham (Nice, 2002)*, *Ann. Inst. Fourier* **53**(4) (2003), 1237-1258.

[40] Raynaud, M., Anneaux locaux henseliens, *Lect. Notes Math.* **169**, Springer-Verlag (1970).

[41] Ribenboim, P., Théorie des valuations (1964), Presse Univ. Montréal.

[42] Shannon, D.L., Monoidal transforms, *Amer. J. Math* **45** (1973), 284-320.

[43] Teissier, B., Valuations, deformations and toric geometry, in *Val. Theory and its applications II (Saskatoon 1999), Fields Inst. Comm.* **33** (2003),361-459.

[44] Zariski, O., The reduction of the singularities of an algebraic surface, *Ann. Math.* **40** (1939) 639-689.

[45] Zariski, O., Local uniformization of algebraic varieties, *Ann. Math.* **41** (1940), 852-896.

[46] Zariski, O., Reduction of the singularities of algebraic three dimensional varieties, *Ann. Math.* **45** (1944), 472-542.

[47] Zariski, O. and Samuel, P., Commutative Algebra II (1960), Van Nostrand, Princeton.