

Hyperdifferential properties of Drinfeld quasi-modular forms

V. Bosser and F. Pellarin

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Abstract. This article is divided in two parts. In the first part we endow a certain ring of “Drinfeld quasi-modular forms” for $\mathbf{GL}_2(\mathbb{F}_q[T])$ (where q is a power of a prime) with a system of “divided derivatives” (or hyperderivations). This ring contains Drinfeld modular forms as defined by Gekeler in [6], and the hyperdifferential ring obtained should be considered as a close analogue in positive characteristic of famous Ramanujan’s differential system relating to the first derivatives of the classical Eisenstein series of weights 2, 4 and 6. In the second part of this article we prove that, when $q \neq 2, 3$, if \mathcal{P} is a non-zero hyperdifferential prime ideal, then it contains the Poincaré series $h = P_{q+1,1}$ of [6]. This last result is the analogue of a crucial property proved by Nesterenko [11] in characteristic zero in order to establish a multiplicity estimate.

1 Introduction

In [6], Gekeler introduced modular forms for $\mathbf{GL}_2(\mathbb{F}_q[T])$, analogues in positive characteristic to the classical modular forms for $\mathbf{SL}_2(\mathbb{Z})$; these functions are frequently called “Drinfeld modular forms”. Both Drinfeld and classical modular forms have interesting differential properties but there is a slight imperfection, as the rings generated by them are not stable under derivation.

To overcome this problem in the classical case, one idea is to weaken a bit the definition of modular forms, allowing slightly more general functional equations, thus leading to the notion of *quasi-modular forms* as in [5, p. 166] and [10, Definition 113] ⁽¹⁾.

The same can be done in the positive characteristic case, as we will see in this article: *Drinfeld quasi-modular forms* can be defined, and the ring

¹More implicitly, Quasi-modular forms already appear in several previous works by Ramanujan and Rankin.

generated by these functions both contains Drinfeld modular forms and is stable under derivation.

To what extent do classical quasi-modular forms and Drinfeld quasi-modular forms resemble each other? This question is the basis for our article. We will find that, apart from some differences arising from the unequal characteristics, the classical theory and Drinfeld's theory of quasi-modular forms bear striking similarities.

The classical framework. Before we describe our results, it is convenient to present the aspects of the classical theory whose analogues in positive characteristic will be discussed in this text.

A holomorphic function $f : \mathcal{H} \rightarrow \mathbb{C}$ (where \mathcal{H} is the complex upper half-plane) is called *quasi-modular form* of weight $w \in \mathbb{N}$ and depth at most $l \in \mathbb{N}$ for $\Gamma_{\mathbb{Q}} := \mathbf{SL}_2(\mathbb{Z})$ if there exist functions f_0, \dots, f_l , which are holomorphic both on \mathcal{H} and “at infinity”, and which satisfy the following condition: for all $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_{\mathbb{Q}}$ and all $z \in \mathcal{H}$,

$$f\left(\frac{az+b}{cz+d}\right) = (cz+d)^w \sum_{r=0}^l f_r(z) \left(\frac{c}{cz+d}\right)^r.$$

After Kaneko and Zagier (cf. [5, Proposition 1], see also [10, Proposition 124]), the ring generated by quasi-modular forms for $\Gamma_{\mathbb{Q}}$ is also the ring of functions $\mathbb{C}[E_2, E_4, E_6]$, where E_2, E_4, E_6 are the classical Eisenstein series of weights 2, 4 and 6 respectively.

Recall that E_4 and E_6 are modular forms (hence, quasimodular forms of depth 0), but that E_2 is a non-modular quasi-modular form of depth 1. Since the derivative of a quasi-modular form of weight w and depth $\leq l$ is again a quasi-modular form, of weight $w + 2$ and depth $\leq l + 1$, and since the functions E_2, E_4, E_6 are algebraically independent over $\mathbb{C}(q)$ by a classical result due to Rankin and Mahler (see the contribution of Bertrand in [12]), $\mathbb{C}[E_2, E_4, E_6]$ is thus isomorphic to a polynomial ring in three indeterminates, endowed with a structure of a differential, filtered (by the depth), graded ring.

Explicitly, if we set $D = (2\pi i)^{-1}d/dz$, the derivation on this ring is described by the following formulas (due to Ramanujan):

$$\begin{cases} DE_2 &= \frac{1}{12}(E_2^2 - E_4) \\ DE_4 &= \frac{1}{3}(E_2E_4 - E_6) \\ DE_6 &= \frac{1}{2}(E_2E_6 - E_4^2). \end{cases} \quad (1)$$

Among several basic questions that we may ask about differential rings, one is to determine their D -differential prime ideals (recall that an ideal \mathcal{I} is said D -differential if $D\mathcal{I} \subset \mathcal{I}$). In [14], the D -differential prime ideals of $\mathbb{C}[E_2, E_4, E_6]$ were all determined applying elementary commutative algebra. There are many others, which we do not describe here, but they all contain Δ . This latter property (with a slight technical hypothesis that we skip here) was used by Nesterenko [11] to prove a *multiplicity estimate*, which was a crucial tool in the proof of his famous theorem on the algebraic independence of values of Eisenstein series: if $z \in \mathcal{H}$, at least three of the four numbers $e^{2\pi iz}, E_2(z), E_4(z), E_6(z)$ are algebraically independent over \mathbb{Q} (see [11], [12, Chapter 3], see also [2, Théorème 1.1]).

Drinfeldian framework. After this survey on quasi-modular forms for the group $\Gamma_{\mathbb{Q}}$, we are ready to introduce the main results of this article.

Let T be an indeterminate, let $A = \mathbb{F}_q[T]$ be the polynomial ring over a finite field \mathbb{F}_q , and let $K = \mathbb{F}_q(T)$ be its field of fractions. Let $K_{\infty} = \mathbb{F}_q((1/T))$ be the completion of K at the infinite prime, and let C be the completion of an algebraic closure of K_{∞} . We will denote by Ω the set $C \setminus K_{\infty}$ and, as usual, by p the characteristic of C .

It is well known that Ω is a connected admissible open subspace of the rigid analytic space $\mathbb{P}_1(C)$. The group $\Gamma_K := \mathbf{GL}_2(A)$ acts discontinuously on Ω by homographies

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_K, \quad z \in \Omega, \quad \gamma(z) = \frac{az + b}{cz + d}.$$

The quotient space can also be given a rigid structure, and it is natural to consider “modular forms” with the straightforward definition.

The foundations of the theory of modular forms on Ω that we need here are essentially contained in [6]. Recall that since Γ_K has the non-trivial character $\det : \Gamma_K \rightarrow \mathbb{F}_q^*$, modular forms naturally have a *type* ⁽²⁾, in addition to a weight. The subring of the ring of holomorphic functions on Ω generated by modular forms is $C[g, h]$, where g and h are two special modular forms, which are defined in [6]. Here, g is the Eisenstein series of weight $q - 1$ and type 0 normalized in a suitable way ⁽³⁾, and h is the Poincaré series $P_{q+1,1}$ of weight $q + 1$ and type 1 (example (5.11) of [6]).

Our first task (Sections 2 to 4) will be to introduce *Drinfeld quasi-modular forms* for the group Γ_K and to study how they behave under the action of the so-called “divided derivatives”.

²It is a class in $\mathbb{Z}/(q-1)\mathbb{Z}$.

³In [6], the notation g changes in the course of the text. In our text we use the notation g for the function g_{new} of [6] (p. 683).

Drinfeld quasi-modular forms will have weights, types and depths, the zero depth corresponding to the case of modular forms. A useful result we will then prove is a structure theorem for these quasi-modular forms, similar to the one of Kaneko and Zagier quoted above. Let E be the “false” Eisenstein series as defined in [6, p. 686]. With our definition, this function will be a quasi-modular form of weight 2, depth 1 and type 1. Denote by $\widetilde{M}_{w,m}^{\leq l}$ the C -vector space of quasi-modular forms of weight w , type m and depth $\leq l$, by \widetilde{M} the ring generated by all quasi-modular forms (see § 2 for the precise definitions), and by $C[E, g, h]_{w,m}^{\leq l}$ the subspace of $C[E, g, h]$ generated by the monomials $E^\alpha g^\beta h^\gamma$ satisfying the conditions $2\alpha + \beta(q-1) + \gamma(q+1) = w$, $\alpha + \gamma \equiv m \pmod{q-1}$ and $\alpha \leq l$.

In Section 2 we prove:

Theorem 1 *The functions E, g, h are algebraically independent over C , and we have*

$$\widetilde{M}_{w,m}^{\leq l} = C[E, g, h]_{w,m}^{\leq l} \quad \text{and} \quad \widetilde{M} = C[E, g, h].$$

Before introducing the differential results of this paper, let us recall that Gekeler proved that the three functions E, g and h satisfy a system of differential equations, namely ⁽⁴⁾:

$$\begin{cases} D_1 E &= E^2 \\ D_1 g &= -(Eg + h) \\ D_1 h &= Eh, \end{cases} \quad (2)$$

where D_1 denotes the operator $(-\bar{\pi})^{-1}d/dz$, $\bar{\pi}$ being a fixed fundamental period of the Carlitz module. Thus, D_1 defines a derivation on the ring $C[E, g, h]$. At first glance, the system (2) seems to be an analogue of (1). But it turns out that the kernel of the derivation D_1 is far too “big” ⁽⁵⁾, and it is hopeless to get results about differential ideals analogous to those mentioned above in the classical case. For instance, the kernel of D_1 contains infinitely many irreducible elements, and so the differential ring $C[E, g, h]$ has infinitely many D_1 -differential principal prime ideals, unlike the characteristic zero case.

⁴See [6]: the last equation corresponds to the definition of E (p. 686), while the first relation corresponds to (8.6) and the second relation is equivalent to Theorem 9.1.

⁵It can be proved that it is equal to the ring

$$C[E^p, g^p, h^p, Eg + h, Eh^{p-1}, E^2 h^{p-2}, \dots, E^{p-1} h].$$

The first important objective of this article is to show that in fact, the ring $C[E, g, h]$ is not only stable under the derivation D_1 , but also under all the “divided derivations”.

Let $f : \Omega \rightarrow C$ be a holomorphic function and $z \in \Omega$ be a point. We define the *divided derivatives* or *hyperderivatives* $(\mathcal{D}_n f)(z)$ of f at z ($n \geq 0$) by the formula

$$f(z + \varepsilon) = \sum_{n \geq 0} (\mathcal{D}_n f)(z) \varepsilon^n, \quad \varepsilon \in C, \quad |\varepsilon| \text{ small.}$$

This defines holomorphic functions $\mathcal{D}_n f : \Omega \rightarrow C$. In order to preserve K -rationality properties, define further $D_n := 1/(-\pi)^n \mathcal{D}_n$ ($n \geq 0$). We then have the following result.

Theorem 2 *Let $n \geq 0$ be an integer. The operators D_n induce C -linear operators*

$$D_n : C[E, g, h] \rightarrow C[E, g, h]$$

such that, for all $(w, l, m) \in \mathbb{N} \times \mathbb{N} \times \mathbb{Z}/(q-1)\mathbb{Z}$,

$$D_n(K[E, g, h]_{w,m}^{\leq l}) \subset K[E, g, h]_{w+2n, m+n}^{\leq l+n}. \quad (3)$$

This Theorem will be proved in Section 3 as a consequence of Proposition 3.1. It is the *hyperdifferential* ring $C[E, g, h]$ (that is, the ring equipped with the operators D_1, D_2, \dots) which should be interpreted as the analogue in the Drinfeldian framework of the differential ring $\mathbb{C}[E_2, E_4, E_6]$.

By Theorem 2, any hyperderivative of a polynomial in the functions E, g, h is again a polynomial in these functions. However, the operators D_n behave erratically and the problem of finding explicit formulas for $D_n E, D_n g$, and $D_n h$ when $n \geq p$ is not a trivial one, in contrast with the classical case, where it suffices to iterate the operator D (see for example the formula (25) of [15]). Some general results concerning the hyperderivatives $D_n E, D_n g$ and $D_n h$ can nevertheless be obtained when n has a particular form. For example, in Proposition 3.7 we determine $D_{p^k} E$ and $D_{p^k} g$ for all k , up to a modular form (see Section 3.5 for other results in the same vein). Also, we give explicit formulas in Section 4 for $D_{p^k} E, D_{p^k} g$ and $D_{p^k} h$ when $p^k \leq q^2$ (Theorem 4.1). As we will see, these formulas allow, at least in principle, to compute $D_n f$ for any $f \in C[E, g, h]$ and any $n \leq q^2$. For instance, in the particular case $q = p$, these formulas (which then determine $D_n f$ even for all $n \leq p^3 - 1$) read

$$\begin{cases} D_p E &= E^{p+1} + \frac{1}{d_1} h^2 \\ D_p g &= E^p g \\ D_p h &= 2E^p h - \frac{1}{d_1} g h^2, \end{cases} \quad (4)$$

$$\begin{cases} D_{p^2}E &= E^{p^2+1} + \frac{1}{d_1^p}g^{p-1}h^{p+1} + \frac{1}{d_2}g^{2p}h^2 \\ D_{p^2}g &= E^{p^2}g - \frac{d_1}{d_2}g^{p+1}h^p + \frac{d_2-d_1^{p+1}}{d_1^{p-1}d_2}h^{2p-1} \\ D_{p^2}h &= E^{p^2}h + \frac{1}{d_1^{p-1}}E^p g^{p-1}h^p - \frac{1}{d_2}g^{2p+1}h^2 - \frac{d_1^{p+1}+d_2}{d_1^p d_2}g^p h^{p+1}, \end{cases} \quad (5)$$

where $d_1 = [1]$ and $d_2 = [2][1]^q$, the notation $[i]$ being defined as usual by $[i] = T^{q^i} - T$.

The second aspect of the theory that we want to develop is, in view of Nesterenko's results quoted before, the study of the ideals which are stable by all the operators D_n . If \mathcal{I} is an ideal of the ring $C[E, g, h]$, we say that \mathcal{I} is *hyperdifferential*, if it is stable under all the operators D_n , that is, if $D_n(\mathcal{I}) \subset \mathcal{I}$ for all $n \geq 0$. Hyperdifferential ideals in $C[E, g, h]$ are natural analogues of D -differential ideals of $\mathbb{C}[E_2, E_4, E_6]$.

Non-trivial hyperdifferential ideals of $C[E, g, h]$ exist: for example, in Proposition 3.6 we will see that the principal ideal (h) is hyperdifferential, in analogy with the ideal (Δ) in the classical case. The next important result we will prove is that in fact, when $q \neq 2, 3$, this ideal is the *only* non-zero hyperdifferential principal prime ideal (again as in the classical case).

More precisely, let us write $F = C(E, g, h)$. The hyperderivatives D_n ($n = 1, 2, \dots$) extend to F in a unique way, we denote these extensions by D_n again. We have:

Theorem 3 *Assume that $q \notin \{2, 3\}$. Let $f \in F^\times$ be such that for all $r \geq 0$ $(D_r f)/f \in C[E, g, h]$. Then, there exists $n \in \mathbb{Z}$ and $c \in C^\times$ such that $f = ch^n$.*

Note that our proof of Theorem 3 really requires that $q \neq 2, 3$, and that we do not know if the result remains true when $q \in \{2, 3\}$. This Theorem is an analogue of Lemma 5.2 of [12, Chapter 10] (or Lemma 4.26 of [2]), but its proof is different, and much more difficult. It requires two steps, contained in Section 5. The first step consists in showing that if f is as in Theorem 3, there exists a p -adic integer $\sigma = s_1 + s_2 p + \dots$ such that for all $k \geq 0$, $f h^{-(s_1 + s_2 p + \dots + s_{k+1} p^k)} \in \ker D_1 \cap \dots \cap \ker D_{p^k}$. Then, with the help of the expansions at infinity of the functions, we show that $\sigma \in \mathbb{Z}$, which yields the result since $\bigcap_{k \geq 0} \ker D_{p^k} = C$.

Somewhat, this approach already appears in the paper [9] by Matzat and van der Put. But these authors work with *linear* hyperdifferential modules over a field, while in our context, the theory is essentially non-linear. This

means that our proofs, even though initially inspired by the work [9], ultimately depend very slightly on it. We should also add that in the proof of Theorem 3, it is not enough to know the formulas (2) (or even the formulas of Section 4). Thus, in this part of our work, the formalism of Kaneko and Zagier becomes very helpful.

As in characteristic zero, one can use Theorem 3 to prove a result for arbitrary non-zero hyperdifferential prime ideals. One gets:

Corollary 4 *Assume that $q \notin \{2, 3\}$. If \mathcal{P} is a non-zero hyperdifferential prime ideal of $C[E, g, h]$, then $h \in \mathcal{P}$.*

Corollary 4 is proved in Section 5.7. It is an analogue of Proposition 5.1 of [12, Chapter 10] (see also Theorem 4.25 of [2]). To prove it, we see from Theorem 3 that we only have to treat the case when the height of the prime ideal \mathcal{P} is ≥ 2 . In characteristic zero, to handle this case the original argument by Nesterenko was a generalisation of an idea of Siegel for the classification of the algebraic solutions of Bessel and Riccati differential equations. This method does not seem to work in our case; to avoid it, we apply a trick appealing to Rankin brackets as in [14], which also works in positive characteristic.

Finally, in Section 6 (Theorem 6.1), we apply Theorem 3 and Corollary 4 to obtain a full description of all the hyperdifferential ideals of $C[E, g, h]$, analogous to Theorem 1.2 of [14].

Remarks about Nesterenko's Theorem. Since it was one of the first motivations of our work, we cannot finish this introduction without a discussion around a possible analogue in the Drinfeldian framework of the above quoted Nesterenko's Theorem. Let $t(z)$ be the usual "parameter at infinity" (see Section 2 for the definition). It seems natural to state the following Conjecture (compare with [4, Conjecture 1]).

Conjecture. *For all $z \in \Omega$, at least three of the four numbers*

$$t(z), E(z), g(z), h(z)$$

are algebraically independent over K .

In the direction of this Conjecture, an important progress would be to prove an analogue of Nesterenko's multiplicity estimate (see [11, Theorem 3], see also [2, Théorème 2.9]). Here too, we could make a Conjecture but we skip it as it merely consists of substituting q and the q -expansions of E_2, E_4, E_6 with t and the t -expansions of E, g, h in Nesterenko's multiplicity estimate (see Section 2 for the precise definition of t -expansion).

As already mentioned, the analogues of Theorem 3 and Corollary 4 in characteristic zero play a decisive role in the proof of Nesterenko’s estimate (see [2, Remarque 4.5 and Théorème 4.22]). However, although the results we obtain here are very similar to those in characteristic zero, they do not suffice to reach a multiplicity estimate: just as the proof of Theorem 3 is much more difficult and need new arguments, it seems that a proof of the multiplicity estimate in characteristic p will also require further ideas (we omit the details about the several occurring obstructions).

Consequently, the above Conjecture seems to be still out of reach. It is worth noting that such a situation is quite atypical: indeed, transcendence (and algebraic independence) theory in the Drinfeldian framework has now strongly been developed, and most of the classical transcendence results have a known analogue in positive characteristic. Recently, the theory has even gone far beyond the classical one in some cases ([1], [13], [3]).

Let us notice that, just as for Nesterenko’s Theorem, the Conjecture does not seem to follow from Grothendieck-like period conjectures in positive characteristic, but only from some variants of André’s conjecture. Hence, it is unclear how it could be handled with the methods of [13].

The results we obtain in this text hopefully constitute a first step towards the Drinfeldian analogue of Nesterenko’s multiplicity estimate. We hope taking others (and maybe all?) in a subsequent paper.

2 Basic properties of quasi-modular forms.

The aim of this Section is to introduce Drinfeld quasi-modular forms and to prove some of their fundamental properties, in particular Theorem 1.

Recall that in the previous Section we have denoted by $\bar{\pi}$ a fixed fundamental period of the Carlitz module. In the whole text, we will further denote by $e_C : C \rightarrow C$ the Carlitz exponential, and by $t : \Omega \rightarrow C$ the usual “parameter at infinity”, that is, $t(z) = 1/e_C(\bar{\pi}z)$. We have $\ker e_C = \bar{\pi}A$ and

$$e_C(z) = \sum_{n \geq 0} \frac{z^{q^n}}{d_n}, \quad (6)$$

where $d_i := [i][i-1]^q \cdots [1]^{q^{i-1}}$ and $[i] := T^{q^i} - T$ (see Section 4 of [6]).

Following the terminology of [6] and [17], we will call *holomorphic function* in Ω an analytic function on Ω in the rigid analytic sense. Holomorphic functions are thus *locally analytic* in the sense of [16] (but the converse is false).

The *imaginary part* of an element $z \in C$ (see Section 5 of [6]) will be denoted by $|z|_i$. If $f : \Omega \rightarrow C$ is holomorphic in Ω and A -periodic (*i.e.* $f(z+a) = f(z)$ for all $a \in A$ and $z \in \Omega$), we will say that it is *holomorphic at infinity* if $f(z)$ is the value, for $|z|_i$ sufficiently large, of a converging series

$$f(z) = \sum_{i \geq 0} c_i t(z)^i, \quad (7)$$

where c_0, c_1, \dots are elements of C . These coefficients uniquely determine f and we will often write, by abuse of notation, $f = \sum_{i \geq 0} c_i t^i$. The series (7) is the *t-expansion* of f .

In the whole text, we will denote by \mathcal{O} the ring of holomorphic functions $f : \Omega \rightarrow C$ which are A -periodic and holomorphic at infinity. This ring contains in particular the modular forms for Γ_K (by Definition 5.9 of [6]). If f is a non-zero element of \mathcal{O} , we will denote by $\nu_\infty(f)$ its *order at infinity*: this is the smallest integer i such that $c_i \neq 0$ in the t -expansion (7). The map ν_∞ extends to a map $\nu_\infty : \text{Frac}(\mathcal{O}) \setminus \{0\} \rightarrow \mathbb{Z}$ in the obvious way.

Definition 2.1 *Let $w \geq 0$ be an integer and $m \in \mathbb{Z}/(q-1)\mathbb{Z}$. A holomorphic function $f : \Omega \rightarrow C$ is called quasi-modular form of weight w and type m if there exist functions $f_0, \dots, f_l \in \mathcal{O}$ such that, for all $z \in \Omega$ and $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_K$:*

$$f(\gamma(z)) = (cz + d)^w (\det \gamma)^{-m} \sum_{i=0}^l f_i(z) \left(\frac{c}{cz + d} \right)^i. \quad (8)$$

The type m of a quasi-modular form is a class in $\mathbb{Z}/(q-1)\mathbb{Z}$. However, to avoid heavy notations, we will sometimes use the same notation m for any representant of the class, thus considering m as an integer. This abuse of notation will not lead to confusion.

Remarks. (i). Let f be a quasi-modular form of weight w and type m , and let $f_0, \dots, f_l \in \mathcal{O}$ be as in the definition. It is easy to see that if $f \neq 0$, then the weight w , the type m and the polynomial $\sum_{i=0}^l f_i X^i \in \mathcal{O}[X]$ are uniquely determined by f . Indeed, if the function f were also a quasi-modular form of weight $w' \geq w$ and type m' , and if an identity analogous to (8) held with functions $g_0, \dots, g_{l'} \in \mathcal{O}$ at the place of f_0, \dots, f_l , then we would have

$$c^{w-w'} (\det \gamma)^{m'-m} A_z \left(\frac{c}{cz + d} \right) - B_z \left(\frac{c}{cz + d} \right) = 0 \quad (9)$$

for all $z \in \Omega$ and all $\gamma \in \Gamma_K$ with $c \neq 0$, where

$$A_z = X^{w'-w} \sum_{i=0}^l f_i(z) X^i \in C[X] \quad \text{and} \quad B_z = \sum_{i=0}^{l'} g_i(z) X^i \in C[X].$$

Choosing now $\gamma = \begin{pmatrix} 1 & d-1 \\ 1 & d \end{pmatrix} \in \Gamma_K$ in (9) (with $d \in A$ arbitrary), we obtain $A_z = B_z$ for all $z \in \Omega$. By fixing an element $z \in \Omega$ such that $A_z \neq 0$, choosing $c \in A$ with $\deg c \geq 1$ such that $A_z(c/(cz+d)) \neq 0$, and taking $\gamma = \begin{pmatrix} \lambda & 0 \\ c & 1 \end{pmatrix}$ with $\lambda \in \mathbb{F}_q^*$ in the equation (9), we get

$$\lambda^{m'-m} = c^{w'-w} \quad \text{for all } \lambda \in \mathbb{F}_q^*.$$

This implies $w = w'$ and $m = m'$ in $\mathbb{Z}/(q-1)\mathbb{Z}$. The equality $A_z = B_z$ for all z then yields the equality between the two polynomials $\sum_{i=0}^l f_i X^i$ and $\sum_{i=0}^{l'} g_i X^i$, as claimed.

In virtue of this remark, if f is any non-zero quasi-modular form, the polynomial

$$P_f := \sum_{i=0}^l f_i X^i$$

of $\mathcal{O}[X]$ is well defined. We call it the *associated polynomial* of f . The degree of P_f is called the *depth* of f and is usually denoted by l ; another notation is $l(f)$. If $f = 0$, we agree that it is a quasi-modular form of weight w , type m and depth l for all w, m, l , and we set $P_f := 0$.

We denote by $\widetilde{M}_{w,m}^l$ (*resp.* by $\widetilde{M}_{w,m}^{\leq l}$) the set (*resp.* the C -vector space) of quasi-modular forms of weight w , type m and depth l (*resp.* depth $\leq l$). Note that $\widetilde{M}_{w,m}^l$ is not a vector space for $l \geq 1$. We will further denote by $\widetilde{M}_{w,m}$ the C -vector space of quasi-modular forms of weight w and type m , and by \widetilde{M} the C -algebra $\widetilde{M} = \sum_{w,m} \widetilde{M}_{w,m}$. In fact, this sum is a direct sum (see Proposition 2.2 below).

(ii). If f is an element of $\widetilde{M}_{w,m}^l$, then the constant term f_0 of the associated polynomial P_f is necessarily equal to f , as follows from the formula (8) by choosing for γ the identity matrix.

(iii). It is clear from the definition that a modular form f of weight w and type m is an element of $\widetilde{M}_{w,m}^0$ with associated polynomial $P_f = f$.

Conversely, Remark (ii) shows that any element f of $\widetilde{M}_{w,m}^0$ is a modular form of weight w and type m , thus we have:

$$M_{w,m} = \widetilde{M}_{w,m}^0, \quad (10)$$

where $M_{w,m}$ denotes the C -vector space of Drinfeld modular forms of weight w and type m (equal to M_w^m , following the notation of [6]).

(iv). If $f_1 \in \widetilde{M}_{w_1,m_1}^{l_1}$ and $f_2 \in \widetilde{M}_{w_2,m_2}^{l_2}$, it is obvious from the definitions that $f_1 f_2 \in \widetilde{M}_{w_1+w_2,m_1+m_2}^{l_1+l_2}$ and that $P_{f_1 f_2} = P_{f_1} P_{f_2}$. Moreover, $P_{\lambda f_1} = \lambda P_{f_1}$ for $\lambda \in C$, and if $w_1 = w_2$, $m_1 = m_2$, then $f_1 + f_2 \in \widetilde{M}_{w_1,m_1}^{\leq \max\{l_1,l_2\}}$ with $P_{f_1+f_2} = P_{f_1} + P_{f_2}$.

An important example of a quasi-modular form which is not a modular form is furnished by the “false” Eisenstein series $E(z)$, which satisfies (see [6, formula (8.4)])

$$E(\gamma(z)) = (cz + d)^2 (\det \gamma)^{-1} \left(E(z) - \frac{c}{\pi(cz + d)} \right) \quad (11)$$

for all $z \in \Omega$ and all $\gamma \in \Gamma_K$. In virtue of the results of Section 8 of [6], we have $E \in \mathcal{O}$. Thus, $E \in \widetilde{M}_{2,1}^1$ and $P_E = E - \overline{\pi}^{-1} X$.

Proposition 2.2 *We have:*

$$\widetilde{M} = \bigoplus_{\substack{w \geq 0 \\ m \in \mathbb{Z}/(q-1)\mathbb{Z}}} \widetilde{M}_{w,m}.$$

Proof. The result can be proved exactly as in [10, proof of Lemma 16] and is thus left to the reader. \square

Throughout this text, we will denote by $C[E, g, h]_{w,m}^{\leq l}$ the space of elements of $C[E, g, h]$ of weight w , type m and depth $\leq l$, that is, the C -vector space of polynomials in E, g, h which are linear combinations with coefficients in C of monomials $E^\alpha g^\beta h^\gamma$ with $2\alpha + \beta(q-1) + \gamma(q+1) = w$, $\alpha + \gamma \equiv m \pmod{q-1}$ and $\alpha \leq l$.

To prove Theorem 1, we will follow a method similar to the one used in [10] in the complex case. We need four Lemmata. In the next Lemma, we will say that a polynomial $P \in C[Y_1, Y_2, Y_3]$ is *isobaric* of weight $w \in \mathbb{N}$ if it is the sum of monomials of the form $\lambda Y_1^\alpha Y_2^\beta Y_3^\gamma$, where $\lambda \in C$ and α, β, γ satisfy $2\alpha + (q-1)\beta + (q+1)\gamma = w$. Thus, an isobaric element of $C[Y_1, Y_2, Y_3]$

is nothing but a homogeneous element for the graduation defined associating to Y_1, Y_2, Y_3 the weights $2, q-1, q+1$.

If all the monomials above are moreover such that $\alpha + \gamma \equiv m \pmod{q-1}$, then we will say that P is of type $m \in \mathbb{Z}/(q-1)\mathbb{Z}$.

Lemma 2.3 *Let $P \in C[Y_1, Y_2, Y_3]$ be a non-zero polynomial isobaric of weight w and type m . Then the function $P(E, g, h)$ is a quasi-modular form of weight w , type m and depth $l = \deg_{Y_1} P$.*

Proof. Write

$$P = \sum_{\alpha=0}^l p_{\alpha}(Y_2, Y_3) Y_1^{\alpha},$$

where $p_l \neq 0$ and p_{α} is an isobaric polynomial of weight $w - 2\alpha$ and type $m - \alpha$ ($0 \leq \alpha \leq l$). Each function $p_{\alpha}(g, h)E^{\alpha}$ is clearly an element of $\widetilde{M}_{w,m}^{\alpha}$. But it is plain that the sum of two non-zero elements of $\widetilde{M}_{w,m}$ of different depths α_1, α_2 is again an element of $\widetilde{M}_{w,m}$ of depth $\max\{\alpha_1, \alpha_2\}$. It follows at once that $P(E, g, h) \in \widetilde{M}_{w,m}^{\deg_{Y_1} P}$. \square

Lemma 2.4 *The functions E, g and h are algebraically independent over C .*

Proof. Let $P \in C[Y_1, Y_2, Y_3]$ be such that $P(E, g, h) = 0$. We can write

$$P = \sum_{w=0}^d \sum_{m \in \mathbb{Z}/(q-1)\mathbb{Z}} P_{w,m},$$

where $P_{w,m}$ is an isobaric polynomial of weight w and type m . From $P(E, g, h) = 0$ and from Proposition 2.2 it follows that $P_{w,m}(E, g, h) = 0$ for each pair (w, m) . We fix w and m and write

$$P_{w,m} = \sum_{\alpha \geq 0} p_{\alpha}(Y_2, Y_3) Y_1^{\alpha},$$

where $p_{\alpha} \in C[Y_2, Y_3]$ is isobaric of weight $w - 2\alpha$ and type $m - \alpha$. Suppose that there is an index $\alpha_0 \geq 1$ such that $p_{\alpha_0}(g, h) \neq 0$, and take α_0 maximal. We have

$$p_{\alpha_0}(g, h)E^{\alpha_0} = - \sum_{0 \leq \alpha < \alpha_0} p_{\alpha}(g, h)E^{\alpha}. \quad (12)$$

Now, $p_{\alpha_0}(g, h)E^{\alpha_0}$ is a non-zero quasi-modular form of depth exactly α_0 , and the right-hand side of (12) is a quasi-modular form of depth at most

$\alpha_0 - 1$ (Lemma 2.3). By uniqueness of the depth, this is impossible. Hence we deduce that $p_\alpha(g, h) = 0$ for all α . Since now g and h are known to be algebraically independent over C , we get $p_\alpha = 0$ for all α , and thus $P_{w,m} = 0$. This being true for all (w, m) , we have $P = 0$, which completes the proof of the Lemma. \square

Lemma 2.5 *Let $f \in \widetilde{M}_{w,m}^l$, and let $P_f = \sum_{i=0}^l f_i X^i$ be its associated polynomial. Then, for all i , $0 \leq i \leq l$, the function f_i is an element of the space $\widetilde{M}_{w-2i, m-i}^{\leq l-i}$, whose associated polynomial P_{f_i} is explicitly given by*

$$P_{f_i} = \sum_{j=i}^l \binom{j}{i} f_j X^{j-i}.$$

In particular, we have $f_l \in M_{w-2l, m-l}$.

Proof. The same proof as in [10, Lemma 119] works. First, an easy computation shows that for $N = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \Gamma_K$, $z \in \Omega$ and $i \in \mathbb{N}$ with $0 \leq i \leq l$, the following equality holds:

$$f_i(z) = \sum_{j=i}^l \binom{j}{i} (-\gamma)^{j-i} (\gamma(z) + \delta)^{i+j-w} (\det N)^{m-j} f_j(Nz).$$

Substituting z by $L(z)$, where $L = \begin{pmatrix} a & b \\ c & d \end{pmatrix} := N^{-1}$, then gives the following transformation formula for $f_i(L(z))$:

$$f_i(L(z)) = (cz + d)^{w-2i} (\det L)^{-(m-i)} \sum_{j=i}^l \binom{j}{i} f_j(z) \left(\frac{c}{cz+d}\right)^{j-i}.$$

The assertions of the Lemma follow at once. \square

Lemma 2.6 *For all integers $w \geq 0$, $l \geq 0$ and for all $m \in \mathbb{Z}/(q-1)\mathbb{Z}$, we have:*

$$\widetilde{M}_{w,m}^{\leq l} = \bigoplus_{0 \leq i \leq l} M_{w-2i, m-i} E^i.$$

Proof. We already know from Lemma 2.4 that the sum

$$\sum_{0 \leq i \leq l} M_{w-2i, m-i} E^i$$

is a direct sum, and that it is a subspace of $\widetilde{M}_{w,m}^l$ (Lemma 2.3). So it remains to prove the opposite inclusion. We argue by induction on l . If $l = 0$, the result is true (it is equality (10)). Suppose now that the result is true for $l - 1$, where $l \geq 1$. Let $f \in \widetilde{M}_{w,m}^l$, $f \neq 0$, and let $P_f = \sum_{i=0}^l f_i X^i$ be its associated polynomial. Since $P_E = -\bar{\pi}^{-1}X + E$ (see formula (11)), we have $P_{E^l} = (-\bar{\pi}^{-1}X + E)^l$. Thus, since we have $f_l \in M_{w-2l,m-l}$ by Lemma 2.5, we get

$$P_{f_l E^l} = (-\bar{\pi})^{-l} f_l X^l + \text{terms of degree} < l.$$

Define now $F := f - (-\bar{\pi})^l f_l E^l$. The function F is a quasi-modular form of weight w and type m with associated polynomial $P_F = P_f - (-\bar{\pi})^l P_{f_l E^l}$, whose degree is by construction at most $l - 1$. Hence $F \in \widetilde{M}_{w,m}^{\leq l-1}$. By induction we can write now $F = \sum_{i=0}^{l-1} F_i E^i$, where $F_i \in M_{w-2i,m-i}$ for all i , from which it follows

$$f = F + (-\bar{\pi})^l f_l E^l \in \bigoplus_{0 \leq i \leq l} M_{w-2i,m-i} E^i.$$

□

Proof of Theorem 1. Since $C[E, g, h]_{\bar{w},m}^{\leq 0} = M_{w,m}$ (see [6]), we obviously have

$$C[E, g, h]_{\bar{w},m}^{\leq l} = \bigoplus_{0 \leq i \leq l} M_{w-2i,m-i} E^i.$$

Therefore Theorem 1 follows from Lemmata 2.4 and 2.6. □

Let L be a subfield of C containing K (recall that $K = \mathbb{F}_q(T)$). Denote by $\widetilde{M}_{\bar{w},m}^{\leq l}(L)$ the L -vector space of elements of $\widetilde{M}_{\bar{w},m}^{\leq l}$ having a t -expansion at infinity with coefficients in L , and define $L[E, g, h]_{\bar{w},m}^{\leq l}$ similarly as before, *i.e.* as the L -vector space $C[E, g, h]_{\bar{w},m}^{\leq l} \cap L[E, g, h]$. Theorem 1 gives rise to the following rationality result.

Corollary 2.7 *Let $L \subset C$ be any subfield of C containing K . Then*

$$\widetilde{M}_{\bar{w},m}^{\leq l}(L) = L[E, g, h]_{\bar{w},m}^{\leq l}.$$

Proof. Since the functions E, g, h are elements of $\widetilde{M}_{\bar{w},m}^{\leq l}(L)$ (the coefficients of their t -expansions are in A , see [6]), we already have the inclusion

$$L[E, g, h]_{\bar{w},m}^{\leq l} \subset \widetilde{M}_{\bar{w},m}^{\leq l}(L). \quad (13)$$

Extending now the scalars from L to C , we get

$$C[E, g, h]_{w,m}^{\leq l} \subset \widetilde{M}_{w,m}^{\leq l}(L) \otimes_L C \subset \widetilde{M}_{w,m}^{\leq l}(C) = \widetilde{M}_{w,m}^{\leq l}.$$

By Theorem 1, these inclusions between C -vector spaces are in fact equalities, and thus

$$\begin{aligned} \dim_L \widetilde{M}_{w,m}^{\leq l}(L) &= \dim_C (\widetilde{M}_{w,m}^{\leq l}(L) \otimes_L C) = \dim_C C[E, g, h]_{w,m}^{\leq l} \\ &= \dim_L L[E, g, h]_{w,m}^{\leq l}. \end{aligned}$$

It then follows from (13) that $L[E, g, h]_{w,m}^{\leq l} = \widetilde{M}_{w,m}^{\leq l}(L)$. \square

3 Hyperderivatives and quasi-modular forms

In this Section, we study the hyperderivatives of quasi-modular forms. After recalling some basic properties on higher derivations and hyperderivatives in Section 3.1, we prove in Section 3.2 a very important proposition (Proposition 3.1), which gives the connection between the polynomial P_f associated to a quasi-modular form f and the polynomial $P_{\mathcal{D}_n f}$ associated to its n -th hyperderivative $\mathcal{D}_n f$. Using this result, we can then prove Theorem 2 in Section 3.3, and the fact that the ideal (h) is hyperdifferential in Section 3.4. Finally, we determine in Section 3.5, up to a modular form, the hyperderivatives $D_{p^i} E$, $D_{p^i} g$ for all integers $i \geq 0$, and we prove a similar result for the $D_{p^i} h$'s.

In what follows, we put as usual $\binom{n}{i} = 0$ if $0 \leq n < i$ and $\binom{n}{i} = (-1)^i \binom{i-n-1}{i}$ if $i \geq 0$, $n < 0$. In particular, we have $\binom{n}{0} = 1$ for all $n \in \mathbb{Z}$.

We also recall the following fact, which will be often used in the sequel: if r is any positive integer and $n = n_0 + n_1 p^r + \dots + n_s p^{r_s}$, $i = i_0 + i_1 p^r + \dots + i_s p^{r_s}$ are two elements of \mathbb{N} written in base p^r , then one has in \mathbb{F}_p :

$$\binom{n}{i} = \binom{n_0}{i_0} \binom{n_1}{i_1} \dots \binom{n_s}{i_s}. \quad (14)$$

The formula (14) was first observed by Lucas in [7] (strictly speaking for $r = 1$) and can be easily derived by identifying the coefficient of X^i in both sides of the following equality of polynomials of $\mathbb{F}_p[X]$ (we expand of course each factor occurring in both sides with the help of the binomial formula):

$$(1 + X)^n = (1 + X)^{n_0} (1 + X^{p^r})^{n_1} \dots (1 + X^{p^{r_s}})^{n_s}.$$

3.1 Background on higher derivations and hyperderivatives

We begin by a short review on iterative higher derivations, following Matsumura [8, Section 27].

After [8], a *higher derivation* $(D_n)_{n \in \mathbb{N}}$ on a C -algebra R is a family of C -linear maps from R to itself, with D_0 the identity map, such that if $f, g \in R$, then the following Leibniz's rule holds

$$D_i(fg) = \sum_{r=0}^i (D_r f)(D_{i-r} g). \quad (15)$$

In particular, the following formula holds for $f \in R$ and for all integers $n, k \geq 0$ (see [8, p. 209])

$$D_{np^k}(f^{p^k}) = (D_n f)^{p^k}. \quad (16)$$

A higher derivation $(D_n)_{n \in \mathbb{N}}$ is *iterative* [8, p. 209] if the following formula holds, for all integers $i, j \geq 0$:

$$D_i \circ D_j = D_j \circ D_i = \binom{i+j}{i} D_{i+j}. \quad (17)$$

It follows from (17) and (14) with $r = 1$ that the operators D_n , $n \geq 0$, are completely determined by the operators D_{p^k} for $k \geq 0$. More precisely, if $n = n_s p^s + \dots + n_1 p + n_0$ is the representation of n in base p (so that $0 \leq n_i \leq p - 1$ for all i), then the following formula holds:

$$D_n = \frac{1}{n_s! \dots n_0!} D_{p^s}^{n_s} \circ \dots \circ D_p^{n_1} \circ D_1^{n_0}. \quad (18)$$

We now review some basic properties of hyperderivatives. Let $f : \Omega \rightarrow C$ be a locally analytic function and $z \in \Omega$ be a point. Recall that we have defined the hyperderivatives $(\mathcal{D}_n f)(z)$ of f at z by the formula

$$f(z + \varepsilon) = \sum_{n \geq 0} (\mathcal{D}_n f)(z) \varepsilon^n, \quad \varepsilon \in C, \quad |\varepsilon| \text{ small}$$

(this corresponds to Definition 2.3 of [16]).

By Corollary 2.5 of [16], the family of operators $(\mathcal{D}_n)_{n \in \mathbb{N}}$ define a higher derivation on the C -algebra of locally analytic functions, and it is also well-known (and easy to check) that this higher derivation is iterative.

Another important feature of the operators \mathcal{D}_n is their good behaviour under holomorphy: If f is holomorphic on Ω and $n \in \mathbb{N}$, then the function

$\mathcal{D}_n f$ is also holomorphic on Ω by [16, Theorem 3.1], and if moreover $f \in \mathcal{O}$, then it follows from [16, Lemma 3.6] that $\mathcal{D}_n f \in \mathcal{O}$.

As explained in Section 1, it will be also convenient to work with the following operators

$$D_n := \frac{1}{(-\bar{\pi})^n} \mathcal{D}_n \quad (n \geq 0). \quad (19)$$

The family $(D_n)_n$ is also an iterative higher derivation on \mathcal{O} , and all the properties quoted above for the operators \mathcal{D}_n remain true for the operators D_n .

3.2 Polynomial associated to a hyperderivative of a quasi-modular form

We prove here Proposition 3.1. This proposition will be a crucial tool in the subsequent sections 3.3, 3.4 and 3.5.

Proposition 3.1 *Let $f \in \widetilde{M}_{w,m}^{\leq l}$ be a quasi-modular form of weight w , type m and depth $\leq l$, and let $P_f = \sum_{i=0}^l f_i X^i$ be its associated polynomial. Then, for all $n \geq 0$, $\mathcal{D}_n f$ is an element of $\widetilde{M}_{w+2n, m+n}^{\leq l+n}$, with associated polynomial*

$$P_{\mathcal{D}_n f} = \sum_{j=0}^{n+l} \left[\sum_{r=0}^n \binom{n+w+r-j-1}{r} \mathcal{D}_{n-r} f_{j-r} \right] X^j,$$

with the convention that $f_i = 0$ if $i < 0$ or $i > l$.

In the sequel, we will adopt the following notations. For $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_K$ and $z \in \Omega$, we write

$$X_\gamma(z) := c/(cz + d), \quad J_\gamma(z) := (cz + d)^2/(\det \gamma).$$

If $f : \Omega \rightarrow C$ is a holomorphic function, we will also denote by $f \circ \gamma$ the function $z \mapsto f(\gamma(z))$.

The proof of Proposition 3.1 requires three Lemmata.

Lemma 3.2 *Let M, N, W be integers with $N \geq 0$. Then we have*

$$\begin{aligned} \sum_{i=0}^N (-1)^i \binom{M}{N-i} \binom{W+i-1}{i} &= \binom{M-W}{N} \\ &= (-1)^N \binom{W+N-M-1}{N}. \end{aligned}$$

Proof. We have in $\mathbb{F}_q[[X]]$:

$$(1 + X)^M = \sum_{i \geq 0} \binom{M}{i} X^i \quad \text{and} \quad (1 + X)^{-W} = \sum_{i \geq 0} (-1)^i \binom{W + i - 1}{i} X^i,$$

so by multiplying we derive

$$(1 + X)^{M-W} = \sum_{N \geq 0} \left[\sum_{i=0}^N (-1)^i \binom{M}{N-i} \binom{W+i-1}{i} \right] X^N.$$

Since on the other hand $(1+X)^{M-W} = \sum_{N \geq 0} \binom{M-W}{N} X^N$, the result follows. \square

Lemma 3.3 *Let $f : \Omega \rightarrow C$ be a holomorphic function. For all $n \geq 1$, $z \in \Omega$ and $\gamma \in \Gamma_K$, we have*

$$(\mathcal{D}_n(f \circ \gamma))(z) = (-1)^n X_\gamma(z)^n \sum_{i=1}^n (-1)^i \binom{n-1}{n-i} (X_\gamma(z) J_\gamma(z))^{-i} (\mathcal{D}_i f)(\gamma(z)). \quad (20)$$

Proof. Let $\varepsilon \in \Omega$ be sufficiently small. We compute easily

$$\gamma(z + \varepsilon) = \frac{\gamma(z) + \frac{a}{cz+d}\varepsilon}{1 + X_\gamma(z)\varepsilon} = \gamma(z) + H,$$

where

$$H = J_\gamma(z)^{-1} \sum_{j \geq 1} (-1)^{j-1} X_\gamma(z)^{j-1} \varepsilon^j.$$

Let us remark that H tends to zero when ε tends to zero. We have for $i \geq 1$

$$H^i = J_\gamma(z)^{-i} \sum_{n \geq i} (-1)^{n-i} X_\gamma(z)^{n-i} \binom{n-1}{n-i} \varepsilon^n,$$

which gives

$$\begin{aligned} f(\gamma(z + \varepsilon)) &= f(\gamma(z) + H) = f(\gamma(z)) + \sum_{i \geq 1} (\mathcal{D}_i f)(\gamma(z)) H^i \\ &= f(\gamma(z)) + \sum_{n \geq 1} (-1)^n \varepsilon^n \sum_{1 \leq i \leq n} (-1)^i \binom{n-1}{n-i} J_\gamma(z)^{-i} X_\gamma(z)^{n-i} (\mathcal{D}_i f)(\gamma(z)). \end{aligned}$$

Lemma 3.3 follows at once. \square

Lemma 3.4 Let f be a non-zero element of $\widetilde{M}_{w,m}^l$, and let $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_K$. Define $\varphi_\gamma : \Omega \rightarrow C$ by $\varphi_\gamma(z) := (\det \gamma)^m (cz + d)^{-w} f(\gamma(z))$. For $n \geq 0$ and $z \in \Omega$, we have

$$(\mathcal{D}_n f)(\gamma(z)) = \frac{(cz + d)^{w+2n}}{(\det \gamma)^{m+n}} \sum_{j=0}^n \binom{w+n-1}{n-j} X_\gamma(z)^{n-j} (\mathcal{D}_j \varphi_\gamma)(z). \quad (21)$$

Proof. We argue by induction on n . For $n = 0$ the Lemma is trivially true. Let us now suppose that $n \geq 1$ and that the Lemma has been proved for all integers $< n$. First of all, rewriting the formula (20) from Lemma 3.3, we obtain

$$\begin{aligned} (\mathcal{D}_n f)(\gamma(z)) &= J_\gamma(z)^n (\mathcal{D}_n (f \circ \gamma))(z) \\ &\quad + (-1)^{n-1} \sum_{i=1}^{n-1} (-1)^i \binom{n-1}{n-i} X_\gamma(z)^{n-i} J_\gamma(z)^{n-i} (\mathcal{D}_i f)(\gamma(z)). \end{aligned}$$

Next, using Lemma 3.4 for $(\mathcal{D}_i f)(\gamma(z))$ (induction hypothesis) and exchanging the priority of the summations thus obtained, we get

$$\begin{aligned} (\mathcal{D}_n f)(\gamma(z)) &= J_\gamma(z)^n (\mathcal{D}_n (f \circ \gamma))(z) \\ &\quad + (cz + d)^w (\det \gamma)^{-m} \sum_{j=0}^{n-1} \left[X_\gamma(z)^{n-j} J_\gamma(z)^n (\mathcal{D}_j \varphi_\gamma)(z) \right. \\ &\quad \left. \times \sum_{i=j}^{n-1} (-1)^{n-1+i} \binom{n-1}{n-i} \binom{w+i-1}{i-j} \right]. \quad (22) \end{aligned}$$

In order to compute $\mathcal{D}_n (f \circ \gamma)$ in this formula, we write

$$(f \circ \gamma)(z) = (\det \gamma)^{-m} (cz + d)^w \varphi_\gamma(z)$$

and apply Leibniz's rule (15). We find

$$\begin{aligned} (\mathcal{D}_n (f \circ \gamma))(z) &= \mathcal{D}_n ((\det \gamma)^{-m} (cz + d)^w \varphi_\gamma(z)) \\ &= (\det \gamma)^{-m} \sum_{j=0}^n \mathcal{D}_{n-j} ((cz + d)^w) (\mathcal{D}_j \varphi_\gamma)(z) \\ &= (\det \gamma)^{-m} (cz + d)^w \sum_{j=0}^n \binom{w}{n-j} X_\gamma(z)^{n-j} (\mathcal{D}_j \varphi_\gamma)(z), \end{aligned}$$

the last line coming from the following formula, which is easy to check:

$$\mathcal{D}_i((cz + d)^w) = \binom{w}{i} X_\gamma(z)^i (cz + d)^{w-i} \quad (i \geq 0).$$

Substituting in (22) the expression obtained for $\mathcal{D}_n(f \circ \gamma)$, we derive

$$\begin{aligned} (\mathcal{D}_n f)(\gamma(z)) &= \frac{(cz + d)^w}{(\det \gamma)^m} J_\gamma(z)^n \sum_{j=0}^n X_\gamma(z)^{n-j} (\mathcal{D}_j \varphi_\gamma)(z) \left[\binom{w}{n-j} \right. \\ &\quad \left. + (-1)^{n-1} \sum_{i=0}^{n-1-j} (-1)^{i+j} \binom{n-1}{n-i-j} \binom{w+i+j-1}{i} \right], \end{aligned}$$

which gives, using Lemma 3.2 with $N = n - j$, $M = n - 1$ and $W = w + j$,

$$(\mathcal{D}_n f)(\gamma(z)) = \frac{(cz + d)^{w+2n}}{(\det \gamma)^{m+n}} \sum_{j=0}^n \binom{w+n-1}{n-j} X_\gamma(z)^{n-j} (\mathcal{D}_j \varphi_\gamma)(z).$$

This is the desired formula for n . □

Proof of Proposition 3.1. Let γ be an element of Γ_K and let φ_γ be defined as in Lemma 3.4. Then we have, for all $z \in \Omega$:

$$\varphi_\gamma(z) = \sum_{i=0}^l f_i(z) X_\gamma(z)^i. \quad (23)$$

The following formula holds for all $i, r \geq 0$, and is easy to check:

$$\mathcal{D}_r(X_\gamma^i) = (-1)^r \binom{i+r-1}{r} X_\gamma^{i+r}. \quad (24)$$

Thus, Leibniz's formula (15) gives, for all j such that $0 \leq j \leq n$,

$$\mathcal{D}_j(f_i X_\gamma^i) = \sum_{r=0}^j (-1)^{j-r} \binom{i+j-r-1}{j-r} X_\gamma^{i+j-r} \mathcal{D}_r f_i. \quad (25)$$

Using now successively (21), (23) and (25), we obtain

$$\begin{aligned}
\frac{(\det \gamma)^{m+n}}{(cz+d)^{w+2n}} (\mathcal{D}_n f)(\gamma(z)) &= \sum_{j=0}^n \binom{k+n-1}{n-j} X_\gamma(z)^{n-j} \sum_{i=0}^l (\mathcal{D}_j(f_i X_\gamma^i))(z) \\
&= \sum_{\substack{0 \leq j \leq n \\ 0 \leq r \leq j \\ 0 \leq i \leq l}} (-1)^{j-r} \binom{k+n-1}{n-j} \binom{i+j-r-1}{j-r} X_\gamma(z)^{n+i-r} (\mathcal{D}_r f_i)(z) \\
&= \sum_{\substack{0 \leq u \leq n+l \\ 0 \leq r \leq n \\ r \leq j \leq n}} (-1)^{j-r} \binom{w+n-1}{n-j} \binom{u-n+j-1}{j-r} X_\gamma(z)^u (\mathcal{D}_r f_{u-n+r})(z).
\end{aligned}$$

This formula shows already that $(\det \gamma)^{m+n} (cz+d)^{-(k+2n)} (\mathcal{D}_n f)(\gamma(z))$ is a polynomial in $X_\gamma(z)$ of degree $\leq n+l$ with coefficients in \mathcal{O} , whence $\mathcal{D}_n f \in \widetilde{M}_{w+2n, m+n}^{\leq l+n}$. Finally, we get the desired formula for $P_{\mathcal{D}_n f}$ by making the change of summation index $r' = n-r$ in the last formula, and then using Lemma 3.2 with $N = r'$, $M = w+n-1$, $W = u-r'$. \square

3.3 Proof of Theorem 2

As a first Corollary of Proposition 3.1, we will now derive Theorem 2. We will need the following Lemma, which will also be useful later.

Lemma 3.5 *Let $f : \Omega \rightarrow C$ be an element of the ring \mathcal{O} having the t -expansion*

$$f(z) = \sum_{n \geq 0} a_n t^n, \quad (26)$$

where $t = t(z)$. Then, for all integers $i \geq 1$, one has

$$(D_i f)(z) = \sum_{n \geq 2} b_{i,n} t^n,$$

where

$$b_{i,n} = \sum_{r=1}^{n-1} (-1)^{i+r} \binom{n-1}{r} \left(\sum_{\substack{i_1, \dots, i_r \geq 0 \\ q^{i_1} + \dots + q^{i_r} = i}} \frac{1}{d_{i_1} \dots d_{i_r}} \right) a_{n-r}.$$

Proof. Fix $z \in \Omega$ with $|z|_i$ sufficiently large, so that f has a t -expansion (26), and let ε be any element of the field K_∞ . Then $|z + \varepsilon|_i = |z|_i$, and thus

$$f(z + \varepsilon) = \sum_{n \geq 0} a_n t(z + \varepsilon)^n. \quad (27)$$

Now, putting as usual $t := t(z)$ and using the formula (6), we have, for $|\varepsilon|$ sufficiently small:

$$\begin{aligned} t(z + \varepsilon)^n &= \frac{1}{(e_{\mathcal{C}}(\overline{\pi}z) + e_{\mathcal{C}}(\overline{\pi}\varepsilon))^n} = \frac{t^n}{(1 + e_{\mathcal{C}}(\overline{\pi}\varepsilon)t)^n} \\ &= \sum_{r \geq 0} (-1)^r \binom{r+n-1}{r} (e_{\mathcal{C}}(\overline{\pi}\varepsilon))^r t^{r+n} \\ &= \sum_{r \geq 0} (-1)^r \binom{r+n-1}{r} t^{r+n} \left(\sum_{i \geq 0} \frac{\overline{\pi}^i \varepsilon^i}{d_i} \right)^r \\ &= \sum_{r \geq 0} (-1)^r \binom{r+n-1}{r} t^{r+n} \sum_{i \geq 0} \alpha_{r,i} \overline{\pi}^i \varepsilon^i, \end{aligned} \quad (28)$$

where

$$\alpha_{r,i} = \sum_{\substack{i_1, \dots, i_r \geq 0 \\ q^{i_1} + \dots + q^{i_r} = i}} \frac{1}{d_{i_1} \cdots d_{i_r}} \quad \text{if } r \geq 1 \text{ and } i \geq 0,$$

and where

$$\alpha_{0,i} = \begin{cases} 1 & \text{for } i = 0 \\ 0 & \text{for } i \neq 0. \end{cases}$$

Substituting (28) in (27) and exchanging the order of the summations, we get

$$f(z + \varepsilon) = \sum_{i \geq 0} \left[\sum_{r \geq 0} \sum_{n \geq 0} (-1)^r a_n \alpha_{r,i} \binom{r+n-1}{r} t^{r+n} \right] \overline{\pi}^i \varepsilon^i.$$

Since on the other hand we have (by virtue of (19))

$$f(z + \varepsilon) = \sum_{i \geq 0} (D_i f)(z) (-1)^i \overline{\pi}^i \varepsilon^i,$$

we obtain

$$(D_i f)(z) = \sum_{r \geq 0} \sum_{n \geq 0} (-1)^{r+i} \binom{r+n-1}{r} \alpha_{r,i} a_n t^{r+n} \quad (i \geq 0). \quad (29)$$

Let now $i \geq 1$. For $r = 0$ we have $\alpha_{r,i} = 0$, and for $r \geq 1$ and $n = 0$ we have $\binom{r+n-1}{r} = 0$. Thus, when $i \geq 1$, we can take $r \geq 1$ and $n \geq 1$ in the formula (29). This gives the Lemma. \square

We are now in position to prove Theorem 2.

Proof of Theorem 2. By Corollary 2.7, it suffices to prove that $D_n(\widetilde{M}_{w,m}^{\leq l}(K)) \subset \widetilde{M}_{w+2n,m+n}^{\leq l+n}(K)$. But if $f \in \widetilde{M}_{w,m}^{\leq l}(K)$, then the function $D_n f$ is an element of $\widetilde{M}_{w+2n,m+n}^{\leq l+n}$ by Proposition 3.1, and the coefficients of its t -expansion all lie in K after Lemma 3.5. Hence $f \in \widetilde{M}_{w+2n,m+n}^{\leq l+n}(K)$ as claimed. \square

3.4 The principal ideal (h) is hyperdifferential.

As a second application of Proposition 3.1, we prove here the following Proposition, which will be very useful in this text.

Proposition 3.6 *The ideal (h) is hyperdifferential.*

Proof. Applying Proposition 3.1 with $f = h \in M_{q+1,1}$ we see that, for all $n \geq 0$, the polynomial $P_{\mathcal{D}_n h}$ associated to $\mathcal{D}_n h$ is:

$$P_{\mathcal{D}_n h} = \sum_{j=0}^n \binom{n+q}{j} (\mathcal{D}_{n-j} h) X^j. \quad (30)$$

Now, the function $\mathcal{D}_{n-j} h$ vanishes at infinity if $0 \leq j < n$ after Lemma 3.5, and of course the same holds for $j = n$ because $\nu_\infty(h) = 1$. Since h does not vanish on Ω (by formula (5.14) of [6]), the functions $f_j := \mathcal{D}_{n-j} h/h$ ($0 \leq j \leq n$) are all holomorphic on Ω and at infinity; consequently, they belong to \mathcal{O} . We deduce from (30) that for all $n \geq 0$, $z \in \Omega$ and $\gamma \in \Gamma_K$:

$$\frac{\mathcal{D}_n h}{h}(\gamma(z)) = (cz + d)^{2n} (\det \gamma)^{-n} \sum_{j=0}^n \binom{n+q}{j} f_j(z) X_\gamma(z)^j,$$

with $f_j \in \mathcal{O}$. In other words, the function $\mathcal{D}_n h/h$ is a quasi-modular form (of weight $2n$, type n and depth $\leq n$), and therefore belongs to the polynomial ring $C[E, g, h]$ by virtue of Theorem 1. \square

3.5 Hyperderivatives of E, g, h .

We end this Section 3 by establishing Proposition 3.7 below, which is another consequence of Proposition 3.1. Roughly speaking, this Proposition

determines, up to a modular form, the functions $D_{p^i}E$, $D_{p^i}g$ and $D_{p^i}h$ for all integers $i \geq 0$. As we will see in Section 4, this result, together with some technical computations, yields a practical way to compute explicitly $D_{p^i}E$, $D_{p^i}g$ and $D_{p^i}h$ when $p^i \leq q^2$. Proposition 3.7 will also be needed in the proof of Lemma 6.2.

Proposition 3.7 (i) *For every integer $i \geq 0$, we have*

$$D_{p^i}E - E^{p^i+1} \in M_{2p^i+2, p^i+1}.$$

(ii) *For every integer $i \geq 1$, we have*

$$\begin{cases} D_{p^i}g = 0 & \text{if } p^i < q \\ D_{p^i}g - E^{p^i}g \in M_{2p^i+q-1, p^i} & \text{if } p^i \geq q. \end{cases}$$

(iii) *For every integer $i \geq 0$, we have*

$$D_{p^i}h - E^{p^i}h - E^q(D_{p^i-q}h) \in M_{2p^i+q+1, p^i+1},$$

where we have used the convention $D_n h = 0$ when $n < 0$.

To prove Proposition 3.7, we will need three Lemmata.

Lemma 3.8 *For every integer $i \geq 0$, we have $D_{p^{i-1}}E = E^{p^i}$.*

Proof. To prove this Lemma, we use the following formula (see [6, page 686]), valid for every $z \in \Omega$:

$$E(z) = \frac{1}{\pi} \sum_{\substack{a \in A \\ a \text{ monic}}} \sum_{b \in A} \frac{a}{az + b}.$$

Since the series are here uniformly convergent (in the above order) on every open ball contained in Ω , we find, by [16, Proposition 2.6] and by using the formula

$$\mathcal{D}_n\left(\frac{a}{az + b}\right) = (-1)^n \left(\frac{a}{az + b}\right)^{n+1} \quad (n \geq 0)$$

which follows from (24),

$$(\mathcal{D}_{p^i-1}E)(z) = \frac{1}{\pi} \sum_{a \text{ monic}} \sum_{b \in A} (-1)^{p^i-1} \left(\frac{a}{az + b}\right)^{p^i} = (-\pi)^{p^i-1} E(z)^{p^i}.$$

Thus, we get $D_{p^i-1}E = E^{p^i}$ as announced. \square

Lemma 3.9 *We have $D_n g = 0$ for every integer n such that $2 \leq n < q$.*

Proof. By [6, formula (6.8)] we have, for all $z \in \Omega$:

$$g(z) = \bar{\pi}^{1-q}(T^q - T) \sum_{\substack{(a,b) \in A^2 \\ (a,b) \neq 0}} \frac{1}{(az + b)^{q-1}}.$$

Using the formula (24), we deduce that for all $n \geq 0$ the following holds:

$$(\mathcal{D}_n g)(z) = \bar{\pi}^{1-q}(T^q - T)(-1)^n \binom{n+q-2}{n} \sum_{\substack{(a,b) \in A^2 \\ (a,b) \neq 0}} \frac{a^n}{(az + b)^{n+q-1}}.$$

Since now for $2 \leq n \leq q-1$ we have in \mathbb{F}_p (for instance by (14) with $p^r = q$) $\binom{n+q-2}{n} = 0$, we get the Lemma. \square

Lemma 3.10 *For every integer $i \geq 0$ such that $p^i \geq q$, we have $D_{p^i-q} h \in M_{2p^i-q+1, p^i}$.*

Proof. Applying Proposition 3.1 we get

$$P_{\mathcal{D}_{p^i-q} h} = \sum_{j=0}^{p^i-q} \binom{p^i}{j} (\mathcal{D}_{p^i-q-j} h) X^j.$$

But in \mathbb{F}_p we have (see (14))

$$\binom{p^i}{j} = \begin{cases} 1 & \text{if } j = 0 \\ 0 & \text{if } 0 < j < p^i. \end{cases}$$

Therefore, we obtain $P_{\mathcal{D}_{p^i-q} h} = \mathcal{D}_{p^i-q} h$. Thus, $\deg(P_{\mathcal{D}_{p^i-q} h}) = 0$ and it follows that $\mathcal{D}_{p^i-q} h$ is a modular form, which proves the Lemma. \square

Proof of Proposition 3.7. (i) For $i = 0$ the assertion (i) is clearly true by the relations (2), so we may assume $i \geq 1$. Since $P_E = -\bar{\pi}^{-1}X + E$, we first note from Proposition 3.1 that the polynomial $P_{\mathcal{D}_{p^i} E}$ associated to $\mathcal{D}_{p^i} E$ equals

$$P_{\mathcal{D}_{p^i} E} = -\frac{1}{\bar{\pi}} X^{p^i+1} + \sum_{j=0}^{p^i} \binom{p^i+1}{j} (\mathcal{D}_{p^i-j} E) X^j.$$

But in \mathbb{F}_p we have

$$\binom{p^i + 1}{j} = \begin{cases} 1 & \text{if } j = 0, j = 1 \text{ or } j = p^i \\ 0 & \text{if } 1 < j < p^i. \end{cases}$$

Therefore, by Lemma 3.8,

$$\begin{aligned} P_{\mathcal{D}_{p^i}E} &= -\frac{1}{\pi}X^{p^i+1} + EX^{p^i} + (\mathcal{D}_{p^i-1}E)X + \mathcal{D}_{p^i}E \\ &= -\frac{1}{\pi}X^{p^i+1} + EX^{p^i} + (-\pi)^{p^i-1}E^{p^i}X + \mathcal{D}_{p^i}E. \end{aligned}$$

Arguing now as in the proof of Lemma 2.6, if we define $f = (-\pi)^{-p^i}\mathcal{D}_{p^i}E - E^{p^i+1}$, we get:

$$\begin{aligned} P_f &= (-\pi)^{-p^i}P_{\mathcal{D}_{p^i}E} - \left(-\frac{1}{\pi}X + E\right)^{p^i+1} \\ &= (-\pi)^{-p^i}P_{\mathcal{D}_{p^i}E} - \left((-\pi)^{-p^i}X^{p^i} + E^{p^i}\right)\left(-\frac{1}{\pi}X + E\right) = f. \end{aligned}$$

Since $\deg P_f = 0$, it follows that f is a modular form, which proves (i).

(ii) Let $i \geq 1$ be an integer. If $p^i < q$, then the assertion (ii) is true by Lemma 3.9, so we may now assume $p^i \geq q$. Applying proposition 3.1, we have

$$P_{\mathcal{D}_{p^i}g} = \sum_{j=0}^{p^i} \binom{p^i + q - 2}{j} (\mathcal{D}_{p^i-j}g)X^j. \quad (31)$$

For every integer j such that $0 \leq j \leq p^i$, write $p^i - j = \alpha q + \beta$ with $\alpha \geq 0$ and $0 \leq \beta \leq q - 1$. If $\beta \geq 2$, then $\mathcal{D}_{\beta}g = 0$ by Lemma 3.9, and thus

$$\mathcal{D}_{p^i-j}g = \mathcal{D}_{\alpha q + \beta}g = \binom{\alpha q + \beta}{\beta} \mathcal{D}_{\alpha q + \beta}g = \mathcal{D}_{\alpha q}(\mathcal{D}_{\beta}g) = 0.$$

If $\beta = 1$, one has

$$\binom{p^i + q - 2}{j} = \binom{p^i + q - 2}{\alpha q + q - 1} = 0,$$

and if $\beta = 0$, one finds easily (recall that q is a power of p)

$$\binom{p^i + q - 2}{j} = \binom{p^i + q - 2}{\alpha q + q - 2} = \begin{cases} 1 & \text{if } \alpha = 0 \text{ or } \alpha = p^i/q \\ 0 & \text{if } 0 < \alpha < p^i/q. \end{cases}$$

Putting these remarks together, (31) leads to

$$P_{\mathcal{D}_{p^i}g} = gX^{p^i} + \mathcal{D}_{p^i}g.$$

Thus, by the same argument as before, $\mathcal{D}_{p^i}g - E^{p^i}g$ is a modular form, and so is an element of M_{2p^i+q-1, p^i} .

(iii) For every integer $i \geq 0$ we have, by Proposition 3.1,

$$P_{\mathcal{D}_{p^i}h} = \sum_{j=0}^{p^i} \binom{p^i+q}{j} (\mathcal{D}_{p^{i-j}}h)X^j. \quad (32)$$

In the case $p^i < q$ one easily computes

$$\binom{p^i+q}{j} = \begin{cases} 0 & \text{if } 0 < j < p^i \\ 1 & \text{if } j = 0 \text{ or } j = p^i. \end{cases}$$

If $p^i = q$, then

$$\binom{p^i+q}{j} = \begin{cases} 1 & \text{if } j = 0 \\ 2 & \text{if } j = p^i \\ 0 & \text{if } 0 < j < p^i, \end{cases}$$

and if $p^i > q$, we find

$$\binom{p^i+q}{j} = \begin{cases} 1 & \text{if } j \in \{0, q, p^i\} \\ 0 & \text{if } 0 < j < p^i, j \neq q. \end{cases}$$

Hence, we deduce in the three cases from (32):

$$P_{\mathcal{D}_{p^i}h} = \mathcal{D}_{p^i}h + (\mathcal{D}_{p^{i-q}}h)X^q + hX^{p^i}.$$

If we now set $f = (-\bar{\pi})^{-p^i}\mathcal{D}_{p^i}h - E^{p^i}h$, then we have

$$\begin{aligned} P_f &= (-\bar{\pi})^{-p^i}P_{\mathcal{D}_{p^i}h} - \left(-\frac{1}{\bar{\pi}}X + E\right)^{p^i}h \\ &= (-\bar{\pi})^{-p^i}(\mathcal{D}_{p^i}h + (\mathcal{D}_{p^{i-q}}h)X^q + hX^{p^i}) - (-\bar{\pi})^{-p^i}hX^{p^i} - E^{p^i}h \\ &= f + (-\bar{\pi})^{-p^i}(\mathcal{D}_{p^{i-q}}h)X^q. \end{aligned}$$

Since the function $\mathcal{D}_{p^{i-q}}h$ is a modular form by Lemma 3.10, it follows that $f - E^q(\mathcal{D}_{p^{i-q}}h)$ is also modular. This means that

$$\mathcal{D}_{p^i}h - E^{p^i}h - E^q(\mathcal{D}_{p^{i-q}}h) \in M_{2p^i+q+1, p^i+1}$$

and part (iii) of Proposition 3.7 is proved. \square

Remark. Proposition 3.7 may be applied to give another proof of Proposition 3.6. Due to the importance of the latter result in this text, we give here all the details.

Alternative proof of Proposition 3.6. By Proposition 3.7 Part (iii), the function $D_{p^i}h - E^{p^i}h - E^q(D_{p^i-q}h)$ is a modular form for every $i \geq 0$. By Lemma 3.5, this modular form vanishes at infinity, hence belongs to the ideal (h) . From this we deduce:

$$\begin{aligned} D_{p^i}h &\in (h) && \text{for } p^i < q \\ D_{p^i}h - E^q D_{p^i-q}h &\in (h) && \text{for } p^i \geq q. \end{aligned}$$

By induction on $i \geq 0$, we now show that $D_n h \in (h)$ for all $0 \leq n < p^i$. The first property above, together with the formulas (15) and (18), easily imply that $D_n h \in (h)$ for all $n < q$, so the assertion is true for i such that $1 \leq p^i \leq q$. Supposing now that the property holds for an integer i such that $p^i \geq q$, we have in particular that $D_{p^i-q}h \in (h)$, which implies $D_{p^i}h \in (h)$. By (15), (18) and the induction hypothesis, we then get $D_n h \in (h)$ for all $0 \leq n < p^{i+1}$, *i.e.* the assertion for $i + 1$. \square

4 Explicit formulas for D_i when $1 \leq i \leq q^2$

In this (more technical) Section, we compute explicitly $D_n E$, $D_n g$ and $D_n h$ as polynomials in E , g , h , when $n \leq q^2$ is a power of p . The formulas obtained allow to compute (at least in principle) $D_n f$ for any $f \in C[E, g, h]$ and any n such that $0 \leq n \leq q^2$, thanks to Leibniz's rule (15) and the formula (18).

We summarise the main results in the following Theorem. Here, e is the integer such that $q = p^e$.

Theorem 4.1 *Let i, n be integers. The following formulas hold.*

- (i) $D_n E = E^{n+1}$ if $0 \leq n < q$.
- (ii) $D_1 g = -(Eg + h)$ and $D_n g = 0$ if $2 \leq n < q$.
- (iii) $D_n h = E^n h$ if $0 \leq n < q$.
- (iv) $D_{p^i} E = E^{p^i+1} + \frac{1}{d_1^{p^i-e}} g^{p^i-e-1} h^{p^i-e+1}$ if $q \leq p^i < q^2$.

- (v) $D_{p^i}g = E^{p^i}g$ if $q \leq p^i < q^2$.
- (vi) $D_{p^i}h = E^{p^i}h + \frac{1}{d_1^{p^i-e-1}}E^q g^{p^i-e-1}h^{p^i-e} - \frac{1}{d_1^{p^i-e}}g^{p^i-e}h^{p^i-e+1}$ if $q \leq p^i < q^2$.
- (vii) $D_{q^2}E = E^{q^2+1} + \frac{1}{d_1^q}g^{q-1}h^{q+1} + \frac{1}{d_2}g^{2q}h^2$.
- (viii) $D_{q^2}g = E^{q^2}g - \frac{d_1}{d_2}g^{q+1}h^q + (\frac{1}{d_1^{q-1}} - \frac{d_2^2}{d_2})h^{2q-1}$.
- (ix) $D_{q^2}h = E^{q^2}h + \frac{1}{d_1^{q-1}}E^q g^{q-1}h^q - \frac{1}{d_2}g^{2q+1}h^2 - (\frac{d_1}{d_2} + \frac{1}{d_1^q})g^q h^{q+1}$.

The proof of Theorem 4.1 requires several technical Lemmata.

Lemma 4.2 *Let i, j be integers such that $q < p^i < q^2$ and $q < p^j \leq q^2$. We have the following t -expansions.*

- (i) $E = t + t^{q^2-2q+2} + \dots$
- (ii) $g = 1 - d_1 t^{q-1} - d_1 t^{q^3-2q^2+2q-1} + \dots$
- (iii) $h = -t - t^{q^2-2q+2} + \dots$
- (iv) $D_q E = \frac{1}{d_1}t^2 + t^{q+1} + \dots$
- (v) $D_q g = t^q + \dots$
- (vi) $D_q h = -\frac{1}{d_1}t^2 - t^{q+1} + \dots$
- (vii) $D_{p^i} E = \frac{1}{d_1^{p^i-e+1}}t^{p^i-e+1} + \dots$
- (viii) $D_{p^i} g = O(t^{p^i-e+1})$
- (ix) $D_{p^i} h = -\frac{1}{d_1^{p^i-e+1}}t^{p^i-e+1} + \dots$
- (x) $D_{p^j-q} h = -\frac{1}{d_1^{p^j-e-1}}t^{p^j-e} + \dots$
- (xi) $D_{q^2} E = \frac{1}{d_2}t^2 + \frac{1}{d_1}t^{q+1} + \dots$
- (xii) $D_{q^2} g = \frac{d_1}{d_2}t^q - \frac{1}{d_1^{q-1}}t^{2q-1} + \dots$
- (xiii) $D_{q^2} h = -\frac{1}{d_2}t^2 - \frac{1}{d_1}t^{q+1} + \dots$

Proof. The formulas (i), (ii) and (iii) follow from [6], Corollaries (10.5), (10.11) and (10.4). To determine the other t -expansions, we apply Lemma 3.5. Define, for $r \geq 1$ and $i \geq 1$,

$$\alpha_{r,i} = \sum_{\substack{i_1, \dots, i_r \geq 0 \\ q^{i_1} + \dots + q^{i_r} = i}} \frac{1}{d_{i_1} \cdots d_{i_r}}.$$

One computes painlessly, for all integers i, j such that $q < p^i < q^2$ and $q < p^j \leq q^2$,

$$\alpha_{r,q} = \begin{cases} 1/d_1 & \text{if } r = 1 \\ 0 & \text{if } 2 \leq r \leq q-1 \\ 1/d_1^q & \text{if } r = q, \end{cases}$$

$$\alpha_{r,p^i} = \begin{cases} 0 & \text{if } 1 \leq r < p^{i-e} \\ 1/d_1^{p^{i-e}} & \text{if } r = p^{i-e}, \end{cases}$$

$$\alpha_{r,p^j-q} = \begin{cases} 0 & \text{if } 1 \leq r < p^{j-e} - 1 \\ 1/d_1^{p^{j-e}-1} & \text{if } r = p^{j-e} - 1. \end{cases}$$

and

$$\alpha_{r,q^2} = \begin{cases} 1/d_2 & \text{if } r = 1 \\ 0 & \text{if } 2 \leq r \leq q-1 \\ 1/d_1^q & \text{if } r = q \\ 0 & \text{if } q < r < 2q-1 \\ 1/d_1^{q-1} & \text{if } r = 2q-1. \end{cases}$$

Thanks to these expressions and the expansions (i), (ii), (iii) already obtained, Lemma 3.5 now yields Lemma 4.2 without difficulty. \square

In the next Lemma, if v_1, \dots, v_n are elements of a vector space, we denote by $\langle v_1, \dots, v_n \rangle$ the subspace they generate.

Lemma 4.3 *Let i, n be integers. The following holds:*

$$(i) \text{ If } p \leq p^i < q, \text{ then } M_{2p^i+2,p^i+1} = \begin{cases} \{0\} & \text{if } (p, q, i) \neq (2, 4, 1) \\ \langle g^2 \rangle & \text{if } (p, q, i) = (2, 4, 1). \end{cases}$$

(ii) *If $q \leq p^i < q^2$, then*

$$M_{2p^i+2,p^i+1} = \begin{cases} \langle g^6, gh^3 \rangle & \text{if } (p, q, i) = (2, 4, 3) \\ \langle g^4, h^2 \rangle & \text{if } (p, q, i) = (3, 3, 1) \\ \langle g^6, g^3h, h^2 \rangle & \text{if } (p, q, i) = (2, 2, 1) \\ \langle g^{p^{i-e}-1}h^{p^{i-e}+1} \rangle & \text{otherwise.} \end{cases}$$

$$(iii) \quad M_{2q^2+2, q^2+1} = \begin{cases} \langle g^{q-1}h^{q+1}, g^{2q}h^2 \rangle & \text{if } q \geq 4 \\ \langle g^2h^4, g^6h^2, g^{10} \rangle & \text{if } q = 3 \\ \langle gh^3, g^4h^2, g^7h, g^{10} \rangle & \text{if } q = 2. \end{cases}$$

(iv) If $q \leq p^i < q^2$, then

$$M_{2p^i+q-1, p^i} = \begin{cases} \langle g^{p^i-e+1}h^{p^i-e} \rangle & \text{if } (p, q, i) \neq (2, 2, 1) \\ \langle g^5, g^2h \rangle & \text{if } (p, q, i) = (2, 2, 1). \end{cases}$$

$$(v) \quad M_{2q^2+q-1, q^2} = \langle g^{2q+2}h, g^{q+1}h^q, h^{2q-1} \rangle.$$

$$(vi) \quad \text{If } p \leq p^i < q, \text{ then } M_{2p^i+q+1, p^i+1} = \begin{cases} \{0\} & \text{if } (p, q, i) \neq (2, 4, 1) \\ \langle g^3 \rangle & \text{if } (p, q, i) = (2, 4, 1) \end{cases}$$

(vii) If $q \leq p^i < q^2$, then

$$M_{2p^i+q+1, p^i+1} = \begin{cases} \langle g^7, g^2h^3 \rangle & \text{if } (p, q, i) = (2, 4, 3) \\ \langle g^5, gh^2 \rangle & \text{if } (p, q, i) = (3, 3, 1) \\ \langle g^7, g^4h, gh^2 \rangle & \text{if } (p, q, i) = (2, 2, 1) \\ \langle g^{p^i-e}h^{p^i-e+1} \rangle & \text{otherwise.} \end{cases}$$

$$(viii) \quad M_{2q^2+q+1, q^2+1} = \begin{cases} \langle g^q h^{q+1}, g^{2q+1} h^2 \rangle & \text{if } q \geq 4 \\ \langle g^3 h^4, g^7 h^2, g^{11} \rangle & \text{if } q = 3 \\ \langle g^2 h^3, g^5 h^2, g^8 h, g^{11} \rangle & \text{if } q = 2. \end{cases}$$

$$(ix) \quad M_{2p^i-q+1, p^i} = \begin{cases} \langle g^{p^i-e-1}h^{p^i-e} \rangle & \text{if } q < p^i < q^2 \\ \langle g^{q-1}h^q, g^{2q}h \rangle & \text{if } p^i = q^2. \end{cases}$$

Proof. We know that a basis of $M_{w,m}$ consists of monomials $g^\alpha h^\beta$, where α and β are non-negative integers satisfying

$$\begin{cases} \alpha(q-1) + \beta(q+1) = w \\ \beta \equiv m \pmod{q-1}. \end{cases}$$

It is now a simple exercise to solve this system of equations for each of the cases (i) to (ix). We leave it to the reader. \square

Proof of Theorem 4.1. (1) We first prove the formulas involving the function E , and we begin by showing (i). Let n be an integer such that $0 \leq n < q$, and suppose first that n has the form $n = p^i$. If $p \leq p^i < q$ and $(p, q, i) \neq (2, 4, 1)$, Proposition 3.7 (i) and Lemma 4.3 (i) yield immediately $D_{p^i}E = E^{p^i+1}$. If $(p, q, i) = (2, 4, 1)$, the same Proposition and the same Lemma now imply the

existence of $\lambda \in C$ such that $D_{p^i}E - E^{p^i+1} = \lambda g^2$. But since $D_{p^i}E - E^{p^i+1}$ vanishes at infinity (Lemma 3.5), whereas g does not, we have $\lambda = 0$ and $D_{p^i}E = E^{p^i+1}$ again. Thus, the formula (i) is proved when n has the form $n = p^i$ (for $i = 0$ this follows from (2)). It is easy to deduce from this, by induction on α , that the formula (i) also holds when $n = \alpha p^i$ with $0 \leq \alpha \leq p-1$ and $1 \leq p^i < q$. For if the formula is true for $(\alpha-1)p^i$ (where $\alpha > 1$), then

$$\begin{aligned} \alpha D_{\alpha p^i}E &= D_{p^i} \circ D_{(\alpha-1)p^i}E = D_{p^i}(E^{(\alpha-1)p^i} \cdot E) \\ &= E^{(\alpha-1)p^i} D_{p^i}E + D_{p^i}(E^{(\alpha-1)p^i})E \\ &= E^{\alpha p^i+1} + (D_1 E^{\alpha-1})^{p^i} E \\ &= E^{\alpha p^i+1} + ((\alpha-1)E^{\alpha-2}(D_1 E))^{p^i} E \\ &= \alpha E^{\alpha p^i+1}. \end{aligned}$$

Now, if n is arbitrary in the range $1 \leq n < q$, we argue by induction on n as follows. Define $i \geq 0$ as the integer such that $p^i \leq n < p^{i+1}$, and write $n = \alpha p^i + \beta$, where $1 \leq \alpha \leq p-1$ and $0 \leq \beta < p^i$. By induction hypothesis we may assume that $D_\beta E = E^{\beta+1}$, from which we deduce

$$D_n E = D_\beta(D_{\alpha p^i}E) = D_\beta(E^{\alpha p^i+1}) = E^{\alpha p^i} D_\beta E = E^{\alpha p^i+\beta+1} = E^{n+1}.$$

To prove (iv), we use Proposition 3.7 (i) and Lemma 4.3 (ii). In the case

$$(p, q, i) \notin \{(2, 4, 3), (3, 3, 1), (2, 2, 1)\},$$

we find that there exists $\lambda \in C$ such that $D_{p^i}E - E^{p^i+1} = \lambda g^{p^i-e-1} h^{p^i-e+1}$. Since by Lemma 4.2

$$D_{p^i}E = \frac{1}{d_1^{p^i-e+1}} t^{p^i-e+1} + \dots$$

and $g = 1 + \dots$, $h = -1 + \dots$, it follows that $\lambda = 1/d_1^{p^i-e}$, which gives (iv). If $(p, q, i) = (2, 4, 3)$, the same method shows that there exist $\lambda, \mu \in C$ such that $D_{p^i}E - E^{p^i+1} = \lambda g^6 + \mu g h^3$. The comparison of the t -expansions in both sides implies $\mu = 0$ and $\lambda = 1/d_1^{p^i-e}$, from which (iv) follows again. The other cases are similar.

To establish (vii), we argue exactly in the same way, but using now Lemmata 4.3 (iii) and 4.2 (xi).

(2) Let us now prove the formulas involving the function g . The formulas (ii) have already been proved (see (2) and Lemma 3.9), so we have to

prove (v) and (viii). The same method as the one used in the proof of (iv) above applies: We simply use Part (ii) of Proposition 3.7 instead of Part (i), Lemma 4.2 for the t -expansions, and Parts (iv), (v) of Lemma 4.3 to know a basis of the space M_{2p^i+q-1,p^i} .

(3) Finally, we prove the formulas (iii), (vi) and (ix). Let n be an integer such that $0 \leq n < q$, and first suppose that n has the form $n = p^i$. If $p \leq p^i < q$ and $(p, q, i) \neq (2, 4, 1)$, then Proposition 3.7 (iii) and Lemma 4.3 (vi) yield at once the formula (iii) for $n = p^i$. If $(p, q, i) = (2, 4, 1)$, arguing similarly and using the fact that $D_{p^i}h - E^{p^i}h$ vanishes at infinity, we obtain again (iii). Thus, (iii) is proved when $n = p^i$ (if $p^i = 1$ it follows from (2)). Arguing now as in the proof of (i), we first deduce from this that (iii) remains valid when $n = \alpha p^i < q$ with $0 \leq \alpha \leq p - 1$, and then, as before, that the formula is true for all n with $1 \leq n < q$.

To prove (vi) and (ix), we apply Proposition 3.7 again: The function

$$D_{p^i}h - E^{p^i}h - E^q(D_{p^i-q}h)$$

is thus a modular form belonging to M_{2p^i+q+1,p^i+1} for all $i \geq e$. But this function has a t -expansion which is clearly an $O(t^2)$ by Lemma 3.5. So, using Lemma 4.3 (parts (vii) and (viii)), we see that there exist elements $\lambda, \mu \in C$ such that

$$D_{p^i}h - E^{p^i}h - E^q(D_{p^i-q}h) = \lambda g^{p^i-e} h^{p^i-e+1} + \mu g^{2q+1} h^2,$$

where $\mu = 0$ if $p^i < q^2$. With the help of Lemma 4.2, we can compute the first terms of the t -expansions on both sides. We find $\lambda = -1/d_1^{p^i-e}$ if $p^i < q^2$ and, if $p^i = q^2$, $\lambda = -(\frac{d_1}{d_2} + \frac{1}{d_1^q})$, $\mu = -1/d_2$. To complete the proof, it remains to express $D_{p^i-q}h$ as a polynomial in E, g, h . If $p^i = q$, we have $D_{p^i-q}h = h$ and we get immediately (vi). If $p^i > q$, then $D_{p^i-q}h = O(t^2)$ and therefore, by Lemma 4.3 (ix) and Lemma 3.10, there exists $\nu \in C$ such that $D_{p^i-q}h = \nu g^{p^i-e-1} h^{p^i-e}$. The t -expansion of $D_{p^i-q}h$ given by Lemma 4.2 (x) now yields $\nu = 1/d_1^{p^i-e-1}$, thus proving (vi) and (ix). \square

5 Proofs of Theorem 3 and Corollary 4.

We consider the subfields of F ⁽⁶⁾

$$F_k = \bigcap_{i=0}^k \text{Ker}(D_{p^i}),$$

⁶By [8, Theorem 27.2], the iterative derivation $(D_n)_{n \geq 0}$ extends to an iterative higher derivation on F in a unique way. We denote this extension by $(D_n)_{n \geq 0}$ again.

with $k \in \mathbb{N}$ and we set $F_{-1} = F$. Clearly, $F_{k-1} \supset F_k$ for all $k \in \mathbb{N}$ and by (15), one sees that $D_{p^k}(fg) = (D_{p^k}f)g + f(D_{p^k}g)$ for $f \in F_{k-1}, g \in F$. In particular, the restriction of D_{p^k} to F_{k-1} is a derivation and thus, if $k \geq 0$, $x, y \in F_{k-1}$ and $y \neq 0$, then:

$$D_{p^k} \left(\frac{x}{y} \right) = \frac{y(D_{p^k}x) - x(D_{p^k}y)}{y^2}$$

(see Proposition 2.2 of [9] for further general properties of these subfields).

We also need to work with two families of subsets of F^\times . If $k \in \mathbb{N}$, we write

$$\Psi_k = \{f \in F^\times \text{ such that } (D_{p^j}f)/f \in \widetilde{M} \text{ for all } 0 \leq j \leq k\},$$

and we define

$$G_k := \Psi_k \cap F_{k-1}^\times.$$

We also write $\Psi_{-1} = G_{-1} = F^\times$. Clearly, $\Psi_{k-1} \supset \Psi_k$ for all $k \in \mathbb{N}$ and $G_k \subset \Psi_k$ for all $k \geq -1$. Moreover, $G_{k-1} \supset F_{k-1}^\times \supset G_k$ for all $k \in \mathbb{N}$. In Lemma 5.1 of Section 5.1 we will prove that the G_k 's and the Ψ_k 's are multiplicative groups.

Plan of Section 5. In order to describe the content of this part, it is worth outlining the structure of proof of Theorem 3 we intend to develop.

By Proposition 3.6, $h \in \cap_{k=0}^\infty \Psi_k$, hence $C^\times h^\mathbb{Z} \subset \cap_{k=0}^\infty \Psi_k$. We also observe that Theorem 3 is equivalent to:

$$\bigcap_{k=0}^\infty \Psi_k = C^\times h^\mathbb{Z}.$$

We will prove that for all $k \geq 0$:

$$\Psi_k = F_k^\times h^\mathbb{Z}. \tag{33}$$

This result, combined with the results of Section 5.2 (where we study t -expansions in F^\times), will easily yield Theorem 3.

The proof of the equalities (33) is by induction on $k \geq -1$; the case $k = -1$ being trivial, let us assume that $\Psi_{k-1} = F_{k-1}^\times h^\mathbb{Z}$ and let f be a quasi-modular form in Ψ_k . We know by induction hypothesis that there exists $n \in \mathbb{Z}$ such that fh^n is in F_{k-1}^\times so that to explain the idea behind the induction step we may restrict our attention to $f \in G_k = \Psi_k \cap F_{k-1}^\times$.

Since $h^s \in \cap_{k=0}^\infty \Psi_k$, for all $s \in \mathbb{Z}$ we have

$$D_{p^k}(fh^{-sp^k}) = a_s fh^{-sp^k}$$

with a_s a quasi-modular form of weight $2p^k$ in F_{k-1}^\times . The depth of a_s is in general $\leq p^k$. In Section 5.3 we determine some properties of the associated polynomials $P_{D_n f}$; by means of them, we observe that since $a_s \in F_{k-1}^\times$, its depth is divisible by p^k . Hence, $l(a_s) \in \{0, p^k\}$.

In section 5.4 we characterise the integers s such that $l(a_s) < p^k$; their set is non-empty. For those integers s , a_s is a modular form because then $l(a_s) = 0$.

In Section 5.5, we observe that modular forms in F_{k-1}^\times are very easy to describe (contrarily to quasi-modular forms). Thanks to these considerations, we see that if s is such that a_s is modular (and we will find such integers), then $a_s = 0$ hence proving (33) for the index k . Finally, Sections 5.6 and 5.7 serve to end the proofs of the Theorem and Corollary in question.

This sketch of proof will hopefully help the reader to access the present part of the text, but it must be considered carefully, as several technical problems arise while taking at the place of f quasi-modular, a more general element of F^\times .

5.1 Preliminaries.

We begin with the following Lemma.

Lemma 5.1 *For all $k \geq -1$, Ψ_k and G_k are multiplicative subgroups of F^\times .*

Proof. This is clear for $k = -1$. Let us write

$$\Theta_n = \{f \in F^\times \text{ such that } (D_j f)/f \in \widetilde{M} \text{ for all } 0 \leq j \leq n\},$$

so that $\Theta_{p^j} = \Psi_j$ for all $j \in \mathbb{N}$. We will prove by induction on n that for all $n \geq 0$, Θ_n is a subgroup of F^\times (this will imply the required property for the Ψ_k 's). It is obviously true for $n = 0$, so we suppose now that $n \geq 1$ and that Θ_j is a subgroup of F^\times for all $j \leq n - 1$. We will show that if $x, y \in \Theta_n$, then $xy^{-1} \in \Theta_n$.

Let x, y be elements of Θ_n . For all $j = 0, \dots, n$ we have $D_j x = \alpha_j x$ and $D_j y = \beta_j y$, with $\alpha_j, \beta_j \in \widetilde{M}$. Now, since Θ_j is a subgroup of F^\times for all $j = 0, \dots, n - 1$, we have $xy^{-1} \in \Theta_j$, and therefore there exists $\gamma_j \in \widetilde{M}$ with $D_j(xy^{-1}) = \gamma_j xy^{-1}$. Thus, it only remains to show that

$D_n(xy^{-1})/(xy^{-1}) \in \widetilde{M}$. But we have

$$\begin{aligned} \alpha_n x = D_n x = D_n(xy^{-1} \cdot y) &= yD_n(xy^{-1}) + \sum_{j=0}^{n-1} D_j(xy^{-1})D_{n-j}y \\ &= yD_n(xy^{-1}) + \sum_{j=0}^{n-1} \gamma_j \beta_{n-j} x, \end{aligned}$$

from which it follows that

$$D_n(xy^{-1})/(xy^{-1}) = \alpha_n - \sum_{j=0}^{n-1} \gamma_j \beta_{n-j} \in \widetilde{M}.$$

Hence, $xy^{-1} \in \Theta_n$ and Θ_n is a multiplicative subgroup of F^\times for all n . The property for G_k follows because $G_k = \Psi_k \cap F_{k-1}^\times$ for all $k \geq 0$. \square

5.2 t -expansions.

We obviously have an embedding $F \subset C((t))$. More generally, we have the following Lemma.

Lemma 5.2 *For all $k \geq 0$ we have an embedding:*

$$F_{k-1} \subset C((t^{p^k})).$$

Proof. We check that for all $k \geq 0$, each element x of F_{k-1} has a convergent t -expansion (in a neighborhood of 0):

$$x = \sum_{m \geq m_0} c_m t^{p^k m}, \quad c_m \in C.$$

We argue by induction on k , the case $k = 0$ being clear. Let k be a strictly positive integer and let x be in F_{k-1} . We then have $x \in F_{k-2}$, and by induction hypothesis:

$$x = \sum_{m \geq m_0} c_m t^{p^{k-1} m}. \quad (34)$$

By (16) and since $D_1 t = t^2$, we have $D_{p^{k-1}}(t^{p^{k-1} m}) = (D_1 t^m)^{p^{k-1}} = m t^{p^{k-1}(m+1)}$. Thus, we get

$$D_{p^{k-1}} x = \sum_{m \geq m_0} m c_m t^{(m+1)p^{k-1}},$$

and $D_{p^{k-1}} x = 0$ if and only if $m c_m = 0$ for all m . This implies $p \mid m$ if $c_m \neq 0$, and hence $x \in C((t^{p^k}))$ by (34). \square

Lemma 5.3 *Let f be a non-zero element of F , and let us suppose that there exists a p -adic integer $s_1 + s_2p + \cdots + s_{k+1}p^k + \cdots$ and a sequence $(f_k)_{k \in \mathbb{N}} \subset F^\times$ such that, for all k , $f_k \in F_k$ and*

$$f = h^{s_1 + s_2p + \cdots + s_{k+1}p^k} f_k. \quad (35)$$

Then there exists $n \in \mathbb{Z}$ and $c \in C^\times$ with $f = ch^n$.

Proof. Since $F_k \subset C((t^{p^{k+1}}))$ (Lemma 5.2), we can write

$$f_k = b_k t^{l_k p^{k+1}} + (\text{terms of higher degree in } t), \quad k = -1, 0, \dots,$$

with $b_k \in C^\times$ and $l_k \in \mathbb{Z}$. By (35), we have

$$f_k = f_{k-1} h^{-s_{k+1}p^k}, \quad k \geq 0.$$

Since $\nu_\infty(h) = 1$, we obtain that:

$$l_k p^{k+1} = l_{k-1} p^k - s_{k+1} p^k, \quad k \geq 0,$$

from which we deduce

$$l_k = \frac{l_{k-1}}{p} - \frac{s_{k+1}}{p}, \quad k \geq 0. \quad (36)$$

First, let us suppose that $\nu_\infty(f) = 0$. Then $l_{-1} = 0$, and (36) implies $l_0 = l_1 = \cdots = 0$ because for all k , l_k and s_k are integers, and $0 \leq s_{k+1}/p < 1$ by hypothesis. This implies $s_1 = s_2 = \cdots = 0$ and $f = f_0 = f_1 = f_2 = \cdots$. Hence:

$$f \in \bigcap_{k=0}^{\infty} F_k = C,$$

which implies $f = b_{-1} = c \in C^\times$.

Let us now suppose that $\nu_\infty(f) \neq 0$. Since $\nu_\infty(h) = 1$, if we set $m = -\nu_\infty(f)$ and $\tilde{f} = h^m f$, then we have $\nu_\infty(\tilde{f}) = 0$. Let us now observe that there exist two sequences of integers $(a_i)_{i \geq 1}$ and $(m_i)_{i \geq 0}$ unique with the property that for all $k \geq 0$, $a_k \in \{0, \dots, p-1\}$ and

$$m + s_1 + s_2p + \cdots + s_{k+1}p^k = a_1 + a_2p + \cdots + a_{k+1}p^k + m_k p^{k+1}.$$

The expression on the right hand side is nothing but the p -adic expansion of the p -adic integer $m + \sum_{k=0}^{\infty} s_{k+1}p^k$, truncated to the k -th digit. Hence, for all $k \geq 0$:

$$\begin{aligned} \tilde{f} = h^m f &= h^{m+s_1+s_2p+\cdots+s_{k+1}p^k} f_k \\ &= h^{a_1+a_2p+\cdots+a_{k+1}p^k} (h^{m_k p^{k+1}} f_k) \\ &= h^{a_1+a_2p+\cdots+a_{k+1}p^k} \tilde{f}_k, \end{aligned}$$

with $\tilde{f}_k := h^{m_k p^{k+1}} f_k$. Since $h^{m_k p^{k+1}} \in F_k^\times$ and $f_k \in F_k^\times$, we get $\tilde{f}_k \in F_k^\times$ for all k . Now, since $\nu_\infty(\tilde{f}) = 0$, we can use the first part of the proof and we obtain $\tilde{f} = h^m f = c \in C^\times$, whence $f = ch^{-m}$. \square

5.3 Computing depths of derivatives.

Terminology. In the following, we will say that a function

$$\Phi : \Gamma_K \times \Omega \rightarrow C$$

is an *error term of degree $\leq l$* if for all $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_K$ and $z \in \Omega$:

$$\Phi(\gamma, z) = \sum_{j=0}^l \sum_{i \in \mathbb{Z}/(q-1)\mathbb{Z}} f_{i,j}(z) \det(\gamma)^{-i} X_\gamma(z)^j,$$

where the $f_{i,j}$'s are holomorphic functions on Ω which do not depend on γ .

To simplify matters, the notation $\mathcal{R}(l)$ is reserved to designate any function which is an error term of degree $\leq l$. Hence, two different functions may be denoted by the same symbol.

Let us denote by $\widetilde{M}_{w,*}^l$ the subset of \widetilde{M} whose elements are linear combinations of quasi-modular forms $f^{(i)} \in \widetilde{M}_{w,i}^{\leq l}$ ($i \in \mathbb{Z}/(q-1)\mathbb{Z}$) such that there exists i with $0 \neq f^{(i)} \in \widetilde{M}_{w,i}^l$; notice that this is not a C -vector space. By Theorem 1, $\widetilde{M}_{w,*}^l$ is also the subset of $C[E, g, h]$ of the polynomials P which are isobaric of weight w , such that $\deg_E(P) = l$. If P is an element of $C[E, g, h]$, we will also write $l(P) = \deg_E(P)$, and we will refer to it as to the depth of P .

Proposition 5.4 *Let f be a quasi-modular form of weight w , depth l and type m (satisfying (8)), let $n \geq 1$ be an integer. Then, for $\gamma \in \Gamma_K$, we have:*

$$\begin{aligned} (\mathcal{D}_n f)(\gamma(z)) &= (cz + d)^{w+2n} \det(\gamma)^{-m-n} \times \\ &\quad \left(\binom{w-l+n-1}{n} f_l(z) X_\gamma(z)^{l+n} + \mathcal{R}(l+n-1) \right). \end{aligned}$$

Let f be a non-zero element of $\widetilde{M}_{w,}^l$. Then:*

$$\begin{aligned} (\mathcal{D}_n f)(\gamma(z)) &= (cz + d)^{w+2n} (\det \gamma)^{-n} \times \\ &\quad \left(\binom{w-l+n-1}{n} \Pi_\gamma(z) X_\gamma(z)^{l+n} + \mathcal{R}(l+n-1) \right), \end{aligned}$$

where for all $\gamma \in \Gamma_K$, Π_γ is a non-zero holomorphic function $\Omega \rightarrow C$ which does not depend on n .

The depth of $\mathcal{D}_n f$ is $< l + n$ if and only if

$$\binom{w - l + n - 1}{n} \equiv 0 \pmod{p}. \quad (37)$$

Proof. The first formula of the Proposition is a simple application of Proposition 3.1. We now prove the second part of the Proposition and we begin by explaining how to construct Π_γ . Let us consider a non-zero element f of $\widetilde{M}_{w,*}^l$. We can write

$$f = \sum_{i \in \mathbb{Z}/(q-1)\mathbb{Z}} f^{(i)} \quad (38)$$

with $f^{(i)}$ quasi-modular of weight w , depth l_i and type i . For $\gamma \in \Gamma_K$ we have:

$$f^{(i)}(\gamma(z)) = (cz + d)^w \det(\gamma)^{-i} \sum_{j=0}^{l_i} f_j^{(i)}(z) X_\gamma(z)^j$$

with $f_j^{(i)}$ quasi-modular for all i, j (Lemma 2.5). Thus, we can write:

$$\begin{aligned} f(\gamma(z)) &= (cz + d)^w X_\gamma(z)^l \sum_i^* \det(\gamma)^{-i} f_l^{(i)}(z) + \\ &\quad (cz + d)^w \sum_{i=0}^{q-2} \det(\gamma)^{-i} \sum_{j=0}^{\min\{l_i, l-1\}} f_j^{(i)}(z) X_\gamma(z)^j \\ &= (cz + d)^w \left(X_\gamma(z)^l \sum_i^* \det(\gamma)^{-i} f_l^{(i)}(z) + \mathcal{R}(l-1) \right), \end{aligned}$$

where the sum \sum^* runs over the indexes $i \in \mathbb{Z}/(q-1)\mathbb{Z}$ such that $l(f^{(i)}) = l(f) = l$ (by hypothesis, this sum is non-empty). By setting

$$\Pi_\gamma(z) = \sum_i^* \det(\gamma)^{-i} f_l^{(i)}(z) \quad (39)$$

we get, for all $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_K$:

$$f(\gamma(z)) = (cz + d)^w \left(X_\gamma(z)^l \Pi_\gamma(z) + \mathcal{R}(l-1) \right). \quad (40)$$

By Lemma 2.5, if $f_l^{(i)} \neq 0$, it is a modular form of weight $w-2l$ and type $i-l$. For all γ , the function $\Pi_\gamma : \Omega \rightarrow C$ is not identically zero because, as follows from Proposition 2.2 (see also Theorem 5.13 of [6]), non-zero modular forms of the same weight but with different types are C -linearly independent.

Let us go back to the expression (38) for f ; we have the identity (40) with $\Pi_\gamma(z)$ as in (39).

More explicitly we have, for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_K$ and for $i \in \mathbb{Z}/(q-1)\mathbb{Z}$:

$$f^{(i)}(\gamma(z)) = (cz + d)^w \det(\gamma)^{-i} (X_\gamma(z)^{l_i} f_{l_i}^{(i)}(z) + \mathcal{R}(l_i - 1)).$$

After the first part of the Proposition:

$$\begin{aligned} & (\mathcal{D}_n f^{(i)})(\gamma(z)) \\ &= (cz + d)^{w+2n} (\det(\gamma))^{-i-n} \times \\ & \quad \left(\binom{w-l_i+n-1}{n} f_{l_i}^{(i)}(z) X_\gamma(z)^{l_i+n} + \mathcal{R}(l_i+n-1) \right), \end{aligned}$$

thus

$$\begin{aligned} & (\mathcal{D}_n f)(\gamma(z)) \\ &= (cz + d)^{w+2n} (\det \gamma)^{-n} \left(\binom{w-l+n-1}{n} X_\gamma(z)^{l+n} \sum_i^* (\det(\gamma))^{-i} f_l^{(i)} + \mathcal{R}(l+n-1) \right) \\ &= (cz + d)^{w+2n} (\det \gamma)^{-n} \left(\binom{w-l+n-1}{n} X_\gamma(z)^{l+n} \Pi_\gamma(z) + \mathcal{R}(l+n-1) \right), \end{aligned}$$

and the second part of the Proposition follows.

In particular, since Π_γ is not identically zero for all γ , the depth of $\mathcal{D}_n f$ is $< l+n$ if and only if (37) holds. \square

5.4 Further properties of elements in $\widetilde{M} \cap F_k^\times$.

Lemma 5.5 *For all $k \geq -1$, if $f \in \widetilde{M}_{w,*}^l \cap F_k^\times$, then $w-l = \alpha p^{k+1}$ with $\alpha \in \mathbb{N}$.*

Proof. We prove the Lemma by induction on k . Let f be as in the hypotheses of the Lemma. For $k = -1$ the result is trivial, so let us now suppose that $k \geq 0$, and that we have already proved the Lemma for the integer $k-1$. As $F_k \subset F_{k-1}$, we have $f \in F_{k-1}$ and by induction hypothesis we get $w-l = \alpha_{k-1} p^k$ with $\alpha_{k-1} \in \mathbb{N}$. Moreover, by condition (37) of Proposition 5.4,

$$0 \equiv \binom{w-l+p^k-1}{p^k} \equiv \binom{(\alpha_{k-1}+1)p^k-1}{p^k} \pmod{p}.$$

Let $\beta = \beta_0 + \beta_1 p + \cdots + \beta_s p^s$ be a rational integer, with $\beta_0, \beta_1, \dots \in \{0, \dots, p-1\}$. Since

$$(1 + \beta)p^k - 1 = (p-1) \sum_{i=0}^{k-1} p^i + p^k(\beta_0 + \beta_1 p + \cdots + \beta_s p^s),$$

taking into account (14) we see that $\binom{(\beta+1)p^k-1}{p^k} \equiv \beta \pmod{p}$. In particular,

$$\binom{(\alpha_{k-1} + 1)p^k - 1}{p^k} \equiv \alpha_{k-1} \pmod{p}. \quad (41)$$

We thus obtain $\alpha_{k-1} = \alpha_k p$ with $\alpha_k \in \mathbb{N}$, and $w - l = \alpha_k p^{k+1}$. \square

Lemma 5.6 *Let $k \geq 0$ be an integer and let us consider elements $f_1 \in \widetilde{M}_{w_1, *}^{l_1} \cap F_{k-1}^\times$ and $f_2 \in \widetilde{M}_{w_2, *}^{l_2} \cap F_{k-1}^\times$. For all $s \in \mathbb{N}$ we have*

$$f := f_2 h^{sp^k} (\mathcal{D}_{p^k} f_1) - f_1 \mathcal{D}_{p^k} (f_2 h^{sp^k}) \in \bigoplus_{m \in \mathbb{Z}/(q-1)\mathbb{Z}} \widetilde{M}_{w_1+w_2+sp^k(q+1)+2p^k, m}^{\leq l_1+l_2+p^k},$$

and there exists a unique integer $s \in \{0, \dots, p-1\}$ such that:

$$l(f) < p^k + l_1 + l_2. \quad (42)$$

Proof. The first property of the Lemma easily follows from Theorem 1.

Since $f_1, f_2 \in F_{k-1}^\times$ by hypothesis, we have $w_1 - l_1 = \beta_1 p^k$ and $w_2 - l_2 = \beta_2 p^k$ with integers $\beta_1, \beta_2 \geq 0$ (Lemma 5.5). For all $\gamma \in \Gamma_K$ we have, after Proposition 5.4 or identity (40):

$$f_j(\gamma(z)) = (cz + d)^{w_j} \left(\Pi_{\gamma, j}(z) X_\gamma(z)^{l_j} + \mathcal{R}(l_j - 1) \right), \quad j = 1, 2,$$

where $\Pi_{\gamma, j}$ ($j = 1, 2$) is defined as in the equality (39). Proposition 5.4 implies, for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_K$:

$$\begin{aligned} (\mathcal{D}_{p^k} f_1)(\gamma(z)) &= (cz + d)^{w_1+2p^k} \times \\ &\quad \left(\binom{w_1 - l_1 + p^k - 1}{p^k} \Pi_{\gamma, 1} X_\gamma(z)^{l_1+p^k} + \mathcal{R}(l_1 + p^k - 1) \right), \\ (\mathcal{D}_{p^k} (f_2 h^{sp^k}))(\gamma(z)) &= (cz + d)^{w_2+sp^k(q+1)+2p^k} \times \\ &\quad \left(\binom{w_2 + sp^k(q+1) - l_2 + p^k - 1}{p^k} \Pi_{\gamma, 2} (\det \gamma)^{-sp^k} h^{sp^k} X_\gamma(z)^{l_2+p^k} + \right. \\ &\quad \left. \mathcal{R}(l_2 + p^k - 1) \right). \end{aligned}$$

Hence,

$$\begin{aligned}
& f_2(\gamma(z))h(\gamma(z))^{sp^k}(\mathcal{D}_{p^k}f_1)(\gamma(z)) = \\
& = (cz+d)^{w(f)} \left[\binom{w_1 - l_1 + p^k - 1}{p^k} \Pi_{\gamma,1} \Pi_{\gamma,2} (\det \gamma)^{-sp^k} h^{sp^k} X_\gamma(z)^{l_1+l_2+p^k} + \right. \\
& \quad \left. \mathcal{R}(l_1 + l_2 + p^k - 1) \right], \\
& f_1(\gamma(z))(\mathcal{D}_{p^k}(f_2 h^{sp^k}))(\gamma(z)) = (cz+d)^{w(f)} \left[\binom{w_2 + sp^k(q+1) - l_2 + p^k - 1}{p^k} \times \right. \\
& \quad \left. \Pi_{\gamma,1} \Pi_{\gamma,2} (\det \gamma)^{-sp^k} h^{sp^k} X_\gamma(z)^{l_1+l_2+p^k} + \mathcal{R}(l_1 + l_2 + p^k - 1) \right].
\end{aligned}$$

This implies that for $\gamma \in \Gamma_K$:

$$f(\gamma(z)) = (cz+d)^{w(f)} \left(A \Pi_{\gamma,1} \Pi_{\gamma,2} (\det \gamma)^{-sp^k} h^{sp^k} X_\gamma(z)^{l_1+l_2+p^k} + \mathcal{R}(l_1 + l_2 + p^k - 1) \right),$$

where

$$\begin{aligned}
A & := \binom{w_1 - l_1 + p^k - 1}{p^k} - \binom{w_2 - l_2 + p^k s(q+1) + p^k - 1}{p^k} \\
& = \binom{p^k(\beta_1 + 1) - 1}{p^k} - \binom{p^k(\beta_2 + s(q+1) + 1) - 1}{p^k} \\
& \equiv \beta_1 - (\beta_2 + s) \pmod{p},
\end{aligned}$$

after (41). Since the product:

$$(cz+d)^{w(f)} \Pi_{\gamma,1} \Pi_{\gamma,2} h^{sp^k} X_\gamma(z)^{l_1+l_2+p^k}$$

is not identically zero over $\Gamma_K \times \Omega$, we have that $l(f) < l_1 + l_2 + p^k$ if and only if $s \equiv \beta_1 - \beta_2 \pmod{p}$, and there exists only one such integer in $\{0, \dots, p-1\}$. \square

Remark. In particular, by choosing $f_1 = 1$ in Lemma 5.6, we find the following property. Let $k \geq 0$ be an integer and let $f \in \widetilde{M}_{w,*}^l \cap F_{k-1}$ be a non-zero quasi-modular form. There exists a unique integer $s \in \{0, \dots, p-1\}$ such that

$$l(\mathcal{D}_{p^k}(h^{p^k s} f)) < p^k + l. \quad (43)$$

5.5 Modular forms in F_k .

We need some information about the modular forms contained in F_k .

Lemma 5.7 *Let $f \in C[g, h]$ be a modular form. If $D_1 f = D_p f = \dots = D_{p^k} f = 0$, then there exists a modular form \tilde{f} such that $f = \tilde{f}^{p^{k+1}}$.*

Proof. Let us consider the case $k = 0$ first. By assumption, f is an element of $M_{w,m}$ for some $w \in \mathbb{N}$ and $m \in \mathbb{Z}/(q-1)\mathbb{Z}$. Let $\partial : C[g, h] \rightarrow C[g, h]$ be the derivation defined by $\partial := -h \frac{\partial}{\partial g}$. It is easily seen that the restriction of ∂ to $M_{w,m}$ coincides with the operator $\partial_w : M_{w,m} \rightarrow M_{w,m}$ (of weight 2 and type 1) defined on p. 687 of [6], that is, we have $\partial f = D_1 f - w E f$. Since ∂f is a modular form and $D_1 f = 0$, we have thus $\partial f = -w E f = 0$, so that $w \equiv 0 \pmod{p}$. On the other hand, we have $\partial f = -h \partial f / \partial g = 0$, so f is of the form

$$f = \sum_{\substack{i, j \\ \text{such that} \\ j \equiv 0 \pmod{p}}} \lambda_{i,j} h^i g^j. \quad (44)$$

Since the weight of h is $q+1$ and $w \equiv 0 \pmod{p}$, we see that $\lambda_{i,j} \neq 0$ in (44) implies $i, j \equiv 0 \pmod{p}$, and f is the p -th power of another modular form.

We continue the proof of the Lemma by induction on k : let us consider a modular form $f \in C[g, h]$ such that $D_1 f = D_p f = \dots = D_{p^k} f = 0$, with $k > 0$. Since $D_1 f = D_p f = \dots = D_{p^{k-1}} f = 0$, we see from the induction hypothesis that there exists $r \in C[g, h]$ such that $f = r^{p^k}$. Now, from (16) immediately follows:

$$0 = D_{p^k} f = D_{p^k}(r^{p^k}) = (D_1 r)^{p^k}$$

and r is a p -th power after the first part of the Lemma: $r = \tilde{f}^p$; thus $f = \tilde{f}^{p^{k+1}}$. \square

Lemma 5.8 *Let us suppose that $q \notin \{2, 3\}$. For all $k \geq 0$ there does not exist a non-zero element of $\widetilde{M}_{2p^k, *}^l \cap F_{k-1}$ with $l < p^k$.*

Proof. The Lemma for $k = 0$ is clear: there does not exist a non-zero modular form of weight 2, because $q \geq 4$.

Let us suppose by contradiction that for $k > 0$ there exists a non-zero isobaric element f of $\widetilde{M} \cap F_{k-1}$ of weight $2p^k$ and depth $l < p^k$. Since $f \in F_{k-1}$ we have, after Lemma 5.5, $p^k(2 - \alpha) = l$ with $\alpha \in \mathbb{N}$, and this condition implies $\alpha = 2$ and $l = 0$. Thus, f is a modular form in F_{k-1} . After

Lemma 5.7, it is a p^k -th power of another modular form \tilde{f} , which must have weight 2, and which is zero after the first part of the Lemma; this implies a contradiction. \square

Remark. For $q = 2, 3$, Lemma 5.8 is false: g has weight 2 for $q = 3$ and g^2 has weight 2 for $q = 2$; Lemma 5.8 is the only tool of the proof of Theorem 3 which needs the hypothesis $q \notin \{2, 3\}$.

5.6 Proof of Theorem 3.

We need the next Lemma.

Lemma 5.9 *For all $k \geq 0$ the following property holds. Let us consider $f \in G_k$: there exists a unique $s \in \{0, \dots, p-1\}$ such that*

$$h^{-p^k s} f \in F_k^\times.$$

In particular,

$$G_k = h^{p^k \mathbb{Z}} F_k^\times.$$

Proof. Write $f = a/b$ with $a, b \in \widetilde{M}$ non-zero. We have $b^{p^k} f = ab^{p^k-1}$ with $f \in F_{k-1}^\times$ (by hypothesis) and $b^{p^k} \in F_{k-1}$ (by (16)), so $x := ab^{p^k-1} \in \widetilde{M} \cap F_{k-1}$ and $y := b^{p^k} \in \widetilde{M} \cap F_{k-1}$. Thus we have $f = x/y$ with $x, y \in \widetilde{M} \cap F_{k-1}^\times$.

Now, since $D_1 h/h \in \widetilde{M}$ and $D_1(h^{p^i}) = 0$ for $i \geq 1$, we easily deduce from (16) that $h^{p^k} \in G_k$. Hence, since G_k is a group by Lemma 5.1, we find that for all $s \in \mathbb{Z}$, there exists $a = a_s \in \widetilde{M}$ such that:

$$D_{p^k}(h^{-sp^k} f) = ah^{-sp^k} f.$$

We claim that a is isobaric of weight $2p^k$. To prove this, we have of course to consider only the case $a \neq 0$. Since D_{p^k} is a derivation on F_{k-1} and $h^{sp^k}, x, y \in \widetilde{M} \cap F_{k-1}$, we have:

$$h^{sp^k} y(D_{p^k} x) - x D_{p^k}(h^{sp^k} y) = axy h^{sp^k}.$$

Let us write:

$$x = \sum_{i=\alpha}^{\beta} x_i, \quad y = \sum_{j=\gamma}^{\delta} y_j, \quad a = \sum_{l=\mu}^{\nu} a_l,$$

with $x_i, y_j, a_l \in \widetilde{M}$ isobaric of weights i, j, l , and $x_\alpha, x_\beta, y_\gamma, y_\delta, a_\mu, a_\nu$ non-zero. We have:

$$\begin{aligned} & \sum_{v=\alpha+\gamma}^{\beta+\delta} \sum_{i+j=v} (y_j h^{sp^k} (D_{p^k} x_i) - x_i D_{p^k} (y_j h^{sp^k})) = \\ & = h^{sp^k} \sum_{w=\alpha+\gamma+\mu}^{\beta+\delta+\nu} \sum_{i+j+l=w} a_l x_i y_j. \end{aligned} \quad (45)$$

If a non-vanishing isobaric summand $y_j h^{sp^k} (D_{p^k} x_i) - x_i D_{p^k} (y_j h^{sp^k})$ on the left-hand side of (45) is non-zero, then its weight $r := 2p^k + sp^k(q+1) + i + j$ satisfies $2p^k + sp^k(q+1) + \alpha + \gamma \leq r \leq 2p^k + sp^k(q+1) + \beta + \delta$. Since $h^{sp^k} a_\mu x_\alpha y_\gamma$ and $h^{sp^k} a_\nu x_\beta y_\delta$ are two non-vanishing isobaric summands in the right-hand side of (45), whose weights are respectively $sp^k(q+1) + \mu + \alpha + \gamma \leq sp^k(q+1) + \nu + \beta + \delta$, we deduce that there exist two non-zero isobaric summands in the sum on the left-hand side. We find:

$$\begin{aligned} 2p^k + sp^k(q+1) + \alpha + \gamma & \leq sp^k(q+1) + \mu + \alpha + \gamma \\ & \leq sp^k(q+1) + \nu + \beta + \delta \\ & \leq 2p^k + sp^k(q+1) + \beta + \delta, \end{aligned}$$

which implies $2p^k \leq \mu \leq \nu \leq 2p^k$, that is, $\mu = \nu = 2p^k$. Hence $a \in \widetilde{M}$ is isobaric of weight $2p^k$, as claimed.

From this we deduce:

$$h^{sp^k} y_\delta (D_{p^k} x_\beta) - x_\beta D_{p^k} (h^{sp^k} y_\delta) = a x_\beta y_\delta h^{sp^k}. \quad (46)$$

After Lemma 5.6, there exists exactly one $s \in \{0, \dots, p-1\}$ such that the term on the left-hand side of (46) has depth $< p^k + l(x_\beta) + l(y_\delta)$. Thus we see that the depth of $a x_\beta y_\delta h^{sp^k}$ is $< p^k + l(x_\beta) + l(y_\delta)$. Since the depth of h is zero, we deduce that a has depth $< p^k$.

Since $a = D_{p^k} (h^{-p^k s} f) / (h^{-p^k s} f)$ and $h^{-p^k s} f, D_{p^k} (h^{-p^k s} f) \in F_{k-1}$, we have $a \in F_{k-1}$. Thus, $a \in \widetilde{M} \cap F_{k-1}$ is isobaric of weight $2p^k$ and depth $< p^k$. After Lemma 5.8, $a = 0$; this implies $D_{p^k} (h^{-p^k s} f) = 0$ and $h^{-p^k s} f \in F_k$. We deduce the equality of multiplicative groups $G_k = h^{p^k \mathbb{Z}} F_k^\times$ as well. \square

We can terminate the proof of Theorem 3. Let $f \in \bigcap_n \Psi_n$: thus, $f \in G_0 = \Psi_1$. After Lemma 5.9, there exists (a unique) $s_1 \in \{0, \dots, p-1\}$ such that $f_0 := h^{-s_1} f \in F_0^\times$; we continue by induction on k .

Let us suppose that

$$f_{k-1} := h^{-(s_1+s_2p+\dots+s_kp^{k-1})} f \in F_{k-1}^\times,$$

for $k > 0$ and $s_1, \dots, s_k \in \{0, \dots, p-1\}$. Proposition 3.6 and Lemma 5.1 imply that

$$h^{-(s_1+s_2p+\dots+s_kp^{k-1})} \in \Psi_k$$

and by hypothesis, $f \in \Psi_k$. Thus, $f_{k-1} \in \Psi_k \cap F_{k-1}^\times = G_k$.

After Lemma 5.9, there exists a unique $s_{k+1} \in \{0, \dots, p-1\}$ such that $f_k := h^{-(s_1+s_2p+\dots+s_{k+1}p^k)} f \in F_k^\times$.

Thanks to this inductive process, for all $k \geq 0$, we can construct a sequence of elements $f_k \in F_k^\times$ and a p -adic integer $s_1+s_2p+\dots+s_{k+1}p^k+\dots$ such that for all k , the equality (35) is satisfied. Lemma 5.3 implies that there exists $n \in \mathbb{Z}$ and $c \in C^\times$, with $f = ch^n$. \square

5.7 Proof of Corollary 4.

If U, V are isobaric polynomials of $C[E, g, h]$ of weights $w(U), w(V)$, their Rankin bracket $[U, V]$ is defined by:

$$[U, V] = w(U)U(D_1V) - w(V)V(D_1U).$$

In the following Lemma we collect the properties of these brackets that we need.

Lemma 5.10 *Let M be an isobaric element of $C[E, g, h]$.*

- *The map $d_M : X \mapsto [X, M]$ defined on the set of isobaric elements of $C[E, g, h]$ satisfies*

$$d_M = d_M(E) \frac{\partial}{\partial E} + d_M(g) \frac{\partial}{\partial g} + d_M(h) \frac{\partial}{\partial h}.$$

- *If \mathcal{I} is an ideal of $C[E, g, h]$ such that $D_1\mathcal{I} \subset \mathcal{I}$, and if $X \in \mathcal{I}$ is isobaric, then $d_M(X) \in \mathcal{I}$.*

The proof of this Lemma is easy and left to the reader. For the following Lemma, we recall that an ideal is *isobaric* if it is generated by isobaric elements of $C[E, g, h]$.

Lemma 5.11 *Every isobaric non-principal prime ideal \mathcal{P} such that $D_1\mathcal{P} \subset \mathcal{P}$ contains h .*

Proof. We closely follow the proof of Lemma 2.3 of [14]. Let \mathcal{P} be a non-principal isobaric prime ideal of $C[E, g, h]$. Eliminating E , we see that \mathcal{P} contains a non-zero isobaric element of $C[g, h]$, say f . We choose f so that its total degree $\deg f$ is minimal. Let us define $f_g = \frac{\partial f}{\partial g}$ and $f_h = \frac{\partial f}{\partial h}$. By minimality of $\deg f$, f is not a p -th power in $C[g, h]$. It follows that $f_g \neq 0$ or $f_h \neq 0$.

We first suppose that $f_g \neq 0$. By minimality of $\deg f$, we have $f_g \notin \mathcal{P}$. Now, by Lemma 5.10, we have $d_h(f) \in \mathcal{P}$ and

$$d_h(f) = f_g d_h(g) + f_h d_h(h) = f_g d_h(g).$$

Since \mathcal{P} is prime, it follows that $d_h(g) \in \mathcal{P}$. But by (2), $d_h(g) = h^2$. Thus, we get $h \in \mathcal{P}$.

If $f_g = 0$ then $f_h \neq 0$. By using the same arguments as before, we find that $f_h \notin \mathcal{P}$, and then $d_g(h) = -h^2 \in \mathcal{P}$, which implies $h \in \mathcal{P}$. \square

Lemma 5.12 *Let \mathcal{I} be an ideal of $C[E, g, h]$ and let us denote by $\tilde{\mathcal{I}}$ the ideal generated by the isobaric elements of \mathcal{I} . Let us suppose that \mathcal{I} is hyperdifferential. Then, we have the following properties*

1. *The ideal $\tilde{\mathcal{I}}$ is hyperdifferential.*
2. *If \mathcal{I} is prime, then $\tilde{\mathcal{I}}$ is prime.*
3. *If \mathcal{I} is prime and non-principal, then $\tilde{\mathcal{I}} \neq (0)$.*

Proof. The first Part easily follows from the fact that the images of any isobaric polynomial by the operators D_n is again isobaric (Theorem 2). The proof of the Parts two and three closely follows the proof of Lemma 5.2 of [14]; we do not need to give more details here. \square

Proof of Corollary 4. Let \mathcal{P} be non-zero hyperdifferential prime ideal. If \mathcal{P} is principal, then Theorem 3 gives immediately $h \in \mathcal{P}$, so we may suppose that \mathcal{P} is non-principal. By Lemma 5.12, \mathcal{P} contains a non-zero isobaric hyperdifferential prime ideal $\tilde{\mathcal{P}}$. If $\tilde{\mathcal{P}}$ is principal, then $\tilde{\mathcal{P}} = (h)$ by Theorem 3. If $\tilde{\mathcal{P}}$ is not principal, then Lemma 5.11 implies $h \in \tilde{\mathcal{P}} \subset \mathcal{P}$. \square

6 Classification of hyperdifferential ideals.

The aim of this Section is to provide a full description of the hyperdifferential ideals of $C[E, g, h]$. Let us write:

$$\begin{aligned}\mathcal{P}_0 &= (E, h) \\ \mathcal{P}_\infty &= (g, h) \\ \mathcal{P}_d &= (h, E^{q-1} - dg), \quad d \in C^\times.\end{aligned}$$

These ideals are all prime, and we have the following diagram of inclusions (here c and d vary in C^\times):

$$\begin{array}{ccccc} (E, g - c, h) & \leftarrow & \mathcal{P}_0 & & \\ & \swarrow & & \nwarrow & \\ (E, g, h) & \leftarrow & \mathcal{P}_d & \leftarrow & (h) \\ & \swarrow & & \nwarrow & \\ & & \mathcal{P}_\infty & & \end{array} \quad (47)$$

In the following Theorem, we prove (under the condition $q \neq 2, 3$) that all the ideals of the diagram (47) are the only non-zero hyperdifferential prime ideals of $C[E, g, h]$.

Theorem 6.1 *Let us assume that $q \neq 2, 3$.*

- (i) *If \mathcal{P} is a principal hyperdifferential non-zero prime ideal of $C[E, g, h]$, then $\mathcal{P} = (h)$.*
- (ii) *For all $d \in C \cup \{\infty\}$, the ideal \mathcal{P}_d is hyperdifferential. If \mathcal{P} is a non-zero hyperdifferential prime ideal of $C[E, g, h]$ of height 2, then there exists $d \in C \cup \{\infty\}$ such that $\mathcal{P} = \mathcal{P}_d$.*
- (iii) *For all $c \in C$, the ideal $(E, g - c, h)$ is hyperdifferential. If \mathcal{P} is a maximal ideal which also is hyperdifferential, then there exists $c \in C$ such that $\mathcal{P} = (E, g - c, h)$.*

Before going on with the proof of this Theorem, we will need two Lemmata. In the next Lemma, we use the notation $f_1 \equiv f_2 \pmod{h}$ for $f_1, f_2 \in C[E, g, h]$, which means that $f_2 - f_1 \in (h)$.

Lemma 6.2 *For $n, \mu, \nu \in \mathbb{N}$ we have*

$$D_n(E^\mu g^\nu) \equiv \binom{\mu + \nu(q-1) + n - 1}{n} E^{\mu+n} g^\nu \pmod{h}.$$

Proof. We have to show that for every monomial f in E and g , and for any $n \in \mathbb{N}$, we have

$$D_n(f) \equiv \binom{w(f) - l(f) + n - 1}{n} E^n f \pmod{h}. \quad (48)$$

First of all, we notice that if this formula holds for f_1 and f_2 (and all n), then it also holds for the product $f_1 f_2$: this follows from Leibniz formula (15) and Lemma 3.2 with $N = n$, $W = w(f) - l(f)$ and $M = l(g) - w(g)$ (notice also that $(-1)^i \binom{M}{n-i} = (-1)^N \binom{N-M-i-1}{N-i}$). Thus, it suffices to prove the lemma for $f = E$ and for $f = g$.

Next, we show that it suffices to prove the formula (48) when n has the form $n = p^k$, $k \geq 0$. Indeed, assuming the formula for $n = p^k$, an easy induction on m first shows that for all $m \in \mathbb{N}$,

$$D_{p^k}^m f \equiv \left[\prod_{1 \leq i \leq m} \binom{w(f) - l(f) + ip^k - 1}{p^k} \right] E^{mp^k} f \pmod{h}$$

(here we need the fact that $f_1 \equiv f_2 \pmod{h}$ implies $D_n f_1 \equiv D_n f_2 \pmod{h}$). Since

$$\begin{aligned} \prod_{1 \leq i \leq m} \binom{w(f) - l(f) + ip^k - 1}{p^k} &= \binom{w(f) - l(f) + mp^k - 1}{mp^k} \prod_{1 \leq i \leq m} \binom{ip^k}{p^k} \\ &= m! \binom{w(f) - l(f) + mp^k - 1}{mp^k} \end{aligned}$$

by (14), we find, using $D_{mp^k} = \frac{1}{m!} D_{p^k}^m$ (for $0 \leq m \leq p-1$), that the formula (48) also holds for $n = mp^k$ with $0 \leq m \leq p-1$. If now n is arbitrary, write $m = n_0 + \dots + n_s p^s$ in base p ($0 \leq n_i \leq p-1$). Then formula (48) for $n_i p^i$ together with (18) gives

$$D_n f = \left[\prod_{1 \leq i \leq s} \binom{w(f) - l(f) + n_0 + \dots + n_i p^i - 1}{n_i p^i} \right] E^{n_0 + \dots + n_s p^s} f.$$

But the expression in brackets is also equal to

$$\binom{w(f) - l(f) + n - 1}{n} \prod_{1 \leq i \leq s} \binom{n_0 + \dots + n_i p^i}{n_i p^i},$$

which is equal to $\binom{w(f) - l(f) + n - 1}{n}$ by (14). So (48) holds for all n .

It follows from the above remarks that we only need to prove the Lemma when $n = p^k$ ($k \geq 0$) and for the functions $f = E$ and $f = g$. But in this case this follows from Proposition 3.7, Parts (i) and (ii). Indeed, by Lemma 3.5, the differences $D_{p^k}E - E^{p^k+1}$ and $D_{p^k}g - E^{p^k}g$ are modular form which vanish at infinity, hence multiples of h . \square

The next Lemma, easy, will also be needed in the Proof of Theorem 6.1.

Lemma 6.3 *Let r, r' be two rational integers such that for all $n \geq 0$:*

$$\binom{r+n-1}{n} \equiv \binom{r'+n-1}{n} \pmod{p}.$$

Then, $r = r'$.

Proof. Let X be an indeterminate. Let us suppose by contradiction that two distinct integers r, r' exist, such that for all n , the congruence of the Lemma holds. Since

$$(1-X)^{-r} = \sum_{n \geq 0} \binom{r+n-1}{n} X^n$$

in $\mathbb{F}_p[[X]]$, and a similar identity holds for $(1-X)^{-r'}$, we have that $(1-X)^{-r} = (1-X)^{-r'}$ in $\mathbb{F}_p[[X]]$, which means that $(1-X)^s = 1$ with $s = r' - r \neq 0$; this is impossible. \square

Proof of Theorem 6.1: Part (i). This follows from Proposition 3.6 and Theorem 3.

Part (ii). We introduce a new graduation on the ring $C[E, g]$, namely the one defined by assigning to E the degree 1 and to g the degree $q-1$. In what follows, the word ‘‘homogeneous’’ will then refer to this new graduation.

Let P be a homogeneous polynomial in $C[E, g]$ of weight r . By Lemma 6.2 we see that for all n ,

$$D_n P \equiv \binom{r+n-1}{n} E^n P \pmod{h}.$$

Thus, for such a polynomial P , the ideal (P, h) is hyperdifferential. In particular, the property above is fulfilled for the polynomial E with $r = 1$, and for the polynomials $E^{q-1} - dg$ ($d \in C^\times$) and g , with $r = q-1$. Hence the ideals $\mathcal{P}_0, \mathcal{P}_d$ (with $d \in C^\times$) and \mathcal{P}_∞ are hyperdifferential.

Let us now consider a hyperdifferential prime ideal \mathcal{P} of height 2. By Theorem 4, $h \in \mathcal{P}$. Thus, there exists an irreducible polynomial $P \in C[E, g]$ such that $\mathcal{P} = (P, h)$; without loss of generality, we can suppose that P is not proportional to E (otherwise, $\mathcal{P} = (E, h) = \mathcal{P}_0$ and we are done). By assumption, for all $n \geq 0$ there exists $A_n \in C[E, g]$ such that

$$D_n P \equiv A_n P \pmod{h}.$$

Since the operator $D_n : C[E, g] \rightarrow C[E, g, h]/(h) = C[E, g]$ is homogeneous of degree n by Lemma 6.2, it is easy to see that A_n must be a linear combination of homogeneous terms of weight $\leq n$.

We can write:

$$P = \sum_{r=0}^s P_r,$$

where $P_s \neq 0$ and $P_r \in C[E, g]$ is homogeneous of weight r for all r . We have, explicitly by Lemma 6.2, ($n \geq 0, 0 \leq r \leq s$)

$$D_n P_r \equiv \binom{r+n-1}{n} E^n P_r \pmod{h}.$$

Thus, E^n divides $A_n P$, which implies that $A_n = \lambda_n E^n$ for some $\lambda_n \in C$. We get:

$$\sum_{r=0}^s \binom{r+n-1}{n} P_r = \lambda_n \sum_{r=0}^s P_r,$$

which implies that for all r such that $P_r \neq 0$,

$$\binom{s+n-1}{n} = \binom{r+n-1}{n}, \quad n \geq 0.$$

By Lemma 6.3, we find that the sum $\sum_{r=0}^s P_r$ has exactly one non-vanishing term, P_s . Therefore, P is homogeneous of degree s , and we have:

$$P = E^\mu g^\nu \prod_{j=1}^a (E^{q-1} - \lambda_j g)^{\mu_j}, \quad (49)$$

for integers $\mu, \nu, a, \mu_1, \dots, \mu_a$ not all zero, and non-zero elements $\lambda_1, \dots, \lambda_a \in C$. Now, P is irreducible. Hence, considering each type of factor in the product on the right hand side of (49), we get $P = E$ (case which is excluded) or $P = g$ or $P = E^{q-1} - dg$ for $d \in C^\times$. If $P = g$ we get $\mathcal{P} = \mathcal{P}_\infty$, and if $P = E^{q-1} - dg$ for some $d \in C^\times$, then we get $\mathcal{P} = \mathcal{P}_d$.

Part (iii). Taking into account the t -expansions (i), (ii) and (iii) of Lemma 4.2, we first notice that $\mathcal{P}_0 = (E, h)$ is also the ideal generated by the quasi-modular forms which vanish at the infinity. After Lemma 3.5, it contains the images of all the operators D_n ($n \geq 1$). Thus, we have $D_n(g - c) \in (E, h)$ for all $n \geq 1, c \in C$ (by the way, this argument gives an alternative way to check that \mathcal{P}_0 is hyperdifferential). Since now \mathcal{P}_0 is hyperdifferential after part (i), it follows that for all $c \in C$, the maximal ideal $(E, g - c, h)$ is hyperdifferential.

Let $\mathcal{P} = (E - c_1, g - c_2, h - c_3)$ be a hyperdifferential maximal ideal. In particular, D_1E, D_1g, D_1h belong to \mathcal{P} , that is, $E^2, -Eg - h, Eh \in \mathcal{P}$, after the formulas (2). Thus $E \in \mathcal{P}$, which implies $c_1 = 0$, and $h \in \mathcal{P}$, which implies $c_3 = 0$. \square

Remark. There is also another way to check that \mathcal{P}_∞ is hyperdifferential. By Theorem 1, \mathcal{P}_∞ is the ideal generated by the quasi-modular forms f which have weights w and depths l satisfying the inequality $l < w/2$. By Theorem 2, this property holds for $D_n f$, for all $n \geq 1$ and for each quasi-modular form $f \in \mathcal{P}_\infty$. Hence, $D_n \mathcal{P}_\infty \subset \mathcal{P}_\infty$ for all n .

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Vincent Bosser,
 Mathematisches Institut, Universität Basel,

Rheinsprung 21,
CH-4051 Basel, Switzerland.
E-mail: Vincent.Bosser@unibas.ch

Federico Pellarin,
L.M.N.O., Université de Caen,
Campus II - Boulevard Maréchal Juin,
BP 5186 - F14032 Caen Cedex, France.
E-mail: pellarin@math.unicaen.fr