ASYMPTOTICS FOR STEADY STATE VOLTAGE POTENTIALS IN A BIDIMENSIONAL HIGHLY CONTRASTED MEDIUM WITH THIN LAYER

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Abstract. We study the behavior of steady state voltage potentials in two kinds of bidimensional media composed of material of complex permittivity equal to 1 (respectively α) surrounded by a thin membrane of thickness h and of complex permittivity α (respectively 1). We provide in both cases a rigorous derivation of the asymptotic expansion of steady state voltage potentials at any order as h tends to zero, when Neumann boundary condition is imposed on the exterior boundary of the thin layer. Our complex parameter α is bounded but may be very small compared to 1, hence our results describe the asymptotics of steady state voltage potentials in all heterogeneous and highly heterogeneous media with thin layer. The terms of the potential in the membrane are given explicitly in local coordinates in terms of the boundary data and of the curvature of the domain, while these of the inner potential are the solutions to the so-called dielectric formulation with appropriate boundary conditions. The error estimates are given explicitly in terms of h and α with appropriate Sobolev norm of the boundary data. We show that the two situations described above lead to completely different asymptotic behaviors of the potentials.

Introduction

We study the behavior of the steady state voltage potentials in highly contrasted media surrounded by a thin layer. The motivation of the present work comes from numerical problems raised by the researchers in computational electromagnetics, who want to compute the quasi-static electric field in highly contrasted materials with thin layer. The thinness of the membrane surrounding an inner domain leads to numerical difficulties, in particular for the meshing.

To avoid these difficulties, we perform an asymptotic expansion of the potentials in terms of the membrane thickness. The approached inner potential is then the finite sum of the solutions to elementary problems in the inner domain with appropriate conditions on its boundary, which approximate the effect of the thin layer. Thereby, the thin membrane does not have to be considered anymore. Our method leads to the construction of so-called "approximated boundary conditions" at any order [11]. We estimate precisely the error performed by this method in terms of an appropriate power of the relative thinness and with a precise Sobolev norm of the boundary data. This method is well-known for non highly contrasted media. It is formally described in some particular cases in [1] and [15]. We also refer to Krähenbühl and Muller [16] for electromagnetic considerations. Usually, when it is estimated (see for example [11]) , the norm of the error involves a imprecise norm of the boundary data (a $\mathscr{C}^\infty$ norm while a weaker norm is enough) and mainly,

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the constant of the estimate depends strongly on the dielectric parameters of the domain.

It is not obvious (it is even false in general!) that such results hold for highly contrasted domains with thin layer, and this is a fact that researchers in computational electromagnetics are often confronted to such media. For example, a simple electric modelization of the biological cell consists of a conducting cytoplasm surrounded by a thin insulating membrane[1]; the modulus of the cytoplasmic complex permittivity divided by the membrane permittivity is around 10^5 while the relative thinness is equal to 10^{-3}. On the other hand, the medium might be a dielectric surrounded by a thin metallic layer. In both cases it is not clear that the usual approximated boundary condition might be used.

We derive asymptotics of the potential steady state voltage in all possible domains with thin layer (heterogeneous or highly heterogeneous). As we will see, the two situations described above lead to different behaviors of the potentials. The membrane relative thickness is equal to h, while the charateristic length of the inner domains is equal to 1. The first medium consists of a conducting inner domain (say that its complex permittivity is equal to 1) surrounded by a thin membrane; we denote by α the membrane complex permittivity. The parameter α is bounded but it may tend to zero. This is the reason why we say that the thin layer is an insulating membrane. The second material consists of an insulating inner domain of permittivity α surrounded by a conducting thin membrane (say that its complex permittivity is equal to 1). In this case, we suppose that α tends to zero. These two kinds of media describe all the possible media with thin layer. The aim of this paper is to derive full rigorous asymptotic expansion of steady state voltage potentials with respect to the small parameter h for bounded α (but it may tend to zero).

Let us write mathematically our problem. Let Ω_h be a smooth bounded bidimensional domain (see Fig. 1), composed of a smooth domain $\mathcal{O}$ surrounded by a thin membrane $\mathcal{O}_h$ with a small constant thickness h:

$$\Omega_h = \mathcal{O} \cup \mathcal{O}_h.$$

Let α be a non null complex parameter with positive real part; α is bounded but it may be very small. Without loss of generality, we suppose that $|\alpha| \leq 1$. Denote by q_h and γ_h the following piecewise constant functions

$$\forall x \in \Omega_h, \quad q_h(x) = \begin{cases} 1, \text{ if } x \in \mathcal{O}, \\ \alpha, \text{ if } x \in \mathcal{O}_h, \end{cases}$$

$$\forall x \in \Omega_h, \quad \gamma_h(x) = \begin{cases} \alpha, \text{ if } x \in \mathcal{O}, \\ 1, \text{ if } x \in \mathcal{O}_h. \end{cases}$$

We would like to understand the behavior for h tending to zero and uniformly with respect to $|\alpha| \leq 1$ of V_h and u_h the respective solutions to the following problems (1) and (2) with Neumann boundary condition; V_h satisfies

$$\nabla \cdot (q_h \nabla V_h) = 0 \text{ in } \Omega_h, \tag{1a}$$

$$\frac{\partial V_h}{\partial n} = \phi \text{ on } \partial\Omega_h, \tag{1b}$$

$$\int_{\partial\mathcal{O}} V_h \, \mathrm{d}\sigma = 0; \tag{1c}$$

[1] We refer to the author thesis [17] for a precise description of the biological cell.

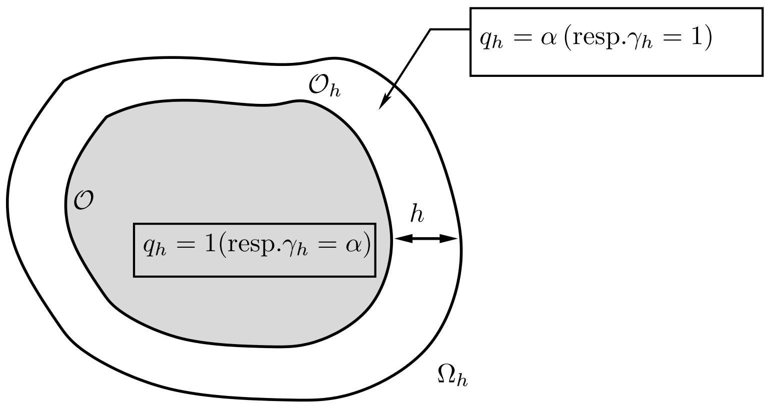


FIGURE 1. Parameters of Ω_h.

and u_h satisfies

(2a) $$\nabla \cdot (\gamma_h \nabla u_h) = 0 \text{ in } \Omega_h,$$

(2b) $$\frac{\partial u_h}{\partial n} = \phi \text{ on } \partial\Omega_h,$$

(2c) $$\int_{\partial\mathcal{O}} u_h \,\mathrm{d}\sigma = 0.$$

Since we impose a Neumann boundary conditions on $\partial\Omega_h$ the boundary data ϕ must satisfy the compatibility condition:

$$\int_{\partial\Omega_h} \phi \,\mathrm{d}\sigma = 0.$$

The above functions V_h and u_h are well-defined and belong to $H^1(\Omega_h)$ as soon as ϕ belongs to $H^{-1/2}(\partial\Omega_h)$.

Several authors have worked on similar problems (see for instance Beretta *et al.* [5] and [6]). They compared the exact solution to the so-called background solution defined by replacing the material of the membrane by the inner material. The difference between these two solutions has then been given through an integral involving the polarization tensor defined for instance in [2], [3], [5], [6], [7], plus some remainder terms. The remainder terms are estimated in terms of the measure of the inhomogeneity. In this paper, we do not use this approach, for several reasons.

The Beretta *et al.* estimate of the remainder terms depends linearly on α and $1/\alpha$: their results are no more valid in a highly contrasted domain (*i.e.* for α very large or very small). Secondly, α is complex-valued, hence differential operators involved in our case are not self-adjoint, so that the Γ-convergence techniques of Beretta *et al.* do not apply. Thirdly, the potential in the membrane is not given explicitly in [5], [6] or [7], while we are definitely interested in this potential, in order to obtain the transmembranar potential (see Fear and Stuchly [12]). Finally, the asymptotics of Beretta *et al.* are valid on the boundary of the domain, while we are interested in the potentials in the inner domain.

The heuristics of this work consist in performing a change of coordinates in the membrane $\mathcal{O}_h$, so as to parameterize it by local coordinates (η, θ), which vary in a domain independently of h; in particular, if we denote by L the length of $\partial\mathcal{O}$ (in the following, without any restriction, we suppose that L is equal to 2π)), the variables (η, θ) should vary in $[0, 1] \times \mathbb{R}/L\mathbb{Z}$. This change of coordinates leads to an expression of the Laplacian in the membrane, which depends on h. Once the transmission conditions of the new problem are derived, we perform a formal asymptotic expansion of the solution to Problem (1) (respectively to Problem (2)) in terms of h. It remains to validate this expansion. In this paper we work with bidimensional domain and we are confident that the same analysis could be perfomed in higher dimensions.

This paper is structured as follows. In Section 1, we make precise our geometric conventions. We perform a change of variables in the membrane, and with the help of some differential geometry results, we write Problem (1) and Problem (2) in the language of differential forms. We refer the reader to Flanders [13] or Dubrovin *et al.* [9] (or [8] for the french version) for courses on differential geometry. We derive transmission and boundary conditions in the intrinsic language of differential forms, and we express these relations in local coordinates.

In Section 3 we study Problem (1). In paragraph 3.1 we derive formally all the terms of the asymptotic expansion of the solution to our problem in terms of h. Paragraph 3.2 is devoted to a proof of the estimate of the error.

Problem (2) is considered in Section 4. We supposed that α tends to zero: a boundary layer phenomenon appears. To obtain our error estimates, we link the parameters h and α. We introduce a complex parameter β such that

$$Re(\beta) > 0, or\,(Re(\beta) = 0, \text{ and } \Im(\beta) \neq 0)\,,$$

and

$$|\beta| = o\left(\frac{1}{h}\right), \quad \text{and} \quad \frac{1}{|\beta|} = o\left(\frac{1}{h}\right).$$

We distinguish two different cases, depending on the convergence of $|\alpha|$ to zero: $\alpha = \beta h^q$, for $q \in N^*$ and $\alpha = o(h^N)$ for all $N \in \mathbb{N}$.

For $q = 1$ we obtain mixed boundary conditions for the asymptotic terms of the inner potential, and as soon as $q \geq 1$, appropriate Dirichlet boundary conditions are obtained. We end this section by error estimates.

In Appendix, we give some useful differential geometry formulae.

Remark 1. *The use of the formalism of differential forms $\delta\,(q_h \mathrm{d})$ could seem futile for the study of the operator $\nabla \cdot (q_h \nabla)$. In particular the expression of Laplace operator in local coordinates is well known. However we wanted to present this point of view to show how simple it is to write a Laplacian in curved coordinates once the metric is known.*

Moreover once this formalism is understood for the functions (or 0-forms), it is easy to study $\delta\,(q_h \mathrm{d})$ applied to 1-forms. This leads directly to the study of the operator $\mathrm{rot}\,(q_h\, \mathrm{rot})$, whose expression in local coordinates is less usual.

We choose to present our two main theorems in this introduction so that the reader interested in our results without their proves might find them easily.

We suppose that $\partial\mathcal{O}$ is smooth. We denote by Φ the $\mathscr{C}^\infty-$ diffeormorphism, which maps a neighborhood of cylinder $\boldsymbol{C} = [0, 1] \times \mathbb{R}/2\pi\mathbb{Z}$ unto a neighborhood of the thin layer. The diffeomorphism $\Phi_0 = \Phi(0, \cdot)$ maps the torus unto the boundary $\partial\mathcal{O}$ of the inner domain while $\Phi_1 = \Phi(1\cdot)$ is the $\mathscr{C}^\infty-$diffeomorphism from the torus unto $\partial\Omega_h$. We denote by κ the curvature of $\partial\mathcal{O}$ written in local coordinates, and let h_0 be such that

$$h_0 < \frac{1}{\sup_{\theta\in\mathbb{R}/2\pi\mathbb{Z}} |\kappa(\theta)|}.$$

Asymptotic for an insulating thin layer. The first theorem gives the asymptotic expansion of the solution V_h of (1), for h tending to zero, for bounded α.

Theorem 1. *Let h belong to $(0, h_0)$. The complex parameter α satisfies*

$$|\alpha| \leq 1, \tag{3}$$

$$\Re(\alpha) > 0 \ \text{ or } \ \Big\{\Re(\alpha) = 0 \text{ and } \Im(\alpha) \neq 0\Big\}. \tag{4}$$

Let $N \in \mathbb{N}$ and ϕ belong to $H^{N+3/2}(\partial\Omega_h)$. Denote by f and f the following functions:

$$\forall \theta \in \mathbb{R}/2\pi\mathbb{Z}, \quad f(\theta) = \phi \,\mathrm{o}\, \Phi_1(\theta),$$
$$\forall x \in \partial\mathcal{O}, \quad \mathrm{f}(x) = \phi \,\mathrm{o}\, \Phi_1 \,\mathrm{o}\, \Phi_0^{-1}(x).$$

Define the sequence of potentials $(V_k^c, V_k^m)_{k=0}^N$ as follows. We impose

$$\forall(\eta, \theta) \in \boldsymbol{C}, \quad \partial_\eta V_0^m = 0,$$

and we use the convention

$$\begin{cases} V_l^c = 0, \ \text{if } l \leq -1, \\ V_l^m = 0, \ \text{if } l \leq -1. \end{cases}$$

For $0 \leq k \leq N$ we define for all $0 \leq s \leq 1$ the function $\partial_\eta V_k^m(s, \cdot)$ on $\mathbb{R}/2\pi\mathbb{Z}$:

$$\begin{aligned} \partial_\eta V_{k+1}^m(s, \cdot) =& \delta_{1,k+1} f + \int_s^1 \Big\{ \kappa \left\{ 3\eta \partial_\eta^2 V_k^m + \partial_\eta V_k^m \right\} \\ &+ 3\eta^2 \kappa^2 \partial_\eta^2 V_{k-1}^m + 2\eta\kappa^2 \partial_\eta V_{k-1}^m + \partial_\theta^2 V_{k-1}^m \\ &+ \eta^3 \kappa^3 \partial_\eta^2 V_{k-2}^m + \eta^2 \kappa^3 \partial_\eta V_{k-2}^m + \eta\kappa \partial_\theta^2 V_{k-2}^m - \eta\kappa' \partial_\theta V_{k-2}^m \Big\} \mathrm{d}\eta, \end{aligned}$$

and the functions V_k^c and V_k^m are then defined by

$$\begin{aligned} &\Delta V_k^c = 0, \\ &\partial_n V_k^c|_{\partial\mathcal{O}} = \alpha \partial_\eta V_{k+1}^m \,\mathrm{o}\, \Phi_0^{-1}, \\ &\int_{\partial\mathcal{O}} V_k^c \mathrm{d}\sigma = 0, \\ &\forall s \in (0,1), \quad V_k^m(s, \cdot) = \int_0^s \partial_\eta V_k^m(\eta, \cdot)\, \mathrm{d}\eta + V_k^c \,\mathrm{o}\, \Phi_0. \end{aligned}$$

Let R_N^c and R_N^m be the functions defined by:

$$\begin{cases} R_N^c = V_h - \sum_{k=0}^N V_k^c h^k, \ \text{in } \mathcal{O}, \\ R_N^m = V_h \,\mathrm{o}\, \Phi - \sum_{k=0}^N V_k^m h^k, \ \text{in } \boldsymbol{C}. \end{cases}$$

Then, there exists a constant $C_{\mathcal{O},N} > 0$ depending only on the domain $\mathcal{O}$ and on N such that

$$\|R_N^c\|_{H^1(\mathcal{O})} \leq C_{\mathcal{O},N} \|\mathrm{f}\|_{H^{N+3/2}(\partial\mathcal{O})} |\alpha| h^{N+1/2}, \tag{5a}$$

$$\|R_N^m\|_{H^1_{\mathfrak{g}}(\boldsymbol{C})} \leq C_{\mathcal{O},N} \|\mathrm{f}\|_{H^{N+3/2}(\partial\mathcal{O})} h^{N+1/2}. \tag{5b}$$

Moreover, if ϕ belongs to $H^{N+5/2}(\partial\Omega_h)$, then we have

$$\|R_N^c\|_{H^1(\mathcal{O})} \leq C_{\mathcal{O},N} \|\mathrm{f}\|_{H^{N+5/2}(\partial\mathcal{O})} |\alpha| h^{N+1}, \tag{6a}$$

$$\|R_N^m\|_{H^1_{\mathfrak{g}}(\boldsymbol{C})} \leq C_{\mathcal{O},N} \|\mathrm{f}\|_{H^{N+5/2}(\partial\mathcal{O})} h^{N+1/2}. \tag{6b}$$

In this theorem, we approach the potential in the inner domain at the order N by solving N elementary problems with appropriate boundary condition. From these results, we may build an approximated boundary condition on $\partial\mathcal{O}$ at any order, in order to solve only one problem. However, this kind of conditions lead to numerical unstabilities, this is the reason why we think that the method to obtain the potential step by step is more useful.

Since it is classical to write approximated boundary conditions we make precise these conditions at the orders 0 and 1. Denote by $\mathfrak{K}$ the curvature of $\partial\mathcal{O}$ in Euclidean coordinates and by V^0_{app} and V^1_{app} the approximated potentials with approximated boundary condition at the order 0 and 1 respectively. We have:

$$\Delta V^0_{app} = 0, \text{ in } \mathcal{O}, \tag{7}$$

$$\partial_n V^0_{app} = \alpha \mathrm{f}, \tag{8}$$

and

$$\Delta V^1_{app} = 0, \text{ in } \mathcal{O}, \tag{9}$$

$$\partial_n V^1_{app} - \alpha h \partial_t^2 V^1_{app} = \alpha(1 + h\mathfrak{K})\mathrm{f}, \tag{10}$$

where ∂_t denotes the tangential derivative on $\partial\mathcal{O}$. The boundary condition (10) imposed to $\partial_n V^1_{app}$ is well-known for non highly contrasted media. It might be found in [16]. With our theorem, we prove that it remains valid for a very insulating membrane, and we give precise norm estimates. Moreover we give complete asymptotic expansion of the potential in both domains (the inner domain and the thin layer).

We perform numerial simulations in a circle of radius 1 surrounded by a thin layer of thickness h. In Fig 2, the left frame illustrates the asymptotic estimates at the orders 0 and 1 of Theorem 1 for an insulating thin layer. However, the right frame shows that as soon as the thin layer becomes very conducting, for example as soon as $\alpha = 1/h$, these asymptotics are no more valid: we have to use the asymptotics of Theorem 2.

Asymptotics for an insulating inner domain. Let β be a complex parameter satisfying:

$$Re(\beta) > 0, or\, (Re(\beta) = 0, \text{ and } \Im(\beta) \neq 0)\,.$$

The modulus of β may tend to infinity, or to zero but it must satisfy:

$$|\beta| = o\left(\frac{1}{h}\right), \quad \text{and} \quad \frac{1}{|\beta|} = o\left(\frac{1}{h}\right).$$

Theorem 2. *Let h belong to $(0, h_0)$. Let $q \in \mathbb{N}^*$ and $N \in \mathbb{N}$. We suppose that α satisfies:*

$$\alpha = \beta h^q. \tag{11}$$

Let ϕ belong to $H^{N+3/2+q}(\partial\Omega_h)$ and denote by f and f the following functions:

$$\forall\theta \in \mathbb{R}/2\pi\mathbb{Z}, \quad f(\theta) = \phi \,\mathrm{o}\, \Phi_1(\theta),$$

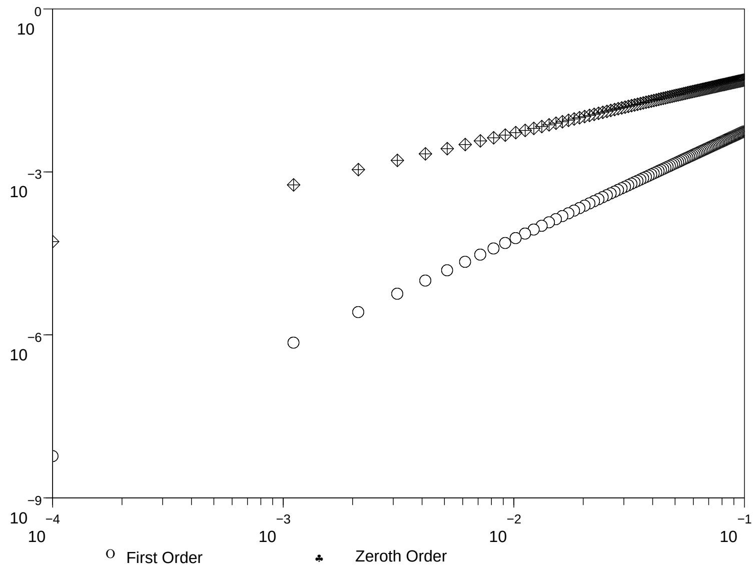
$$\forall x \in \partial\mathcal{O}, \quad \mathrm{f}(x) = \phi \,\mathrm{o}\, \Phi_1 \,\mathrm{o}\, \Phi_0^{-1}(x).$$

Define the function $(u^{c,q}_k, u^{m,q}_k)_{k=-1}^N$ by induction as follows, with the convention

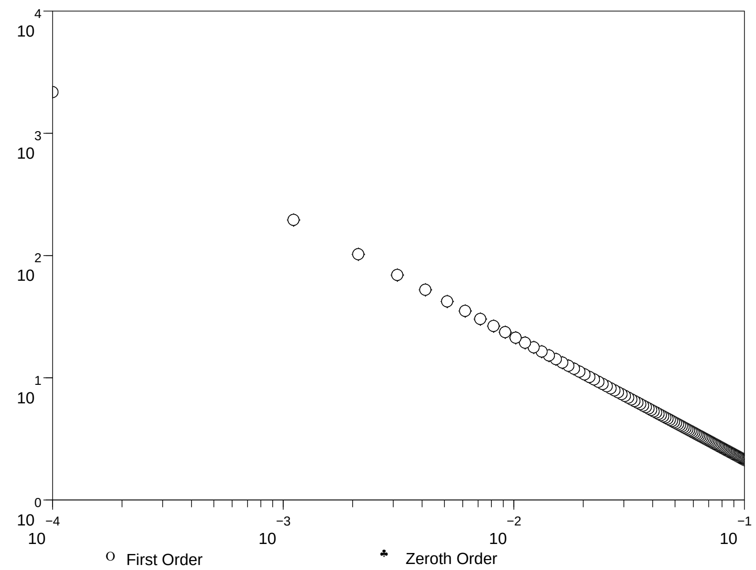
$$\begin{cases} u^{c,q}_l = 0, \ \text{if}\, l \leq -2, \\ u^{m,q}_l = 0, \ \text{if}\, l \leq -2. \end{cases}$$

- *If $q = 1$*

$$\begin{cases} \Delta u^{c,1}_{-1} = 0, \ \text{in } \mathcal{O}, \\ -\left.\partial_t^2 u^{c,1}_{-1}\right|_{\partial\mathcal{O}} + \beta\, \left.\partial_n u^{c,1}_{-1}\right|_{\partial\mathcal{O}} = \mathrm{f}, \\ \displaystyle\int_{\partial\mathcal{O}} u^{c,1}_{-1} \mathrm{d}_{\partial\mathcal{O}} = 0. \end{cases}$$



(a) $\alpha = i$



(b) $\alpha = i/h$

FIGURE 2. H^1 norm of the error at the orders 0 and 1.

$$\forall(\eta,\theta)\in \boldsymbol{C},\quad u_{-1}^{m,1}=u_{-1}^{c,1}|_{\partial\mathcal{O}}\,\mathrm{o}\,\Phi_0.$$

Moreover,

$$\partial_\eta u_0^{m,1}=0,\quad \partial_\eta u_1^{m,1}=(1-\eta)\partial_\theta^2 u_{-1}^{m,1}+f.$$

For $0\leq k\leq N$*, denote by* ϕ_k^1 *the following function:*

$$\phi_k^1=\int_0^1\left(\kappa\left(3\eta\partial_\eta^2 u_{k+1}^{m,1}+\partial_\eta u_{k+1}^{m,1}\right)+\eta\kappa\partial_\theta^2 u_{k-1}^{m,1}-\eta\kappa'\partial_\theta u_{k-1}^{m,1}\right)\mathrm{d}\eta.$$

and define $u_k^{c,1}$ *by*

$$\begin{cases}\Delta u_k^{c,1}=0,\ \text{in }\mathcal{O},\\ -\partial_t^2 u_k^{c,1}\Big|_{\partial\mathcal{O}}+\beta\,\partial_n u_k^{c,1}\Big|_{\partial\mathcal{O}}=\left(\phi_k^1-\displaystyle\int_0^1(\eta-1)\partial_\theta^2\partial_\eta u_k^{m,1}\mathrm{d}\eta\right)\mathrm{o}\,\Phi_0^{-1},\\ \displaystyle\int_{\partial\mathcal{O}}u_k^{c,1}\mathrm{d}_{\partial\mathcal{O}}=0.\end{cases}$$

In the membrane $u_k^{m,1}$ *is defined by*

$$u_k^{m,1}=\int_0^s\partial_\eta u_k^{m,q}\mathrm{d}\eta+u_k^{c,q}\,\mathrm{o}\,\Phi_0,$$

and $\partial_\eta u_{k+i}^{m,1}$ *for* $i=1,2$ *is determined by:*

$$\begin{aligned}\partial_\eta u_{k+i}^{m,1}=\int_1^s\Bigg(&-\kappa\left(3\eta\partial_\eta^2 u_{k+i-1}^{m,1}+\partial_\eta u_{k+i-1}^{m,1}\right)\\ &-\partial_\theta^2 u_{k+i-2}^{m,1}-\eta\kappa\partial_\theta^2 u_{k+i-3}^{m,1}+\eta\kappa'\partial_\theta u_{k+i-3}^{m,1}\Bigg)\mathrm{d}\eta.\end{aligned}\tag{12}$$

- *If* $q\geq 2$*. The function* $u_1^{m,q}$ *is defined by*

$$\int_{\mathbb{T}}u_{-1}^{m,q}\mathrm{d}\theta=0,$$
$$-\partial_\theta^2 u_{-1}^{m,q}=f.$$

The potential $u_{-1}^{c,q}$ *is solution to the following problem:*

$$\begin{cases}\Delta u_{-1}^{c,q}=0,\ \text{in }\mathcal{O},\\ u_{-1}^{c,q}\big|_{\partial\mathcal{O}}=u_{-1}^{m,q}\,\mathrm{o}\,\Phi_0^{-1}.\end{cases}$$

Moreover,

$$\partial_\eta u_0^{m,q}=0,\quad \partial_\eta u_1^{m,q}=(1-\eta)\partial_\theta^2 u_{-1}^{m,q}+f.$$

For $0\leq k\leq N$*, denote by* ϕ_k^q *the following function:*

$$\phi_k^q=\int_0^1\left(\kappa\left(3\eta\partial_\eta^2 u_{k+1}^{m,q}+\partial_\eta u_{k+1}^{m,q}\right)+\eta\kappa\partial_\theta^2 u_{k-1}^{m,q}-\eta\kappa'\partial_\theta u_{k-1}^{m,q}\right)\mathrm{d}\eta.$$

$u_k^{m,q}|_{\eta=1}$ *is entirely determined by the equality:*

$$-\partial_\theta^2 u_k^{m,q}|_{\eta=1}=\beta\partial_n u_{k+1-q}^{c,q}\,\mathrm{o}\,\Phi_0+\phi_k^q-\int_0^1\eta\partial_\theta^2\partial_\eta u_k^{m,q}\mathrm{d}\eta,$$

hence

$$u_k^{m,q}(s,\theta)=\int_1^s\partial_\eta u_k^{m,q}\mathrm{d}\eta+u_k^{m,q}|_{\eta=1}.$$

The potential $u_k^{c,q}$ satisfies the following boundary value problem:

$$\begin{cases} \Delta u_k^{c,q} = 0, \ \text{in } \mathcal{O}, \\ u_k^{c,q}|_{\partial\mathcal{O}} = u_k^{m,q} \circ \Phi_0^{-1}. \end{cases}$$

The functions $\left(\partial_\eta u_{k+i}^{m,q}\right)_{i=1,2}$ satisfies equation (12)*, in which $u^{m,1}$ is replaced by $u^{m,q}$.*

Let $r_N^{c,q}$ and $r_N^{m,q}$ be the functions defined by:

$$\begin{cases} r_N^{c,q} = u_h - \sum_{k=-1}^N u_k^{c,q} h^k, \ \text{in } \mathcal{O}, \\ r_N^{m,q} = u_h \circ \Phi - \sum_{k=-1}^N u_k^{m,q} h^k, \ \text{in } \boldsymbol{C}. \end{cases}$$

Then, there exists a constant $C_{\mathcal{O},N} > 0$ depending only on the domain $\mathcal{O}$ and on N such that

$$\|r_N^{c,q}\|_{H^1(\mathcal{O})} \leq C_{\mathcal{O},N} \|\mathrm{f}\|_{H^{N+3/2+q}(\partial\mathcal{O})} \max\left(\sqrt{\frac{h}{|\beta|}}, \sqrt{h}\right) h^{N+1/2},$$

$$\|r_N^{m,q}\|_{H^1_{\mathfrak{g}}(\boldsymbol{C})} \leq C_{\mathcal{O},N} \|\mathrm{f}\|_{H^{N+3/2}(\partial\mathcal{O})} h^{N+1/2}.$$

If ϕ belongs to $H^{N+5/2+q}(\partial\Omega_h)$, we have

$$\|r_N^{c,q}\|_{H^1(\mathcal{O})} \leq C_{\mathcal{O},N} \|f\|_{H^{N+5/2+q}(\mathbb{T})} h^{N+1}.$$

We observe that if $q = 1$ and $N = 0$, the approximated boundary condition at the order 0 is given by:

$$-(1 - h\mathfrak{K}/2)\partial_t^2 u_{0,app}^1 + \frac{h\partial_t\mathfrak{K}}{2}\partial_t u_{0,app}^1 + \beta\partial_n u_{0,app}^1 = \frac{1 + h\mathfrak{K}}{h} \phi \circ \Phi_1 \circ \Phi_0^{-1}.$$

Thus it is very different from the approximated boundary condition (8) imposed to V_{app}^0 in the case of an insulating membrane. This is a feature of the conducting thin layer. Observe on Fig 3 that the numerical computations in a circle confirm our theorical results.

Thanks to our previous results by comparing the parameters $|\alpha|$ and h of a heterogeneous medium with thin layer, we know *a priori*, which asymptotic formula (Theorem 1 or Theorem 2) has to be computed. We emphasize that our method might be easily implemented by iterative process as soon as the geometry of the domain is precisely known.

In the following, we show how the potentials $(V_k^c, V_k^m)_{k\geq 0}$ of the previous theorems are built, and then we prove these theorems. Let us now make precise the geometric conventions.

1. Geometry

The boundary of the domain $\mathcal{O}$ is assumed to be smooth. The orientation of the boundary $\partial\mathcal{O}$ is the trigonometric orientation. To simplify, we suppose that the length of $\partial\mathcal{O}$ is equal to 2π. We denote by $\mathbb{T}$ the flat torus:

$$\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}.$$

Since $\partial\mathcal{O}$ is smooth, we can parameterize it by a function Ψ of class $\mathscr{C}^\infty$ from $\mathbb{T}$ to $\mathbb{R}^2$ satisfying:

$$\forall\theta \in \mathbb{T}, \quad |\Psi'(\theta)| = 1.$$

Since the boundary $\partial\Omega_h$ of the cell is parallel to the boundary $\partial\mathcal{O}$ of the inner domain the following identities hold:

$$\partial\mathcal{O} = \{\Psi(\theta), \theta \in \mathbb{T}\},$$

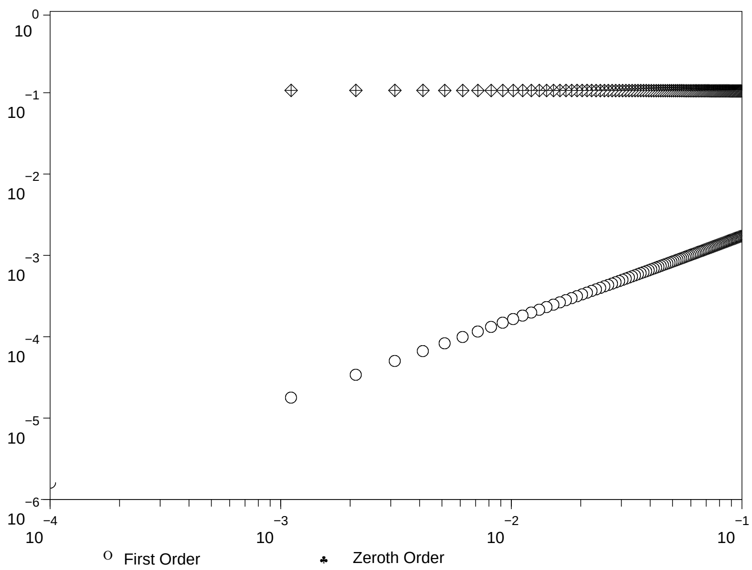


FIGURE 3. H^1 Norm of the error at the order -1 and 0 for an insulating inner domain : $\alpha = ih$.

and

$$\partial\Omega_h = \{\Psi(\theta) + hn(\theta), \theta \in \mathbb{T}\}.$$

Here $n(\theta)$ is the unitary exterior normal at $\Psi(\theta)$ to $\partial\mathcal{O}$. Therefore the membrane $\mathcal{O}_h$ is parameterized by:

$$\mathcal{O}_h = \{\Phi(\eta, \theta),\ (\eta,\ \theta) \in]0,\ 1[\times\mathbb{T}\},$$

where

$$\Phi(\eta, \theta) = \Psi(\theta) + h\eta n(\theta).$$

Denote by κ the curvature of $\partial\mathcal{O}$. Let h_0 belong to $(0, 1)$ such that:

$$h_0 < \frac{1}{\|\kappa\|_\infty}. \tag{13}$$

Thus for all h in $[0, h_0]$, there exists an open intervall I containing $(0, 1)$ such that Φ is a smooth diffeomorphism from $I \times \mathbb{R}/2\pi\mathbb{Z}$ to its image, which is a neighborhood of the membrane. The metric in $\mathcal{O}_h$ is:

$$h^2 \mathrm{d}\eta^2 + (1 + h\eta\kappa)^2 \mathrm{d}\theta^2. \tag{14}$$

Thus, we use two systems of coordinates, depending on the domains $\mathcal{O}$ and $\mathcal{O}_h$: in the interior domain $\mathcal{O}$, we use Euclidean coordinates (x, y) and in the membrane $\mathcal{O}_h$, we use local (η, θ) coordinates with metric (14).

We translate into the language of differential forms Problem (1) and Problem (2). We refer the reader to Dubrovin, Fomenko and Novikov [9] or Flanders [13] for the definition of the exterior derivative denoted by d, the exterior product denoted by ext, the interior derivative denoted by δ and the interior product denoted by int. In Appendix we give the formulae describing these operators in the case of a general 2D metric. Our aim, while rewriting our problems (1) and (2) is to take into account nicely the change of coordinates in the thin membrane.

Let $\mathbb{V}$ be the 0-form on Ω_h such that, in the Euclidean coordinates (x, y), $\mathbb{V}$ is equal to V, and let F be the 0-form, which is equal to ϕ on $\partial\Omega_h$. We denote by $\mathbb{N}$ the 1-form corresponding to the inward unit normal on the boundary Ω_h (see for instance Gilkey *et al.* [14] p.33):

$$\begin{aligned}\mathbb{N} &= N_x \mathrm{d}x + N_y \mathrm{d}y,\\ &= N_\eta \mathrm{d}\eta.\end{aligned}$$

$\mathbb{N}^*$ is the inward unit normal 1-form. Problem (1) takes now the intrinsic form:

$$\delta\left(q_h \mathrm{d}\mathbb{V}\right) = 0, \text{ in } \Omega_h, \tag{15a}$$

$$\mathrm{int}(\mathbb{N}^*)\mathrm{d}\mathbb{V} = F, \text{ on } \partial\Omega_h. \tag{15b}$$

According to Green's formula (Lemma 1.5.1 of [14]), we obtain the following transmission conditions for $\mathbb{V}$ along $\partial\mathcal{O}$:

$$\begin{aligned}\mathrm{int}(\mathbb{N}^*)\mathrm{d}\mathbb{V}|_{\partial\mathcal{O}} &= \alpha\,\mathrm{int}(\mathbb{N}^*)\mathrm{d}\mathbb{V}|_{\partial\mathcal{O}_h\setminus\partial\Omega_h},\\ \mathrm{ext}(\mathbb{N}^*)\mathbb{V}|_{\partial\mathcal{O}} &= \mathrm{ext}(\mathbb{N}^*)\mathbb{V}|_{\partial\mathcal{O}_h\setminus\partial\Omega_h}.\end{aligned} \tag{15c}$$

Similarly, denoting by $\mathbb{U}$ the 0-form equal to u in Euclidean coordinates we rewrite Problem (2) as follows:

$$\delta\left(\gamma_h \mathrm{d}\mathbb{U}\right) = 0, \text{ in } \Omega_h, \tag{16a}$$

$$\mathrm{int}(\mathbb{N}^*)\mathrm{d}\mathbb{U} = F, \text{ on } \partial\Omega_h; \tag{16b}$$

the following transmission conditions hold on $\partial\mathcal{O}$:

$$\begin{aligned}\alpha\,\mathrm{int}(\mathbb{N}^*)\mathrm{d}\mathbb{U}|_{\partial\mathcal{O}} &= \mathrm{int}(\mathbb{N}^*)\mathrm{d}\mathbb{U}|_{\partial\mathcal{O}_h\setminus\partial\Omega_h},\\ \mathrm{ext}(\mathbb{N}^*)\mathbb{U}|_{\partial\mathcal{O}} &= \mathrm{ext}(\mathbb{N}^*)\mathbb{U}|_{\partial\mathcal{O}_h\setminus\partial\Omega_h}.\end{aligned} \tag{16c}$$

2. Statement of the problem

In this section, we write Problem (15) and Problem (16) in local coordinates, with the help of differential forms. It is convenient to write:

$$\forall\theta \in \mathbb{T}, \quad \Phi_0\left(\theta\right) = \Phi\left(0, \theta\right), \Phi_1\left(\theta\right) = \Phi\left(1, \theta\right),$$

and to denote by $\boldsymbol{C}$ the cylinder:

$$\boldsymbol{C} = [0, 1] \times \mathbb{T}.$$

We denote by $\mathfrak{K}$, f and f the following functions:

$$\forall(x, y) \in \partial\mathcal{O}, \quad \mathfrak{K}(x, y) = \kappa \,\mathrm{o}\, \Phi_0^{-1}(x, y), \tag{17}$$

$$\forall\theta \in \mathbb{T}, \quad f(\theta) = \phi \,\mathrm{o}\, \Phi_1(\theta), \tag{18}$$

$$\forall x \in \partial\mathcal{O}, \quad \mathrm{f} = f \,\mathrm{o}\, \Phi_0^{-1}(x). \tag{19}$$

Using the expressions of the differential operators d and δ, which are respectively the exterior and the interior derivatives (see Appendix), applied to the metric (14), the Laplacian in the membrane is given in the local coordinates (η, θ) by:

$$\begin{aligned}&\forall\left(\eta, \theta\right) \in \boldsymbol{C},\\ &\Delta|_{\Phi(\eta,\theta)} = \frac{1}{h(1 + h\eta\kappa)}\partial_\eta\left(\frac{1 + h\eta\kappa}{h}\partial_\eta\right) + \frac{1}{1 + h\eta\kappa}\partial_\theta\left(\frac{1}{1 + h\eta\kappa}\partial_\theta\right).\end{aligned} \tag{20}$$

Moreover, for a 0-form z defined in $\mathcal{O}_h$, we have:

$$\begin{aligned} \mathrm{int}(\mathbb{N}^*)\mathrm{d}z|_{\partial\mathcal{O}} &= \frac{1}{h}\partial_\eta z|_{\eta=0}, \\ \mathrm{int}(\mathbb{N}^*)\mathrm{d}z|_{\partial\Omega_h} &= \frac{1}{h}\partial_\eta z|_{\eta=1}. \end{aligned}$$

Denote by

$$\begin{aligned} V^c &= V, \text{ in } \mathcal{O}, \\ V^m &= V \,\mathrm{o}\, \Phi, \text{ in } \boldsymbol{C}, \end{aligned}$$

and by

$$\begin{aligned} u^c &= u, \text{ in } \mathcal{O}, \\ u^m &= u \,\mathrm{o}\, \Phi, \text{ in } \boldsymbol{C}. \end{aligned}$$

We infer that Problem (15) may be rewritten as follows:

$$\Delta V^c = 0, \text{ in } \mathcal{O}, \tag{21a}$$

$$\forall\,(\eta,\theta)\in \boldsymbol{C}, \quad \frac{1}{h^2}\partial_\eta\left((1+h\eta\kappa)\partial_\eta V^m\right) + \partial_\theta\left(\frac{1}{1+h\eta\kappa}\partial_\theta V^m\right) = 0, \tag{21b}$$

$$\partial_n V^c \,\mathrm{o}\, \Phi_0 = \left.\frac{\alpha}{h}\partial_\eta V^m\right|_{\eta=0}, \tag{21c}$$

$$V^c \,\mathrm{o}\, \Phi_0 = \left. V^m\right|_{\eta=0}, \tag{21d}$$

$$\left.\partial_\eta V^m\right|_{\eta=1} = hf. \tag{21e}$$

$$\int_{\partial\mathcal{O}} V\,\mathrm{d}\sigma = 0.$$

Similarly the couple (u^c, u^m) satisfies

$$\Delta u^c = 0, \text{ in } \mathcal{O}, \tag{22a}$$

$$\forall\,(\eta,\theta)\in \boldsymbol{C}, \quad \frac{1}{h^2}\partial_\eta\Big((1+h\eta\kappa)\partial_\eta u^m\Big) + \partial_\theta\left(\frac{1}{1+h\eta\kappa}\partial_\theta u^m\right) = 0, \tag{22b}$$

$$\alpha\partial_n u^c \,\mathrm{o}\, \Phi_0 = \left.\frac{1}{h}\partial_\eta u^m\right|_{\eta=0}, \tag{22c}$$

$$u^c \,\mathrm{o}\, \Phi_0 = \left. u^m\right|_{\eta=0}, \tag{22d}$$

$$\left.\partial_\eta u^m\right|_{\eta=1} = hf, \tag{22e}$$

$$\int_{\partial\mathcal{O}} u\,\mathrm{d}\sigma = 0.$$

Remark 3. *In the following, the parameter α is such that:*

$$\Re(\alpha) > 0 \;\; or \;\; \Big\{\Re(\alpha) = 0 \; and \; \Im(\alpha) \neq 0\Big\}.$$

Since α represents a complex permittivity it may be written (see Balanis and Constantine [4]*) as follows:*

$$\alpha = \varepsilon - \mathrm{i}\sigma/\omega,$$

with ε, σ, and ω positive. Thus this hypothesis is always satisfied for dielectric materials.

Notation 4. *We provide $\boldsymbol{C}$ with the metric* (14). *The L^2 norm of a* 0*-form u in $\boldsymbol{C}$, denoted by $\|u\|_{\Lambda^0 L^2_m(\boldsymbol{C})}$, is equal to:*

$$\|u\|_{\Lambda^0 L^2_{\mathfrak{g}}(\boldsymbol{C})} = \left(\int_0^1 \int_0^{2\pi} h(1+h\eta\kappa)|u(\eta,\theta)|^2 \,\mathrm{d}\eta\,\mathrm{d}\theta\right)^{1/2}, \\ = \|u\|_{L^2(\mathcal{O}_h)},$$

and the L^2 norm of its exterior derivative du*, denoted by $\|\mathrm{d}u\|_{\Lambda^1 L^2_m}$ is equal to*

$$\|\mathrm{d}u\|_{\Lambda^1 L^2_{\mathfrak{g}}(\boldsymbol{C})} = \left(\int_0^1 \int_0^{2\pi} \frac{1+h\eta\kappa}{h}\left|\partial_\eta u(\eta,\theta)\right|^2 + \frac{h}{1+h\eta\kappa}\left|\partial_\theta u(\eta,\theta)\right|^2 \,\mathrm{d}\eta\,\mathrm{d}\theta\right)^{1/2}, \\ = \|\operatorname{grad} u\|_{L^2(\mathcal{O}_h)}.$$

To simplify our notations, for a 0-form u defined on $\boldsymbol{C}$, we define by $\|u\|_{H^1_{\mathfrak{g}}(\boldsymbol{C})}$ the following quantity

$$\|u\|_{H^1_{\mathfrak{g}}(\boldsymbol{C})} = \|u\|_{\Lambda^0 L^2_m(\boldsymbol{C})} + \|\mathrm{d}u\|_{\Lambda^1 L^2_m(\boldsymbol{C})},$$

when the above integrals are well-defined. Observe that for a function $u \in H^1(\mathcal{O}_h)$, we have:

$$\|u\|_{H^1(\mathcal{O}_h)} = \|u \circ \Phi\|_{H^1_{\mathfrak{g}}(\boldsymbol{C})}.$$

Remark 5 (Poincaré inequality in the thin layer)**.** *Let z belong to $H^1_{\mathfrak{g}}(\boldsymbol{C})$, such that*

$$\int_0^{2\pi} z(0,\theta)\,\mathrm{d}\theta = 0. \tag{23}$$

Then, there exists an h-independant constant $C_{\mathcal{O}}$ such that

$$\|z\|_{\Lambda^0 L^2_{\mathfrak{g}}(\boldsymbol{C})} \le C_{\mathcal{O}} \|\mathrm{d}z\|_{\Lambda^1 L^2_{\mathfrak{g}}(\boldsymbol{C})}. \tag{24}$$

We prove (24) *using Fourier analysis. According to the definition* (13) *of h_0 there exists two constants $C_{\mathcal{O}}$ and $c_{\mathcal{O}}$ depending on the domain $\mathcal{O}$ such that the following inequalities hold:*

$$\|z\|^2_{\Lambda^0 L^2_{\mathfrak{g}}(D)} \le C_{\mathcal{O}} h \int_0^1 \int_0^{2\pi} |z(\eta,\theta)|^2 \,\mathrm{d}\theta\,\mathrm{d}\eta, \tag{25a}$$

$$\|\mathrm{d}z\|^2_{\Lambda^1 L^2_{\mathfrak{g}}(D)} \ge c_{\mathcal{O}} \left(\int_0^1 \int_0^{2\pi} \frac{|\partial_\eta z(\eta,\theta)|^2}{h} + h\left|\partial_\theta z\right|^2 \,\mathrm{d}\theta\,\mathrm{d}\eta\right). \tag{25b}$$

For $k \in \mathbb{Z}$, we denote by $\widehat{z}_k$ the k^{th}-Fourier coefficient (with respect to θ) of z:

$$\widehat{z}_k = \frac{1}{2\pi}\int_0^{2\pi} z(\theta)\, e^{-\mathrm{i}k\theta}\,\mathrm{d}\theta.$$

Since $\left(\widehat{\partial_\theta z}\right)_k = \mathrm{i}k\widehat{z}_k$, we infer:

$$\forall k \neq 0, \quad \int_0^1 |\widehat{z}_k(\eta)|^2 \,\mathrm{d}\eta \le \int_0^1 \left|\left(\widehat{\partial_\theta z}\right)_k(\eta)\right|^2 \mathrm{d}\eta.$$

According to gauge condition (23)*, we have:*

$$\widehat{z}_0(0) = 0,$$

thus, using the equality

$$\widehat{z}_0(\eta) = \int_0^\eta \left(\widehat{\partial_\eta z}\right)_0(s)\mathrm{d}s,$$

we infer

$$\int_0^1 |\widehat{z}_0(\eta)|^2 \,\mathrm{d}\eta \leq \int_0^1 \left|\left(\widehat{\partial_\eta z}\right)_0(\eta)\right|^2 \,\mathrm{d}\eta.$$

Therefore,

$$\sum_{k\in\mathbb{Z}} \int_0^1 |\widehat{z}_k(\eta,\theta)|^2 \,\mathrm{d}\eta \leq \sum_{k\in\mathbb{Z}} \left\{ \int_0^1 \left|\left(\widehat{\partial_\theta z}\right)_k(\eta)\right|^2 \,\mathrm{d}\eta + \int_0^1 \left|\left(\widehat{\partial_\eta z}\right)_k(\eta)\right|^2 \right\}.$$

We end the proof of (24) *by using Parseval inequality and inequalities* (25).

3. Asymptotic expansion of the steady state potential for an insulating membrane

We derive asymptotic expansions with respect to h of the potentials (V^c, V^m) solution to Problem (21). The membrane is insulating since the modulus of α is supposed to be smaller than 1. However, our results are still valid if $|\alpha|$ is bounded by a constant C_0 greater than 1. We emphasize that the following results are valid for α tending to zero.

3.1. **Formal asymptotic expansion.** We write the following ansatz:

$$V^c = V_0^c + hV_1^c + h^2V_2^c + \cdots, \tag{26a}$$

$$V^m = V_0^m + hV_1^m + h^2V_2^m + \cdots. \tag{26b}$$

We multiply (21b) by $h^2(1+h\eta\kappa)^2$ and we order the powers of h to obtain:

$$\begin{aligned} &\forall(\eta,\theta)\in[0,1]\times\mathbb{T},\\ &\partial_\eta^2 V^m + h\kappa\left\{3\eta\partial_\eta^2 V^m + \partial_\eta V^m\right\} + h^2\left\{3\eta^2\kappa^2\partial_\eta^2 V^m + 2\eta\kappa^2\partial_\eta V^m + \partial_\theta^2 V^m\right\}\\ &\quad + h^3\left\{\eta^3\kappa^3\partial_\eta^2 V^m + \eta^2\kappa^3\partial_\eta V^m + \eta\kappa\partial_\theta^2 V^m - \eta\kappa'\partial_\theta V^m\right\} = 0 \end{aligned} \tag{27}$$

We are now ready to derive formally the terms of the asymptotic expansions of V^c and V^m by identifying the terms of the same power in h.

Recall that for (m,n) in $\mathbb{N}^2$, $\delta_{m,n}$ is Kronecker symbol equal to 1 if $m=n$ and to 0 if $m\neq n$. By identifying the powers of h, we infer that for $l\in\mathbb{N}$, V_l^c and V_l^m satisfy the following equations:

$$\Delta V_l^c = 0, \text{ in } \mathcal{O}, \tag{28a}$$

for all $(\eta,\theta)\in\boldsymbol{C}$,

$$\begin{aligned} \partial_\eta^2 V_l^m = -\Big\{&\kappa\left\{3\eta\partial_\eta^2 V_{l-1}^m + \partial_\eta V_{l-1}^m\right\}\\ &+ 3\eta^2\kappa^2\partial_\eta^2 V_{l-2}^m + 2\eta\kappa^2\partial_\eta V_{l-2}^m + \partial_\theta^2 V_{l-2}^m\\ &+ \eta^3\kappa^3\partial_\eta^2 V_{l-3}^m + \eta^2\kappa^3\partial_\eta V_{l-3}^m + \eta\kappa\partial_\theta^2 V_{l-3}^m - \eta\kappa'\partial_\theta V_{l-3}^m\Big\}, \end{aligned} \tag{28b}$$

with transmission conditions

$$\partial_n V_l^c \,\mathrm{o}\, \Phi_0 = \alpha\, \partial_\eta V_{l+1}^m\big|_{\eta=0}, \tag{28c}$$

$$V_l^c \,\mathrm{o}\, \Phi_0 = V_l^m\big|_{\eta=0}, \tag{28d}$$

with boundary condition

$$\partial_\eta V_l^m\big|_{\eta=1} = \delta_{l,1} f, \tag{28e}$$

and with gauge condition

$$\int_{\partial\mathcal{O}} V_l^c \mathrm{d}\sigma = 0. \tag{28f}$$

In equations (28), we have implicitly imposed

$$\begin{cases} V_l^c = 0, \text{ if } l \leq -1, \\ V_l^m = 0, \text{ if } l \leq -1. \end{cases} \tag{29}$$

The next lemma ensures that for each non null integer N, the functions V_N^c and V_N^m are entirely determined if the boundary condition ϕ is enough regular.

Notation 6. *For $s \in \mathbb{R}$, we denote by $\mathscr{C}^\infty([0,1]; H^s(\mathbb{T}))$ the space of functions u defined for $(\eta,\theta) \in [0,1] \times \mathbb{T}$, such that for almost all $\theta \in \mathbb{T}$, $u(\cdot,\theta)$ belongs to $\mathscr{C}^\infty([0,1])$, and such that for all $\eta \in [0,1]$, $u(\eta,\cdot)$ belongs to $H^s(\mathbb{T})$.*

Lemma 7. *We suppose that $\partial\mathcal{O}$ is smooth.*

For $N \in \mathbb{N}$ and $p \geq 0$ we suppose that ϕ belongs to $H^{N+p-1/2}(\partial\Omega_h)$ and let $|\alpha| \leq 1$.

Then the functions $V_0^m, \cdots, V_N^m$ and $V_0^c, \cdots, V_N^c$ are uniquely determined and they belong to the respective functional spaces:

$$\forall k = 0, \cdots, N,$$

$$V_k^m \in \mathscr{C}^\infty\left([0,1]; H^{N+p-k+1/2}(\mathbb{T})\right), \tag{30a}$$

$$V_k^c \in H^{N+p-k+1}(\mathcal{O}). \tag{30b}$$

Moreover, there exists a constant $C_{N,\mathcal{O},p}$ such that:

$$\forall k = 0, \cdots, N,$$

$$\sup_{\eta\in[0,1]} \|V_k^m(\eta,\cdot)\|_{H^{N+p-k+1/2}(\mathbb{T})} \leq C_{N,\mathcal{O},p} \|\mathrm{f}\|_{H^{N+p-1/2}(\partial\mathcal{O})}, \tag{31a}$$

$$\|V_k^c\|_{H^{N+p-k+1}(\mathcal{O})} \leq |\alpha| C_{N,\mathcal{O},p} \|\mathrm{f}\|_{H^{N+p-1/2}(\partial\mathcal{O})}. \tag{31b}$$

Remark 8. *To simplify, we suppose that $|\alpha| \leq 1$, but the same result may be obtained if there exists $C_0 > 1$ such that $|\alpha| \leq C_0$. In this case, the constant $C_{N,\mathcal{O},p}$ would also depends on C_0.*

Proof. Since $\partial\mathcal{O}$ is smooth and since ϕ belongs to $H^{N+p-1/2}(\partial\Omega_h)$, for $N \geq 0$ and $p \geq 0$, then the functions f and f defined by (18) and by (19) belong respectively to $H^{N+p-1/2}(\mathbb{T})$ and to $H^{N+p-1/2}(\partial\mathcal{O})$. We prove this lemma by recursive process.

- $N = 0$. Let $p \geq 0$ and let ϕ belong to $H^{p-1/2}(\partial\Omega_h)$.

Thus f and f belong respectively to $H^{p-1/2}(\mathbb{T})$ and $H^{p-1/2}(\partial\mathcal{O})$. Using (28b) and (28e), we infer:

$$\begin{cases} \partial_\eta^2 V_0^m = 0\,, \\ \partial_\eta V_0^m|_{\eta=1} = 0, \end{cases} \tag{32}$$

hence, $\partial_\eta V_0^m = 0$. According to (28b) and to (28e), we straight infer

$$\partial_\eta V_1^m = f.$$

Therefore by (28a) and (28c) the function V_0^c satisfies the following Laplace problem:

$$\Delta V_0^c = 0, \tag{33a}$$

$$\partial_n V_0^c|_{\partial\mathcal{O}} = \alpha \mathrm{f}, \tag{33b}$$

with gauge condition

$$\int_{\partial\mathcal{O}} V_0^c \mathrm{d}\sigma = 0. \tag{33c}$$

According to (28d), we infer

$$V_0^m = V_0^c \,\mathrm{o}\, \Phi_0, \tag{34}$$

hence V_0^c and V_0^m are entirely determined and they belong to the following spaces:

$$V_0^m \in \mathscr{C}^\infty\left([0,1]; H^{p+1/2}(\mathbb{T})\right),$$
$$V_0^c \in H^{p+1}(\mathcal{O}).$$

Observe also that there exists a constant $C_{\mathcal{O},p}$ such that

$$\sup_{\eta\in[0,1]} \|V_0^m(\eta,\cdot)\|_{H^{p+1/2}(\mathbb{T})} \leq C_{\mathcal{O},p}\|\mathrm{f}\|_{H^{p-1/2}(\partial\mathcal{O})},$$
$$\|V_0^c\|_{H^{p+1}(\mathcal{O})} \leq |\alpha| C_{\mathcal{O},p}\|\mathrm{f}\|_{H^{p-1/2}(\partial\mathcal{O})}.$$

- Induction.

Let $N \geq 0$. Suppose that for all $p \geq 0$, for all $\phi \in H^{N+p-1/2}(\partial\Omega_h)$ and for $M = 0,\cdots,N$ the functions V_M^c and V_M^m are known. Suppose that they belong respectively to $H^{N+p-M+1}(\mathcal{O})$ and to $V_M^m \in \mathscr{C}^\infty\left([0,1]; H^{N+p-M+1/2}(\mathbb{T})\right)$ and that estimates (31) hold.

Let ϕ belong to $H^{N+p+1/2}(\partial\Omega_h)$. Therefore, for $M = 0,\cdots,N$, the functions V_M^c and V_M^m are known, they belong respectively to $H^{N+p-M+2}(\mathcal{O})$ and to $V_M^m \in \mathscr{C}^\infty\left([0,1]; H^{N+p-M+3/2}(\mathbb{T})\right)$ and the following estimates hold:

$$\forall M = 0,\cdots,N,$$
$$\sup_{\eta\in[0,1]} \|V_M^m(\eta,\cdot)\|_{H^{N+p-M+3/2}(\mathbb{T})} \leq C_{N,\mathcal{O},p}\|\mathrm{f}\|_{H^{N+p+1/2}(\partial\mathcal{O})},$$
$$\|V_M^c\|_{H^{N+p-M+2}(\mathcal{O})} \leq |\alpha| C_{N,\mathcal{O},p}\|\mathrm{f}\|_{H^{N+p+1/2}(\partial\mathcal{O})}.$$

We are going to build V_{N+1}^c and V_{N+1}^m. From (28b) and (28e), we infer, for all $(\eta,\theta) \in \boldsymbol{C}$,

$$\begin{aligned}
\partial_\eta^2 V_{N+1}^m = -\Bigg\{&\kappa\left\{3\eta\partial_\eta^2 V_N^m + \partial_\eta V_N^m\right\}\\
&+ 3\eta^2\kappa^2\partial_\eta^2 V_{N-1}^m + 2\eta\kappa^2\partial_\eta V_{N-1}^m + \partial_\theta^2 V_{N-1}^m\\
&+ \eta^3\kappa^3\partial_\eta^2 V_{N-2}^m + \eta^2\kappa^3\partial_\eta V_{N-2}^m + \eta\kappa\partial_\theta^2 V_{N-2}^m - \eta\kappa'\partial_\theta V_{N-2}^m\Bigg\},\\
\partial_\eta V_{N+1}^m\big|_{\eta=1} = 0.&
\end{aligned}$$

Recall that we use convention (29). Since we have supposed that V_M^m is known for $M \leq N$ and belongs to $\mathscr{C}^\infty\left([0,1]; H^{N+1+p-M-1/2}(\mathbb{T})\right)$, we infer that:

$$\forall (s,\theta) \in \boldsymbol{C},$$

$$\begin{aligned}
\partial_\eta V_{N+1}^m(s,\cdot) = \int_s^1 \Bigg\{&\kappa\left\{3\eta\partial_\eta^2 V_N^m + \partial_\eta V_N^m\right\}\\
&+ 3\eta^2\kappa^2\partial_\eta^2 V_{N-1}^m + 2\eta\kappa^2\partial_\eta V_{N-1}^m + \partial_\theta^2 V_{N-1}^m\\
&+ \eta^3\kappa^3\partial_\eta^2 V_{N-2}^m + \eta^2\kappa^3\partial_\eta V_{N-2}^m + \eta\kappa\partial_\theta^2 V_{N-2}^m - \eta\kappa'\partial_\theta V_{N-2}^m\Bigg\}\mathrm{d}\eta,
\end{aligned} \tag{35}$$

is entirely determined and belongs to $\mathscr{C}^\infty\left([0,1];H^{p+1/2}(\mathbb{T})\right)$. Moreover, since $\partial_\eta V^m_{N+1}$ is known, we infer exactly by the same way that $\partial_\eta V^m_{N+2}$ is also determined. Actually, it is equal to

$$\forall (s,\theta)\in \boldsymbol{C},$$
$$\begin{aligned}\partial_\eta V^m_{N+2}(s,\cdot) = \int_s^1 \Big\{ &\kappa\left\{3\eta\partial^2_\eta V^m_{N+1} + \partial_\eta V^m_{N+1}\right\}\\ &+ 3\eta^2\kappa^2\partial^2_\eta V^m_N + 2\eta\kappa^2\partial_\eta V^m_N + \partial^2_\theta V^m_N\\ &+ \eta^3\kappa^3\partial^2_\eta V^m_{N-1} + \eta^2\kappa^3\partial_\eta V^m_{N-1} + \eta\kappa\partial^2_\theta V^m_{N-1} - \eta\kappa'\partial_\theta V^m_{N-1}\Big\}\mathrm{d}\eta,\end{aligned}$$

and it belongs to $\mathscr{C}^\infty\left([0,1];H^{p+1/2}(\mathbb{T})\right)$. According to (28c), the function V^c_{N+1} is then uniquely determined by

$$\Delta V^c_{N+1} = 0, \tag{36a}$$

$$\partial_n V^c_{N+1}\big|_{\partial\mathcal{O}} = \alpha\partial_\eta V^m_{N+2} \circ \Phi_0^{-1}, \tag{36b}$$

with gauge condition

$$\int_{\partial\mathcal{O}} V^c_{N+1}\mathrm{d}\sigma = 0. \tag{36c}$$

Moreover, it belongs to $H^{p+1}(\mathcal{O})$. Transmission condition (28d) implies the following expression of V^m_{N+1}:

$$\forall s\in(0,1),\quad V^m_{N+1}(s,\cdot) = \int_0^s \partial_\eta V^m_{N+1}(\eta,\cdot)\,\mathrm{d}\eta + V^c_{N+1}\circ\Phi_0,$$

where $\partial_\eta V^m_{N+1}$ is given by (35) and belongs to $\mathscr{C}^\infty\left([0,1];H^{p+1/2}(\mathbb{T})\right)$. We infer also that there exists $C_{N+1,\mathcal{O},p}>0$ such that

$$\sup_{\eta\in[0,1]}\left\|V^m_{N+1}(\eta,\cdot)\right\|_{H^{p+1/2}(\mathbb{T})} \le C_{N+1,\mathcal{O},p}\|\mathrm{f}\|_{H^{N+p+1/2}(\partial\mathcal{O})},$$
$$\left\|V^c_{N+1}\right\|_{H^{p+1}(\mathcal{O})} \le |\alpha| C_{N+1,\mathcal{O},p}\|\mathrm{f}\|_{H^{N+p+1/2}(\partial\mathcal{O})},$$

hence the lemma. □

Observe that the functions (V^c_k, V^m_k) are these given in Theorem 1.

3.2. **Error Estimates of Theorem 1.** Let us prove now the estimates of Theorem 1. Let $N\in\mathbb{N}$ and ϕ belong to $H^{N+3/2}(\partial\Omega_h)$. The function f is defined by (19). Let R^c_N and R^m_N be the functions defined by:

$$\begin{cases} R^c_N = V_h - \sum_{k=0}^N V^c_k h^k, \text{ in } \mathcal{O},\\ R^m_N = V_h\circ\Phi - \sum_{k=0}^N V^m_k h^k, \text{ in } \boldsymbol{C}.\end{cases}$$

We have to prove that there exists a constant $C_{\mathcal{O},N}>0$ depending only on the domain $\mathcal{O}$ and on N such that

$$\|R^c_N\|_{H^1(\mathcal{O})} \le C_{\mathcal{O},N}\|\mathrm{f}\|_{H^{N+3/2}(\partial\mathcal{O})}|\alpha| h^{N+1/2}, \tag{37a}$$

$$\|R^m_N\|_{H^1_{\mathfrak{g}}(\boldsymbol{C})} \le C_{\mathcal{O},N}\|\mathrm{f}\|_{H^{N+3/2}(\partial\mathcal{O})} h^{N+1/2}. \tag{37b}$$

Moreover, if ϕ belongs to $H^{N+5/2}(\partial\Omega_h)$, then we have

$$\|R^c_N\|_{H^1(\mathcal{O})} \le C_{\mathcal{O},N}\|\mathrm{f}\|_{H^{N+5/2}(\partial\mathcal{O})}|\alpha| h^{N+1}, \tag{38a}$$

$$\|R^m_N\|_{H^1_{\mathfrak{g}}(\boldsymbol{C})} \le C_{\mathcal{O},N}\|\mathrm{f}\|_{H^{N+5/2}(\partial\mathcal{O})} h^{N+1/2}. \tag{38b}$$

Proof of Theorem 1. Since ϕ belongs to $H^{N+3/2}(\partial\Omega_h)$, according to the previous lemma, the couples of functions (R^c_N, R^m_N) and $\left(R^c_{N+1}, R^m_{N+1}\right)$ are well defined and belong to $H^1(\mathcal{O}) \times H^1_{\mathfrak{g}}(\boldsymbol{C})$. The Sobolev space $H^1_{\mathfrak{g}}(\boldsymbol{C})$ is defined in Notation 4.

Denote by g_N the following function defined on $\boldsymbol{C}$:

$$\begin{aligned} g_N = & \kappa\left(3\eta\partial^2_\eta V^m_N + \partial_\eta V^m_N\right) + 3\eta^2\kappa^2\partial^2_\eta V^m_{N-1} + 2\eta\kappa^2\partial_\eta V^m_{N-1} + \partial^2_\theta V^m_{N-1} \\ & + \eta^3\kappa^3\partial^2_\eta V^m_{N-2} + \eta^2\kappa^3\partial_\eta V^m_{N-2} + \eta\kappa\partial^2_\theta V^m_{N-2} - \eta\kappa'\partial_\theta V^m_{N-2} \\ & + h\Bigg(3\eta^2\kappa^2\partial^2_\eta V^m_N + 2\eta\kappa^2\partial_\eta V^m_N + \partial^2_\theta V^m_N \\ & + \eta^3\kappa^3\partial^2_\eta V^m_{N-1} + \eta^2\kappa^3\partial_\eta V^m_{N-1} + \eta\kappa\partial^2_\theta V^m_{N-1} - \eta\kappa'\partial_\theta V^m_{N-1}\Bigg) \\ & + h^2\left(\eta^3\kappa^3\partial^2_\eta V^m_N + \eta^2\kappa^3\partial_\eta V^m_N + \eta\kappa\partial^2_\theta V^m_N - \eta\kappa'\partial_\theta V^m_N\right) \end{aligned} \tag{39}$$

According to the previous lemma and since ϕ belongs to $H^{N+1/2}(\partial\Omega_h)$, the above function g_N belongs to $\mathscr{C}^\infty\left([0,1]; H^{-1/2}(\mathbb{T})\right)$ and the function $\partial_\eta V^m_N$ belongs to $\mathscr{C}^\infty\left([0,1]; H^{3/2}(\mathbb{T})\right)$. Moreover, there exists a constant $C_{N,\mathcal{O}}$ such that

$$\begin{cases} \sup_{\eta\in[0,1]} \|g_N(\eta,\cdot)\|_{H^{-1/2}(\mathbb{T})} \leq C_{N,\mathcal{O}}\|f\|_{H^{N+1/2}(\mathbb{T})}, \\ \sup_{\eta\in[0,1]} \|\partial_\eta V^m_N(\eta,\cdot)\|_{H^{3/2}(\mathbb{T})} \leq C_{N,\mathcal{O}}\|f\|_{H^{N+1/2}(\mathbb{T})}. \end{cases} \tag{40}$$

The functions R^c_N and R^m_N satisfy the following problem:

$$\Delta R^c_N = 0, \text{ in } \mathcal{O},$$

$$\partial_\eta\left(\frac{1+h\eta\kappa}{h}\partial_\eta R^m_N\right) + \partial_\theta\left(\frac{h}{1+h\eta\kappa}\partial_\theta R^m_N\right) = \frac{-h^N}{(1+h\eta\kappa)}g_N,$$

with transmission conditions:

$$\partial_n R^c_N \,\mathrm{o}\, \Phi_0 = \frac{\alpha}{h}\left(\partial_\eta R^m_N|_{\eta=0} + h^{N+1}\,\partial_\eta V^m_N|_{\eta=0}\right),$$

$$R^c_N \,\mathrm{o}\, \Phi_0 = R^m_N|_{\eta=0},$$

with boundary condition

$$\partial_\eta R^m_N|_{\eta=1} = 0,$$

and with gauge condition

$$\int_{\partial\mathcal{O}} R^c_N \,\mathrm{d}\sigma = 0.$$

By multiplying the above equality by $\overline{R_N}$ and by integration by parts, we infer that:

$$\begin{aligned} \|\mathrm{d}R^c_N\|^2_{\Lambda^1 L^2(\mathcal{O})} + \alpha\,\|\mathrm{d}R^m_N\|^2_{\Lambda^1 L^2_{\mathfrak{g}}(\boldsymbol{C})} = & -\alpha h^N \int_{\boldsymbol{C}} g_N(\eta,\theta)\overline{R^m_N}(\eta,\theta)\,\mathrm{d}\eta\,\mathrm{d}\theta \\ & + \alpha h^{N+1}\int_{\mathbb{T}} \partial_\eta V^m_N|_{\eta=0}\,\overline{R^m_N}\big|_{\eta=0}\,\mathrm{d}\theta \\ & - \alpha h^{N+1}\int_{\mathbb{T}} \kappa\,\partial_\eta V^m_N|_{\eta=1}\,\overline{R^m_N}\big|_{\eta=1}\,\mathrm{d}\theta. \end{aligned} \tag{41}$$

By hypothesis (4), and using by Cauchy-Schwarz inequality and estimates (40), we infer that there exists a constant $C_{\mathcal{O},N} > 0$ such that

$$\Re(\alpha)\,\|\mathrm{d}R^m_N\|^2_{\Lambda^1 L^2_{\mathfrak{g}}(\boldsymbol{C})} \leq |\alpha|C_{\mathcal{O},N}h^{N-1/2}\|R^m_N\|_{H^1_{\mathfrak{g}}(\boldsymbol{C})}\|f\|_{H^{N+1/2}(\mathbb{T})}\|R^m_N\|_{H^1_{\mathfrak{g}}(\boldsymbol{C})},$$

and

$$|\Im(\alpha)|\,\|\mathrm{d}R_N^m\|^2_{\Lambda^1 L^2_{\mathfrak{g}}(\boldsymbol{C})} \leq |\alpha| C_{\mathcal{O},N} h^{N-1/2} \|R_N^m\|_{H^1_{\mathfrak{g}}(\boldsymbol{C})} \|f\|_{H^{N+1/2}(\mathbb{T})} \|R_N^m\|_{H^1_{\mathfrak{g}}(\boldsymbol{C})},$$

hence

$$\|\mathrm{d}R_N^m\|^2_{\Lambda^1 L^2_{\mathfrak{g}}(\boldsymbol{C})} \leq C_{\mathcal{O},N} h^{N-1/2} \|f\|_{H^{N+1/2}(\mathbb{T})} \|R_N^m\|_{H^1_{\mathfrak{g}}(\boldsymbol{C})}.$$

Since $\int_{\mathbb{T}} R_N^m|_{\eta=0}\,\mathrm{d}\theta = 0$, by Poincaré inequality (24), there exists a strictly positive constant $C_{\mathcal{O}}$, which does not depend on h such that

$$\|R_N^m\|_{\Lambda^0 L^2_{\mathfrak{g}}(\boldsymbol{C})} \leq C_{\mathcal{O}}\, \|\mathrm{d}R_N^m\|_{\Lambda^1 L^2_{\mathfrak{g}}(\boldsymbol{C})}\,,$$

hence

$$\|R_N^m\|_{H^1_{\mathfrak{g}}(\boldsymbol{C})} \leq C_{\mathcal{O},N} \|f\|_{H^{N+1/2}(\mathbb{T})} h^{N-1/2},$$

and therefore we deduce directly from the above estimate and from (41),

$$\|R_N^c\|_{H^1(\mathcal{O})} \leq C_{\mathcal{O},N} \|f\|_{H^{N+1/2}(\mathbb{T})} |\alpha| h^{N-1/2}.$$

The above estimate holds for $\phi \in H^{N+1/2}(\partial\Omega_h)$. Since ϕ belongs to $H^{N+3/2}(\partial\Omega_h)$, we obtain the same result by replacing N by $N+1$:

$$\left\{ \begin{aligned} &\left\|R_{N+1}^m\right\|_{H^1_{\mathfrak{g}}(\boldsymbol{C})} \leq C_{\mathcal{O},N+1} \|f\|_{H^{N+3/2}(\mathbb{T})} h^{N+1/2}, \\ &\left\|R_{N+1}^c\right\|_{H^1(\mathcal{O})} \leq C_{\mathcal{O},N+1} \|f\|_{H^{N+3/2}(\mathbb{T})} |\alpha| h^{N+1/2}. \end{aligned} \right. \tag{42}$$

According to the previous lemma, the functions V_{N+1}^c and V_{N+1}^m are well-defined and there exists a constant $C_{N,\mathcal{O}}$ such that:

$$\|V_{N+1}^c\|_{H^1(\mathcal{O})} \leq |\alpha| C_{N,\mathcal{O}} \|\mathrm{f}\|_{H^{N+3/2}(\partial\mathcal{O})},$$
$$\|V_{N+1}^m\|_{H^1_{\mathfrak{g}}(\boldsymbol{C})} \leq \frac{C_{N,\mathcal{O}}}{\sqrt{h}} \|\mathrm{f}\|_{H^{N+3/2}(\partial\mathcal{O})}.$$

Writing

$$R_N^c = R_{N+1}^c + V_{N+1}^c h^{N+1},$$

and

$$R_N^m = R_{N+1}^m + V_{N+1}^m h^{N+1},$$

we infer that

$$\|R_N^c\|_{H^1(\mathcal{O})} \leq C_{\mathcal{O},N} \|f\|_{H^{N+3/2}(\mathbb{T})} |\alpha| h^{N+1/2},$$

and

$$\|R_N^m\|_{H^1_{\mathfrak{g}}(\boldsymbol{C})} \leq C_{\mathcal{O},N} \|f\|_{H^{N+3/2}(\mathbb{T})} h^{N+1/2}.$$

If ϕ belongs to $H^{N+5/2}(\partial\Omega_h)$, we write

$$R_N^c = R_{N+2}^c + V_{N+1}^c h^{N+1} + V_{N+2}^c h^{N+2},$$

and

$$R_N^m = R_{N+1}^m + V_{N+1}^m h^{N+1} + V_{N+2}^m h^{N+2},$$

to obtain estimates (38), hence Theorem 1. □

Remark 9 (The case of an insulating inner domain)**.** *Consider Problem* (2)*:*

$$\operatorname{div}(\gamma_h \operatorname{grad} u) = 0 \ \textit{in}\ \Omega_h,$$
$$\frac{\partial u}{\partial n} = \phi \ \textit{on}\ \partial\Omega_h,$$
$$\int_{\partial\mathcal{O}} u\,\mathrm{d}\sigma = 0.$$

If the inner domain is perfectly insulating (i.e. if γ_h vanishes in $\mathcal{O}$), the steady state potential in the membrane satisfies:

$$\frac{1}{h^2}\partial_\eta\left((1+h\eta\kappa)\partial_\eta u^m\right)+\partial_\theta\left(\frac{1}{1+h\eta\kappa}\partial_\theta u^m\right)=0,\ \textit{in } \boldsymbol{C},$$

with the following boundary conditions:

$$\partial_\eta u^m|_{\eta=0}=0,\quad \partial_\eta u^m|_{\eta=1}=hf,$$

and with gauge condition

$$\int_0^{2\pi} u^m|_{\eta=0}\mathrm{d}\theta=0.$$

By identifying the terms of the same power of h we would obtain: $u_0^m=0$, and u_1^m would satisfy:

$$\begin{aligned}&\partial_\eta^2 u_1^m=0,\ \textit{in } \boldsymbol{C},\\ &\partial_\eta u_1^m|_{\eta=0}=0,\ \partial_\eta u_1^m|_{\eta=1}=f,\\ &\int_0^{2\pi} u_1^m|_{\eta=0}\mathrm{d}\theta=0,\end{aligned}$$

which is a non sense as soon as $f\neq 0$. Our ansatz (26) fails. Actually, the asymptotic expansion of u^m begins at the order -1: a boundary layer phenomenon appears.This is described in the next section.

4. Asymptotic expansion of the steady state potential for an insulating inner domain

Consider now the solution u_h to Problem (2). In this section, we suppose

(43a) $$|\alpha| \text{ tends to zero,}$$

(43b) $$\Re(\alpha)>0 \text{ or } \Big\{\Re(\alpha)=0 \text{ and } \Im(\alpha)\neq 0\Big\}.$$

Thus the inner domain is insulating. Let β be a complex parameter satisfying:

$$Re(\beta)>0, or\,(Re(\beta)=0, \text{ and } \Im(\beta)\neq 0)\,.$$

The modulus of β may tend to infinity, or to zero but it must satisfy:

$$|\beta|=o\left(\frac{1}{h}\right),\quad \text{and}\quad \frac{1}{|\beta|}=o\left(\frac{1}{h}\right).$$

We suppose that u may be written as follows:

$$u_h=\frac{1}{h}u_{-1}+u_0+hu_1+\cdots.$$

We denote by u^c and $u^m\,\mathrm{o}\,\Phi^{-1}$ the respective restrictions of u_h to $\mathcal{O}$ and to $\mathcal{O}_h$. One of the two following cases holds.

Hypothesis 10 ($\alpha=\beta h^q$)**.** *There exists $q\geq 1$ such that:*

(44) $$\alpha=\beta h^q.$$

Hypothesis 11 ($\alpha=o(h^N)$, $\forall N\in\mathbb{N}$)**.** *The complex parameter α satisfies* (43) *and for all $N\in\mathbb{N}$,*

(45) $$\forall N\in\mathbb{N},\quad |\alpha|=o(h^N).$$

First we suppose that Hypothesis 10 holds: we will discuss on Hypothesis 11 later on. We denote by $(u^{c,q}, u^{m,q})$ the solution to Problem 2 under the Hypothesis 10.

According to (22), by ordering and identifying the terms of the same power of h, for $k \in \mathbb{N} \cup -1$, for $q \in \mathbb{N}^*$, $u_k^{c,q}$ and $u_k^{m,q}$ satisfy:

$$\Delta u_l^{c,q} = 0, \text{ in } \mathcal{O}, \tag{46a}$$

for all $(\eta, \theta) \in \boldsymbol{C}$,

$$\begin{aligned} \partial_\eta^2 u_l^{m,q} = - \Big\{ & \kappa \left\{ 3\eta \partial_\eta^2 u_{l-1}^{m,q} + \partial_\eta u_{l-1}^{m,q} \right\} \\ & + 3\eta^2 \kappa^2 \partial_\eta^2 u_{l-2}^{m,q} + 2\eta\kappa^2 \partial_\eta u_{l-2}^{m,q} + \partial_\theta^2 u_{l-2}^{m,q} \\ & + \eta^3 \kappa^3 \partial_\eta^2 u_{l-3}^{m,q} + \eta^2 \kappa^3 \partial_\eta u_{l-3}^{m,q} + \eta\kappa \partial_\theta^2 u_{l-3}^{m,q} - \eta\kappa' \partial_\theta u_{l-3}^{m,q} \Big\}, \end{aligned} \tag{46b}$$

$$u_l^{c,q} \,\mathrm{o}\, \Phi_0 = u_l^{m,q}|_{\eta=0}\,, \tag{46c}$$

$$\partial_\eta u_l^{m,q}|_{\eta=1} = \delta_{l,1} f, \tag{46d}$$

$$\int_{\partial\mathcal{O}} u_l^{c,q} \mathrm{d}\sigma = 0. \tag{46e}$$

Transmission condition (22c) coupled with Hypothesis 10 implies:

$$\beta \partial_n u_{l-1-q}^{c,q} \,\mathrm{o}\, \Phi_0 = \partial_\eta u_l^{m,q}|_{\eta=0}\,, \tag{46f}$$

In equations (46), we have implicitly imposed

$$\begin{cases} u_l^{c,q} = 0, \text{ if } l \leq -2, \\ u_l^{m,q} = 0, \text{ if } l \leq -2. \end{cases} \tag{47}$$

Let us now derive formal asymptotics of u when Hypothesis 10 holds.

4.1. **Formal asymptotics.**

- $N = -1$.

The functions $u_{-1}^{m,q}$ satisfies

$$\begin{cases} \partial_\eta^2 u_{-1}^{m,q} = 0, \text{in } \boldsymbol{C}, \\ \partial_\eta u_{-1}^{m,q}|_{\eta=0} = 0, \quad \partial_\eta u_{-1}^{m,q}|_{\eta=1} = 0, \end{cases}$$

hence $u_{-1}^{m,q}$ depends only on the variable θ. Observe that we have, for almost all $\theta \in \mathbb{T}$ the following equality:

$$u_{-1}^{m,q}(\theta) = u_{-1}^{c,q} \,\mathrm{o}\, \Phi_0(\theta).$$

- $N = 0$.

The function $u_0^{m,q}$ satisfies:

$$\begin{cases} \partial_\eta^2 u_0^{m,q} = 0, \text{in } \boldsymbol{C}, \\ \partial_\eta u_0^{m,q}|_{\eta=0} = 0, \quad \partial_\eta u_0^{m,q}|_{\eta=1} = 0, \end{cases}$$

hence, $\partial_\eta u_0^{m,q}$ vanishes identically in $\boldsymbol{C}$.

- $N = 1$.

The functions $u_1^{m,q}$ satisfy:

$$\begin{cases} \partial_\eta^2 u_1^{m,q} = -\partial_\theta^2 u_{-1}^{m,q}, \text{in } \boldsymbol{C}, \\ \partial_\eta u_1^{m,q}|_{\eta=0} = \beta \partial_n u_{-1-q}^{c,q} \,\mathrm{o}\, \Phi_0, \quad \partial_\eta u_1^{m,q}|_{\eta=1} = f. \end{cases}$$

Therefore for $q = 1$ we obtain the following equality:

$$-\partial_\theta^2 u_{-1}^{m,1} + \beta \partial_n u_{-1}^{c,1} \,\mathrm{o}\, \Phi_0 = f,$$

hence the following boundary condition imposed to $u_{-1}^{c,1}$ on $\partial\mathcal{O}$:

$$-\left.\partial_\theta^2 u_{-1}^{c,1}\right|_{\partial\mathcal{O}} + \beta \left.\partial_n u_{-1}^{c,1}\right|_{\partial\mathcal{O}} = \mathrm{f}.$$

Therefore, the function $u_{-1}^{c,1}$ is solution to the following problem:

$$\begin{cases} \Delta u_{-1}^{c,1} = 0, \text{ in } \mathcal{O}, \\ -\left.\partial_\theta^2 u_{-1}^{c,1}\right|_{\partial\mathcal{O}} + \beta \left.\partial_n u_{-1}^{c,1}\right|_{\partial\mathcal{O}} = \mathrm{f}, \\ \displaystyle\int_{\partial\mathcal{O}} u_{-1}^{c,1} \mathrm{d}\partial\mathcal{O} = 0. \end{cases}, \tag{48}$$

$$\forall(\eta,\theta) \in \boldsymbol{C}, \quad u_{-1}^{m,1} = u_{-1}^{c,1}|_{\partial\mathcal{O}} \,\mathrm{o}\, \Phi_0. \tag{49}$$

Since $\Re(\beta) > 0$, a straight application of Lax-Milgram theorem ensures that $u^{c,1}$ is uniquely determined and belongs to $H^1(\mathcal{O})$ as soon as the boundary data belongs to $H^{-3/2}(\partial\mathcal{O})$.

If $q \geq 2$, the function $u_1^{m,q}$ satisfies:

$$-\partial_\theta^2 u_{-1}^{m,q} = f. \tag{50}$$

Since $\int_{\mathbb{T}} u_{-1}^{m,q} \mathrm{d}\theta = 0$, equality (50) defines uniquely $u_{-1}^{m,q}$. We infer that $u_{-1}^{c,q}$ is solution to the following problem:

$$\begin{cases} \Delta u_{-1}^{c,q} = 0, \text{ in } \mathcal{O}, \\ -\left. u_{-1}^{c,q}\right|_{\partial\mathcal{O}} = u_{-1}^{m,q} \,\mathrm{o}\, \Phi_0^{-1}. \end{cases} \tag{51}$$

Hence we have determined $u_{-1}^{m,q}$ and $u_{-1}^{c,q}$ for $q \in \mathbb{N}^*$. Observe that $u_{-1}^{c,1}$ is solution to Laplace equation with mixed boundary condition, and for $q \geq 2$ the potential $u_{-1}^{c,q}$ is the solution to Laplace equation with Dirichlet boundary condition, while for an insulating membrane, we obtained Neumann conditions for the approximated steady state potentials.

Let us now determined $u_N^{m,q}$ and $u_N^{c,q}$ for $q \in \mathbb{N}^*$ by recurrence.

- Induction.

Suppose that for $N \geq 0$, the functions $u_{N-1}^{m,q}$, $u_{N-1}^{c,q}$, $\partial_\eta u_N^{m,q}$ and $\partial_\eta u_{N+1}^{m,q}$ are built.

The function $u_{N+2}^{m,q}$ satisfies:

$$\begin{cases} \partial_\eta^2 u_{N+2}^{m,q} = -\kappa\left(3\eta\partial_\eta^2 u_{N+1}^{m,q} + \partial_\eta u_{N+1}^{m,q}\right) - \partial_\theta^2 u_N^{m,q} - \eta\kappa\partial_\theta^2 u_{N-1}^{m,q} + \eta\kappa'\partial_\theta u_{N-1}^{m,q}, \text{in } \boldsymbol{C}, \\ \partial_\eta u_{N+2}^{m,q}|_{\eta=0} = \beta\partial_n u_{N+1-q}^{c,q} \,\mathrm{o}\, \Phi_0, \quad \partial_\eta u_{N+2}^{m,q}|_{\eta=1} = 0. \end{cases}$$

Denote by ϕ_N^q the following function:

$$\phi_N^q = \int_0^1 \left(\kappa\left(3\eta\partial_\eta^2 u_{N+1}^{m,q} + \partial_\eta u_{N+1}^{m,q}\right) + \eta\kappa\partial_\theta^2 u_{N-1}^{m,q} - \eta\kappa'\partial_\theta u_{-1}^{m,q}\right)\mathrm{d}\eta.$$

Since $\partial_\eta^2 u_{N+1}^{m,q}$ and $\partial_\eta u_{N+1}^{m,q}$ are supposed to be known, the function ϕ_N^q is entirely determined. Observe that if $q = 1$, $\partial_\eta u_{N+2}^{m,1}|_{\eta=0}$ is unknown since $\partial_n u_N^{c,1}$ is not yet determined, while as soon as $q \geq 2$, $\partial_\eta u_{N+2}^{m,q}|_{\eta=0}$ is known.

Using transmission condition (46c) , we infer the following equality satisfied by $u_N^{m,1}$ in $\eta = 0$:

$$-\partial_\theta^2 u_N^{m,1}|_{\eta=0} + \beta\partial_n u_N^{c,1} \,\mathrm{o}\, \Phi_0 = \phi_N^1 - \int_0^1 (\eta - 1)\partial_\theta^2\partial_\eta u_N^{m,1}\mathrm{d}\eta,$$

hence the boundary condition imposed to $u_N^{c,1}$ on $\partial\mathcal{O}$:

$$\beta\,\partial_n u_N^{c,1}\Big|_{\partial\mathcal{O}} - \partial_\theta^2 u_N^{c,1}\Big|_{\partial\mathcal{O}} = \left(\phi_N^1 - \int_0^1 (\eta-1)\partial_\theta^2\partial_\eta u_N^{m,1}\mathrm{d}\eta\right) \mathrm{o}\,\Phi_0^{-1}.$$

Thus the function $u_N^{c,1}$ is solution to the following problem:

$$\left\{\begin{aligned} &\Delta u_N^{c,1} = 0, \text{ in } \mathcal{O},\\ &-\partial_\theta^2 u_N^{c,1}\Big|_{\partial\mathcal{O}} + \beta\,\partial_n u_N^{c,1}\Big|_{\partial\mathcal{O}} = \left(\phi_N^1 - \int_0^1 (\eta-1)\partial_\theta^2\partial_\eta u_N^{m,1}\mathrm{d}\eta\right) \mathrm{o}\,\Phi_0^{-1},\\ &\int_{\partial\mathcal{O}} u_N^{c,1}\mathrm{d}_{\partial\mathcal{O}} = 0. \end{aligned}\right. \tag{52}$$

In the membrane $u_N^{m,1}$ is defined by

$$u_N^{m,1} = \int_0^s \partial_\eta u_N^{m,q}\mathrm{d}\eta + u_N^{c,q}\,\mathrm{o}\,\Phi_0. \tag{53}$$

If $q \geq 2$, $u_N^{m,q}|_{\eta=1}$ is entirely determined by the equality:

$$-\partial_\theta^2 u_N^{m,q}|_{\eta=1} = \beta\partial_n u_{N+1-q}^{c,q}\,\mathrm{o}\,\Phi_0 + \phi_N^q - \int_0^1 (\eta-1)\partial_\theta^2\partial_\eta u_N^{m,q}\mathrm{d}\eta,$$

hence

$$u_N^{m,q}(s,\theta) = \int_1^s \partial_\eta u_N^{m,q}\mathrm{d}\eta + u_N^{m,q}|_{\eta=1}.$$

The potential $u_N^{c,q}$ satisfies the following boundary value problem:

$$\left\{\begin{aligned} &\Delta u_N^{c,q} = 0, \text{ in } \mathcal{O},\\ &u_N^{c,q}|_{\partial\mathcal{O}} = u_N^{m,q}\,\mathrm{o}\,\Phi_0^{-1}. \end{aligned}\right. \tag{54}$$

Observe that for $q \geq 1$, $\partial_\eta u_{N+2}^{m,q}$ is then entirely determined by:

$$\begin{aligned}\partial_\eta u_{N+2}^{m,q} = \int_1^s \Bigg(&-\kappa\left(3\eta\partial_\eta^2 u_{N+1}^{m,q} + \partial_\eta u_{N+1}^{m,q}\right)\\ &- \partial_\theta^2 u_N^{m,q} - \eta\kappa\partial_\theta^2 u_{N-1}^{m,q} + \eta\kappa'\partial_\theta u_{N-1}^{m,q}\Bigg)\mathrm{d}\eta.\end{aligned}$$

Therefore, we have proved that for all $N \geq -1$, for $q \in \mathbb{N}^*$, the functions $u_N^{c,q}$ and $u_N^{m,q}$ are uniquely determined.

Remark 12 (Regularity)**.** *Observe that these functions are the potentials given in Theorem 2. We leave the reader verify by induction that the following regularities hold. Let $q \in \mathbb{N}^*$, $N \geq -1$ and $p \geq 1$. Let ϕ belong to $H^{N+p-3/2}(\partial\Omega_h)$.*

$$\begin{aligned} &u_{-1}^{c,q} \in H^{1+N+p}(\mathcal{O}),\\ &u_{-1}^{m,q} \in \mathscr{C}^\infty\left([0,1]; H^{1/2+N+p}(\mathbb{T})\right),\\ \forall k = 0,\cdots,N,&\\ &u_k^{c,q} \in H^{1+N+p-2[k/2]}(\mathcal{O}), \qquad (55a)\\ &u_k^{m,q} \in \mathscr{C}^\infty\left([0,1]; H^{1/2+N+p-2[(k+1)/2]}(\mathbb{T})\right). \qquad (55b)\end{aligned}$$

Moreover, there exists a constant $C_{N,\mathcal{O},p}$ independant on h and β such that:

$$\sup_{\eta\in[0,1]} \left\| u_{-1}^{m,q}(\eta,\cdot)\right\|_{H^{1/2+N+p}(\mathbb{T})} \leq C_{N,\mathcal{O},p}\|\mathrm{f}\|_{H^{N+p-3/2}(\partial\mathcal{O})}, \tag{56a}$$

$$\left\| u_{-1}^{c}\right\|_{H^{1+N+p}(\mathcal{O})} \leq C_{N,\mathcal{O},p}\|\mathrm{f}\|_{H^{N+p-3/2}(\partial\mathcal{O})}, \tag{56b}$$

$$\forall k = 0,\cdots,N,$$

$$\sup_{\eta\in[0,1]} \| u_{k}^{m,q}(\eta,\cdot)\|_{H^{1/2+N+p-2[(k+1)/2]}(\mathbb{T})} \leq C_{N,\mathcal{O},p}\|\mathrm{f}\|_{H^{N+p-3/2}(\partial\mathcal{O})}, \tag{56c}$$

$$\| u_{k}^{c,q}\|_{H^{1+N+p-[k/2]}(\mathcal{O})} \leq C_{N,\mathcal{O},p}\|\mathrm{f}\|_{H^{N+p-3/2}(\partial\mathcal{O})}. \tag{56d}$$

4.2. **Error estimates of Theorem 2.** Let us now prove Theorem 2. Let $q \in \mathbb{N}^*$ and $N \in \mathbb{N}$. The complex parameter α satisfies (43) with Hypothesis 10. Let ϕ belong to $H^{N+3/2+q}(\partial\Omega_h)$. Let $r_N^{c,q}$ and $r_N^{m,q}$ be the functions defined by:

$$\begin{cases} r_N^{c,q} = u - \sum_{k=-1}^{N} u_k^{c,q} h^k, \text{ in } \mathcal{O}, \\ r_N^{m,q} = u \,\mathrm{o}\, \Phi - \sum_{k=-1}^{N} u_k^{m,q} h^k, \text{ in } \boldsymbol{C}. \end{cases}$$

We have to prove that there exists a constant $C_{\mathcal{O},N} > 0$ depending only on the domain $\mathcal{O}$ and on N such that

$$\| r_N^{c,q}\|_{H^1(\mathcal{O})} \leq C_{\mathcal{O},N}\|\mathrm{f}\|_{H^{N+3/2+q}(\partial\mathcal{O})} \max\left(\sqrt{\frac{h}{|\beta|}}, \sqrt{h}\right) h^{N+1/2}, \tag{57a}$$

$$\| r_N^{m,q}\|_{H^1_{\mathfrak{g}}(\boldsymbol{C})} \leq C_{\mathcal{O},N}\|\mathrm{f}\|_{H^{N+3/2}(\partial\mathcal{O})} h^{N+1/2}. \tag{57b}$$

If ϕ belongs to $H^{N+5/2+q}(\partial\Omega_h)$, we have

$$\| r_N^{c,q}\|_{H^1(\mathcal{O})} \leq C_{\mathcal{O},N}\| f\|_{H^{N+5/2+q}(\mathbb{T})} h^{N+1}.$$

Proof. The proof of Theorem 2 is similar to the proof of Theorem 1. Since ϕ belongs to $H^{N-1/2}(\partial\Omega_h)$, according to the previous lemma, the couples of functions $(r_N^{c,q}, r_N^{m,q})$ and $\left(r_{N+1}^{c,q}, r_{N+1}^{m,q}\right)$ are well defined and belong to $H^1(\mathcal{O}) \times H^1_{\mathfrak{g}}(\boldsymbol{C})$.

Denote by $\tilde{g}_N$ the following function defined on $\boldsymbol{C}$:

$$\begin{aligned} \tilde{g}_N =& \kappa\left(3\eta\partial_\eta^2 u_N^{m,q} + \partial_\eta u_N^{m,q}\right) + 3\eta^2\kappa^2\partial_\eta^2 u_{N-1}^{m,q} + 2\eta\kappa^2\partial_\eta u_{N-1}^{m,q} + \partial_\theta^2 u_{N-1}^{m,q} \\ &+ \eta^3\kappa^3\partial_\eta^2 u_{N-2}^{m,q} + \eta^2\kappa^3\partial_\eta u_{N-2}^{m,q} + \eta\kappa\partial_\theta^2 u_{N-2}^{m,q} - \eta\kappa'\partial_\theta u_{N-2}^{m,q} \\ &+ h\Bigg(3\eta^2\kappa^2\partial_\eta^2 u_N^{m,q} + 2\eta\kappa^2\partial_\eta u_N^{m,q} + \partial_\theta^2 u_N^{m,q} \\ &+ \eta^3\kappa^3\partial_\eta^2 u_{N-1}^{m,q} + \eta^2\kappa^3\partial_\eta u_{N-1}^{m,q} + \eta\kappa\partial_\theta^2 u_{N-1}^{m,q} - \eta\kappa'\partial_\theta u_{N-1}^{m,q}\Bigg) \\ &+ h^2\left(\eta^3\kappa^3\partial_\eta^2 u_N^{m,q} + \eta^2\kappa^3\partial_\eta u_N^{m,q} + \eta\kappa\partial_\theta^2 u_N^{m,q} - \eta\kappa'\partial_\theta u_N^{m,q}\right) \end{aligned} \tag{58}$$

According to the previous lemma and since ϕ belongs to $H^{N-1/2}(\partial\Omega_h)$, the above function $\tilde{g}_N$ belongs to $\mathscr{C}^\infty\left([0,1]; H^{-1/2}(\mathbb{T})\right)$ and the function $\partial_\eta V_N^{m,q}$ belongs to $\mathscr{C}^\infty\left([0,1]; H^{3/2}(\mathbb{T})\right)$. Moreover, there exists a constant $C_{N,\mathcal{O}}$ such that

$$\begin{cases} \sup_{\eta\in[0,1]} \|\tilde{g}_N(\eta,\cdot)\|_{H^{-1/2}(\mathbb{T})} \leq C_{N,\mathcal{O}}\|f\|_{H^{N-1/2}(\mathbb{T})}, \\ \sup_{\eta\in[0,1]} \|\partial_\eta u_N^{m,q}(\eta,\cdot)\|_{H^{3/2}(\mathbb{T})} \leq C_{N,\mathcal{O}}\|f\|_{H^{N-1/2}(\mathbb{T})}. \end{cases} \tag{59}$$

The functions $r_N^{c,q}$ and $r_N^{m,q}$ satisfy the following problem:

$$\Delta r_N^{c,q} = 0, \text{ in } \mathcal{O},$$

$$\partial_\eta \left(\frac{1+h\eta\kappa}{h} \partial_\eta r_N^{m,q} \right) + \partial_\theta \left(\frac{h}{1+h\eta\kappa} \partial_\theta r_N^{m,q} \right) = \frac{-h^N}{(1+h\eta\kappa)} \tilde{g}_N,$$

with transmission conditions:

$$\beta h^{1+q} \partial_n r_N^{c,q} \,\mathrm{o}\, \Phi_0 = \frac{1}{h} \left(\partial_\eta r_N^{m,q}|_{\eta=0} + \beta h^{N+1+q} \left(\partial_n u_{N-1}^{c,q} \,\mathrm{o}\, \Phi_0 + h \partial_\eta u_N^{m,q} \,\mathrm{o}\, \Phi_0 \right) \right),$$

$$r_N^{c,q} \,\mathrm{o}\, \Phi_0 = r_N^{m,q}|_{\eta=0},$$

with boundary condition

$$\partial_\eta r_N^{m,q}|_{\eta=1} = 0,$$

and with gauge condition

$$\int_{\partial\mathcal{O}} r_N^{c,q} \,\mathrm{d}\sigma = 0.$$

By multiplying the above equality by $\overline{r_N}$ and by integration by parts, we infer that:

(60)

$$\beta h^{1+q} \left\| \mathrm{d} r_N^{c,q} \right\|^2_{\Lambda^1 L^2(\mathcal{O})} + \left\| \mathrm{d} r_N^{m,q} \right\|^2_{\Lambda^1 L^2_{\mathfrak{g}}(\boldsymbol{C})} = -h^N \int_{\boldsymbol{C}} \tilde{g}_N(\eta,\theta) \overline{r_N^{m,q}}(\eta,\theta) \,\mathrm{d}\eta \,\mathrm{d}\theta + \beta h^{N+1+q} \int_{\mathbb{T}} \left(\partial_\eta u_{N-1}^{c,q} \,\mathrm{o}\, \Phi_0 + h \partial_\eta u_N^{c,q} \,\mathrm{o}\, \Phi_0 \right) \overline{r_N^{m,q}} \Big|_{\eta=0} \mathrm{d}\theta.$$

The end of the proof is similar to Theorem 1. Using the positivity of $\Re(\beta)$ we straight infer estimate (57b) of $r_N^{m,q}$. To obtain the estimates of $r_N^{c,q}$, we write:

$$r_N^{c,q} = r_{N+q}^{c,q} + \sum_{k=1}^{q} u_{N+k}^{c,q} h^{N+k}.$$

□

4.3. **The case $\alpha = o(h^N)$, $\forall N \in \mathbb{N}$.** Now, we suppose that Hypothesis 11 holds. In this case, we prove that u^c and u^m may be approximated by U^c and U^m, which are solution to:

(61a) $$\Delta U^m = 0, \text{ in } \mathcal{O}_h,$$

(61b) $$\partial_\eta U^m|_{\partial\mathcal{O}} = 0, \quad \partial_\eta U^m|_{\partial\Omega_h} = \phi,$$

(61c) $$\int_{\partial\mathcal{O}} U^m \mathrm{d}\sigma = 0.$$

and

(62a) $$\Delta U^c = 0, \text{ in } \mathcal{O},$$

(62b) $$U^c|_{\partial\mathcal{O}} = U^m|_{\partial\mathcal{O}}.$$

Actually, we have the following lemma:

Lemma 13. *Let ϕ belong to $H^{-1/2}(\partial\Omega_h)$. Let (u^c, u^m) be the solution to Problem* (2)*, and U^m and U^c be defined respectively by* (61) *and* (62)*. Then, we have:*

(63) $$\|u^m - U^m\|_{H^1(\mathcal{O}_h)} \leq C_{\mathcal{O}} |\alpha| \, |\phi|_{H^{-1/2}(\partial\Omega_h)},$$

(64) $$\|u^c - U^c\|_{H^1(\mathcal{O})} \leq C_{\mathcal{O}} \sqrt{|\alpha|} \, |\phi|_{H^{-1/2}(\partial\Omega_h)}.$$

Proof. Denote by w^c and w^m the following functions:

$$w^c = u^c - U^c, \quad w^m = u^m - U^m,$$

and let ϕ belong to $H^{-1/2}(\partial\Omega_h)$. We have:

$$\Delta w^c = 0, \text{ in } \mathcal{O}, \tag{65a}$$

$$\Delta w^m = 0, \text{ in } \mathcal{O}_h, \tag{65b}$$

$$\alpha\, \partial_n w^c|_{\partial\mathcal{O}} = \partial_n w^m|_{\partial\mathcal{O}} - \alpha\, \partial_n U^c|_{\partial\mathcal{O}}\,, \tag{65c}$$

$$w^c|_{\partial\mathcal{O}} = w^m|_{\partial\mathcal{O}}\,, \tag{65d}$$

$$\partial_\eta w^m|_{\partial\Omega_h} = 0, \tag{65e}$$

$$\int_{\partial\mathcal{O}} w^m \mathrm{d}\sigma = 0. \tag{65f}$$

Thus we infer:

$$\alpha \int_{\mathcal{O}} |\nabla w^c|^2 \,\mathrm{dvol}_{\mathcal{O}} + \int_{\mathcal{O}_h} |\nabla w^m|^2 \,\mathrm{dvol}_{\mathcal{O}_h} = \alpha \int_{\partial\mathcal{O}} \partial_n U^c|_{\partial\mathcal{O}}\, \overline{w^m} \mathrm{d}\sigma. \tag{66}$$

It is well-known that :

$$\|U^m\|_{H^1(\mathcal{O}_h)} \leq C_{\mathcal{O}}\, |\phi|_{H^{-1/2}(\partial\Omega_h)}\,,$$

and

$$\|U^c\|_{H^1(\mathcal{O}_h)} \leq C_{\mathcal{O}}\, |U^m|_{\partial\mathcal{O}}|_{H^{1/2}(\partial\mathcal{O})}\,.$$

Since α satisfies (43) we infer,

$$\|w^m\|_{H^1(\mathcal{O}_h)} \leq C_{\mathcal{O}}|\alpha|\, |\phi|_{H^{-1/2}(\partial\Omega_h)}\,,$$

and thereby

$$\|w^c\|_{H^1(\mathcal{O})} \leq C_{\mathcal{O}}\sqrt{|\alpha|}\, |\phi|_{H^{-1/2}(\partial\Omega_h)}\,.$$

□

It remains to derive asymptotics of U^m and then these of U^c. They are similar to asymptotics of $u^{m,q}$ for $q \geq 2$: we just have to replace β by zero. We think the reader may easily derive these asymptotics from our previous results.

Conclusion

In this paper, we have studied the steady state potentials in a highly contrasted domain with thin layer when Neumann boundary condition is imposed on the exterior boundary. We derived rigorous asymptotics with respect to the thickness of the potentials in each domain and we gave error estimate in terms of appropriate Sobolev norm of the boundary data, electromagnetic parameters of our domain and a constant depending only on the geometry of the domain. It has to be mentionned that for an insulating inner domain (or equivalently a conducting membrane), the asymptotic expansions start at the order -1 and mixed or Dirichlet boundary conditions has to be imposed on the asymptotic terms of the inner domain.

To illustrate these asymptotics, numerical simulations using FEM are forthcomig work with Patrick Dular from Université de Liège and Ronan Perrussel from Ampère laboratory of Lyon. Few results have been shown at the conference NUMELEC [18] with GetDP[10]. The main difficulty in illustrating the convergence of our asymptotic consists in the geometrical approximation of the domain: high-order geometric elements seem to be necessary.

Appendix

Let $\star$ denote the Hodge star operator, which maps 0-forms to 2-forms, 1-forms to 1-forms and 2-forms to 0-forms (see Flanders [13]). We give explicit formulae for the operators d, δ, ext and int. These formulae are straightforward consequences of the definition of the operators $\star$, d and $\delta = \star^{-1}\mathrm{d}\,\star$. We refer the reader to Dubrovin, Fomenko and Novikov [9].

We consider the metric given by the following matrix G

$$G = \begin{pmatrix} g_{11} & g_{12} \\ g_{12} & g_{22} \end{pmatrix}. \tag{67}$$

We denote by $|G|$ the determinant of G. The inverse of G is denoted by G^{-1}

$$G^{-1} = (g^{ij})_{ij},$$

and we suppose that the signature of G is equal to 1. Thereby, the operator $\star^2$ is equal to Id on the space of 0-forms and 2-forms and it is equal to $-\,\mathrm{Id}$ on 1-forms.

4.4. **Star operator in $\mathbb{R}^2$.**

4.4.1. *On 0-forms and on 2-forms.* Let T be a 0-form and let S be the 2-form $\nu\,\mathrm{d}y^1\mathrm{d}y^2$. Then $\star T$ is the 2-form $\mu\,\mathrm{d}y^1\mathrm{d}y^2$ and $\star S$ is the 0-form f. The following identities hold:

$$\mu = \sqrt{|G|}T,$$
$$f = \frac{1}{\sqrt{|G|}}\nu.$$

4.4.2. *On 1-forms.* Let T be the 1-form $T_1\,\mathrm{d}y^1 + T_2\,\mathrm{d}y^2$. Then $\star T$ is the 1-form $\mu_1\,\mathrm{d}y^1 + \mu_2\,\mathrm{d}y^2$, and we have the following formulae:

$$\mu_1 = -\sqrt{|G|}\left(g^{12}T_1 + g^{22}T_2\right),$$
$$\mu_2 = \sqrt{|G|}\left(g^{11}T_1 + g^{12}T_2\right).$$

4.5. **The action of d acting on 0-forms in $\mathbb{R}^2$.** Let μ be a 0 form, then $\mathrm{d}\mu$ has the following expression:

$$\mathrm{d}\mu = \frac{\partial\mu}{\partial y^1}\mathrm{d}y^1 + \frac{\partial\mu}{\partial y^2}\mathrm{d}y^2.$$

4.6. **The action of δ acting on 1-forms on $\mathbb{R}^2$.** Let μ be the 1-form $\mu_1\mathrm{d}y^1 + \mu_2\mathrm{d}y^2$, and define $\delta\mu = \alpha$. The 0-form α is equal to:

$$\alpha = -\frac{1}{\sqrt{|G|}}\left\{\frac{\partial}{\partial y_1}\left(\sqrt{|G|}\left(g^{11}\mu_1 + g^{12}\mu_2\right)\right) + \frac{\partial}{\partial y_2}\left(\sqrt{|G|}\left(g^{12}\mu_1 + g^{22}\mu_2\right)\right)\right\}.$$

4.7. **The exterior product of a 1-form with a 0-form.** Let N be the 1-form $N_1\mathrm{d}y^1 + N_2\mathrm{d}y^2$ and f be a 0-form. The exterior product of $\mathrm{ext}(N)f$ is:

$$\mathrm{ext}(N)f = fN_1\mathrm{d}y^1 + fN_2\mathrm{d}y^2.$$

4.8. **The interior product of a 1-form with a 1-form.** Let N and μ be the 1-forms $N_1\mathrm{d}y^1 + N_2\mathrm{d}y^2$, and $\mu_1\mathrm{d}y^1 + \mu_2\mathrm{d}y^2$. Then 0-form $\mathrm{int}(N)\mu$ has the following expression:

$$\mathrm{int}(N)\mu = N_1\left(\mu_1 g^{11} + \mu_2 g^{12}\right) + N_2\left(\mu_1 g^{12} + \mu_2 g^{22}\right).$$

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