

Solvability of monotone systems of fully nonlinear elliptic PDE's

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In this note we study quasimonotone systems of fully nonlinear equations, in the form

$$\begin{cases} H_1(D^2u_1, Du_1, u_1, \dots, u_n, x) = f_1(x) \\ H_2(D^2u_2, Du_2, u_1, \dots, u_n, x) = f_2(x) \\ \dots \\ H_n(D^2u_n, Du_n, u_1, \dots, u_n, x) = f_n(x), \end{cases} \quad (1)$$

in a bounded domain $\Omega \subset \mathbb{R}^N$; $n, N \geq 1$. Here H_i are uniformly elliptic fully nonlinear operators of Isaac's type.

Our work is motivated by the well-known paper [IK], where the theory of viscosity solutions was applied to get existence and uniqueness results for this type of systems. More precisely, in that paper conditions were given under which the system satisfies a comparison principle (and so it cannot have more than one solution), and it was shown that whenever the system has a subsolution and a supersolution, which are ordered, then Perron's method guarantees the existence of a solution. It is our goal here, on one hand, to extend as far as possible the uniqueness result from [IK] in the uniformly elliptic case, that is, to give a necessary and sufficient condition under which the system satisfies a comparison principle, and on the other hand, to give explicit conditions under which ordered sub- and supersolutions can be found. For a large class of systems our results are optimal. Further, we aim at proving results of existence of eigenvalues of vector fully nonlinear operators.

We shall make essential use of the results in the two recent papers [QS], where corresponding results for scalar equations were obtained, and [BS], where an Alexandrov-Bakelman-Pucci and Harnack type estimates for systems of type (1) were proved.

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We intend to consider nonlinear operators of Isaac's type, that is, H_k will be allowed to be a sup-inf of coupled linear operators

$$\sup_{\alpha \in \mathcal{A}_k} \inf_{\beta \in \mathcal{B}_k} \left\{ \sum_{i,j=1}^N a_{ij,k}^{\alpha,\beta}(x) \partial_{ij} u_k + \sum_{i=1}^N b_{i,k}^{\alpha,\beta}(x) \partial_i u_k + \sum_{j=1}^n c_{j,k}^{\alpha,\beta}(x) u_j \right\}. \quad (2)$$

where $\mathcal{A}_k, \mathcal{B}_k$ are arbitrary index sets. When $\mathcal{B}_k = \emptyset$ in (2) the corresponding sup-operator is usually referred to as Hamilton-Jacobi-Bellman (HJB) operator. These operators are essential tools in control theory and in theory of large deviations, while Isaac's operators are basic in game theory. We refer to [BS], [IK], and the references in these papers, for a larger list of problems, where systems of type (1) appear.

Before stating our general hypotheses on H_i in (1), we note for the reader's convenience that they are satisfied by operators of type (2) for instance if the matrices $(a_{ij,k}^{\alpha,\beta})$ are continuous and uniformly positive definite, and if $b_{i,k}^{\alpha,\beta}, c_{i,k}^{\alpha,\beta} \in L^\infty(\Omega)$, $c_{j,k}^{\alpha,\beta} \geq 0$ for $j \neq k$, for all α, β, i, j, k .

Next, we list our hypotheses on (1). We shall use the following notation. We define, for all $M, N \in \mathcal{S}_N$, $p, q \in \mathbb{R}^N$, the extremal operators $\mathcal{L}^-, \mathcal{L}^+$ by

$$\mathcal{L}^-(M, p) = \mathcal{M}_{\lambda, \Lambda}^-(M) - \gamma |p|, \quad \mathcal{L}^+(M, p) = \mathcal{M}_{\lambda, \Lambda}^+(M) + \gamma |p|,$$

for some constants $0 < \lambda \leq \Lambda$; $\gamma \geq 0$. Here $\mathcal{M}^+, \mathcal{M}^-$ denote the Pucci operators $\mathcal{M}_{\lambda, \Lambda}^+(M) = \sup_{A \in \mathcal{A}} \text{tr}(AM)$, $\mathcal{M}_{\lambda, \Lambda}^-(M) = \inf_{A \in \mathcal{A}} \text{tr}(AM)$, where $\mathcal{A} \subset \mathcal{S}_N$ denotes the set of matrices whose eigenvalues lie in $[\lambda, \Lambda]$.

For a function f we denote $f_+ = \max\{f, 0\}$, $f = f_+ - f_-$. We set

$$m_i^-(u) := \delta_i |u_i| + \sum_{j \neq i} \delta_j (u_j)_-, \quad m_i^+(u) := \delta_i |u_i| + \sum_{j \neq i} \delta_j (u_j)_+,$$

$$F_i^*(M, p, u) = \mathcal{L}^-(M, p) + m_i^+(u), \quad i = 1, \dots, n.$$

for some $\delta_1, \dots, \delta_n \geq 0$. Further, we shall make use of the following condition, which permits to compare operators. Given two operators $F(M, p, u, x)$, $H(M, p, u, x)$, we say that H satisfies condition (D_F) provided

$$(D_F) \quad \begin{aligned} -F(N - M, q - p, v - u, x) &\leq H(M, p, u, x) - H(N, q, v, x) \\ &\leq F(M - N, p - q, u - v, x). \end{aligned}$$

We shall use this condition for operators F which are convex in (M, p, u) (this is equivalent to supposing that F itself satisfies (D_F)). Note F_i^* are convex. In general, HJB operators are convex.

We suppose that the operators H_i are uniformly elliptic with bounded measurable coefficients, that is, for some $\gamma, \delta > 0$ and all $M, N \in \mathcal{S}_N$, $p, q \in \mathbb{R}^N$, $u, v \in \mathbb{R}^n$, $x \in \Omega$,

$$(H_1) \quad H_i \text{ satisfies } (D_{F_i^*}), \quad i = 1, \dots, n$$

Note the definition of m_i and (H_1) imply that the function $H_i(M, p, u, x)$ is nondecreasing in all variables u_j with $j \neq i$. Such a system is called quasimonotone.

Since we want to prove existence results, we are going to suppose that

$$(H_2) \quad H_i(M, 0, 0, x) \text{ is continuous in } \mathcal{S}_N \times \bar{\Omega}.$$

Viewing to get results on existence of eigenvalues, we are led to suppose that these operators are positively homogeneous.

$$(H_3) \quad H_i(tM, tp, tu, x) = t H_i(M, p, u, x), \quad \text{for all } t \geq 0, \quad i = 1, \dots, n.$$

In order to obtain precise results, it will be important to describe the coupling in the system, that is, the way it relates the functions u_i to each other. It turns out that this description can be done in terms of the bounded measurable functions $c_{ij}(x) := F_i(0, 0, e_j, x)$, where $e_j \in \mathbb{R}^n$ is the vector whose j -th coordinate is 1 and all other coordinates are zero. We set $\mathcal{C}(x) := (c_{ij}(x))_{i,j=1}^n$. As explained in [BS], any such matrix can have its lines and columns renumbered in such a way that it is in block triangular form, with each block on the main diagonal being *irreducible*. We recall a $n \times n$ matrix \mathcal{C} is called irreducible provided for any non-empty sets $I, J \subset \{1, \dots, n\}$ such that $I \cap J = \emptyset$ and $I \cup J = \{1, \dots, n\}$, there exist $i_0 \in I$ and $j_0 \in J$ for which

$$\text{meas}\{x \in \Omega \mid c_{i_0 j_0}(x) > 0\} > 0. \quad (3)$$

For simplicity, when (3) holds we write $c_{i_0 j_0} \not\equiv 0$ in Ω . Whenever \mathcal{C} is irreducible, we will say that (1) is *fully coupled*. Simply speaking, a system is fully coupled provided it cannot be split into two subsystems, one of which does not depend on the other.

More precisely, we can always renumber the equations in (1) and the components of u in such a way that we can write $\mathcal{C} = (\mathcal{C}_{kl})_{k,l=1}^m$, where $1 \leq m \leq n$, \mathcal{C}_{kl} are $t_k \times t_l$ matrices for some $t_k \leq n$ with $\sum_{k=1}^m t_k = n$, \mathcal{C}_{kk} is an *irreducible* matrix for all $k = 1, \dots, m$, and $\mathcal{C}_{kl} \equiv 0$ in Ω , for all $k, l \in \{1, \dots, m\}$ with $k < l$. Note that $m = 1$ means \mathcal{C} itself is irreducible, while $m = n$ means \mathcal{C} is in triangular form. We set $s_0 = 0$, $s_k = \sum_{j=1}^k t_j$, and $S_k = \{s_{k-1} + 1, \dots, s_k\}$.

For instance, any 1×1 matrix is irreducible. Up to renumbering, when $n = 2$ we divide the set of 2×2 matrices \mathcal{C} into two parts : matrices of

the form $\begin{pmatrix} * & a \\ b & * \end{pmatrix}$ and matrices of the form $\begin{pmatrix} * & 0 \\ * & * \end{pmatrix}$, where $a, b \neq 0$ and $*$ stands for an arbitrary function. The first of these matrices is irreducible, the second is not. Respectively, for $n = 3$, there are four types of matrices (of course, up to renumbering again) :

$$\begin{pmatrix} * & a & * \\ * & * & b \\ c & * & * \end{pmatrix}, \quad \begin{pmatrix} * & a & 0 \\ b & * & c \\ 0 & d & * \end{pmatrix}, \quad \begin{pmatrix} * & a & 0 \\ b & * & 0 \\ * & * & * \end{pmatrix}, \quad \begin{pmatrix} * & 0 & 0 \\ * & * & 0 \\ * & * & * \end{pmatrix}, \quad a, b, c, d \neq 0.$$

The first two of these matrices are irreducible, the third has one 2×2 and one 1×1 irreducible blocks, and the fourth has three 1×1 irreducible blocks.

We make the convention that all (in)equalities between vectors will be understood to hold component-wise. Also, any time we speak of a solution (supersolution, subsolution) we shall mean L^N -viscosity solutions, unless otherwise stated. We refer to [CCKS] and [BS] for a general review of L^N -viscosity solutions. Note that viscosity solutions are continuous and that any function in $W_{loc}^{2,N}(\Omega)$ satisfies (1) almost everywhere – such a solution is called strong – if and only if it is a L^N -viscosity solution.

In the following we denote with $\mathcal{F}[\psi]$ the vector $(F_i(D^2\psi_i, \psi_i, \psi, x))_{i=1}^n$, for any vector function ψ , respectively we denote with $\mathcal{H}[\psi]$ the vector $(H_i(D^2\psi_i, \psi_i, \psi, x))_{i=1}^n$.

Our first result concerns existence of eigenvalues for fully coupled systems. We set

$$\lambda_1^+ = \lambda_1^+(\mathcal{F}, \Omega) = \sup\{\lambda \in \mathbb{R} \mid \text{there exists } \psi \in C(\Omega, \mathbb{R}^n) \text{ such that } \psi > 0 \text{ and } \mathcal{F}[\psi] + \lambda\psi \leq 0 \text{ in } \Omega\}.$$

$$\lambda_1^- = \lambda_1^-(\mathcal{F}, \Omega) = \sup\{\lambda \in \mathbb{R} \mid \text{there exists } \psi \in C(\Omega, \mathbb{R}^n) \text{ such that } \psi < 0 \text{ and } \mathcal{F}[\psi] + \lambda\psi \geq 0 \text{ in } \Omega\}.$$

Theorem 1 *Suppose that $F_i(M, p, u, x)$ satisfies (H_0) – (H_2) and is convex in (M, p, u) , for all $1 \leq i \leq n$. Assume $\mathcal{C} = (F_i(0, 0, e_j, x))_{i,j=1}^n$ is an irreducible matrix. Then*

(a) *there exists vectors $\varphi_1^+, \varphi_1^- \in W_{loc}^{2,q}(\Omega, \mathbb{R}^n) \cap C(\bar{\Omega}, \mathbb{R}^n)$, $\forall q < \infty$, such that*

$$\begin{cases} \mathcal{F}[\varphi_1^\pm] + \lambda_1^\pm \varphi_1^\pm = 0, & \pm \varphi_1 > 0 & \text{in } \Omega \\ \varphi_1 = 0 & & \text{on } \partial\Omega. \end{cases}$$

(b) we have $\lambda_1^+ \leq \lambda_1^-$ and λ_1^+ can be the only eigenvalue of \mathcal{F} in the interval $(-\infty, \lambda_1^-)$. If $\lambda_1^+ = \lambda_1^-$ then $\varphi_1^+ \equiv \varphi_1^-$; the vector φ_1^\pm spans $\text{Ker}(\mathcal{F} + \lambda_1^\pm)$ in $C(\overline{\Omega}, \mathbb{R}^n)$ under the Dirichlet boundary condition.

(c) assume there is a vector $\Psi \in C(\Omega, \mathbb{R}^n)$ such that $\Psi > 0$ and $\mathcal{F}[\Psi] \leq 0$ in Ω . Then either $\lambda_1^+ > 0$ or $\lambda_1^+ = 0$ and $\Psi = \text{const.} \varphi_1^+$.

(d) if we normalize $\varphi_1^\pm = (\phi_{1,1}^\pm, \dots, \phi_{1,n}^\pm)$ in such a way that $\min_{1 \leq i \leq n} \phi_{1,i}^\pm(x_0) = 1$ for some $x_0 \in \Omega$, then

$$\sup_{\Omega} \varphi_1^\pm := \sup_{\Omega} \max_{1 \leq i \leq n} \phi_{1,i}^\pm \leq C,$$

where C depends only on $x_0, \Omega, \lambda, \Lambda, \gamma, \delta$, and a lower bound on the measures of the subsets of Ω on which $c_{ij} > 0$, where c_{ij} are the functions used to define the full coupling in \mathcal{C} .

It follows from this theorem that to any \mathcal{F} such that $F_i(M, p, u, x)$ satisfies $(H_0) - (H_2)$ and is convex in (M, p, u) we can associate a set of numbers $\lambda_{11}^\pm, \dots, \lambda_{1m}^\pm$, where m is the number of irreducible blocks which appear in the decomposition of \mathcal{C} , and λ_{1k}^\pm are the eigenvalues, given by Theorem 1, of the subsystem containing only the operators (F_i) with indices $i \in S_k$, in which all $u_j, j \notin S_k$, are set to zero. We denote

$$\lambda_1^\pm := \min_{1 \leq k \leq m} \lambda_{1k}^\pm.$$

The positivity of the eigenvalues turns out to be a necessary and sufficient condition for the vector operator \mathcal{F} to satisfy the comparison principle and the Alexandrov-Bakelman-Pucci inequality, and to be a sufficient condition for the Dirichlet problem (1) to be solvable.

We recall that a second order operator \mathcal{H} satisfies the *comparison principle* (CP), provided for any $u, v \in C(\overline{\Omega}, \mathbb{R}^n)$, one of which is in $W_{\text{loc}}^{2,N}(\Omega)$, such that

$$\begin{cases} \mathcal{H}[u] \geq \mathcal{H}[v] & \text{in } \Omega \\ u \leq v & \text{on } \partial\Omega, \end{cases}$$

we have $u \leq v$ in Ω . A particular case of (CP) is the *maximum principle*, when one of u, v is set to zero (and $H(0, 0, 0, x) \equiv 0$, as we always assume).

Theorem 2 Suppose a second-order operator \mathcal{F} is such that $F_i(M, p, u, x)$ satisfies $(H_0) - (H_2)$ and is convex in (M, p, u) , for all $1 \leq i \leq n$. Then $\lambda_1^+(\mathcal{F}) > 0$ is necessary and sufficient for \mathcal{F} to satisfy (CP).

Hence, if a second-order operator \mathcal{H} satisfies $(D_{\mathcal{F}})$, then $\lambda_1^+(\mathcal{F}) > 0$ is sufficient for H to satisfy (CP).

Theorem 3 Suppose the second-order operator \mathcal{F} is such that $F_i(M, p, u, x)$ satisfies $(H_0) - (H_2)$ and is convex in (M, p, u) , for all $1 \leq i \leq n$.

I. If $\lambda_1^-(\mathcal{F}) > 0$ then for any $u \in C(\overline{\Omega}, \mathbb{R})$, $f \in L^N(\Omega, \mathbb{R})$, the inequality $\mathcal{F}[u] \leq f$ implies

$$\sup_{\Omega} \max_{1 \leq i \leq n} (u_i)_- \leq C(\sup_{\partial\Omega} \max_{1 \leq i \leq n} (u_i)_- + \|\max_{1 \leq i \leq n} (f_i)_+\|_{L^N(\Omega)}),$$

where C depends on Ω , N , λ , Λ , γ , δ , and $\lambda_1^-(\mathcal{F})$.

II. In addition, if $\lambda_1^+(\mathcal{F}) > 0$ then $\mathcal{F}[u] \geq f$ implies

$$\sup_{\Omega} \max_{1 \leq i \leq n} u_i \leq C(\sup_{\partial\Omega} \max_{1 \leq i \leq n} (u_i)_+ + \|\max_{1 \leq i \leq n} (f_i)_+\|_{L^N(\Omega)}).$$

We have the following general result on solvability of (1).

Theorem 4 Suppose the second-order operator \mathcal{F} is such that $F_i(M, p, u, x)$ satisfies $(H_0) - (H_2)$ and is convex in (M, p, u) , for all $1 \leq i \leq n$, and suppose that the second-order operator \mathcal{H} satisfies $(D_{\mathcal{F}})$ (recall that \mathcal{F} always satisfies $(D_{\mathcal{F}})$).

I. If $\lambda_1^+(\mathcal{F}) > 0$ then for any $f \in L^p(\Omega, \mathbb{R}^n)$, $p \geq N$, there exists a viscosity solution $u \in C(\overline{\Omega}, \mathbb{R}^n)$ of

$$\begin{aligned} \mathcal{H}[u] &= f \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega. \end{aligned}$$

If in addition all $H_i(M, p, u, x)$ are convex in M then u is the unique viscosity solution of this problem, and $u \in W_{\text{loc}}^{2,p}(\Omega, \mathbb{R}^n)$, $p < \infty$.

II. If we only know that $\lambda_1^-(\mathcal{F}) > 0$ then for any $f \in L^p(\Omega, \mathbb{R}^n)$, $p \geq N$, such that $f \geq 0$ in Ω , there exists a solution $u \in W_{\text{loc}}^{2,p}(\Omega, \mathbb{R}^n) \cap C(\overline{\Omega}, \mathbb{R}^n)$, such that $u \leq 0$ in Ω , of

$$\begin{aligned} \mathcal{F}[u] &= f \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega. \end{aligned}$$

Note that we could always take \mathcal{F} in the last theorem to be the extremal operator from (H_1) , if $\lambda_1^+(\mathcal{F}^*) > 0$; however using $(D_{\mathcal{F}})$ gives a more precise result, in the sense that we may have $\lambda_1^+(\mathcal{F}^*) \leq 0$ but $\lambda_1^+(\mathcal{F}) > 0$ for some "intermediate" \mathcal{F} between \mathcal{H} and \mathcal{F}^* .

It is important for applications to have bounds on the eigenvalues in terms of the operator and the domain. These bounds permit to verify the hypotheses of the previous theorems.

Theorem 5 *Suppose the second-order operator \mathcal{F} is such that $F_i(M, p, u, x)$ satisfies $(H_0) - (H_2)$ and is convex in (M, p, u) , for all $1 \leq i \leq n$. Set $\delta = \sum_{i=1}^n \delta_i$, where δ_i are the numbers from (H_1) .*

Let d and $|\Omega|$ denote respectively the diameter and the Lebesgue measure of Ω , and let R be the radius of the largest ball inscribed in Ω . There exists a positive constant C_1 , which depends only on $N, n, \lambda, \Lambda, \gamma, d$, and a positive constant C_2 , which depends only on $N, n, \lambda, \Lambda, \gamma$ (C_1, C_2 are bounded when these quantities are bounded), such that

$$\frac{1}{C_1 d |\Omega|^{1/N}} - \delta \leq \lambda_1^+(\mathcal{F}) \leq \lambda_1^-(\mathcal{F}) \leq \frac{C_2(1 + \delta)}{R^2}.$$

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