Travelling-wave analysis and identification
A scattering theory framework

Taous-Meriem Laleg, Emmanuelle Crépeau and Michel Sorine

***Abstract*— This article presents a new travelling waves analysis and identification method based on scattering theory. This inverse scattering technique consists in solving the spectral problem associated to a one-dimensional Schrödinger operator perturbed by a potential depending upon the wave to analyze, and optimized in order to approximate this wave by an isospectral flow in the sense of Lax. In this method, the interacting components of an N-soliton are the elementary travelling waves for the approximation. These N solitons play an analogous role to linear superpositions of sinus and cosinus in the Fourier analysis of standing waves. In the proposed analysis of travelling waves, low and high frequency components are replaced by low and high velocity components.**

Two applications of the method are presented. The first one concerns the identification of an N-soliton and is illustrated with $N = 3$. The second one consists in the analysis of the arterial blood pressure waves during the systolic phase (pulse transit time) and the diastolic phase (low velocity flow).

Key-words : Solitons, identification, scattering theory, Schrödinger operator, arterial blood pressure.

I. INTRODUCTION

Travelling waves are particular linear or nonlinear waves arising in many phenomena in physics, chemistry or biology. They have been studied in depth as particular solutions of some nonlinear partial differential equations (NPDE) used as models in these fields [20]. They have some spatio-temporal coherence during their propagation, being composed of interacting elementary waves that keep the same kind of shapes in some moving frames : pulse-shaped waves for the so-called "solitons" or front-shaped waves for the "kinks". From a mathematical point of view they may arise from a complex interplay between nonlinearities and dispersion or dissipation of waves by the propagation media : amplitude-dependent velocity and dispersion for solitons in fluids, reaction and diffusion for the travelling action potentials in electrophysiology or combustion wavefronts in reactive flows. They can be observed in many signals and their analysis may give some insight in the underlying complex phenomena, but there is a lack of adapted signal analysis or identification techniques.

In this paper we propose to use some techniques from the theory of solitons for the analysis and identification of travelling waves. We consider the scattering-theory framework that offers interesting tools to solve some NPDE of evolution, the Direct and Inverse Scattering Transforms (DST and IST) [2], [11], [16]. They have been proposed initially by Gardner et al [5] to solve a Korteweg-de Vries equation (KdV) and have been soon generalized to many other nonlinear evolution equations thanks to the Lax formalism [12]. This theory based on the resolution of a spectral problem associated with a one-dimensional Schrödinger operator, constitutes in some sense an extension in a nonlinear context of the operational calculus based on the Fourier Transform [2]. Solitons with N components interacting nonlinearly (N-solitons) play here, a similar role to linear superpositions of N sinus and cosinus in the Fourier analysis of standing waves.

The concept of soliton refers to a solitary wave emerging unchanged in shape and speed from the collision with other solitary waves. Since their discovery in 1834 by Scott Russell [2], [15], they fascinate the scientists with their very interesting coherent-structure characteristics. They are used in many fields for modelling natural phenomena. The simplest example of a soliton is a travelling pulse solution of a dispersionless linear wave equation. They are also solution of nonlinear dispersive equations like the ubiquitous KdV equation arising in a variety of physical problems, for example to describe wave motion in shallow canals. This third order NPDE includes both nonlinear and dispersive effects and solitons result here from a stable equilibrium between these effects [17], [22].

The new travelling wave analysis and identification method we propose here is based on the scattering theory for the KdV equation. It consists in solving the spectral problem associated to a one-dimensional Schrödinger operator perturbed by a potential depending upon the wave to analyze, and optimized in order to approximate this wave by an isospectral flow in the sense of Lax. In this method, the interacting components of N-soliton solutions of KdV equation are the elementary travelling waves for the approximation.

In the next section, we recall elements of the scattering theory used later in the paper : the Lax formalism and the central notion of isospectral flows [11], [12], [17] ; the DST and IST techniques that will be our basic computational tools. Section III focuses on the scattering theory and N-solitons of the KdV equation. Section IV describes how to use the scattering technique as a signal analysis and identification technique. Section V presents an example, the identification of an N-soliton. We illustrate the procedure numerically with $N = 3$. In section VI, the previous results are used to revisit the analysis of the arterial blood pressure (ABP) proposed recently in [4], [9], [10]. In particular it is shown how the decomposition of the wave into fast and slow parts can be used to analyze the systolic and diastolic phases,

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a technique that may have interesting clinical applications. Finally a conclusion summarizes the different results.

II. The Scattering theory framework

We recall the scattering theory framework introduced to solve some NPDE and that will be used for signal analysis.

A. Lax formalism and isospectral flows

We consider solutions y of a nonlinear wave equation :

$$\frac{\partial y(x,t)}{\partial t} = D(y, \frac{\partial y}{\partial x}, \frac{\partial^2 y}{\partial x^2}, \frac{\partial^3 y}{\partial x^3}, ...), \quad (1)$$

where D is a nonlinear function of y and its successive derivatives with respect to $x \in \mathbb{R}$. We suppose the existence of two linear operators $L(y)$ and $M(y)$, the so-called Lax pair, defined on an appropriate Hilbert space $\mathcal{H}$, verifying :

$$\frac{\partial L(y)}{\partial t} - [M(y), L(y)] = 0, \quad M(y) + M^*(y) = 0, \quad (2)$$

where $M^*(y)$ denotes the adjoint operator of $M(y)$ and

$$[M(y), L(y)] = M(y)L(y) - L(y)M(y). \quad (3)$$

The main property is that $L(y(.,t))$ is unitarily equivalent to $L(y(.,0))$ ($y(.,t)$ denotes $x \to y(x,t)$) :

$$U^*(t)L(y(.,t))U(t) = L(y(.,0)) \text{ with } U^*(t)U(t) = I. \quad (4)$$

So, although $L(y)$ changes with time according to equation (2), its eigenvalues λ are independent of time [11], [12], [17]. It follows from (2) and (4) that

$$\frac{\partial U(t)}{\partial t} = M(y)U(t), \quad U(0) = I. \quad (5)$$

The discrete eigenvalues and eigenfunctions of $L(y(.,t))$ can be computed from those of $L(y(.,0))$, using :

$$L(y(.,t))\psi = \lambda\psi, \ \ \lambda(t) = \lambda(0), \quad \frac{\partial \psi}{\partial t} = M(y)\psi, \quad (6)$$
$$\parallel \psi(t) \parallel_{L^2(\mathbb{R})} = \parallel \psi(0) \parallel_{L^2(R)} .$$

A flow y verifying (4) is said isospectral.

For our purpose, a convenient choice will be a Schrödinger operator on $\mathcal{H} = L^2(\mathbb{R})$ with a time-dependent potential y :

$$L(y) = -\frac{\partial^2}{\partial x^2} + y. \quad (7)$$

In this operator notation, the time is a parameter and $L(y(.,t)) : \phi \to -\frac{\partial^2 \phi}{\partial x^2} + y(.,t)\phi$ for $\phi \in H^2(\mathbb{R})$.
A first example for $M(y)$ is :

$$M(y) = -c\frac{\partial}{\partial x}, \quad (8)$$

then, equation (2) leads to :

$$\frac{\partial y}{\partial t} + c\frac{\partial y}{\partial x} = 0, \quad (9)$$

which is a transport equation that has travelling wave solutions $Y(x - ct)$ moving to the right with the speed c.

Another important example for $M(y)$ is :

$$M(y) = -4\frac{\partial^3}{\partial x^3} + 3y\frac{\partial}{\partial x} + 3\frac{\partial}{\partial x}y, \quad (10)$$

then we get the KdV equation (11) for which the scattering theory has been first developed by Gardner et al [5].

$$\frac{\partial y}{\partial t} - 6y\frac{\partial y}{\partial x} + \frac{\partial^3 y}{\partial x^3} = 0. \quad (11)$$

The N-solitons described later are travelling wave solutions of (11) that will be used as our basic isospectral flows.

B. The Direct Scattering Transform

Let V be a function in the Schwartz space $\mathcal{S}(\mathbb{R})$ of regular and rapidly decreasing functions on $\mathbb{R}$. The DST of the potential V is the solution of the spectral problem for $L(V)$:

$$-\frac{\partial^2 \psi}{\partial x^2} + V\psi = \lambda\psi, \quad -\infty < x < +\infty. \quad (12)$$

The spectrum of this operator has two components : a discrete spectrum with negative eigenvalues and a continuous spectrum including positive eigenvalues [2], [5].

Denoting the positive eigenvalues by $\lambda = k^2$, the continuous spectrum is characterized by the following asymptotic boundary conditions where $T^V(k)$ and $R^V(k)$ are respectively the transmission and reflection coefficients associated to V :

$$\psi(x,k) \to T^V(k)\exp(-ikx), \quad x \to -\infty, \quad (13)$$

$$\psi(x,k) \to \exp(-ikx) + R^V(k)\exp(ikx), \ \ x \to +\infty. \quad (14)$$

Conservation of energy leads to :

$$|T^V(k)|^2 + |R^V(k)|^2 = 1.$$

The discrete spectrum has N^V negative eigenvalues and normalized eigenfunctions. We note $\lambda_n^V = -(\kappa_n^V)^2$, and :

$$\Sigma_d(V) = \{-(\kappa_n^V)^2, \ \psi_n^V, \ n = 1 \cdots N^V\}.$$

The eigenfunctions ψ_n^V behave as $\psi_n^V(x) \sim c_n^V \exp(-\kappa_n^V x)$ in the limit of large x where the coefficients c_n^V are defined by :

$$c_n^V = \lim_{x \to +\infty} \exp(\kappa_n^V x)\psi_n^V(x), \ \ n = 1, \cdots, N^V. \quad (15)$$

The DST of V is the collection of data $S(V)$ defined by :

$$\mathcal{S}(\mathbb{R}) \to \mathcal{T} = \mathcal{S}(\mathbb{R}) \times (\mathbb{R}^+)^N \times \mathbb{R}^N$$
$$V \to S(V) = (R^V, \ \kappa_n^V, \ c_n^V, \ \ n = 1, \cdots, N^V) \quad (16)$$

C. The Inverse Scattering Transform

The IST is the transformation $S^{-1}(\alpha)$ reconstructing a potential in $\mathcal{S}(\mathbb{R})$ from the scattered data $\alpha = (R, \kappa_n, c_n, n = 1 \cdots N) \in \mathcal{T}$. It proceeds in three steps [5]. We first define the function B :

$$B(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \exp(ikx)R(k)dk + \sum_{n=1}^{N} c_n^2 \exp(-\kappa_n x). \quad (17)$$

The first term in $B(x)$ is the inverse Fourier transform of the reflection coefficient R and represents the contribution of the continuous spectrum. The second term is the contribution of the discrete spectrum in the potential.

Next, we consider the Gel'fand-Levitan-Marchenko (GLM) integral equation where $K(x,z)$ is the unknown which is determined uniquely. Here, x is a parameter.

$$K(x,z)+B(x+z)+\int_x^{+\infty} K(x,z')B(z+z')dz' = 0, \quad \text{for } z > x. \quad (18)$$

Finally, V is given by the simple formula :

$$V(x) \quad = \quad -2\frac{\partial K(x,x)}{\partial x}. \quad (19)$$

D. Case of a reflectionless potential

The case of a reflectionless potential V of the Schrödinger equation (12) corresponds to $R^V = 0$. It is of particular interest as V can be computed analytically knowing the parameters κ_n^V and c_n^V, $n = 1, \cdots, N^V$. In fact $-V$ is an N-soliton solution of the KdV equation (11) [5] and is described in the next section. It furthermore verifies :

$$V = -4\sum_{n=1}^{N^V} \kappa_n^V (\psi_n^V)^2. \quad (20)$$

To summarize this section, we have introduced the scattering theory which gives a correspondence between a potential V and its spectral transform $S(V)$. The DST ($V \to S(V)$) and the IST ($S(V) \to V$) involve the solution of respectively, the spectral problem associated with a Schrödinger operator $L(V)$, and the GLM integral equation associated with the scattered data. Reflectionless potentials are N-soliton solutions of KdV equation and are known explicitly.

III. Scattering theory and N-solitons of KdV

The solution y of (11) can be constructed using the DST and IST. This procedure is a nonlinear version of the technique of resolution of linear PDE via the Fourier transform. This construction proposed in [5] can be summarized by the following formula :

$$y(.,t) = S^{-1} \circ \mathbb{L}(t) \circ S(y(.,0)), \quad (21)$$

where, for $\alpha = (R(.,0),\ \kappa_n,\ c_n,\ n = 1, \cdots, N)$,

$$\mathbb{L}(t)\alpha = (R(.,t),\ \kappa_n,\ c_n(t),\ n = 1, \cdots, N) \quad (22)$$

with :

$$R(k,t)=R(k,0)(k)\exp(8ik^3t),\ c_n(t)=c_n\exp(4\kappa_n^3 t) \quad (23)$$

Remark the linear evolution of the scattered data in (21).

If the potential is reflectionless at $t = 0$ then it remains reflectionless for all t and the solution of the KdV equation is given by an N-soliton and can be written as follows [5] :

$$y(x,t) = -2\frac{\partial^2}{\partial x^2}\log\det(I + A(x,t)), \quad (24)$$

where $A(x,t)$ is an $N{\times}N$ matrix with coefficients given by :

$$a_{mn} = \frac{c_m c_n}{\kappa_m+\kappa_n}\exp\left(-(\kappa_m+\kappa_n)x + 4(\kappa_m^3+\kappa_n^3)t\right). \quad (25)$$

The κ_n^2 and c_n characterize respectively the speeds and positions of the soliton components.

IV. Scattering theory and signal processing

As we have seen, the scattering theory framework offers a tool, the DST, for associating a spectral description $S(V(.,t))$ to each value of a time-dependent potential V. For isospectral flows, $V = -y$, and $S(-y)$ evolves linearly in time, with constant eigenvalues. Reflectionless isospectral flows are the N-soliton solutions of KdV and are good candidates to approximate flows "looking like travelling pulses" ("pulse" meaning here that they are in the Schwartz class) and to associate to such a flow y, a potential $V(y)$ and then a spectral description.

A. Problem formulation

The main step in this procedure is to determine V such that the reflectionless part of this potential,

$$y_d(V,y)) = V^{-1} \circ S^{-1} \circ \Pi_d \circ S \circ V(y) \quad (26)$$

is closed to y, where Π_d is the projector zeroing the R-component of $S(V)$. Using $\Sigma_d(V)$, we have, with (20) :

$$y_d(V,y) = V^{-1}\left(4\sum_{n=1}^{N^{V(y)}} \kappa_n^{V(y)}(\psi_n^{V(y)})^2\right). \quad (27)$$

Therefore, a problem formulation is to find V minimizing

$$J(V) = \| y_d(V,y) - y \|_{L^2(\mathbb{R})}. \quad (28)$$

B. Determination of the order N^V

The main idea here, is based on results of T. Kato [8] analyzing the spectrum of $L(\chi q)$ as a function of χ. For a square-well potential q of depth -1, the number $N(\chi)$ of discrete eigenvalues increases with the depth χ of the square-well χq. In fact, by changing χ, it is possible to move discrete eigenvalues to and from the continuous spectrum. The situation is quite similar here and the idea is then to optimise $J(V)$ in the class

$$V(y) = -\chi(y - y_{min}), \quad (29)$$

where y_{min} is a constant such that $y - y_{min} > 0$, so that V is like a well of variable depth. In the numerical experiments we have done, it has been sufficient to fix y_{min} and to optimise over χ. So, to simplify the notations, we suppose $y_{min} = 0$ and write $J(\chi)$ instead of $J(V)$. The constant χ will then be used to determine the number of the solitons components and hence the order of the isospectral flow approximation.

As $L(-\chi y)$ is linear in χ, its eigenvalues $\lambda_n(\chi)$, $n = 1, \cdots, N(\chi)$ are decreasing branches of algebraic functions of χ having constant multiplicity [8]. Furthermore, the number of negative eigenvalues of $L(-\chi y)$ is a nondecreasing function of the parameter χ and there is an infinite unbounded sequence of values of χ at which the number of the eigenvalues is incremented by one, the new eigenvalue being born from the continuous spectrum [8], [14].

In practice, this technique enables us to express the wave y in a soliton base. Therefore, the method can be seen as a nonlinear Fourier Transform where solitons play the role of the sinus and cosinus and the parameter χ can be considered as the analogous of the frequency. Here, χ is proportional to the speed of the solitons components. Hence, low and high frequency components are replaced by low and high velocity components. The next sections will illustrate this technique.

V. Identification of an N-soliton

The scattering theory is used here to identify an N-soliton initial condition y_0 of the KdV equation (11).

We suppose that y_0 is measured and $S(y_0)$ unknown. The formulae (24), (25) for an N-soliton are used to generate the test data. In order to determine the evolution of this N-soliton, we compute the spectrum of $L(y_0)$. Several methods have been proposed to solve the spectral problem of a Schrödinger operator. Recent codes, SLEIGN2 [1] or MATSLISE [13] based on CP method [6], [7] are adapted to large N values. Here, N remains small and we use a Fourier pseudo-spectral method [18]. The potential is defined at N_Λ equidistant points $(x_1, x_2, \cdots, x_{N_\Lambda})$ of the interval $[-\Lambda, \Lambda]$. We note ψ^e this approximation of ψ in $[-\Lambda, \Lambda]$ then the Schrödinger spectral problem becomes :

$$(-D^{(2)}_{N_\Lambda} + Y_0)\psi^e = \lambda\psi^e, \quad \int_{-\Lambda}^{\Lambda} (\psi^e)^2(x)dx = 1. \qquad (30)$$

where $D^{(2)}_{N_\Lambda}$ is the second order differentiation matrix in space and Y_0 the following diagonal matrix :

$$Y_0 = diag(y_0(x_1), y_0(x_2), \cdots, y_0(x_{N_L}))$$

Note that Λ must be chosen large enough to avoid harmful effects of truncation errors. The determination of the parameters κ_n and c_n is severely affected by the truncation, even in this case of rapidly decreasing N-solitons. To solve this problem, we use an optimisation procedure to determine the c_n so that $S(y_0)$ is well approximated by the isospectral flow in the sense of (28). We have used the Nelder-Mead simplex method implemented in Matlab (fminsearch command).

The proposed method was tested with the 3-soliton shown in Fig. 1, which is defined by the following parameters :

$$\kappa_1=9, \quad \kappa_2=7, \quad \kappa_3=5, \quad s_1=2, \quad s_2=2.05, \quad s_3=2.1 \qquad (31)$$

where the parameters s_n determine the relative position of each component of an N-soliton and are given by :

$$c_n = \sqrt{2\kappa_n \exp(2\kappa_n s_n)}, \quad n = 1, \cdots, N. \qquad (32)$$

With the procedure described above the eigenvalues were recovered with a high precision and the initial potential estimated using (27) is in good agreement with real data as shown Fig. 2. Using equation (24), we get the evolution of the potential and the results are illustrated in Fig. 3 and Fig. 4.

VI. Soliton-based decomposition of ABP waves

The scattering theory is now used to analyse a measured ABP signal, $P(t)$. This example is based on [4], [9], [10] where, following a suggestion made in [21], a KdV equation is used as a physical model of the arterial flow.

We solve the spectral problem of $L(-\chi P)$ as described before, while optimizing the parameter χ to tune the speeds

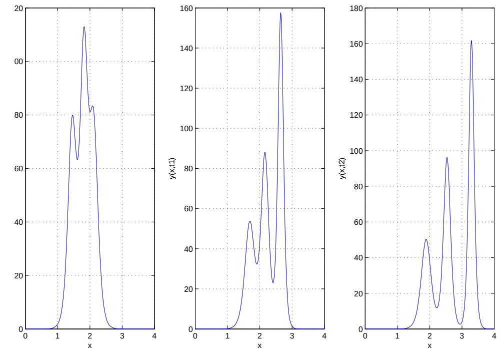

Fig. 1. A 3-soliton at three successive times 0, $t1$, $t2$

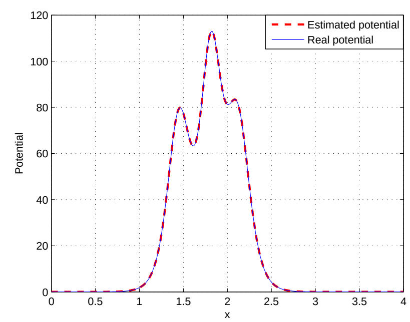

Fig. 2. Estimated and real potentials at $t = 0$

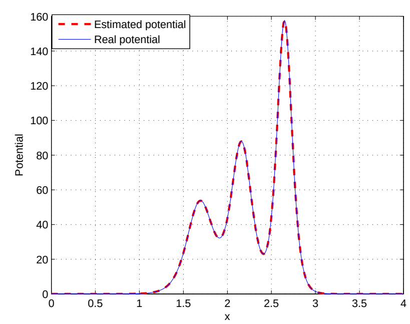

Fig. 3. Estimated and real potentials at $t = t1$

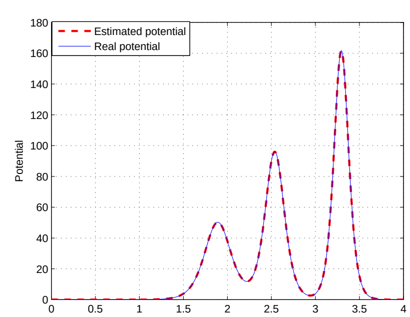

Fig. 4. Estimated and real potentials at $t = t2$

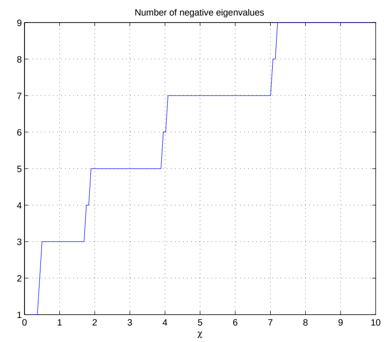

Fig. 5. The number of negative eigenvalues with χ

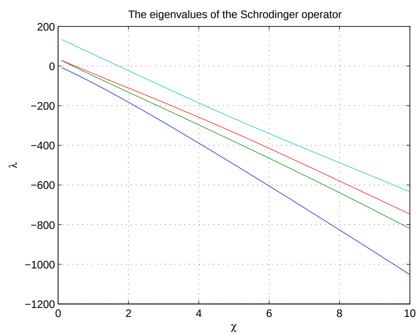

Fig. 6. The evolution of the forth first eigenvalues with χ

of the soliton components. The approximation of the measured pressure by an isospectral flow is then :

$$\hat{P} = -\hat{\chi}^{-1} S^{-1} \circ \Pi_d \circ S(-\hat{\chi} P). \tag{33}$$

The influence of the parameter χ on spectral properties of $L(-\chi P)$ is illustrated Fig. 5 and Fig. 6 (evolutions of $N(\chi)$ and of the forth first eigenvalues with χ respectively). The results confirm the theoretical description given section IV : $N(\chi)$ is nondecreasing with χ and the eigenvalues are decreasing branches of algebraic functions of χ.

Fig. 7 and Fig. 8 compare measured and estimated pressures P and $\hat{P}$ respectively at the aorta and at the finger levels for different values of χ, and hence of the order $N(\chi)$ of the solitons. Only 5 to 8 components are sufficient for a good approximation of the ABP. In Fig. 9 several beats of measured and reconstructed pressures are shown.

Now, we exploit the relation between χ and the speed of the soliton components to separate the ABP into fast and slow components. We define the projectors Π_i, zeroing R and the N_c largest (resp. $N(\chi) - N_c$ smallest) κ_n^2 of $S(V)$ for $i = s$ (resp. $i = f$) for $0 < N_c < N$. Then, the following

$$P_i = -\chi_{N_c}^{-1} S^{-1} \circ \Pi_i \circ S(-\chi_{N_c} P), \quad i = f, s. \tag{34}$$

are good candidates to represent the fast ($i = f$) and slow ($i = s$) phenomena in the ABP. This is confirmed by the experimental results. For $N_c = 2$ and $N = 10$, Fig. 10 and Fig. 11 show that P_f and P_s are respectively localized during the systole and the diastole, as expected.

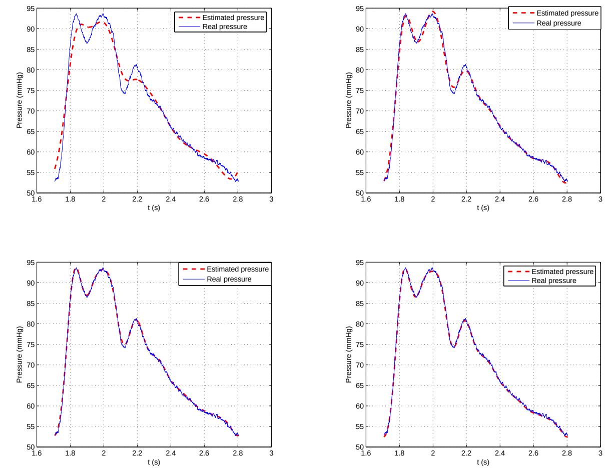

Fig. 7. Measured and reconstructed pressures at the Aorta with N-solitons. From Left to Right : $N = 4$, $N = 6$ (Up). $N = 8$, $N = 10$ (Down)

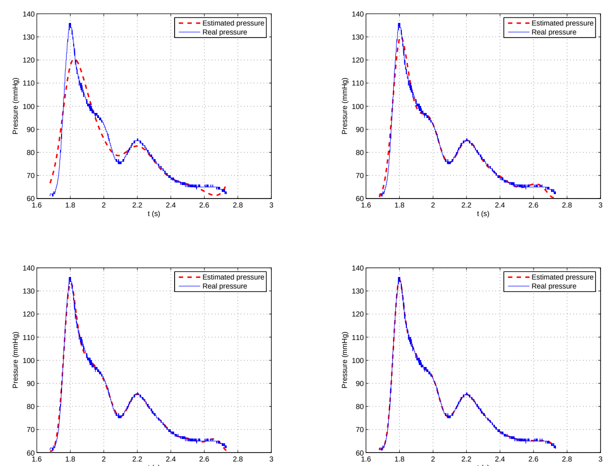

Fig. 8. Measured and reconstructed pressures at the Finger with N-solitons. From Left to Right : $N = 3$, $N = 5$ (Up). $N = 7$, $N = 9$ (Down)

This decomposition of the ABP into fast and slow parts complete the results in [3], [4], [9], [10].

This approach has to be compared to the standard description of the ABP waves as linear superposition of harmonic waves using the Fourier Transform [19]. The promising results obtained here lead us to consider clinical applications of this technique.

VII. CONCLUSION

A new signal analysis method based on scattering theory has been presented. A discrete spectrum is associated to the signal through an iso-spectral flow approximation using N-solitons. The two examples studied show that travelling pulses are well estimated. A decomposition into fast and slow components has been proposed. This analogue for travelling waves of the high and low frequency decomposition for standing waves seems promising, in particular for some biological signal analysis. Many theoretical as well as ex-

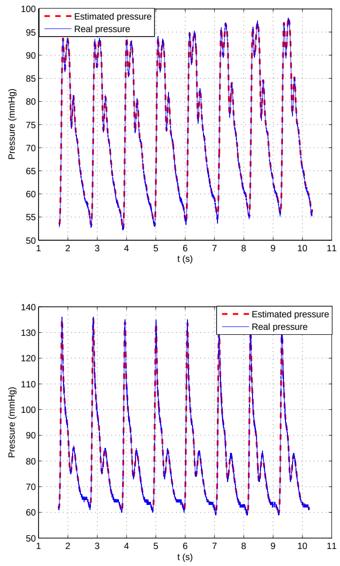

Fig. 9. Multi-beat measures and estimates : Aorta (up) and Finger (down)

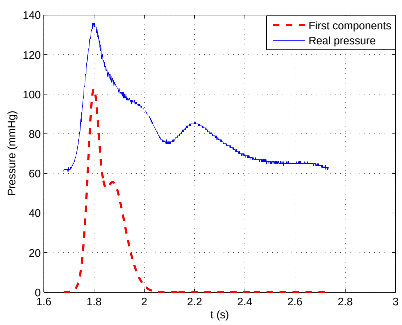

Fig. 10. P_f and fast systolic phenomena

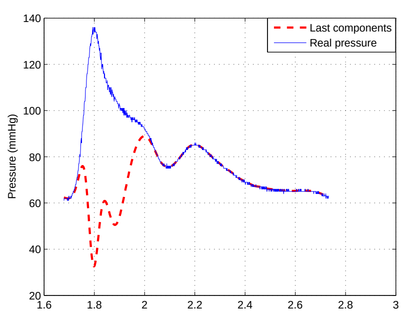

Fig. 11. P_s and slow diastolic phenomena

perimental questions are raised by the proposed method. In particular the analysis of the projection on solitons is under current research.

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