

# STABLE NORMS OF NON-ORIENTABLE SURFACES

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ABSTRACT. We study the stable norm on the first homology of a closed, non-orientable surface equipped with a Riemannian metric. We prove that in every conformal class there exists a metric whose stable norm is polyhedral. Furthermore the stable norm is never strictly convex if the first Betti number of the surface is greater than two.

## 1. INTRODUCTION

Given a compact Riemannian manifold  $(M, g)$  with first Betti number  $b_1(M) > 0$ , the stable norm  $\| \cdot \|$  on  $H_1(M, \mathbb{R})$  is defined in [Gr81] (see also [Fe74]) as

$$\begin{aligned} H_1(M, \mathbb{R}) &\longrightarrow \mathbb{R} \\ h &\longmapsto \|h\| := \inf \left\{ \sum_{i=1}^n |r_i| l_g(\gamma_i) \right\} \end{aligned}$$

where

- $l_g$  denotes the length with respect to  $g$
- the  $r_i$  are real numbers
- the  $\gamma_i$  are Lipschitz 1-cycles
- $h = \sum_{i=1}^n r_i [\gamma_i]$ .

Note that since we want to minimize the length we may assume from the start that the  $\gamma_i$  are closed geodesics that minimize the length in their free homotopy class.

In general the infimum may not be reached. It is remarkable that when the dimension of  $M$  is two, it is reached for every integer homology class (Proposition 14). When the infimum is actually a minimum, we may wonder whether the minimizing cycles are connected. Note that every component  $\gamma_i, i = 1 \dots n$  of a minimizing cycle  $\sum_{i=1}^n r_i \gamma_i$  is itself minimizing in its own homology class. A minimizing cycle whose connected components have distinct homology classes yields a flat region in the unit sphere  $\mathcal{S}_1$  of the stable norm, containing the convex hull of the  $\{[\gamma_i]/l_g(\gamma_i)\}_{i=1}^n$ . So we may ask how often does it occur, how many components do the minimizing cycles have and what is the dimension of the corresponding flat (that is the dimension of the affine subspace it spans). In this paper we give some answers when  $M$  is a closed non-orientable surface. Our first result is similar to Theorem 7 of [Mt97] which addresses the orientable case. We denote by  $[x]$  the integer part of a real number  $x$ .

**Theorem A** *Assume  $M$  is a closed non-orientable surface endowed with a Riemannian metric. Then every connected minimizing cycle is a component*

of a minimizing cycle with at least  $\lfloor (b_1(M) + 1)/2 \rfloor - 1$  homologically independent components, and at most  $2b_1(M) - 1$  non pairwise homologically proportional components.

So the stable norm is never strictly convex for  $b_1(M) > 2$ . The difference with Theorem 7 of [Mt97] is that the dimension of the corresponding flat may be more than  $\lfloor (b_1(M) + 1)/2 \rfloor - 1$ . Observe that if  $b_1(M) = 2$  the stable norm may be strictly convex. For instance, take a hyperbolic punctured torus, cut off a sufficiently thin neighborhood of the cusp, and glue a projective plane.

Let  $\pi: M_o \rightarrow M$  be the orientation cover of a non-orientable surface  $M$ . A simple closed curve  $\gamma$  of  $M$  is said of *type I* (resp. *of type II*) if its inverse image  $\pi^{-1}(\gamma)$  consists of either one curve or two homologous curves (resp. two non-homologous curves). Remark that one-sided simple closed curves on  $M$  (curves whose tubular neighbourhood is homeomorphic to a Möbius strip) are of type I while two-sided simple closed curves on  $M$  (curves whose tubular neighbourhood is homeomorphic to an annulus) may be of type I or II. The following theorem states that the local geometry of the unit sphere  $\mathcal{S}_1$  is special near homology classes whose minimizing cycles consist of curves of type I. Specifically, the intersection of the unit ball with a neighborhood of such a class is a cone .

**Theorem B** *Assume  $M$  is a closed non-orientable surface endowed with a Riemannian metric. Let  $h_0$  be an integer homology class all of whose minimizing cycles consist of geodesics of type I. Then for all  $h \in H_1(M, \mathbb{R})$ , there exists  $s(h_0, h) > 0$  such that the subset of the unit sphere  $\mathcal{S}_1$*

$$\left\{ \frac{h_0 + sh}{\|h_0 + sh\|} : s \in [0, s(h_0, h)] \right\}$$

*is a straight segment.*

Apart from surfaces little is known about minimizing cycles. For flat tori they exist and are connected in every integer homology class (or multiple thereof). Other homology classes do not have minimizing cycles. Furthermore the stable norm of a flat torus is Euclidean. Apart from [MR95] which deals with hyperbolic metrics on a punctured torus, the only other examples ([Ban90], [BB06]) where the stable norm is actually computed have very few connected minimizing cycles : the unit ball of the stable norm is a polyhedron. So there is but a finite number of connected minimizing cycles, corresponding to the vertices of the polyhedron. In every homology class there is a minimizing cycle which is a linear combination of the connected ones. All such examples assume  $\dim M \geq 3$  ; if  $\dim M = 2$  and  $M$  is orientable, [Mt97] rules out the unit ball being a polyhedron. The situation is different when  $\dim M = 2$  and  $M$  is not orientable :

**Theorem C** *Assume  $M$  is a closed non-orientable surface. Then in every conformal class there exists a metric whose stable norm has a polyhedron as its unit ball.*

Now we briefly describe the contents of the paper. In Sections 2 contains basic facts about non-orientable surfaces. In Sections 3 and 4 we have gathered prerequisites about minimizing measures (in the sense of [Mr91]) and stable norms. Some material from [Mt96] and [Mt97] has been included, either because it was not published, or because we found the redaction to be wanting. In Section 5 we prove the technical lemmas we need for our main theorems. Some consequences are derived, among which Proposition 14, which says a minimizing measure with a rational homology class is supported on periodic orbits, and Lemma 13, which says a geodesic asymptotic to a closed geodesic is not in the support of any minimizing measure. In the last section we prove our main theorems.

## 2. PRELIMINARIES : NON-ORIENTABLE SURFACES

Let  $(M, g)$  be a smooth, closed, non-orientable Riemannian manifold of dimension two.

**2.1. First homology group.** By the classical Surface Classification Theorem, any orientable closed surface is a connected sum of tori, any non-orientable closed surface is a connected sum of tori and projective planes. Since the connected sum of three projective planes is homeomorphic to the connected sum of a torus and a projective plane, in fact any non-orientable surface is a connected sum of tori and one or two projective planes. Recall that the connected sum of two projective planes is the Klein bottle  $\mathbb{K}$ .

Denote by  $\Sigma_k$  an orientable surface of genus  $k$  (that is,  $\Sigma_k \cong \#^k \mathbb{T}^2$ ). We have

$$H_1(\Sigma_k \# \mathbb{K}, \mathbb{R}) \cong H_1(\Sigma_k, \mathbb{R}) \oplus H_1(\mathbb{K}, \mathbb{R}) \cong \mathbb{R}^{2k} \oplus \mathbb{R}$$

whence the first Betti number  $b_1(\Sigma_k \# \mathbb{K})$  of  $\Sigma_k \# \mathbb{K}$  is  $2k + 1$ . Likewise,

$$H_1(\Sigma_k \# \mathbb{R}P^2, \mathbb{R}) \cong H_1(\Sigma_k, \mathbb{R}) \cong \mathbb{R}^{2k}$$

and  $b_1(\Sigma_k \# \mathbb{R}P^2) = 2k$ .

Similarly, we have

$$H_1(\Sigma_k \# \mathbb{R}P^2, \mathbb{Z}) \cong H_1(\Sigma_k, \mathbb{Z}) \oplus H_1(\mathbb{R}P^2, \mathbb{Z}) \cong \mathbb{Z}^{2k} \oplus \mathbb{Z}/2\mathbb{Z}$$

and

$$H_1(\Sigma_k \# \mathbb{K}, \mathbb{Z}) \cong H_1(\Sigma_k, \mathbb{Z}) \oplus H_1(\mathbb{K}, \mathbb{Z}) \cong \mathbb{Z}^{2k} \oplus \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}.$$

For any manifold  $M$ , the torsion-free part of  $H_1(M, \mathbb{Z})$  embeds as a lattice  $\Lambda$  in  $H_1(M, \mathbb{R})$ . We say

- an element of  $H_1(M, \mathbb{R})$  is *integer* if it belongs to  $\Lambda$
- a subspace of  $H_1(M, \mathbb{R})$  is *integer* if it is generated by integer classes
- an element  $h$  of  $H_1(M, \mathbb{R})$  is *rational* if  $rh$  belongs to  $\Lambda$  for some real number  $r$ .

**2.2. Orientation cover.** Let  $\pi: M_o \rightarrow M$  be the orientation cover of  $M$ . Then  $M_o$  is an orientable surface endowed with a fixed-point free, orientation-reversing involution  $I$ . Let  $I_*$  be the involution of  $H_1(M_o, \mathbb{R})$  induced by  $I$ , and let  $E_1$  (resp  $E_{-1}$ ) be the eigenspace of  $I_*$  for the eigenvalue 1 (resp  $-1$ ). First observe that

**Proposition 1.**  *$E_1$  and  $E_{-1}$  are Lagrangian for the (symplectic) intersection form  $Int$  on  $H_1(M_o, \mathbb{R})$ .*

*Proof.* Take  $x, y \in E_1$  (resp  $E_{-1}$ ). We have  $\text{Int}(I_*(x), I_*(y)) = \text{Int}(x, y)$  but on the other hand, since  $I$  reverses the orientation of  $M_o$ ,  $\text{Int}(I_*(x), I_*(y)) = -\text{Int}(x, y)$  whence  $\text{Int}(x, y) = 0$ , which proves that  $E_1$  (resp  $E_{-1}$ ) is isotropic. In particular  $\dim E_1 \leq 2^{-1}b_1(M_o)$  and  $\dim E_{-1} \leq 2^{-1}b_1(M_o)$ . Now since  $I_*$  is a linear involution,  $\dim E_1 + \dim E_{-1} = b_1(M_o)$  whence  $\dim E_1 = \dim E_{-1} = 2^{-1}b_1(M_o)$  that is,  $E_1$  (resp  $E_{-1}$ ) is Lagrangian for the symplectic form  $\text{Int}$ .  $\square$

Furthermore

**Proposition 2.**  $\ker \pi_* = E_{-1}$ .

*Proof.* Let  $\gamma$  be a 1-cycle in  $M_o$  such that  $\pi_*([\gamma]) = 0$ . That is,  $\pi(\gamma)$  bounds a 2-chain  $C$  in  $M$ . Then  $\pi^{-1}(\pi(\gamma))$  bounds the 2-chain  $\pi^{-1}(C)$  in  $M_o$ . But  $\pi^{-1}(\pi(\gamma)) = \gamma \cup I(\gamma)$ , so  $[\gamma] + [I(\gamma)] = 0$ . Conversely, if  $\gamma$  is a 1-cycle in  $M_o$  such that  $[\gamma] + [I(\gamma)] = 0$ , then  $\gamma$  and  $I(\gamma)$  together bound a two-chain  $C$  in  $M_o$ , so  $\pi(\gamma) = \pi(I(\gamma))$  bounds the two-chain  $\pi(C)$  in  $M$ , thus  $[\pi(\gamma)] = 0$  in  $H_1(M, \mathbb{R})$ .  $\square$

Consequently  $\pi_*$  identifies  $H_1(M, \mathbb{R})$  with  $E_1$ .

### 3. PRELIMINARIES : MINIMIZING MEASURES AND STABLE NORM

The material of this section is taken from [Mt96] and was not published. Most of the ideas therein were presented to the second author by Albert Fathi.

Because we like our problems with some compactity, we introduce an alternative definition of the stable norm. It relies on invariant measures of the geodesic flow and is inspired by Mather's theory for Lagrangian systems. Then minimizing objects, in the form of measures (or asymptotic cycles as in [S57]), exist in every homology class. The question of whether a minimizing cycle exists becomes "are minimizing measures supported on closed geodesics?".

Let  $(M, g)$  be a compact Riemannian manifold of any dimension with first Betti number  $b_1(M) > 0$ . Denote by

- $T^1M$  the unit tangent bundle of  $(M, g)$
- $p$  the canonical projection  $T^1M \rightarrow M$
- $\phi_t$  the geodesic flow in  $T^1M$

**3.1. Minimizing measures.** Define  $\mathcal{M}$  as the set of all probability measures on  $T^1M$ , endowed with the weak\* topology. Then  $\mathcal{M}$  is compact and metrizable ([Dieu], 13.4.2). Besides it embeds homeomorphically as a convex subset of the dual to the vector space  $C^0(T^1M)$  of continuous functions on  $T^1M$ . Let  $\mathcal{M}_g$  be the subset of  $\mathcal{M}$  that consists of  $\phi_t$ -invariant measures. Then  $\mathcal{M}_g$  is closed in  $\mathcal{M}$ , hence compact, and convex. Fix an element  $\mu$  of  $\mathcal{M}_g$ . By [Mr91], for any  $C^1$  function  $f$  on  $M$ , we have

$$\int_{T^1M} df(x) \cdot v \, d\mu(x, v) = 0.$$

Thus, if  $\omega$  is a smooth closed one-form on  $M$ , the integral

$$\int_{T^1M} \omega_x(v) \, d\mu(x, v)$$

only depends on the cohomology class of  $\omega$ . By duality this endows  $\mu$  with a homology class :  $[\mu]$  is the unique element of  $H_1(M, \mathbb{R})$  such that

$$\langle [\mu], [\omega] \rangle = \int_{T^1 M} \omega d\mu$$

for any smooth closed one-form  $\omega$  on  $M$ . The map

$$\begin{aligned} [\cdot] : \mathcal{M}_g &\longrightarrow H_1(M, \mathbb{R}) \\ \mu &\longmapsto [\mu] \end{aligned}$$

is continuous and affine, so the image  $\mathcal{B}_1$  of  $\mathcal{M}_g$  in  $H_1(M, \mathbb{R})$  is compact and convex.

**Proposition 3.**  $\mathcal{B}_1$  is the unit ball of the stable norm.

*Proof.* We first prove that  $\mathcal{B}_1$  is the unit ball of some norm  $N$ .

Denote by

$$\begin{aligned} \mathcal{I} : T^1 M &\longrightarrow T^1 M \\ (x, v) &\longmapsto (x, -v) \end{aligned}$$

the canonical involution of  $T^1 M$ . We have, for any  $(x, v)$  in  $T^1 M$ ,

$$\phi_t(x, -v) = \phi_{-t}(x, v)$$

so if  $\mu$  is in  $\mathcal{M}_g$ , then  $\mathcal{I}_*\mu$  is again in  $\mathcal{M}_g$ . Let  $\omega$  be a smooth closed one-form on  $M$ . We have

$$\begin{aligned} \langle [\mathcal{I}_*\mu], [\omega] \rangle &= \int_{T^1 M} \omega_x(v) d\mathcal{I}_*\mu(x, v) \\ &= \int_{T^1 M} \omega_x(-v) d\mu(x, v) \\ &= - \int_{T^1 M} \omega_x(v) d\mu(x, v) \\ &= - \langle [\mu], [\omega] \rangle \end{aligned}$$

whence

$$[\mathcal{I}_*\mu] = -[\mu],$$

so  $\mathcal{B}_1$  is centrally symmetric.

Now let us show that  $\mathcal{B}_1$  contains the origin in its interior. Fix a basis  $h_1, \dots, h_n$  of  $H_1(M, \mathbb{R})$  such that  $h_1, \dots, h_n$  are integer elements of  $H_1(M, \mathbb{R})$ . Let  $\gamma_1, \dots, \gamma_n$  be closed geodesics parametrized by arc length such that  $[\gamma_i] = h_i$ ,  $i = 1, \dots, n$  and let  $\mu_1, \dots, \mu_n$  be the probability measures defined by

$$\int_{T^1 M} f(x, v) d\mu_i(x, v) := \frac{1}{l_g(\gamma_i)} \int_0^{l_g(\gamma_i)} f(\gamma_i(t), \dot{\gamma}_i(t)) dt, \quad i = 1, \dots, n$$

where  $f \in C^0(T^1 M)$ . We have, for any smooth closed one-form  $\omega$  on  $M$

$$\langle [\mu_i], [\omega] \rangle = \frac{1}{l_g(\gamma_i)} \langle [\gamma_i], [\omega] \rangle, \quad i = 1, \dots, n$$

whence

$$[\mu_i] = \frac{1}{l_g(\gamma_i)} [\gamma_i], \quad i = 1, \dots, n.$$

Therefore  $\mathcal{B}_1$  contains the points  $\pm l_g(\gamma_i)^{-1} [\gamma_i]$ ,  $i = 1, \dots, n$ , so it contains their convex hull, which contains the origin in its interior because  $[\gamma_i]$ ,  $i = 1, \dots, n$  generate  $H_1(M, \mathbb{R})$ .

So  $\mathcal{B}_1$  is the unit ball for some norm  $N$  in  $H_1(M, \mathbb{R})$ , which justifies the notation  $\mathcal{B}_1$ .

Let us show this norm  $N$  is no other than the stable norm. First we show  $\| \cdot \| \geq N$ . Take  $h$  in  $H_1(M, \mathbb{R})$  and  $\epsilon > 0$ . Let  $\sum_i r_i \gamma_i$  be a cycle such that the  $\gamma_i$  are closed geodesics,  $[\sum_i r_i \gamma_i] = h$ , and  $\sum_i |r_i| l_g(\gamma_i) \leq \|h\| + \epsilon$ . Reorienting the  $\gamma_i$  if need be, we may assume that the  $r_i$  are non-negative. Then the formula

$$\int_{T^1 M} f(x, v) d\mu(x, v) := \frac{\sum_i r_i \int_0^{l_g(\gamma_i)} f(\gamma_i(t), \dot{\gamma}_i(t)) dt}{\sum_i r_i l_g(\gamma_i)}$$

defines an element of  $\mathcal{M}_g$ , with homology

$$[\mu] = \frac{[\sum_i r_i \gamma_i]}{\sum_i r_i l_g(\gamma_i)}$$

By definition we have  $N([\mu]) \leq 1$ , whence, since  $N$  is a norm

$$N\left(\left[\sum_i r_i \gamma_i\right]\right) \leq \sum_i |r_i| l_g(\gamma_i) \leq \|h\| + \epsilon.$$

Since  $\epsilon$  is arbitrary, we conclude that  $\| \cdot \| \geq N$ .

Now let us show that  $\| \cdot \| \leq N$ . It suffices to show that for any  $\mu \in \mathcal{M}_g$ , we have  $\|[\mu]\| \leq 1$ . Here we use the dual stable norm (see [Gr81], 4.35). A norm  $\| \cdot \|_0$  is defined on the space of  $C^1$  closed one-forms on  $M$  by

$$\|\omega\|_0 := \max \{ \omega_x(v) : (x, v) \in T^1 M \}.$$

This norm induces a norm on  $H^1(M, \mathbb{R}) : \forall c \in H^1(M, \mathbb{R})$ ,

$$\|c\|_0 := \inf \{ \|\omega\|_0 : [\omega] = c \}.$$

**Lemma 4** ([Gr81]). *The norm  $\| \cdot \|_0$  on  $H^1(M, \mathbb{R})$  is dual to the stable norm, that is, for any  $h \in H_1(M, \mathbb{R})$ ,*

$$\|h\| = \max \{ \langle c, h \rangle : c \in H^1(M, \mathbb{R}), \|c\|_0 \leq 1 \}.$$

In view of the above Lemma, what we need to show is that for any  $c \in H^1(M, \mathbb{R})$  such that  $\|c\|_0 \leq 1$ , we have  $\langle c, [\mu] \rangle \leq 1$ . As  $\|c\|_0 \leq 1$ , for all  $\epsilon > 0$  there exists a closed one-form  $\omega$  such that  $[\omega] = c$  and  $|\omega_x(v)| \leq 1 + \epsilon$  for all  $(x, v) \in T^1 M$ . By the Ergodic Decomposition Theorem ([Mn83], Theorem 6.4 p. 170) we have

$$\int_{T^1 M} \omega d\mu = \int_{T^1 M} \left\{ \int_{T^1 M} \omega d\mu_{x,v} \right\} d\mu(x, v)$$

where, for  $\mu$ -almost every  $(x, v)$ ,

$$\int_{T^1 M} \omega d\mu_{x,v} = \lim_{T \rightarrow +\infty} \frac{1}{T} \int_0^T \omega(\phi_t(x, v)) dt.$$

Since  $\phi_t(x, v)$  is in  $T^1 M$  for all  $t$ , the above expression is  $\leq 1 + \epsilon$ , which proves that  $\langle [\omega], [\mu] \rangle \leq 1 + \epsilon$ . Thus  $\langle c, [\mu] \rangle \leq 1$  for any  $c \in H^1(M, \mathbb{R})$  such that  $\|c\|_0 \leq 1$  so  $\| \cdot \| \leq N$ .

Finally  $\| \cdot \| = N$  and  $\mathcal{B}_1 = \{h \in H_1(M, \mathbb{R}) : \|h\| \leq 1\}$ .  $\square$

We say an element  $\mu$  of  $\mathcal{M}_g$  is *minimizing* if its homology class lies on the boundary  $\mathcal{S}_1$  of  $\mathcal{B}_1$ , that is, if there exists a cohomology class  $c$  such that  $\langle c, [\mu] \rangle = 1$  and  $\langle c, h \rangle \leq 1$  for all  $h \in \mathcal{B}_1$ .

**3.2. Link with Mather's theory.** In this paragraph we prove that the minimizing measures just defined are minimizing in the sense of Mather ([Mr91]), which allows us to use Mather's Graph Theorem.

Consider the set  $\mathcal{M}'_g$  of all compactly supported,  $\phi_t$ -invariant probability measures on the tangent bundle  $TM$  of  $M$  and not just  $T^1M$  (here  $\phi_t$  denote the geodesic flow in  $TM$ ). We can define the homology class of an element of  $\mathcal{M}'_g$  just like we do for an element of  $\mathcal{M}_g$ . Mather's  $\beta$ -function is defined in [Mr91] as

$$\begin{aligned} \beta: H_1(M, \mathbb{R}) &\longrightarrow \mathbb{R} \\ h &\longmapsto \min \left\{ \int_{TM} \frac{1}{2} \| \cdot \|_g^2 d\mu : \mu \in \mathcal{M}'_g, [\mu] = h \right\} \end{aligned}$$

where  $\| (x, v) \|_g^2 := g_x(v, v)$  for all  $(x, v) \in TM$ .

The measures achieving the minimum for some  $h$  are called *h-minimizing*. Next we show that this definition of minimizing agrees with ours.

**Proposition 5.** *A minimizing measure  $\mu \in \mathcal{M}_g$  is  $[\mu]$ -minimizing. Conversely an  $h$ -minimizing measure in  $\mathcal{M}'_g$  with  $h \in \mathcal{S}_1$  is in  $\mathcal{M}_g$ ; in particular, it is minimizing. Furthermore,  $2\beta = \| \cdot \|^2$ .*

*Proof.* Let us begin by showing that  $\beta$  is quadratic (i.e. 2-homogeneous). Take  $h \in H_1(M, \mathbb{R})$ ,  $\mu$  an  $h$ -minimizing measure,  $\lambda$  a real number. The formula

$$f \longmapsto \int_{TM} f(x, \lambda v) d\mu(x, v)$$

defines a probability measure on  $TM$ , whose homology class is  $\lambda h$ . Therefore we have  $\beta(\lambda h) \leq \lambda^2 \beta(h)$  and likewise,

$$\beta(h) = \beta\left(\frac{1}{\lambda} \lambda h\right) \leq \frac{1}{\lambda^2} \beta(\lambda h)$$

whence  $\beta(\lambda h) = \lambda^2 \beta(h)$ .

Now, since  $2\beta$  and  $\| \cdot \|^2$  are both quadratic, proving that

$$\mathcal{B}_1 = \left\{ h \in H_1(M, \mathbb{R}) : \beta(h) \leq \frac{1}{2} \right\}$$

suffices to prove that  $2\beta = \| \cdot \|^2$ . Note that

$$\mathcal{B}_1 \subset \left\{ h \in H_1(M, \mathbb{R}) : \beta(h) \leq \frac{1}{2} \right\}$$

for if  $\mu$  is an element of  $\mathcal{M}_g$ , we can view it as a measure on  $TM$  supported on  $T^1M$ , thus

$$\int_{TM} \frac{1}{2} \| \cdot \|_g^2 d\mu = \int_{T^1M} \frac{1}{2} d\mu = \frac{1}{2}$$

whence  $\beta([\mu]) \leq 1/2$ .

Conversely, let  $h \in H_1(M, \mathbb{R})$  be such that  $\beta(h) = 1/2$ , and let  $\mu$  be an  $h$ -minimizing measure. Then by [C95] the support of  $\mu$  is contained in the energy level one half, that is,  $T^1M$ . Thus  $[\mu] \in \mathcal{B}_1$ , and

$$\mathcal{B}_1 = \left\{ h \in H_1(M, \mathbb{R}) : \beta(h) \leq \frac{1}{2} \right\} \text{ whence } \beta = \frac{1}{2} \| \cdot \|^2.$$

Besides, since  $\beta(h) = 1/2$ ,  $h$  lies on the boundary of  $\mathcal{B}_1$ , hence  $\mu$  is minimizing in our sense. Now we would like to prove that a minimizing measure in our sense minimizes in the sense of Mather. Let  $\mu \in \mathcal{M}_g$  be such that  $[\mu] \in \mathcal{S}_1$ . Then as we have just seen  $\beta([\mu]) = 1/2$  so

$$\int_{TM} \frac{1}{2} \| \cdot \|^2_g d\mu \geq \frac{1}{2}.$$

On the other hand since  $\mu$  is supported in  $T^1M$ , we have

$$\int_{TM} \frac{1}{2} \| \cdot \|^2_g d\mu \leq \frac{1}{2}$$

whence

$$\int_{TM} \frac{1}{2} \| \cdot \|^2_g d\mu = \frac{1}{2} = \beta([\mu])$$

that is,  $\mu$  is minimizing in the sense of Mather.  $\square$

The main reward of our efforts is that we may use Mather's Graph Theorem. Let  $c$  be a cohomology class such that  $\langle c, h \rangle \leq 1$  for all  $h \in \mathcal{B}_1$  and  $\langle c, h_0 \rangle = 1$  for some  $h_0 \in \mathcal{B}_1$  (thus  $\|c\|_0 = 1$ ). We say a measure  $\mu \in \mathcal{M}_g$  is  $c$ -minimizing if  $\langle c, [\mu] \rangle = 1$ . Let  $\mathcal{M}_c \subset T^1M$  be the union of the supports of all  $c$ -minimizing measures.

**Theorem 6** (Mather). *The restriction of  $p$  to  $\mathcal{M}_c$  is injective, and its inverse is Lipschitz.*

This means that minimizing measures can be identified with measured geodesic laminations.

#### 4. FLATS OF THE UNIT BALL

Let  $(M, g)$  be a closed Riemannian manifold of any dimension. We call

- supporting subspace to the unit ball of the stable norm, any affine subspace of  $H_1(M, \mathbb{R})$  that meets the unit sphere but not the open unit ball
- flat of the unit ball, the intersection of the unit sphere with a supporting subspace
- dimension of a flat, the dimension of the affine subspace it generates in  $H_1(M, \mathbb{R})$
- interior of a flat, its interior in the affine subspace it generates.

As a trivial example, all points of the unit sphere are zero-dimensional flats. If  $c$  is a cohomology class of dual stable norm one, that is,  $\langle c, h \rangle \leq 1$  for all  $h \in \mathcal{B}_1$  and  $\langle c, h_0 \rangle = 1$  for some  $h_0 \in \mathcal{B}_1$ , then

$$\{h \in H_1(M, \mathbb{R}) : \langle c, h \rangle = 1\}$$

is a supporting hyperplane to the unit ball of the stable norm, and

$$\{h \in \mathcal{B}_1 : \langle c, h \rangle = 1\}$$

is a flat, which may or may not be trivial. Note that by the Hahn-Banach Theorem, any supporting subspace is contained in a supporting hyperplane. So for any flat  $F$ , there exists  $c \in H^1(M, \mathbb{R})$  such that

$$\langle c, h \rangle \leq 1 \quad \forall h \in \mathcal{B}_1 \quad \text{and} \quad F \subset \{h \in \mathcal{B}_1 : \langle c, h \rangle = 1\}.$$

By [Mn92], if a minimizing measure  $\mu$  is ergodic, then  $[\mu]$  is an extremal point of  $\mathcal{B}_1$ , hence it cannot be in the interior of any non-trivial flat. In particular non connected minimizing cycles, when they exist, are the simplest examples of non-trivial flats. Recall Proposition 4 of [Mt97] :

**Lemma 7.** *Let  $F_1$  and  $F_2$  be two flats of the unit ball, both containing a point  $h_0$  such that  $h_0$  is an interior point of  $F_1$ . Then there exists a flat  $F$  containing  $F_1 \cup F_2$ .*

*Proof.* Let  $c \in H^1(M, \mathbb{R})$  be such that

$$\langle c, h \rangle \leq 1 \quad \forall h \in \mathcal{B}_1 \quad \text{and} \quad F_2 \subset \{h \in \mathcal{B}_1 : \langle c, h \rangle = 1\}.$$

Then  $c$  restricted to the convex set  $F_1$  has a maximum at the interior point  $h_0$ . Since  $c$  is linear, this implies that  $c$  is constant on  $F_1$ . Hence

$$\{h \in \mathcal{B}_1 : \langle c, h \rangle = 1\}$$

is a flat containing  $F_1 \cup F_2$ .  $\square$

**Lemma 8.** *Let  $F_1$  and  $F_2$  be two flats of the unit ball, both containing a point  $h_0$  in their interiors. Then there exists a flat  $F$  containing  $F_1 \cup F_2$  such that  $h_0$  is an interior point of  $F$ .*

*Proof.* Let

- $V_i, i = 1, 2$  be the underlying vector space of the affine space generated by  $F_i$
- $V := V_1 + V_2$
- $A$  be the affine subspace  $h_0 + V$
- $F := A \cap \mathcal{B}_1$
- $c \in H^1(M, \mathbb{R})$  be given by the previous Lemma such that

$$\langle c, h \rangle \leq 1 \quad \forall h \in \mathcal{B}_1 \quad \text{and} \quad F_1, F_2 \subset \{h \in \mathcal{B}_1 : \langle c, h \rangle = 1\}.$$

Since  $F_1, F_2 \subset \{h \in \mathcal{B}_1 : \langle c, h \rangle = 1\}$  we have  $A \subset \{h \in \mathcal{B}_1 : \langle c, h \rangle = 1\}$  so  $A$  is a supporting subspace whence  $F$  is a flat. Besides, since  $F$  is convex and contains  $F_1$  and  $F_2$ , it contains the convex hull  $C$  of  $F_1$  and  $F_2$ . Now, since  $h_0$  is interior to both  $F_1$  and  $F_2$ , there exist open neighborhoods of zero  $U_1, U_2$  in  $V_1, V_2$  respectively such that  $h_0 + U_i \subset F_i, i = 1, 2$ . So the convex hull of  $h_0 + U_1$  and  $h_0 + U_2$  is open in  $A$ , and contained in  $C$ , hence in  $F$ . Thus  $h_0$  is an interior point of  $F$ .  $\square$

The former Lemma means that for any homology class  $h$ , there exists a unique maximal flat containing  $h$  in its interior.

### Orientable surfaces.

Assume, for the remainder of this section, that  $M$  is an orientable surface. If  $F$  is a flat of the unit ball of the stable norm,  $h_1, h_2 \in F$  and  $\mu_1, \mu_2$

are minimizing measures such that  $[\mu_i] = h_i$ ,  $i = 1, 2$ , then by Mather's Graph Theorem the supports of  $\mu_1$  and  $\mu_2$  do not intersect transversally so  $\text{Int}(h_1, h_2) = 0$ . Thus the vector space generated by  $F$  in  $H_1(M, \mathbb{R})$  is isotropic with respect to the symplectic intersection form. In particular its dimension is  $\leq 2^{-1}b_1(M)$  so  $\dim F \leq 2^{-1}b_1(M) - 1$ . This was first observed by M.J. Carneiro ([C95]). In [Mt97] this upper bound is proved to be optimal, so non-trivial flats always exists for orientable surfaces of genus  $\geq 2$  (see Theorem 16 and its corollary).

It is proved in [Mt97] (Proposition 6, which surprisingly we have seen no need to reprove) that a flat containing a rational point in its interior is a finite polyhedron with at most  $3(2^{-1}b_1(M) - 1)$  vertices. Furthermore the vertices are rational homology classes which have connected minimizing cycles.

## 5. TECHNICAL LEMMAS

**5.1. Key lemma, one-sided case.** After writing this lemma we came across reference [Sc82], where a similar result is proved in a topological setting. Lemma 9 and its orientable companion Lemma 10 are improved versions of Lemmas 15, 16, 17 of [Mt97]. The purpose of Lemma 9 was to mimic the approach of [Mt97]. Later on, inspired by [Fa98] we realized it is simpler to use the orientation cover. So Lemma 9 won't be used in the proofs of our main theorems. Still we believe it is interesting in its own right. We do use it, however, in the proof of Lemma 13.

Let  $\gamma_1$  be a closed, one-sided, simple geodesic on a non-orientable surface  $M$ .

**Lemma 9.** *There exists a neighborhood  $V_1$  of  $(\gamma_1, \dot{\gamma}_1)$  in  $T^1M$  such that, for any simple geodesic  $\gamma$ , if  $(\gamma, \dot{\gamma})$  enters (resp. leaves)  $V_1$  then  $\gamma$  is forever trapped in  $p(V_1)$  in the future (resp. past), that is*

$$\exists t_1 \in \mathbb{R}, \forall t \geq (\text{resp } \leq)t_1, \gamma(t) \in p(V_1).$$

*Proof.* Let  $U_1$  be a neighborhood of  $\gamma_1$  in  $M$  homeomorphic to a Möbius strip. Let  $P := \gamma_1(0)$  be a point of  $\gamma_1$ , and let  $\delta$  be a smooth open arc transverse at  $P$  to  $\gamma_1$ , such that  $U_1 \setminus \delta$  is simply connected. Let  $V_1$  be the neighborhood of  $(\gamma_1, \dot{\gamma}_1)$  in  $T^1M$  defined by

- (1)  $p(V_1) = U_1$
- (2)  $\forall (x, v) \in V_1, p(\phi_t(x, v))$  intersects transversally  $\delta$  at least three times  $t_1 < t_2 < t_3$  (the points  $p(\phi_{t_i}(x, v)), i = 1, 2, 3$  may coincide if  $p(\phi_t(x, v))$  is a closed geodesic)
- (3) we would like a condition along the lines of "the geodesics that enter  $V$  always cross  $\delta$  in the same direction as  $\gamma_1$ ". This takes some precaution because  $M$  is not orientable. So we choose a smooth vector field  $X$  in  $U_1$ , transverse to  $\delta$ , which has  $\gamma_1$  as a trajectory and such that every other trajectory is closed and homotopic to the boundary of  $U_1$ , that is, bounds a Möbius strip containing  $\gamma_1$ . We require that

$$\forall (x, v) \in V_1, g(X(x), v) > 0.$$

These conditions define an open set of  $T^1M$  because  $\delta$  is an open arc and we demand that the intersections be transverse.

Now consider a simple geodesic  $\gamma$  such that for some  $t \in \mathbb{R}$ ,  $(\gamma(t), \dot{\gamma}(t)) \in V_1$ . Let  $t_1 < t_2 < t_3$  be such that  $\gamma(t_1), \gamma(t_2), \gamma(t_3)$  are the transverse intersection points of  $\gamma$  and  $\delta$  given by the definition of  $V_1$ .

5.1.1. *First case.* Assume  $\gamma(t_3)$  is farther away from  $P$  than  $\gamma(t_1)$  with respect to the distance on  $\delta$  induced by the metric of  $M$ . The domain  $U'_1$  bounded by  $\gamma([t_1, t_3])$  and the subsegment of  $\delta$  joining  $\gamma(t_1)$  with  $\gamma(t_3)$  is homeomorphic to a Möbius strip. The geodesic  $\gamma$  does not self-intersect, hence it can only cut the boundary of  $U'_1$  along  $\delta$ . By Condition 3 of the definition of  $V_1$ ,  $\gamma$  can only intersect  $\delta$  from left to right as pictured in Figure 1, that is, outwards of  $U'_1$ . Therefore  $\gamma$  is trapped in  $U'_1$  in the past.

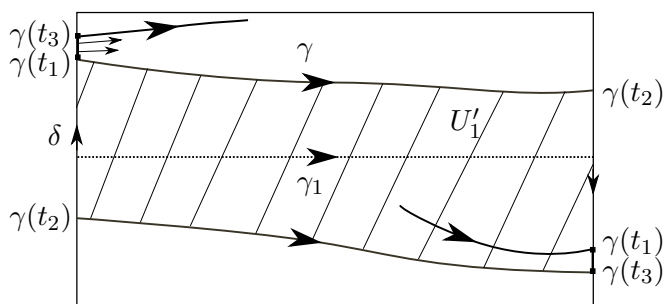


FIGURE 1.

5.1.2. *Second case.* Assume  $\gamma(t_1)$  is farther away from  $P$  than  $\gamma(t_3)$  with respect to the distance on  $\delta$  induced by the metric of  $M$ . We prove as in the first case that  $\gamma$  is trapped in  $U'_1$  in the future.

5.1.3. *Third case.* Assume  $\gamma(t_1) = \gamma(t_3)$ . Then, since  $\gamma$  doesn't self-intersect, it must be a closed geodesic, hence is trapped in  $U_1$  in both past and future.  $\square$

5.2. **Key lemma, two-sided case.** Let  $\gamma_2$  be a closed, two-sided, simple geodesic on a surface  $M$ , orientable or not. Again, we shall only use the orientable case here, but the extra generality comes for free. Let  $U_2$  be a neighborhood of  $\gamma_2$  in  $M$  homeomorphic to an annulus. Choose a symplectic form  $\omega$  in  $U_2$ , yielding a local orientation of  $U_2$ .

**Lemma 10.** *There exists a neighborhood  $V_2$  of  $(\gamma_2, \dot{\gamma}_2)$  in  $T^1M$  such that any simple geodesic  $\gamma$ , if  $(\gamma, \dot{\gamma})$  enters (resp. leaves)  $V_2$  then either  $\gamma$  intersects  $\gamma_2$  or  $\gamma$  is forever trapped in  $p(V_2)$  in the future (resp. past), that is*

$$\exists t_2 \in \mathbb{R}, \forall t \geq (\text{resp. } \leq) t_2, \gamma(t) \in p(V_2).$$

Besides, all intersections with  $\gamma_2$  have the same sign with respect to  $\omega$ .

*Proof.* Let  $P := \gamma_2(0)$  be a point of  $\gamma_2$ , and let  $\delta$  be an smooth open arc transverse at  $P$  to  $\gamma_2$ , such that  $U_2 \setminus \delta$  is simply connected. Assume  $\delta$  is oriented so that  $\omega(P)(\dot{\gamma}_2, \dot{\delta}) > 0$ . Let  $V_2$  be the neighborhood of  $(\gamma_2, \dot{\gamma}_2)$  in  $T^1M$  defined by

- (1)  $p(V_2) = U_2$
- (2)  $\forall (x, v) \in V_2$ ,  $p(\phi_t(x, v))$  intersects transversally  $\delta$  at least twice before intersecting  $\gamma_2$ , if it intersects  $\gamma_2$  at all ; and if it does, it must intersect transversally  $\delta$  at least twice more before either leaving  $U_2$  or meeting  $\gamma_2$  again
- (3) the geodesics that enter  $V_2$  always cross  $\delta$  in the same direction as  $\gamma_2$ , that is,

$$\forall x \in \delta, \forall v \in T_x^1M \text{ such that } (x, v) \in V_2, \omega(x)(v, \dot{\delta}(x)) > 0.$$

These conditions define an open set of  $T^1M$  because  $\delta$  is an open arc and we demand that the intersections be transverse.

Now consider a simple geodesic  $\gamma$  such that for some  $t \in \mathbb{R}$ ,  $(\gamma(t), \dot{\gamma}(t)) \in V_2$ . Let  $t_1 < t_2$  be such that  $\gamma(t_1), \gamma(t_2)$  are the first two transverse intersection points of  $\gamma$  and  $\delta$  given by the definition of  $V_2$ .

5.2.1. *First case.* Assume  $\gamma(t_2)$  is farther away from  $P$  than  $\gamma(t_1)$  with respect to the distance on  $\delta$  induced by the metric of  $M$ . The domain  $U'_2$  bounded by  $\gamma([t_1, t_2])$  and the subarc of  $\delta$  joining  $\gamma(t_1)$  with  $\gamma(t_2)$  on one side, and by  $\gamma_2$  on the other side is homeomorphic to an annulus. The geodesic  $\gamma$  is simple so it cannot self-intersect, hence it can only cut the boundary of  $U'_2$  along  $\delta$  or  $\gamma_2$ . By Condition 3 of the definition of  $V_2$ ,  $\gamma$  can only intersect  $\delta$  from left to right as pictured in Figure 2, that is, outwards of  $U'_2$ . Therefore  $\gamma$  either intersects  $\gamma_2$  or is trapped in  $U'_2$  in the past.

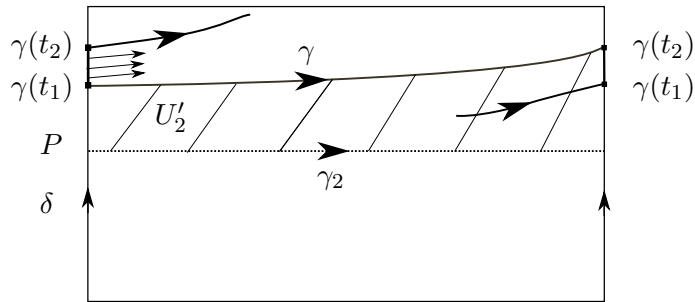


FIGURE 2.

5.2.2. *Second case.* Assume  $\gamma(t_1)$  is farther away from  $P$  than  $\gamma(t_2)$  with respect to the distance on  $\delta$  induced by the metric of  $M$ . Likewise we prove that  $\gamma$  either intersects  $\gamma_2$  or is trapped in  $U'_2$  in the future.

5.2.3. *Third case.* Assume  $\gamma(t_1) = \gamma(t_2)$ . Then, since  $\gamma$  doesn't self-intersect, it must be a closed geodesic, and the conclusion readily follows.

We still have to prove the statement about the sign of the intersections. Assume  $\gamma$  cuts  $\gamma_2$  once with positive sign, that is, downwards in Figure 3. Assume for convenience that the intersection point is  $\gamma(0)$ . Let  $t_1 < t_2 < 0 < t_3 < t_4$  be such that  $\gamma(t_1), \gamma(t_2)$  are the last two transverse intersection points of  $\gamma$  and  $\delta$  before  $\gamma$  meets  $\gamma_2$ , and  $\gamma(t_3), \gamma(t_4)$  are the first two transverse intersection points of  $\gamma$  and  $\delta$  after  $\gamma$  meets  $\gamma_2$ . The domain  $U_2''$  bounded by  $\gamma([t_1, t_2])$  and the subarc of  $\delta$  joining  $\gamma(t_1)$  with  $\gamma(t_2)$  on one side, and by  $\gamma([t_3, t_4])$  and the subarc of  $\delta$  joining  $\gamma(t_3)$  with  $\gamma(t_4)$  on the other side, is homeomorphic to an annulus and contains  $\gamma_2$  in its interior.

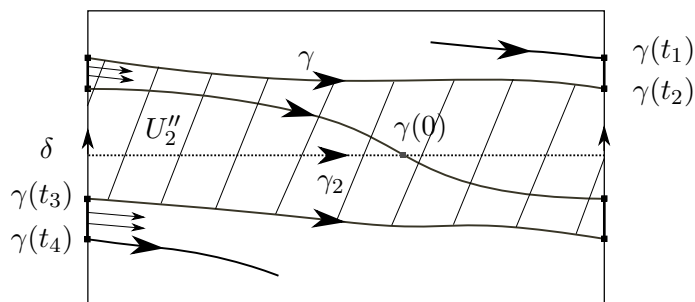


FIGURE 3.

The geodesic  $\gamma$  does not self intersect so it cannot enter  $U_2''$  through segments of  $\gamma$ . It only intersects  $\delta$  from left to right, that is, either between  $\gamma(t_3)$  and  $\gamma(t_4)$  and outwards of  $U_2''$ , or between  $\gamma(t_1)$  and  $\gamma(t_2)$  and inwards of  $U_2''$ . So it can only enter  $U_2''$  through  $\delta$  between  $\gamma(t_1)$  and  $\gamma(t_2)$ , that is, from above in Figure 3. Therefore it always cut  $\gamma_2$  with positive sign.  $\square$

5.3. **Consequences of the key lemmas.** For the ease of the reader, rather than loading up the sentences with “resp.” we have split the next proposition in two, one part for the one-sided case and the other for the two-sided case.

**Proposition 11.** *Let  $\gamma_1$  be a closed, simple, one-sided geodesic on a surface  $M$ , orientable or not. There exists a neighborhood  $V_1$  of  $(\gamma_1, \dot{\gamma}_1)$  in  $T^1M$  such that for any simple geodesic  $\gamma$ , if  $(\gamma, \dot{\gamma})$  enters (resp. leaves)  $V_1$ , then*

- $\gamma$  is a closed geodesic homotopic to  $\gamma_1$  or  $\gamma_1 \cdot \gamma_1$ , the latter being meant as a product in  $\pi_1(M)$
- or  $\gamma$  is positively (resp. negatively) asymptotic to a closed geodesic homotopic to  $\gamma_1$  or  $\gamma_1 \cdot \gamma_1$ .

**Proposition 12.** *Let  $\gamma_2$  be a closed, simple, two-sided geodesic. There exists a neighborhood  $V_2$  of  $(\gamma_2, \dot{\gamma}_2)$  in  $T^1M$  such that for any simple geodesic  $\gamma$ , if  $\gamma$  enters (resp. leaves)  $V_2$ , then*

- either  $\gamma$  is a closed geodesic homotopic to  $\gamma_2$
- or  $\gamma$  is asymptotic to a closed geodesic homotopic to  $\gamma_2$

- or  $\gamma$  intersects  $\gamma_2$ , and all intersections have the same sign with respect to some orientation of  $p(V_2)$ .

*Proof.* Let us prove Proposition 12. Let  $V_1$  be a neighborhood of  $(\gamma_1, \dot{\gamma}_1)$  in  $T^1M$  given by Lemma 10, and small enough so it does not contain any contractible closed geodesic. Let  $\gamma$  be a simple geodesic such that  $(\gamma, \dot{\gamma})$  enters  $V_1$ . Let  $t_1$  be such that  $(\gamma, \dot{\gamma})([t_1, +\infty[) \subset V_1$ . Extend  $\dot{\gamma}([t_1, +\infty[)$  to a smooth vector field in the annulus  $p(V_1)$ . Since the annulus may be embedded in the two-sphere, the Poincaré-Bendixon Theorem applies. So  $\gamma$  is either a fixed point, a cycle of fixed points and heteroclinic orbits, or a closed orbit, or asymptotic to one of the former. Four out of six cases are impossible here because  $\gamma$  is a geodesic so its velocity is constant, hence cannot go to zero. Besides, a closed orbit of a vector field must be a simple closed curve, and a non-contractible simple closed curve in an annulus is homotopic to the boundary of the annulus. This proves Proposition 12. The proof of Proposition 11 is identical, mutatis mutandis.  $\square$

#### 5.4. Geodesics asymptotic to closed geodesics.

**Lemma 13.** *If a geodesic  $\gamma$  is asymptotic to a simple closed geodesic, then  $(\gamma, \dot{\gamma})$  is not in the support of any minimizing measure.*

*Proof.* Let

- $\gamma_0$  be a simple closed geodesic
- $\gamma$  a geodesic asymptotic to  $\gamma_0$
- $V_0$  be a neighborhood of  $(\gamma_0, \dot{\gamma}_0)$  in  $T^1M$  given by Lemma 9 or 10 depending on whether  $\gamma_0$  is one-sided or two-sided, and such that  $(\gamma(0), \dot{\gamma}(0)) \notin V_0$
- $V$  a neighborhood of  $(\gamma(0), \dot{\gamma}(0))$  such that  $V$  is disjoint from  $V_0$  but for large enough  $t$ ,  $\phi_t(V) \subset V_0$
- $\mu$  a minimizing measure.

Assume  $(\gamma, \dot{\gamma})$  is contained in the support of  $\mu$ . Then since  $\text{supp}(\mu)$  is closed and invariant under the geodesic flow, it contains the  $\alpha$ - and  $\omega$ -limit sets of  $(\gamma, \dot{\gamma})$ , in particular it contains  $(\gamma_0, \dot{\gamma}_0)$ . Besides,  $\mu(V) > 0$ . By the Ergodic Decomposition Theorem ([Mn83], Theorem 6.4 p. 170), we have

$$\mu(V) = \int_{T^1M} \mu_{x,v}(V) d\mu(x, v)$$

where, for  $\mu$ -almost every  $x, v$ ,

$$\mu_{x,v}(V) = \lim_{T \rightarrow +\infty} \frac{1}{T} \int_0^T \chi_V(\phi_t(x, v)) dt$$

denoting by  $\chi_V$  the characteristic function of  $V$ . Thus for some  $(x, v)$  we have  $\mu_{x,v}(V) > 0$ . So for some  $t$  in  $\mathbb{R}$ ,  $\phi_t(x, v) \in V$ . By our hypothesis on  $V$ , this implies that for  $t$  large enough  $\phi_t(x, v) \in V_0$ . But since  $(\gamma_0, \dot{\gamma}_0)$  is contained in the support of  $\mu$ , by Mather's Graph Theorem  $p(\phi_t(x, v))$  cannot intersect  $\gamma_0$ . Thus by Proposition 11 or 12 the geodesic  $p(\phi_t(x, v))$  is asymptotic to a geodesic homotopic to  $\gamma_0$  whose lift to  $T^1M$  is contained in  $V_0$ . Therefore  $\phi_t(x, v)$  never comes back to  $V$ , whence

$$\lim_{T \rightarrow +\infty} \frac{1}{T} \int_0^T \chi_V(\phi_t(x, v)) dt = 0$$

which contradicts the fact that  $\mu_{x,v}(V) > 0$ .  $\square$

**5.5. Minimizing measures with rational homology classes.** The Proposition below was stated as Lemma 2.1.6 in [Mt96] and Proposition 5 of [Mt97], although the announcement of its proof was greatly exaggerated.

**Proposition 14.** *Let  $M$  be a closed surface, orientable or not, with a Riemannian metric. If  $h$  is a rational homology class and  $\mu$  is an  $h$ -minimizing measure, then the support of  $\mu$  consists of periodic orbits.*

First we need a

**Lemma 15.** *Let  $M$  be a closed non-orientable surface with a Riemannian metric  $g$  and  $\pi : (M_o, \tilde{g}) \rightarrow (M, g)$  denote the Riemannian orientation cover.*

*If  $\mu \in \mathcal{M}_g$  is a  $c$ -minimizing measure where  $c$  is a cohomology class with  $\|c\|_0 = 1$ , then there exists  $\nu \in \mathcal{M}_{\tilde{g}}$  such that  $\pi_*(\nu) = \mu$ ,  $\nu$  is  $I_*$ -invariant and  $\pi^*(c)$ -minimizing.*

*Proof of the Lemma.* Let us assume for the time being that  $\mu$  is ergodic. That is, there exists  $(x, v) \in T^1M$  such that for any continuous function  $F$  on  $T^1M$ ,

$$\int_{T^1M} F d\mu = \lim_{T \rightarrow +\infty} \frac{1}{T} \int_0^T F(\phi_t(x, v)) dt$$

Let us lift the orbit  $\phi_t(x, v)$  to an orbit  $\tilde{\phi}_t(x, v)$  of the geodesic flow of  $(M_o, \tilde{g})$ . Let  $\nu_T$  be the probability measure on  $T^1M_o$  defined by

$$\int_{T^1M_o} F d\nu_T = \frac{1}{T} \int_0^T F(\tilde{\phi}_t(x, v)) dt$$

for any continuous function  $F$  on  $T^1M_o$ . Since the set of probability measures on  $T^1M_o$  is compact for the weak\* topology, there exists a sequence  $T_n \rightarrow +\infty$  such that  $\nu_{T_n}$  converges to some measure  $\nu$ . Then  $\nu$  is invariant by the geodesic flow on  $M_o$ , that is,  $\nu \in \mathcal{M}_{\tilde{g}}$ . Besides  $\pi_*(\nu) = \mu$  since, for any continuous function  $F$  on  $T^1M$

$$\begin{aligned} \int_{T^1M} F d\pi_*(\nu) &= \int_{T^1M_o} F \circ \pi d\nu = \lim_{n \rightarrow +\infty} \frac{1}{T_n} \int_0^{T_n} F(\pi(\tilde{\phi}_t(x, v))) dt \\ &= \lim_{n \rightarrow +\infty} \frac{1}{T_n} \int_0^{T_n} F(\phi_t(x, v)) dt = \int_{T^1M} F d\mu. \end{aligned}$$

Furthermore, since  $\mu$  is a  $c$ -minimizing measure, for all  $\epsilon > 0$  there exists a closed one-form  $\omega$  such that  $[\omega] = c$ ,  $|\omega_x(v)| \leq 1 + \epsilon$  for all  $x \in M$ ,  $v \in T_x^1M$ , and  $\int \omega d\mu = 1$ . Set  $\tilde{\omega} := \pi^*(\omega)$ , then  $\tilde{c} = [\tilde{\omega}] = \pi^*(c)$  and  $|\tilde{\omega}_x(v)| \leq 1 + \epsilon$  for all  $x \in M_o$ ,  $v \in T_x^1M_o$ , and  $\int \tilde{\omega} d\nu = 1$ . So  $\nu$  is  $\pi^*(c)$ -minimizing. Notice that  $\tilde{\omega}$  is  $I^*$ -invariant, so  $I_*\nu$  is also  $\pi^*(c)$ -minimizing. Then so is  $2^{-1}(\nu + I_*\nu)$ , which is  $I_*$ -invariant. This proves the ergodic case of the Lemma as  $\pi_*(I_*\nu) = \mu$ .

Now consider the map  $\pi_*$  between the two compact convex sets

$$\{\mu \in \mathcal{M}_{\tilde{g}} : I_*(\mu) = \mu\}$$

and  $\mathcal{M}_g$ . It is affine and surjective onto the extremal points of  $\mathcal{M}_g$ , hence surjective onto  $\mathcal{M}_g$ .  $\square$

*Proof of the Proposition.* First let us address the case when  $M$  is orientable.

Let  $h$  be a rational homology class and  $\mu$  be an  $h$ -minimizing measure. Then  $\text{Int}(h, H_1(M, \mathbb{Z}))$  is a discrete subgroup of  $\mathbb{R}$ . Assume the projection  $p(\text{supp}\mu)$  of the support of  $\mu$  to  $M$  contains a non-closed geodesic  $\gamma$ . Since  $M$  is compact  $\gamma$  has a limit point, say  $x_\gamma$  in  $M$ . Let  $t_n$  be an increasing sequence of real numbers such that  $\gamma(t_n) \rightarrow x_\gamma$  when  $n \rightarrow \infty$ . Denote by  $\gamma_n$  the closed curve obtained by closing up  $\gamma([t_n, t_{n+1}])$  with a geodesic segment  $\delta_n$  of length  $d(\gamma(t_n), \gamma(t_{n+1}))$ . Such a segment is unique for  $n$  large enough because  $d(\gamma(t_n), \gamma(t_{n+1}))$  tends to zero. We claim that  $\text{Int}(h, [\gamma_n])$  is not zero for  $n$  large enough, and tends to zero, which contradicts the discreteness of  $\text{Int}(h, H_1(M, \mathbb{Z}))$ .

By Mather's Graph Theorem ([Mr91]), for any  $x$  in  $p(\text{supp}\mu)$ , there exists a unique geodesic, denoted  $\gamma_x$ , which is the projection of an orbit in  $\text{supp}\mu$  and such that  $\gamma_x(0) = x$ . To clear up the notations we denote by  $\gamma_\gamma$  the orbit  $\gamma_x$  with  $x = x_\gamma$ . Call

$$R_n := \{\gamma_x(t) : x \in p(\text{supp}\mu) \cap \delta_n, t \in [0, 1]\}.$$

This is a closed subset of  $M$ .

First let us show that

$$(1) \quad p_*\mu(R_n) \rightarrow 0$$

Denote by  $\chi_n$  the characteristic function of  $R_n$ . The sequence of functions  $\chi_n$  converges pointwise to the characteristic function of  $\gamma_\gamma([0, 1])$ , so

$$p_*\mu(R_n) \rightarrow p_*\mu(\gamma_\gamma([0, 1])).$$

Now the latter cannot be positive unless the geodesic  $\gamma_\gamma$  is closed, for otherwise, since  $\mu$  is invariant by the geodesic flow, the total mass of  $\gamma_\gamma$  would be infinite, contradicting the fact that  $\mu$  is a probability measure. Assume  $\gamma_\gamma$  is closed. It is two-sided because we are assuming  $M$  to be orientable for the time being. Since  $\text{supp}\mu$  is closed,  $x_\gamma$  is in  $\text{supp}\mu$ . Since  $\text{supp}\mu$  is invariant by the geodesic flow  $\gamma_\gamma$  is contained in  $\text{supp}\mu$ . Therefore by Mather's Graph Theorem  $\gamma_\gamma$  and  $\gamma$  do not intersect. Thus by Proposition 12,  $\gamma$  is asymptotic to a closed geodesic, hence cannot be in the support of a minimizing measure by Lemma 13. This proves Equation (1). Besides, since  $\gamma$  is in the support of  $\mu$ ,

$$(2) \quad p_*\mu(R_n) > 0.$$

Next we evaluate  $\text{Int}(h, [\gamma_n])$  and find it equals  $p_*\mu(R_n)$ , which combines with the previous paragraph to prove the Proposition.

First note that by the Ergodic Decomposition Theorem ([Mn83], Theorem 6.4 p. 170)

$$(3) \quad p_*\mu(R_n) = \int_M \left\{ \int \chi_n d\mu_x \right\} dp_*\mu(x)$$

where, for  $p_*\mu$ -almost every  $x$  in  $M$

$$\begin{aligned} \int \chi_n d\mu_x &= \lim_{T \rightarrow +\infty} \frac{1}{T} \int_0^T \chi_n(\gamma_x(t)) dt \\ &= \lim_{T \rightarrow +\infty} \frac{1}{T} \#\{t \in [0, T] : \gamma_x(t) \in \delta_n\} \end{aligned}$$

by the definition of  $R_n$ , denoting  $\sharp$  the cardinal of a set.

For  $x$  in  $p(\text{supp}\mu) \cap \delta_n$ , let  $\gamma_{x,T}$  be a closed curve obtained by closing up  $\gamma_x([0, T])$  with a geodesic segment  $\delta_{x,T}$  of length  $\leq \text{diam}M$ . By Birkhoff's Ergodic Theorem, for  $p_*\mu$ -almost every  $x$ , for any closed one-form  $\omega$  on  $M$ ,

$$\int \omega d\mu_x = \lim_{T \rightarrow +\infty} \frac{1}{T} \int_0^T \omega_{\gamma_x(t)}(\dot{\gamma}_x(t)) dt = \lim_{T \rightarrow +\infty} \frac{1}{T} \langle [\omega], [\gamma_{x,T}] \rangle.$$

Thus, for  $p_*\mu$ -almost every  $x$ ,

$$[\mu_x] = \lim_{T \rightarrow +\infty} \frac{1}{T} [\gamma_{x,T}].$$

Since the dimension of  $H_1(M, \mathbb{R})$  is finite, the bilinear form  $\text{Int}(\cdot, \cdot)$  is continuous so for  $p_*\mu$ -almost every  $x$ ,

$$\text{Int}([\mu_x], [\gamma_n]) = \lim_{T \rightarrow +\infty} \frac{1}{T} \text{Int}([\gamma_{x,T}], [\gamma_n]).$$

Observe that since both  $\gamma$  and  $\gamma_x$  are in the support of  $\mu$ , by the Graph Theorem they cannot intersect transversally. So the transverse intersections of  $\gamma_{x,T}$  and  $\gamma_n$ , if any, occur along  $\delta_n$  or  $\delta_{x,T}$ . Note that for fixed  $n$  the number  $n_{x,T}$  of intersections (counted with sign) of  $\delta_{x,T}$  with  $\gamma$  is bounded independantly of  $T$  because the length of  $\delta_{x,T}$  is bounded independantly of  $T$ .

Furthermore, by the Graph Theorem, all intersections of  $\gamma_x([0, T])$  with  $\delta_n$  have the same sign if  $\delta_n$  is small enough. This is where we need the orientability assumption.

By smoothing the corners one can make  $\gamma_{x,T}$  and  $\gamma_n$  of class  $C^1$  without modifying their transverse intersections. The curve thus obtained are transverse unless  $\gamma = \gamma_x$ . In the latter case one moves  $\gamma_x$  slightly away from  $\gamma$  without modifying the transverse intersections of  $\gamma_{x,T}$  and  $\gamma_n$ . Since all intersections of  $\gamma_x([0, T])$  with  $\delta_n$  have the same sign, we get

$$(4) \quad \text{Int}([\gamma_{x,T}], [\gamma_n]) = \sharp \{t \in [0, T] : \gamma_x(t) \in \delta_n\} + n_{x,T}$$

whence, since  $n_{x,T}$  is bounded independantly of  $T$

$$\text{Int}([\mu_x], [\gamma_n]) = \lim_{T \rightarrow +\infty} \frac{1}{T} \sharp \{t \in [0, T] : \gamma_x(t) \in \delta_n\}$$

so, using Equation (3),

$$\text{Int}(h, [\gamma_n]) = p_*\mu(R_n)$$

which finishes the proof of the orientable case of the Proposition.

Assume now that  $M$  is not orientable. Let  $\mu$  be a minimizing measure such that  $[\mu] = rh$  with  $h \in \Lambda$  and  $r \in \mathbb{R}$ . Let  $\nu$  be an  $I_*$ -invariant minimizing measure given by Lemma 15. Let  $c_1, \dots, c_b$  be an integer basis of  $H^1(M, \mathbb{R})$  and let  $\omega_1, \dots, \omega_b$  be closed one-forms such that  $[\omega_i] = c_i$ ,  $i = 1 \dots b$ . Then  $\int \omega_i d\mu \in r\mathbb{Z}$  for  $i = 1 \dots b$ .

Let  $\tilde{\omega}_1, \dots, \tilde{\omega}_b$  be the lifts of  $\omega_1, \dots, \omega_b$  to  $M_o$ . They are integer one-forms and  $[\tilde{\omega}_1], \dots, [\tilde{\omega}_b]$  is a basis of  $E_1 = \{c \in H^1(M_o, \mathbb{R}) : I^*c = c\}$ . Besides,

$$\int \tilde{\omega}_i d\nu = \int \omega_i d\mu \in r\mathbb{Z}, \quad i = 1, \dots, b.$$

Let us take an integer basis  $c_{b+1}, \dots, c_{2b}$  of

$$E_{-1} = \{c \in H^1(M_o, \mathbb{R}) : I^*c = -c\}$$

and closed one-forms  $\tilde{\omega}_1, \dots, \tilde{\omega}_b$  such that  $[\tilde{\omega}_i] = c_i$  for  $i = b+1, \dots, 2b$ . Since  $\nu$  is  $I_*$ -invariant we have

$$\int \tilde{\omega}_i d\nu = 0, \quad i = b+1, \dots, 2b.$$

Let  $x_1, \dots, x_{2b}$  be the coordinates of  $[\nu]$  in the basis of  $H_1(M_o, \mathbb{R})$  dual to the integer basis  $[\tilde{\omega}_1], \dots, [\tilde{\omega}_{2b}]$  of  $H^1(M_o, \mathbb{Z})$ . We have just seen that  $x_1, \dots, x_{2b}$  are all in  $r\mathbb{Z}$ , so  $[\nu]$  is rational. Thus, using the orientable case of the Proposition, we conclude that  $\nu$ , hence  $\mu$ , is supported on periodic orbits.  $\square$

## 6. PROOFS OF THE MAIN THEOREMS

**6.1. Local results - Orientable case.** Let  $h$  be a rational homology class of a surface, orientable or not. Then by Proposition 14 any  $h$ -minimizing measure is supported on periodic orbits. Call  $\mathcal{P}_h$  the union of the projections on  $M$  of the supports of all  $h$ -minimizing measures. By Mather's Graph Theorem  $\mathcal{P}_h$  is a union of pairwise disjoint closed geodesics. Denote by  $\mathcal{VP}_h$  the vector subspace of  $H_1(M, \mathbb{R})$  generated by all homology classes of geodesics contained in  $\mathcal{P}_h$ . Note that the convex hull of all homology classes of curves in  $\mathcal{P}_h$  is contained in a flat of the unit ball containing  $h$  in its interior.

The following theorem proved in [Mt97] describes the local geometry of the unit ball of the stable norm near a rational homology class in the orientable case.

**Theorem 16.** [Mt97] *Let  $M$  be an orientable closed surface endowed with a Riemannian metric. Let  $h_0$  be a rational point of  $\mathcal{S}_1$ . For all  $h \in \mathcal{VP}_{h_0}^\perp$ , there exists  $s(h_0, h) > 0$  such that the subset of the unit sphere  $\mathcal{S}_1$*

$$\left\{ \frac{h_0 + sh}{\|h_0 + sh\|} : s \in [0, s(h_0, h)] \right\}$$

*is a straight segment.*

*Proof.* For any  $n \in \mathbb{N}^*$  let us denote

$$h_n := \frac{h_0 + \frac{1}{n}h}{\|h_0 + \frac{1}{n}h\|}.$$

Let

- $\mu_n$  be an  $h_n$ -minimizing measure
- $\mu_0$  be a limit point, in the weak-\* topology, of the sequence  $\mu_n$ .

Then  $\mu_0$  is an  $h_0$ -minimizing measure. By Proposition 14,  $\mu_0$  is supported on periodic orbits  $\gamma_i, i \in I$  where  $I$  is some set. Note that for all  $i \in I$  the class  $[\gamma_i]$  belongs to  $\mathcal{VP}_{h_0}$ . For each  $i \in I$  let  $V_i$  be the neighborhood of  $(\gamma_i, \dot{\gamma}_i)$  given by Proposition 12. Let  $V$  be the union over  $i \in I$  of the  $V_i$ . First let us prove that  $V \cap \text{supp}(\mu_n)$  is  $\phi_t$ -invariant and consists of periodic orbits homotopic to some or all of the  $\gamma_i$ . Indeed by Proposition 12 a minimizing geodesic that enters  $V$  is either

- asymptotic to one of the  $\gamma_i$ , which is ruled out by Lemma 13
- homotopic to one of the  $\gamma_i$
- or cuts one of the  $\gamma_i$  with constant sign, which is ruled out by hypothesis.

Suppose  $\mu_n(V) \neq 0$ . Set, for any measurable subset  $A$  of  $T^1M$

$$\begin{aligned}\alpha_n(A) &:= \frac{\mu_n(A \cap V)}{\mu_n(V)} \\ \beta_n(A) &:= \frac{\mu_n(A \setminus V)}{\mu_n(T^1M \setminus V)} \\ \lambda_n &:= \mu_n(V).\end{aligned}$$

Then  $\alpha_n$  and  $\beta_n$  are two probability measures on  $T^1M$ . They are invariant by the geodesic flow because  $V \cap \text{supp}(\mu_n)$ , as well as its complement in  $\text{supp}(\mu_n)$ , is  $\phi_t$ -invariant. In case  $\mu_n(V) = 0$ , set  $\alpha_n(A) := \mu_n$  and  $\lambda_n := 1$ . Since the support of  $\alpha_n$  consists of periodic orbits homotopic to some or all of the  $\gamma_i$ , the homology class of  $\alpha_n$  is contained in the convex hull of the  $[\gamma_i]/l_g(\gamma_i)$ . Note that since the support of  $\mu_0$  consists of all of the  $\gamma_i$ , the homology class of  $\mu_0$  is contained in the relative interior of the convex hull of the  $[\gamma_i]/l_g(\gamma_i)$ .

We have

$$\mu_n = \lambda_n \alpha_n + (1 - \lambda_n) \beta_n$$

and  $\lambda_n$  tends to one as  $n$  tends to infinity, so the homology class of  $\alpha_n$  tends to  $h_0$ . Therefore, when  $n$  is large enough, the homology class of  $\alpha_n$  is contained in the relative interior of the convex hull of the  $[\gamma_i]/l_g(\gamma_i)$ . Thus any supporting cohomology class  $c$  to  $\mathcal{S}_1$  at  $[\alpha_n]$ , i.e. such that  $\langle c, [\alpha_n] \rangle = 1$  and  $\langle c, h \rangle \leq 1$  for all  $h \in \mathcal{B}_1$ , is also a supporting cohomology class to  $\mathcal{S}_1$  at  $h_0$ . In other words, any flat of  $\mathcal{S}_1$  that contains  $[\alpha_n]$  also contains  $h_0$ .

Let  $c$  be a supporting cohomology class to  $\mathcal{S}_1$  at  $h_N$ . We have  $\langle c, h_N \rangle = 1$  and  $|\langle c, h \rangle| \leq 1 \forall h \in \mathcal{S}_1$ . Therefore

$$\lambda_N \langle c, [\alpha_N] \rangle + (1 - \lambda_N) \langle c, [\beta_N] \rangle = 1.$$

Since  $\langle c, [\alpha_N] \rangle \leq 1$ ,  $\langle c, [\beta_N] \rangle \geq 1$ ,  $\lambda_N \in [0, 1]$ , this implies

$$\langle c, [\alpha_N] \rangle = \langle c, h_N \rangle = 1$$

that is,  $[\alpha_N]$  and  $h_N$  are in the same flat of  $\mathcal{S}_1$ , whence  $h_0$  and  $h_N$  are in the same flat of  $\mathcal{S}_1$ .  $\square$

Recall from [Mt97] the orientable analogue of the first part of Theorem **A**:

**Corollary 17.** *Assume  $M$  is a closed orientable surface endowed with a Riemannian metric. Then every rational homology class contained in  $\mathcal{S}_1$  lies in a flat of  $\mathcal{S}_1$  of dimension at least  $b_1(M)/2 - 1$ .*

*Proof.* Let  $h_0$  be a rational point of  $\mathcal{S}_1(M, g)$ . Set

$$p := \dim \mathcal{VP}_{h_0}$$

and assume  $p < b_1(M)/2$ . Choose curves  $\gamma_i$  in  $\mathcal{P}_{h_0}$  for  $i = 1, \dots, p$ , such that  $\{[\gamma_i] \mid i = 1, \dots, p\}$  generate  $\mathcal{VP}_h$ . Since  $p < b_1(M)/2$ , there exists

$h \in H_1(M, \mathbb{R})$  such that

$$h \notin \mathcal{VP}_{h_0} \text{ and } \text{Int}(h, [\gamma_i]) = 0 \ \forall i = 1, \dots, p.$$

By Theorem 16 there exists  $s > 0$  such that

$$F_1 := \left\{ \frac{h_0 + sh}{\|h_0 + sh\|} : s \in [0, s(h_0, h)] \right\}$$

is a straight segment contained in  $\mathcal{S}_1(M, g)$ . On the other hand, the convex hull of  $[\gamma_i]$  for  $i = 1, \dots, p+q$  is contained in a flat  $F_0$  of  $\mathcal{S}_1(M, g)$  of dimension  $p$  that contains  $h_0$  in its interior. From Lemma 7 we deduce that there exists a flat containing  $F_0$  and  $F_1$ . The dimension of said flat is greater than  $p = \dim F_0$  because  $h \notin \mathcal{VP}_{h_0}$ .  $\square$

**6.2. Local results - Non-orientable case.** In this section, we assume  $M$  is a closed non-orientable surface and prove Theorems **A** and **B**. The proofs combine basic facts about the orientation cover of a non orientable surface and Theorem 16.

**Proposition 18.** *Assume  $M$  is a closed non-orientable surface endowed with a Riemannian metric  $g$  and  $\pi : (M_o, \tilde{g}) \rightarrow (M, g)$  its orientation cover. Then*

$$\pi_* \mathcal{B}_1(M_o, \tilde{g}) = \mathcal{B}_1(M, g).$$

and furthermore the vector space  $E_1$  endowed with the restriction of the stable norm of  $(M_o, \tilde{g})$  is isometric to  $H_1(M, \mathbb{R})$  endowed with the stable norm of  $(M, g)$ .

*Proof.* Let  $\mu_o$  be an element of  $\mathcal{M}_{\tilde{g}}$ . Then  $\pi_* \mu_o$  is an element of  $\mathcal{M}_g$ . So  $\pi_* \mathcal{B}_1(M_o, \tilde{g}) \subset \mathcal{B}_1(M, g)$ . Conversely, let  $\mu$  be a minimizing measure of  $M$ . Let  $\nu \in \mathcal{M}_{\tilde{g}}$  be given by Lemma 15. We have  $\pi_*(\nu) = \mu$  so  $\pi_*([\nu]) = [\mu]$ . Thus  $\pi_*$  restricted to  $E_1 \cap \mathcal{S}_1(M_o, \tilde{g})$  is surjective onto  $\mathcal{S}_1(M, g)$ . Since  $\pi_*$  is linear, it must then be surjective from  $E_1 \cap \mathcal{B}_1(M_o, \tilde{g})$  onto  $\mathcal{B}_1(M, g)$ . Besides, since the dimensions of  $E_1$  and  $H_1(M, \mathbb{R})$  are equal,  $\pi_*$  restricted to  $E_1$  must be injective. So  $\pi_*$  restricted to  $E_1$  is a linear isomorphism sending  $E_1 \cap \mathcal{B}_1(M_o, \tilde{g})$  to  $\mathcal{B}_1(M, g)$ .  $\square$

The purpose of the next Proposition is to evaluate the maximal dimension of a flat containing a rational class  $h$  (not necessarily as an interior point), depending on the topological properties of  $h$ -minimizing curves. Recall that a simple closed curve  $\gamma$  of  $M$  is said of *type I* (resp. of *type II*) if its inverse image  $\pi^{-1}(\gamma)$  consists of either one curve or two homologous curves (resp. two non-homologous curves). Let  $h$  be a rational point of  $\mathcal{S}_1(M, g)$ . Partition  $\mathcal{P}_h$  in two subsets  $\mathcal{P}_h^1$  and  $\mathcal{P}_h^2$ , the former consisting only of curves of type I and the latter only of curves of type II. Let  $\mathcal{VP}_h^2$  be the vector subspace of  $H_1(M, \mathbb{R})$  generated by all homology classes of geodesics contained in  $\mathcal{P}_h^2$ . Let  $\mathcal{VP}_h^1$  be such that  $\mathcal{VP}_h^1$  is generated by homology classes of curves of type I and

$$\mathcal{VP}_h^2 \oplus \mathcal{VP}_h^1 = \mathcal{VP}_h.$$

**Proposition 19.** *Let  $M$  be a closed non-orientable surface and let  $h_0$  be a rational point of  $\mathcal{S}_1(M, g)$ . Set*

$$p := \dim \mathcal{VP}_{h_0}^1 \text{ and } q := \dim \mathcal{VP}_{h_0}^2$$

*and assume  $p + 2q < b_1(M)$ . Then there exists a flat of  $\mathcal{B}_1(M, g)$  containing  $h_0$ , of dimension  $> p + q$ .*

*Proof.* Choose curves

- $\gamma_i$  in  $\mathcal{P}_{h_0}^1$  for  $i = 1, \dots, p$ , such that  $\{[\gamma_i] \mid i = 1, \dots, p\}$  generate  $\mathcal{VP}_h^1$
- $\gamma_i$  in  $\mathcal{P}_{h_0}^2$  for  $i = p+1, \dots, p+q$ , such that  $\{[\gamma_i] \mid i = p+1, \dots, p+q\}$  generate  $\mathcal{VP}_h^2$ .

Denote by  $\mu_i$  the  $\phi_t$ -invariant probability measure supported on  $\gamma_i$  for  $i = 1, \dots, p+q$ . Let  $c \in H^1(M, \mathbb{R})$  be such that  $h_0$  is  $c$ -minimizing. Then each  $\mu_i$ , and each convex combination thereof, is also  $c$ -minimizing. Let  $\lambda_i \in ]0, 1[$ ,  $i = 1, \dots, p+q$  be such that  $\sum_i \lambda_i = 1$  and  $\sum_i \lambda_i [\mu_i] = h_0$ .

If  $i \in \{1, \dots, p\}$ , choose a closed geodesic  $\tilde{\gamma}_i$  in  $M_o$  such that

- $\pi(\tilde{\gamma}_i) = \gamma_i$
- $[\tilde{\gamma}_i]$  lies in the eigenspace  $E_1$  for the involution  $I$ .

If  $i \in \{p+1, \dots, p+q\}$ , choose two closed geodesics  $\tilde{\gamma}_i$  and  $\tilde{\gamma}_{i+q}$  in  $M_o$  such that

- $\pi(\tilde{\gamma}_i) = \gamma_i$
- $I(\tilde{\gamma}_i) = \tilde{\gamma}_{i+q}$
- $[\tilde{\gamma}_i] \neq [\tilde{\gamma}_{i+q}]$ .

Define

- $\tilde{\mu}_i$  the  $\tilde{\phi}_t$ -invariant probability measure supported on  $\tilde{\gamma}_i$  for  $i = 1, \dots, p+2q$
- $\tilde{\lambda}_i := \lambda_i$  if  $i = 1, \dots, p$
- $\tilde{\lambda}_i := \lambda_i/2$  if  $i = p+1, \dots, p+q$
- $\tilde{\lambda}_{i+q} := \lambda_i/2$  if  $i = p+q+1, \dots, p+2q$
- $\tilde{\mu} := \sum_{i=1}^{p+2q} \tilde{\lambda}_i \tilde{\mu}_i$
- $\tilde{h}_0 = [\tilde{\mu}]$ .

We have

- $I_*(\tilde{\mu}) = \tilde{\mu}$  whence  $I_*(\tilde{h}_0) = \tilde{h}_0$
- $\pi_*(\tilde{\mu}) = \mu$  whence  $\pi_*(\tilde{h}_0) = h_0$
- $\tilde{\mu}$  is  $\pi^*(c)$ -minimizing
- the vector space generated by  $[\tilde{\gamma}_i]$  for  $i = 1, \dots, p+2q$  equals  $\mathcal{VP}_{\tilde{h}_0}$ .

The last equality stands because of Proposition 18. To clear up the notation, call  $V$  the vector subspace of  $H_1(M_o, \mathbb{R})$  generated by the integer classes  $[\tilde{\gamma}_i]$  for  $i = 1, \dots, p+2q$ . Note that  $I(V) = V$ , so  $V = V_1 \oplus V_{-1}$  where  $V_i = E_i \cap V$ ,  $i = \pm 1$ . Also,  $V = \mathcal{VP}_{\tilde{h}_0}$ . We have

$$\begin{aligned} V_1 &= \text{Vect}(\{[\tilde{\gamma}_i] : i = 1, \dots, p\} \cup \{[\tilde{\gamma}_i] + [\tilde{\gamma}_{i+q}] : i = p+1, \dots, p+q\}) \\ V_{-1} &= \text{Vect}(\{[\tilde{\gamma}_i] - [\tilde{\gamma}_{i+q}] : i = p+1, \dots, p+q\}) \end{aligned}$$

We would like to use Theorem 16 with  $\tilde{h}_0$  playing the part of  $h_0$  and some  $h$  in  $E_1 \cap V^\perp$  but not in  $V$ . Observe that

$$\begin{aligned} \dim V_{-1}^\perp &= b_1(M_o) - q \text{ so} \\ \dim V_{-1}^\perp \cap E_1 &\geq b_1(M_o) - q + b_1(M) - b_1(M_0) \\ &= b_1(M) - q > p + q = \dim V_1 \end{aligned}$$

since we assume  $b_1(M) > p + 2q$ . So there exists  $h \in V_{-1}^\perp \cap E_1$  such that  $h \notin V_1$ . Since  $V_1 \subset E_1$ , we have  $E_1 = E_1^\perp \subset V_1^\perp$  thus  $h \in V_1^\perp$  and

$$h \in V_1^\perp \cap V_{-1}^\perp = (V_1 \oplus V_{-1})^\perp = V^\perp.$$

So  $h \in E_1 \cap V^\perp$ . Furthermore  $h \notin V$  since  $h \in E_1$  and  $h \notin V_1 = E_1 \cap V$ .

By Theorem 16 there exists  $s > 0$  such that

$$F_1 := \left\{ \frac{\tilde{h}_0 + sh}{\|\tilde{h}_0 + sh\|} : s \in [0, s(h_0, h)] \right\}$$

is a straight segment contained in  $\mathcal{S}_1(M_o, \tilde{g}) \cap E_1 \cong \mathcal{S}_1(M, g)$ . On the other hand, the convex hull of  $[\gamma_i]$  for  $i = 1, \dots, p + q$  is contained in a flat  $F_0$  of  $\mathcal{S}_1(M, g)$  that contains  $h_0$  in its interior. From Lemma 7 we deduce that there exists a flat containing  $F_0$  and  $F_1$ . The dimension of such a flat is greater than  $p + q = \dim F_0$  because  $h \notin V$ . □

Taking a rational  $h$  in the proof of Proposition 19, we deduce the first part of Theorem **A**.

**Corollary 20.** *Assume  $M$  is a closed non-orientable surface endowed with a Riemannian metric. Then every connected minimizing cycle is a component of a minimizing cycle whose homology class lies in a flat of  $\mathcal{S}_1$  of dimension at least  $[(b_1(M) + 1)/2] - 1$ .*

Let  $\Gamma$  be a minimizing cycle whose connected components are not pairwise proportional in homology. Then its connected components form a system of disjoint non pairwise homologically proportional simple closed curves. We have the following result.

**Proposition 21.** *A system of disjoint non pairwise homologically proportional simple closed curves has its cardinality bounded from above by  $2b_1(M) - 1$ .*

*Proof.* The argument is classical. Let  $\alpha_1, \dots, \alpha_{p+q}$  be a maximal system of disjoint pairwise non homologically proportional simple closed curves of  $M$ . Suppose that  $p$  is the number of one-sided curves of this system and  $q$  the number of two-sided curves. By cutting  $M$  along these simple closed curves we obtain an union of  $b_1(M) - 1$  pair of pants. So we must have  $3(b_1(M) - 1) = p + 2q$ . This implies  $2(p + q) = 3b_1(M) + p - 3$ . As  $p \leq b_1(M) + 1$ , the assertion follows. □

So  $\Gamma$  has at most  $2b_1(M) - 1$  components. The second part of theorem **A** is proved.

By specializing Proposition 19 to the case when  $q = 0$ , we now deduce Theorem **B** which describes the local geometry of the unit ball of the stable norm near a rational homology class for which the connected components of minimizing cycles are curves of type I.

**Corollary 22.** *Assume  $M$  is a closed non-orientable surface endowed with a Riemannian metric. Let  $h_0$  be an integer homology class all of whose minimizing cycles consist of curves of type I. Then for all  $h \in H_1(M, \mathbb{R})$ , there exists  $s(h_0, h) > 0$  such that the subset of the unit sphere  $\mathcal{S}_1$*

$$\left\{ \frac{h_0 + sh}{\|h_0 + sh\|} : s \in [0, s(h_0, h)] \right\}$$

*is a straight segment.*

**6.3. Global result - Proof of Theorem C.** We first prove the following theorem.

**Theorem 23.** *Let  $M$  be a closed surface (orientable or not) endowed with a Riemannian metric  $g$  and  $c_1, \dots, c_l$  a family of disjoint smooth, simple, closed curves whose homology classes are not pairwise proportional (that is  $[c_i] \notin \mathbb{R}[c_j]$  for  $i \neq j$ ).*

*For all sequence  $\{r_i\}_{i=1}^l$  of positive real numbers, there exists a smooth metric  $g^*$  conformal to  $g$  such that the intersection of  $\mathcal{S}_1(g^*)$  with the subspace spanned by the curves  $[c_1], \dots, [c_l]$  coincides with the polyhedron*

$$\text{Conv}_s \left( \frac{[c_1]}{r_1}, \dots, \frac{[c_l]}{r_l} \right)$$

*where  $\text{Conv}_s$  denote the convex hull of the symmetrized of a set.*

*Proof.* Let  $P$  denote the polyhedron generated as the convex hull

$$\text{Conv}_s \left( \frac{[c_1]}{r_1}, \dots, \frac{[c_l]}{r_l} \right).$$

We can suppose that each curve  $[c_i]$  corresponds to an exposed point of the polyhedron (if not we can discard this curve and the polyhedron  $P$  remains unchanged).

**Lemma 24.** *There exists a smooth metric  $\bar{g}$  conformal to  $g$  and an open neighborhood  $V_i$  of each  $c_i$  such that  $c_i$  is the unique closed  $\bar{g}$ -geodesic of  $V_i$  and  $l_{\bar{g}}(c_i) = r_i$ .*

*Proof of Lemma 24.* For  $\epsilon$  small enough, the  $\epsilon$ -tubular neighborhoods  $U_\epsilon(c_i)$  are pairwise disjoint and the  $g$ -orthogonal projections  $p_i : U_\epsilon(c_i) \rightarrow c_i$  are well defined. For each  $x \in U_\epsilon(c_i)$ , there exists two  $g$ -unitary vectors  $\pm v(x) \in T_x M$  orthogonal to the fiber  $p_i^{-1}(p_i(x))$ . The function  $\alpha_i : U_i \rightarrow ]0, \infty[$  given by the formula

$$\alpha_i(x) = g(Dp_i(x)v(x), Dp_i(x)v(x))$$

is smooth and such that  $\alpha_i \circ c_i = 1$ . We define a new metric  $g'$  conformal to  $g$  by  $\alpha_i g$  on  $U_\epsilon(c_i)$  and by extending the local conformal factors  $\{\alpha_i\}_{i=1}^k$  into a smooth positive function  $\alpha$  on  $M$ . We claim that the projections

$p_i : U_\epsilon(c_i) \rightarrow c_i$  do not increase the lengths with respect to  $g'$ . Indeed, take  $x \in U_i$  and  $w \in T_x U_i$ . Write  $w = \lambda v(x) + \mu v'$ , with  $v' \in T_x p_i^{-1}(p_i(x))$ . Note that  $v$  and  $v'$  are  $g$ -orthogonal ; since  $g'$  is conformal to  $g$ ,  $v$  and  $v'$  are  $g'$ -orthogonal. Orthogonal projections do not increase distances, so

$$\lambda^2 g'(v(x), v(x)) = g'(\lambda v(x), \lambda v(x)) \leq g'(w, w).$$

Now

$$g'(Dp_i(x)w, Dp_i(x)w) = g'(Dp_i(x)\lambda v, Dp_i(x)\lambda v)$$

because  $p_i$  is the orthogonal projection to  $c_i$ , whence

$$\begin{aligned} g'(Dp_i(x)w, Dp_i(x)w) &= \lambda^2 \alpha_i(x) \\ &= \lambda^2 \alpha_i(x) g(v, v) \\ &= \lambda^2 g'(v, v) \leq g'(w, w) \end{aligned}$$

which proves the claim.

Choose a function  $f \in \mathcal{C}^\infty(M)$  null on  $\cup_{i=1}^k c_i$ , positive elsewhere and such that  $\Delta(f) < -\text{Scal}_{g'}$  where  $\text{Scal}_{g'}$  denote the scalar curvature of  $g'$ . Let  $g'' := \exp(f)g'$ . We can easily verify that each projection  $p_i : U_\epsilon(c_i) \rightarrow c_i$  now strictly contracts the lengths. So  $c_i$  is a  $g''$ -geodesic and the negativity of  $\text{Scal}_{g''} = \exp(f)\Delta(f) + \text{Scal}_{g'}$  ensures that the orbit  $c_i$  of the geodesic flow associated to  $g''$  is hyperbolic, hence isolated in a neighborhood  $V_i$ .

We extend the functions  $\lambda_i = r_i/l_g(c_i)$  defined on each neighborhood  $V_i$  into a smooth function  $\lambda$  defined on the whole surface and set  $\bar{g} := \lambda^2 g''$ . The lemma is proved.  $\square$

For any sequence  $\bar{\epsilon}$  such that  $\epsilon_i = 0, \pm 1$  we denote by  $\gamma(\bar{\epsilon})$  the multicurve  $\cup_{i=1}^l \epsilon_i \cdot c_i$  of  $\cup_{i=1}^l V_i$  minimizing the length in the class  $\sum_{i=1}^l \epsilon_i [c_i]$ .

**Lemma 25.** *There exists a smooth metric  $g^*$  conformal to  $\bar{g}$  such that for any sequence  $\bar{\epsilon} := \{\epsilon_i\}_{i=1}^l$  with  $\epsilon_i = 0, \pm 1$ ,*

$$\left\| \sum_{i=1}^l \epsilon_i [c_i] \right\|_s^{g^*} = l_{g^*}(\gamma(\bar{\epsilon})).$$

*Proof of Lemma 25.* Set

$$\delta(\bar{\epsilon}) := l_{\bar{g}}(\gamma(\bar{\epsilon})) - \left\| \sum_{i=1}^l \epsilon_i [c_i] \right\|_s^{\bar{g}}.$$

The set  $\Gamma(\bar{\epsilon})$  of unions of closed geodesics different from  $\gamma(\bar{\epsilon})$  homologous to  $\sum_{i=1}^l \epsilon_i [c_i]$  such that their length is bounded from above by  $l_{\bar{g}}(\gamma(\bar{\epsilon}))$  is compact. It is clear that no multicurve  $\gamma$  in  $\Gamma(\bar{\epsilon})$  can be totally contained in  $\cup_{i=1}^l V_i$ . So, if  $\delta(\bar{\epsilon}) > 0$ , the infimum  $t(\bar{\epsilon})$  of time  $t$  such that there exists a multicurve  $\gamma \in \Gamma(\bar{\epsilon})$  spending a time  $t$  outside  $\cup_{i=1}^l V_i$  is reached and not zero.

Since there is but a finite number of sequences  $\bar{\epsilon}$  we may choose  $\beta$  so big that for all  $\bar{\epsilon}$  with  $\delta(\bar{\epsilon}) > 0$

$$\beta > \log \left[ 1 + \frac{\delta(\bar{\epsilon})}{t(\bar{\epsilon})} \right].$$

Now we choose a function  $f' \in C^\infty(M)$  null on  $\cup_{i=1}^l c_i$ , positive elsewhere and such that  $f' > \beta$  outside  $\cup_{i=1}^l V_i$ . For any multicurve  $\gamma$  spending some time  $t$  outside  $\cup_{i=1}^l V_i$ , we have

$$l_{\exp(f')\bar{g}}(\gamma) > (\exp(\beta) - 1)t + l_{\bar{g}}(\gamma).$$

Now let  $g^* = \exp(f')\bar{g}$ . All the lengths except those of the  $c_i$ 's increase for  $g^*$  so for all  $\bar{\epsilon}$  such that  $\delta(\bar{\epsilon}) > 0$  and for all multicurve  $\gamma$  in the class  $\sum_{i=1}^l \epsilon_i [c_i]$ ,

$$l_{g^*}(\gamma) \geq l_{g^*}(\gamma(\bar{\epsilon})).$$

□

Recall that  $l_{g^*}(c_i) = r_i$  as the lengths of the  $c_i$ 's do not increase. Thus each exposed point of  $P$  belongs to the unit sphere  $\mathcal{S}_1(g^*)$  of the stable norm. Furthermore by lemma 25 each face of  $P$  contains an interior point that belongs to  $\mathcal{S}_1(g^*)$ . This proves the theorem by convexity of the unit sphere of the stable norm. □

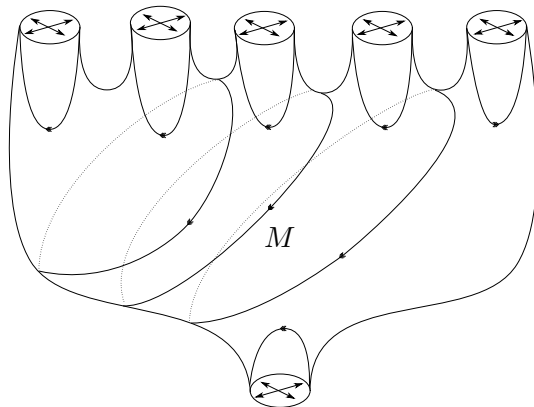


FIGURE 4.

**Corollary 26.** *Let  $M$  be a closed and orientable surface endowed with a Riemannian metric  $g$ . For each isotropic subspace  $L$  of  $H_1(M, \mathbb{R})$  (with respect to  $\text{Int}$ ), there exists a metric  $g^*$  conformal to  $g$  such that the restriction of  $\mathcal{S}_1(g^*)$  to  $L$  is a polyhedra with rational vertices.*

*Proof.* There exists a system of disjoint, smooth, simple and non-pairwise homotopic closed curves  $c_1, \dots, c_l$  that span  $L$ . We apply Theorem 23 to obtain the claim. □

**Remark.** It is a classical result that such a system has cardinality at most  $(3/2)b_1(M) - 3$  (same argument as in the proof of Proposition 21), thus this bounds the number of vertices of the polyhedra obtained that way by  $3b_1(M) - 6$ .

We now deduce, as a corollary of Theorem 23, Theorem C as stated in the introduction :

**Corollary 27.** *Let  $M$  be a closed and non-orientable surface endowed with a Riemannian metric  $g$ . There exists a metric  $g^*$  conformal to  $g$  such that  $\mathcal{S}_1(g^*)$  is a polyhedra with rational vertices.*

*Proof.* There exists a system  $c_1, \dots, c_l$  of smooth, simple, closed curves such that  $c_i \cap c_j = \emptyset$ ,  $[c_i] \notin \mathbb{R}[c_j]$  for  $i \neq j$  and  $H_1(M, \mathbb{R}) = \text{Vect}([c_1], \dots, [c_l])$  (see figure 4 for an example of such a system with  $l = 2b_1(M) - 1$ ). We apply Theorem 23 to obtain the claim.  $\square$

**Remark.** Such a system has cardinality at most  $2b_1(M) - 1$  (proposition 21), thus this bounds the number of vertices of the polyhedra obtained that way by  $4b_1(M) - 2$ .

**Acknowledgements.** The authors are grateful to Ivan Babenko for several helpful conversations.

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